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On the rate of convergence of semigroups of holomorphic functions at the Denjoy–Wolff point

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Abstract. Let $\{\phi_t\}$ be a semigroup of holomorphic self-maps of the unit disc $\mathbb{D}$ with Denjoy–Wolff point $\tau \in \partial\mathbb{D}$. We study the rate of convergence of the semigroup to τ, that is, given $z \in \overline{\mathbb{D}}$, we discuss the behavior of $|\phi_t(z) - \tau|$ as t goes to $+\infty$.

1. Introduction

One-parameter continuous semigroups of holomorphic self-maps of $\mathbb{D}$ –for short, semigroups in $\mathbb{D}$– have been widely studied, see, e.g., [1], [3], [23], [25]. In particular, the behavior of semigroups at the boundary from a dynamical point of view, with special attention to boundary regular fixed points, is a subject that has been addressed in a number of recent papers [4],[6], [7], [13], [16]. The semigroup version of the Denjoy–Wolff theorem [3] guarantees that if the semigroup is non-elliptic, then there is a point $\tau \in \partial\mathbb{D}$ such that $\phi_t(z)$ converges to τ, as t goes to $+\infty$, for all $z \in \mathbb{D}$. The aim of this paper is to study the rate of such convergence. This problem has been studied extensively during the last two decades. We refer to the books [23], Chapter 4, [13], Chapter 7, and the surveys [17], [11], Section 6.4, for various related results; see also [14], [4] and references therein.

The rate of convergence for hyperbolic semigroups is the aim of Section 4 of this paper. Namely, we show that

$$\lim_{t\to+\infty} \frac{1}{t} \log\left(1 - \overline{\tau}\,\phi_t(z)\right) = -\lambda,$$

for all $z \in \mathbb{D}$, where $\lambda > 0$ is the spectral value of the semigroup (Proposition 4.1).

The main result of Section 4 shows that for some hyperbolic semigroups it is possible to get a tighter rate of convergence to the Denjoy–Wolff point, namely, in

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Theorem 4.2 we characterize hyperbolic semigroups for which, for each $z \in \mathbb{D}$, the limit

$$\lim_{t\to+\infty} e^{\lambda t}(1-\overline{\tau}\phi_t(z)) \in \mathbb{C}\setminus\{0\}.$$

We end this section with an example (see Example 4.5) where

$$\lim_{t\to+\infty} e^{\lambda t}(1-\overline{\tau}\phi_t(z)) = 0, \quad z\in\mathbb{D}.$$

Section 5 is devoted to parabolic semigroups. Theorem 5.3 shows that if the range of the Koenigs function h of the parabolic semigroup (ϕ_t) is contained in a sector $S_p(\alpha,\beta)$ where $0 \le \alpha,\beta \le 1$ with $\alpha+\beta>0$ (see Section 3 for the notation) then, for every $z\in\mathbb{D}$, there exists a positive constant C such that, for all $t>0$,

$$|\phi_t(z)-\tau| \le \frac{C}{t^{1/(\alpha+\beta)}},$$

where τ is the Denjoy–Wolff point of the semigroup. The particular cases when $\alpha,\beta\in\{0,1\}$ were obtained in [4].

Conversely, if $h(\mathbb{D})$ contains a sector $S_p(\alpha,\beta)$, with $\min\{\alpha,\beta\}>0$, then, for every $z\in\mathbb{D}$, there exist $C>0$ and $T>0$ such that

$$|\phi_t(z)-\tau| \ge \frac{C}{t^{1/(\alpha+\beta)}},$$

for all $t>T$ (see Theorem 5.6).

We provide examples showing that these results are sharp.

A remarkable result due to Gumenyuk (Proposition 3.2 in [16]) shows that for every $z\in\partial\mathbb{D}$, the function

$$[0,+\infty)\ni t\mapsto \phi_t(z)$$

is continuous and we can also analyze the rate of convergence to the Denjoy–Wolff point when $\phi_t(z)$ remains on the boundary of the unit disc and it converges to the Denjoy–Wolff point, as $t\to+\infty$. This behavior is the subject matter of Theorem 6.1.

The main tool in the proofs of Theorems 5.3 and 6.1 is harmonic measure theory. In Section 3 we review some basic facts about harmonic measure and we prove some results needed throughout the paper.

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2. Notation and preliminaries

Fixed points. For the unproven statements, we refer the reader to, e.g., [1], [10] or [22]. Let $f\colon\mathbb{D}\to\mathbb{D}$ be holomorphic, $x\in\partial\mathbb{D}$, and let

$$\alpha_f(x) := \liminf_{z\to x}\frac{1-|f(z)|}{1-|z|}.$$

It is well known that $\alpha_f(x) > 0$. The number $\alpha_f(x)$ is called the *boundary dilatation coefficient* of f. Julia's lemma shows that if $\alpha_f(x) < +\infty$, then there exists a unique $\eta \in \partial\mathbb{D}$ such that for all $R > 0$ it holds

$$f(E(x, R)) \subseteq E(\eta, \alpha_f(x)R),$$

where $E(y, R) := \{z \in \mathbb{D} : \frac{|y-z|^2}{1-|z|^2} < R\}$ is the *horocycle* of *center* $y \in \partial\mathbb{D}$ and (*hyperbolic*) *radius* $R > 0$. We write $\angle\lim_{z\to x} f(z)$ for the non-tangential (or angular) limit of f at x. If this limit exists and no confusion arises, we denote its value simply by $f(x)$.

Definition 2.1. Let $f\colon \mathbb{D} \to \mathbb{D}$ be holomorphic. A point $x \in \partial\mathbb{D}$ is said to be a *boundary fixed point* of f if $f(x) = x$. If in addition

$$\alpha_f(x) < +\infty, \tag{2.1}$$

then x is called a *boundary regular fixed point of f.*

If $f\colon \mathbb{D} \to \mathbb{D}$ is holomorphic, neither the identity nor an elliptic automorphism, by the Denjoy–Wolff theorem, there exists a unique point $x \in \overline{\mathbb{D}}$, called the *Denjoy–Wolff point* of f, such that $f(x) = x$ and the sequence of iterates (f_n) converges uniformly on compacta of $\mathbb{D}$ to the constant map $z \mapsto x$. Moreover, if $x \in \partial\mathbb{D}$ then x is a boundary regular fixed point of f and $\alpha_f(x) \in (0, 1]$. If, in addition, $\alpha_f(x) = 1$, then f is called *parabolic*.

Semigroups. A (one-parameter) semigroup (ϕ_t) of holomorphic self-maps of $\mathbb{D}$ (for short, semigroups in $\mathbb{D}$) is a continuous homomorphism $t \mapsto \phi_t$ from the additive semigroup of non-negative real numbers to the semigroup of holomorphic self-maps of $\mathbb{D}$ with respect to composition, endowed with the topology of uniform convergence on compacta.

It is known that if (ϕ_t) is a semigroup in $\mathbb{D}$ and ϕ_{t_0} is an automorphism of $\mathbb{D}$ for some $t_0 > 0$, then (ϕ_t) can be extended to a group in $\mathsf{Aut}(\mathbb{D})$.

Definition 2.2. A *boundary* (*regular*) *fixed point* for a semigroup (ϕ_t) in $\mathbb{D}$ is a point $x \in \partial\mathbb{D}$ which is a boundary (regular) fixed point of ϕ_t for all $t > 0$.

Let (ϕ_t) be a semigroup in $\mathbb{D}$. It is well known that ϕ_{t_0} has a fixed point in $\mathbb{D}$ for some $t_0 > 0$ if and only if there exists $x \in \mathbb{D}$ such that $\phi_t(x) = x$ for all $t \geq 0$. In such a case, the semigroup is called *elliptic*. Moreover, there exists $\lambda \in \mathbb{C}$ with $\mathsf{Re}\,\lambda \geq 0$ such that $\phi_t'(x) = e^{-\lambda t}$ for all $t \geq 0$. The elliptic semigroup (ϕ_t) is a group if and only if $\mathsf{Re}\,\lambda = 0$. The number λ is called the *spectral value* of the elliptic semigroup.

If the semigroup (ϕ_t) is not elliptic, then there exists a unique $x \in \partial\mathbb{D}$ which is the Denjoy–Wolff point of ϕ_t for all $t > 0$. Moreover, there exists $\lambda \geq 0$, the *spectral value of* (ϕ_t), such that

$$\alpha_{\phi_t}(x) = e^{-\lambda t}, \quad t \geq 0.$$

A non-elliptic semigroup is said *hyperbolic* if its spectral value is non-zero, while it is said *parabolic* if the spectral value is 0.

Parabolic semigroups can be divided in two sub-types: a parabolic holomorphic semigroup in $\mathbb{D}$ is *of positive hyperbolic step* if $\lim_{t\to+\infty} k_{\mathbb{D}}(\phi_{t+1}(0), \phi_t(0)) > 0$ (here $k_{\mathbb{D}}(z, w)$ is the hyperbolic distance in $\mathbb{D}$ between $z \in \mathbb{D}$ and $w \in \mathbb{D}$). Otherwise, it is called *of zero hyperbolic step*.

By Berkson and Porta's theorem (Theorem (1.1) in [3]), if (ϕ_t) is a semigroup in $\mathbb{D}$, then $t \mapsto \phi_t(z)$ is real-analytic and there exists a unique holomorphic vector field $G\colon \mathbb{D} \to \mathbb{C}$ such that

$$\frac{\partial \phi_t(z)}{\partial t} = G(\phi_t(z)) = G(z)\,\phi_t'(z), \quad \text{for all } z \in \mathbb{D} \text{ and all } t \geq 0.$$

This vector field G, called the *infinitesimal generator* of (ϕ_t), is *semicomplete* in the sense that the Cauchy problem

$$\begin{cases} dx(t)/dt = G(x(t)), \\ x(0) = z, \end{cases}$$

has a solution $x^z\colon [0, +\infty) \to \mathbb{D}$ for every $z \in \mathbb{D}$. Conversely, any semicomplete holomorphic vector field in $\mathbb{D}$ generates a semigroup in $\mathbb{D}$.

A deep result due to Gumenyuk [16] guarantees that there exists the non-tangential limit at every $\sigma \in \partial\mathbb{D}$ for the member functions of a semigroup:

Theorem 2.3. *Let (ϕ_t) be a semigroup in $\mathbb{D}$. Then for any $t \geq 0$ and any $\sigma \in \partial\mathbb{D}$ there exists the non-tangential limit $\phi_t(\sigma) := \angle\lim_{z\to\sigma} \phi_t(z)$. Moreover, for each $\sigma \in \partial\mathbb{D}$ and each Stolz region S of vertex σ, the convergence $\phi_t(z) \to \phi_t(\sigma)$ as $S \ni z \to \sigma$ is locally uniform in $t \in [0, +\infty)$, i.e., for every $\epsilon > 0$ and $T > 0$ there exists $\delta > 0$ such that*

$$|\phi_t(z) - \phi_t(\sigma)| < \epsilon,$$

for all $t \in [0, T]$ and $z \in S$ such that $|z - \sigma| < \delta$.

Holomorphic models. A key notion associated to each semigroup is its model and its Koenigs function. The idea of the model appeared already in [3]. It was further developed in [9], [25], and recently in [2].

Definition 2.4. Let (ϕ_t) be a semigroup in $\mathbb{D}$. A (*holomorphic*) *model* for (ϕ_t) is a triple (Ω, h, Φ_t) such that Ω is an open subset of $\mathbb{C}$, Φ_t is a group of (holomorphic) automorphisms of Ω and $h\colon \mathbb{D} \to h(\mathbb{D}) \subset \Omega$ is univalent on the image, $h\circ\phi_t = \Phi_t\circ h$ and

$$\bigcup_{t\geq 0} \Phi_t^{-1}(h(\mathbb{D})) = \Omega. \tag{2.2}$$

Note that the domain $h(\mathbb{D})$ possesses the remarkable property that $\Phi_t(h(\mathbb{D})) \subset h(\mathbb{D})$ for all $t \geq 0$.

It was proved in [2] that every semigroup of holomorphic self-maps of any complex manifold admits a holomorphic model, unique up to "holomorphic equivalence". Moreover, a model is "universal" in the sense that every other conjugation of the semigroup to a group of automorphisms factorize through the model (see Section 6 in [2] for more details).

Notice that given a model (Ω, h, Φ_t) for a semigroup (ϕ_t) in $\mathbb{D}$, (ϕ_t) is a group if and only if $h(\mathbb{D}) = \Omega$.

In what follows we denote by $\Pi^+ := \{\zeta \in \mathbb{C} : \mathsf{Im}\,\zeta > 0\}$, $\Pi^- := \{\zeta \in \mathbb{C} : \mathsf{Im}\,\zeta < 0\}$ and, given $\rho > 0$, $\mathbb{S}_\rho := \{\zeta \in \mathbb{C} : 0 < \mathsf{Im}\,\zeta < \rho\}$. We simply write $\mathbb{S} := \mathbb{S}_1$. The following result sums up the results in [2], [9], [25], see also [1] and references therein.

Theorem 2.5. *Let (ϕ_t) be a semigroup in $\mathbb{D}$. Then:*

(1) *(ϕ_t) is the trivial semigroup if and only if (ϕ_t) has a holomorphic model $(\mathbb{D}, \mathsf{id}_{\mathbb{D}}, z \mapsto z)$.*

(2) *(ϕ_t) is a group of elliptic automorphisms with spectral value $i\theta$, for $\theta \in \mathbb{R}\setminus\{0\}$, if and only if (ϕ_t) has a holomorphic model $(\mathbb{D}, h, z \mapsto e^{-i\theta t} z)$.*

(3) *(ϕ_t) is elliptic, not a group, with spectral value λ, for $\lambda \in \mathbb{C}$ with $\mathsf{Re}\,\lambda > 0$, if and only if (ϕ_t) has a holomorphic model $(\mathbb{C}, h, z \mapsto e^{-\lambda t} z)$.*

(4) *(ϕ_t) is hyperbolic with spectral value $\lambda > 0$ if and only if it has a holomorphic model $(\mathbb{S}_{\frac{\pi}{\lambda}}, h, z \mapsto z + t)$.*

(5) *(ϕ_t) is parabolic of positive hyperbolic step if and only if it has a holomorphic model either of the form $(\Pi^+, h, z \mapsto z + t)$ or of the form $(\Pi^-, h, z \mapsto z + t)$.*

(6) *(ϕ_t) is parabolic of zero hyperbolic step if and only if it has a holomorphic model $(\mathbb{C}, h, z \mapsto z + t)$.*

The function h in the previous model is called the *Koenigs function* of the semigroup. All the previous models are holomorphically non-equivalent.

Let us recall that a domain $\Omega \subset \mathbb{C}$ is call *starlike at infinity* if $\Omega + t \subset \Omega$ for all $t \geq 0$. Notice that if h is the Koenigs function of a non-elliptic semigroup, then $h(\mathbb{D})$ is starlike at infinity.

In any of the non-elliptic cases, the model is given by $(\Omega = \mathbb{R} \times I, h, z + t)$, where I is any of the intervals $(-\infty, 0)$, $(0, +\infty)$, $(0, \rho)$, for some $\rho > 0$, or $\mathbb{R}$. This interval I is a holomorphic invariant. In particular, in the hyperbolic case, ρ depends on the spectral value λ of the semigroup.

Hyperbolic distance. Given a hyperbolic planar domain Ω (for example, a simply connected domain), we denote by $k_\Omega(z, w)$, $z, w \in \Omega$, the *hyperbolic distance* in Ω. It is well known that

$$k_{\mathbb{D}}(z, w) = \frac{1}{2} \log \frac{1 + |T_w(z)|}{1 - |T_w(z)|}, \quad z, w \in \mathbb{D}, \tag{2.3}$$

where T_w is the automorphism of the unit disc given by $T_w(z) := (w - z)/(1 - \overline{w} z)$. We will also use

$$k_{\mathbb{H}}(w_1, w_2) = \frac{1}{2} \log \frac{1 + \left|\frac{w_1 - w_2}{w_1 + \overline{w_2}}\right|}{1 - \left|\frac{w_1 - w_2}{w_1 + \overline{w_2}}\right|}, \quad w_1, w_2 \in \mathbb{H}, \tag{2.4}$$

where $\mathbb{H} := \{z \in \mathbb{C} : \mathsf{Re}\, z > 0\}$.

Given a holomorphic self-map of the unit disc f and a point $x \in \partial\mathbb{D}$, it is possible to prove (see Proposition 1.2.2, Theorem 1.2.5, and Proposition 1.2.6 in [1]) that

$$\frac{1}{2}\log\alpha_f(x) = \liminf_{w\to x}\,[k_{\mathbb{D}}(0,w) - k_{\mathbb{D}}(0,f(w))] \in (0,+\infty]. \tag{2.5}$$

3. Estimates for harmonic measure

We will need some properties of harmonic measures. We first give the definition; see e.g. Section 3 in Chapter 4 of [20]. Suppose that Ω is a domain in the complex plane with non-polar boundary and E is a Borel subset of $\partial_\infty\Omega$. The harmonic measure of E relative to Ω is the generalized Perron–Wiener solution u of the Dirichlet problem for the Laplacian in Ω with boundary values 1 on E and 0 on $\partial_\infty\Omega\setminus E$. We will use the standard notation

$$u(z) = \omega(z,E,\Omega), \quad z\in\Omega.$$

The boundary of a simply connected domain Ω contains a continuum. Since every continuum is a non-polar set ([20], Corollary 3.8.5), harmonic measures are defined for Ω.

For the sake of clearness and easy reference, we state Harnack's inequality (see Theorem 1.3.1 in Chapter 1 of [20]) for harmonic measures in the context of general simply connected domains with nice boundary.

Theorem 3.1. *Let $\Omega \subsetneq \mathbb{C}$ be a simply connected domain such that $\partial_\infty\Omega$ is locally connected. Let E be a Borel subset of $\partial_\infty\Omega$. Then for every $z_1, z_2 \in \Omega$,*

$$\omega(z_1,E,\Omega) \le e^{2k_\Omega(z_1,z_2)}\,\omega(z_2,E,\Omega).$$

Another basic property of harmonic measure is its conformal invariance (Theorem 4.3.8 in [20]). Using this property, one can compute explicitly some harmonic measures on simple domains such as angular sectors.

For $0\le\alpha,\beta\le 1$ with $\alpha+\beta>0$ and $p\in\mathbb{C}$, consider

$$S_p(\alpha,\beta) := \{p + re^{i\theta} : r>0,\ -\alpha\pi<\theta<\beta\pi\}.$$

This family of angular sectors is exactly formed by those (open) angular sectors of the complex plane which are starlike at infinity. If $0<\alpha\le 1$ and $p\in\mathbb{C}$, we denote $S_p(\alpha) := S_p(\alpha,\alpha)$.

Proposition 3.2. *Let $0<t_0<t$ and $0<\alpha\le 1$. Then*

$$\omega(t_0,[t,+\infty),S_0(\alpha)\setminus[t,+\infty)) \le t_0^{1/(2\alpha)}\,t^{-1/(2\alpha)}.$$

Proof. Let J be the Joukowsky function which maps conformally the first quadrant of the plane onto $\mathbb{H}\setminus[1,+\infty)$. Hence, if we precompose J with the principal branch of the square root, we have a conformal map g which sends the upper half plane Π^+ onto $\mathbb{H}\setminus[1,+\infty)$.

Moreover, $g((0,+\infty)) = [1,+\infty)$ and $g((-\infty,0))$ is exactly the imaginary axis. On the other hand, using the principal branch of the logarithm, the power $z \mapsto z^{1/(2\alpha)}$ maps conformally $S_0(\alpha)$ onto $\mathbb{H}$. Then, by conformal invariance of harmonic measure,

$$\begin{aligned}\omega(t_0,[t,+\infty),S_0(\alpha)\setminus[t,+\infty)) &= \omega(t_0^{1/(2\alpha)},[t^{1/(2\alpha)},+\infty),\mathbb{H}\setminus[t^{1/(2\alpha)},+\infty))\\ &= \omega((t_0/t)^{1/(2\alpha)},[1,+\infty),\mathbb{H}\setminus[1,+\infty))\\ &= \omega(g^{-1}((t_0/t)^{1/(2\alpha)}),(0,+\infty),\Pi^+).\end{aligned}$$

Note that $x := (t_0/t)^{1/(2\alpha)} \in (0,1)$. Thus, $J^{-1}(x) = \exp(i\arccos(x))$. Hence, applying Theorem 4.3.13 in [20],

$$\begin{aligned}\omega(t_0,[t,+\infty),S_0(\alpha)\setminus[t,+\infty)) &= \omega(e^{i\,2\arccos(x)},(0,+\infty),\Pi^+)\\ &= 1-\frac{1}{\pi}\operatorname{Arg}(e^{i\,2\arccos(x)}) = 1-\frac{2}{\pi}\arccos(x)\\ &= \frac{2}{\pi}\arcsin(x) \le x.\end{aligned}$$ □

We will also need some variants of the above proposition when the slit is vertically translated or rotated. The proof of these forthcoming results use the method of polarization (Theorem 2 in [26]), a very useful and deep tool for the proof of geometric estimates for harmonic measure. Namely, one of the consequences of this method is the following fact:

Theorem 3.3. *Let ℓ be a line of the complex plane and denote by Π_1 and Π_2 the two (open) half-planes determined by ℓ. Moreover, denote by $\mathcal{R}$ the reflection with respect to ℓ.*

Let Ω be a simply connected domain of the complex plane with non-polar boundary and set $\Omega_1 := \Omega\cap\Pi_1$ and $\Omega_2 := \Omega\cap\Pi_2$. Assume that $\mathcal{R}\Omega_1\subset\Omega_2$. If E is a Borel subset of $\partial_\infty\Omega$ which is included in Π_1 and $\mathcal{R}E\subset\partial_\infty\Omega$, then

$$\omega(z,E,\Omega)\le\omega(z,\mathcal{R}E,\Omega),\quad \textit{for every } z\in\Omega_2. \tag{3.1}$$

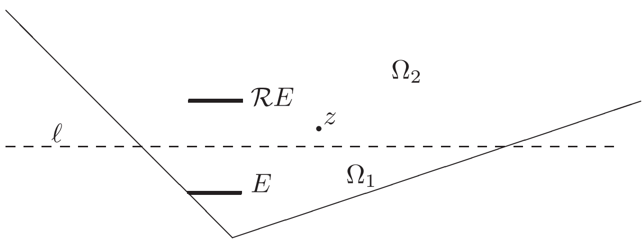

FIGURE 1. An illustration of Theorem 3.3.

Proposition 3.4. *Let $0<\alpha<1$, $a\in\mathbb{R}$ different from zero and $0<t_0$. Then, there exist $C=C(\alpha,a,t_0)>0$ and $T=T(\alpha,a,t_0)>0$ such that, for every $t>T$ the subset*

$$A_{t,a}:=\{s+ia : s\geq t\}.$$

is included in $S_0(\alpha)$ and

$$\omega(t_0,A_{t,a},S_0(\alpha)\setminus A_{t,a})\leq C\,t^{-1/(2\alpha)}.$$

Proof. Fix $0<\alpha<1$, $0<t_0$, and without lose of generality, assume $a>0$. Consider

$$T:=\begin{cases}\max\{t_0,|a|/\tan(\alpha\pi)\}, & 0<\alpha<1/2,\\ t_0, & 1/2\leq\alpha<1,\end{cases}$$

which is a positive constant depending on t_0,a,α. Moreover, for every $t\geq T$, we have

$$A_{t,a}\subset S_0(\alpha).$$

At this point, we distinguish two cases. In both of them, we denote $A_t:=[t,+\infty)$ for $t>0$.

Case (1), $0<\alpha\leq 1/2$.

Let $\ell:=\{s+ia/2 : s\in\mathbb{R}\}$ and, for $t\geq T$, consider $S_1:=\Pi_1\cap S_0(\alpha)\setminus(A_{t,a}\cup A_t)$ (resp. $S_2:=\Pi_2\cap S_0(\alpha)\setminus(A_{t,a}\cup A_t)$), where Π_1 (resp. Π_2) is the open upper (resp. lower) half-plane with respect to ℓ. Note $t_0\in S_2$. Moreover, if $\mathcal{R}$ is the reflection with respect to ℓ, then $\mathcal{R}(S_1)\subset S_2$, $A_{t,a}\subset\Pi_1$ and $\mathcal{R}(A_{t,a})=A_t$. Therefore, by the maximum principle, Theorem 3.3 (with respect to ℓ, S_1 and S_2), the domain monotonicity of harmonic measure and Proposition 3.2, we have that, for $t\geq T$,

$$\begin{aligned}\omega(t_0,&A_{t,a},S_0(\alpha)\setminus A_{t,a})\\ &\leq\omega(t_0,A_{t,a},S_0(\alpha)\setminus(A_{t,a}\cup A_t))+\omega(t_0,A_t,S_0(\alpha)\setminus(A_{t,a}\cup A_t))\\ &\leq 2\omega(t_0,A_t,S_0(\alpha)\setminus(A_{t,a}\cup A_t))\leq 2\omega(t_0,A_t,S_0(\alpha)\setminus A_t)\leq 2\,t_0^{1/(2\alpha)}\,t^{-1/(2\alpha)}.\end{aligned}$$

Case (2), $1/2<\alpha<1$.

Let $L_1:=\{s+ia : s\in\mathbb{R}\}$. Since $\alpha<1$, the line L_1 cuts the half-line $\{\rho e^{i\alpha\pi} : \rho>0\}$ at a certain point p. Indeed, $p=x_1+ia$, where $x_1=x_1(\alpha,a)<0$. Using the conformal invariance and the domain monotonicity of harmonic measure, we have, for $t\geq T$,

$$\begin{aligned}\omega(t_0,A_{t,a},S_0(\alpha)\setminus A_{t,a})&\leq\omega(t_0,A_{t,a},S_p(\alpha)\setminus A_{t,a})\\ &\leq\omega(t_0-p,A_{t,a}-p,S_0(\alpha)\setminus(A_{t,a}-p)).\end{aligned}$$

Note $A_{t,a}-p=[t-x_1,+\infty)$ and $A_t-p=\{s-ia : s\geq t-x_1\}$. For $t\geq T$, denote $S:=S_0(\alpha)\setminus((A_{t,a}-p)\cup(A_t-p)))$. Moreover, let $\ell:=\{s-ia/2 : s\in\mathbb{R}\}$ and $S_1:=\Pi_1\cap S$ (resp. $S_2:=\Pi_2\cap S$), where Π_1 (resp. Π_2) is the open upper (resp. lower) half-plane with respect to ℓ. Note $t_0-p\in S_2$. Moreover, if $\mathcal{R}$ is the reflection with respect to ℓ, then $\mathcal{R}(S_1)\subset S_2$, $A_{t,a}-p\subset\Pi_1$ and $\mathcal{R}(A_{t,a}-p)=A_t-p$.

Therefore, by the maximum principle, Theorem 3.3 (with respect to ℓ, S_1 and S_2) and the domain monotonicity of harmonic measure, we have that, for $t \geq T$,

$$\begin{aligned}\omega(t_0, A_{t,a}, S_0(\alpha) \setminus A_{t,a}) &\leq \omega(t_0 - p, A_{t,a} - p, S) + \omega(t_0 - p, A_t - p, S) \\ &\leq 2\omega(t_0 - p, A_t - p, S) \leq 2\omega(t_0 - p, A_t - p, S_0(\alpha) \setminus (A_t - p)).\end{aligned}$$

Now, let $L_2 := \{s - ia : s \in \mathbb{R}\}$. Since $\alpha < 1$, the line L_2 cuts the half-line $\{\rho e^{-i\alpha\pi} : \rho > 0\}$ at the point $q = x_1 - ia$. Using that x_1 is negative, the conformal invariance and the domain monotonicity of harmonic measure and Proposition 3.2, we have that, for $t \geq T$,

$$\begin{aligned}\omega(t_0, A_{t,a}, S_0(\alpha)\setminus A_{t,a}) &\leq 2\omega(t_0 - p, A_t - p, S_q(\alpha) \setminus (A_t - p)) \\ &= 2\omega(t_0 - (p+q), A_t - (p+q), S_0(\alpha) \setminus (A_t - (p+q))) \\ &= 2\omega(t_0 - 2x_1, [t - 2x_1, +\infty), S_0(\alpha) \setminus [t - 2x_1, +\infty)) \\ &\leq 2\,(t_0 - 2x_1)^{1/(2\alpha)} \left(\frac{1}{t - 2x_1}\right)^{1/(2\alpha)} \leq 2\,(t_0 - 2x_1)^{1/(2\alpha)}\, t^{-1/(2\alpha)}.\end{aligned}$$

Since the quantity $(t_0 - 2x_1)^{1/(2\alpha)}$ depends only on t_0, a and α, the proof is done. □

Proposition 3.5. *Let $0 < \alpha \leq 1$, $0 \leq |\delta| \leq \alpha$ and $0 < t_0 < t_1$. Then, for every $t \geq 0$, the set*

$$A_{t,t_1,\delta} := \{t_1 + s e^{i\delta\pi} : s \geq t\}$$

is included in $S_0(\alpha)$ and there exist $C = C(\alpha, \delta, t_0, t_1) > 0$ and $T = T(\alpha, \delta, t_0, t_1) > 0$ such that, for every $t > T$,

$$\omega(t_0, A_{t,t_1,\delta}, S_0(\alpha) \setminus A_{t,t_1,\delta}) \leq C t^{-1/(2\alpha)}.$$

Proof. The case $\delta = 0$ is included in Proposition 3.2. Fix $0 < \alpha < 1$ and $0 < t_0 < t_1$ and, without loss of generality, assume $\delta > 0$. Since $0 < \delta \leq \alpha$, we have $A_{t,t_1,\delta} \subset S_0(\alpha)$, for $t \geq 0$. We distinguish three cases depending on the value of α and, in all cases, we denote $A_t := [t, +\infty)$ for $t > 0$.

Case (1), $0 < \alpha \leq 1/2$.

Consider the lines $L := \{r e^{i\alpha\pi} : r \in \mathbb{R}\}$ and $\ell_1 := \{t_1 + r e^{i\frac{\delta}{2}\pi} : r \in \mathbb{R}\}$. Since $\delta/2 < \alpha$, L intersects ℓ_1 at a certain point $p = x + iy$. Note $x = x(t_1, \alpha, \delta)$ and $y = y(t_1, \alpha, \delta)$ are negative.

Using the conformal invariance and the domain monotonicity of harmonic measure, we have, for $t > 0$,

$$\begin{aligned}\omega(t_0, A_{t,t_1,\delta}, S_0(\alpha) \setminus A_{t,t_1,\delta}) &\leq \omega(t_0, A_{t,t_1,\delta}, S_p(\alpha) \setminus A_{t,t_1,\delta}) \\ &\leq \omega(t_0 - p, A_{t,t_1,\delta} - p, S_0(\alpha) \setminus (A_{t,t_1,\delta} - p)).\end{aligned}$$

For $t \geq 1 + t_1$, denote $S := S_0(\alpha) \setminus ((A_{t,t_1,\delta} - p) \cup (A_t - p)))$. Moreover, let $\ell := \ell_1 - p$ and $S_1 := \Pi_1 \cap S$ (resp. $S_2 := \Pi_2 \cap S$), where Π_1 (resp. Π_2) is the open upper (resp. lower) half-plane with respect to the line ℓ, that is, the half-plane which includes the points it (resp. $-it$) for t large enough. Note $t_0 - p \in S_1$ but $t_0 - x \in S_2$. Moreover, consider the domain

$$\widetilde{S} := S_0(\alpha) \setminus (A_{1+t_1,t_1,\delta} - p).$$

Since $t_0 - p, t_0 - x \in \widetilde{S}$, we can define

$$C_t := \exp\left(2\, k_{S_0(\alpha)\setminus((A_{t,t_1,\delta}-p)}(t_0 - p, t_0 - x)\right), \quad t \geq 1 + t_1,$$

and $\widetilde{C} := \exp\left(2\, k_{\widetilde{S}}(t_0 - p, t_0 - x)\right)$. Moreover, these constants are well-defined and positive, $C_t \leq \widetilde{C}$ and $\widetilde{C}$ depends only on t_0, t_1, α and δ. On the other hand, if $\mathcal{R}$ is the reflection with respect to the line ℓ, then $\mathcal{R}(S_1) \subset S_2$, $A_{t,t_1,\delta} - p \subset \Pi_1$ and $\mathcal{R}(A_{t,t_1,\delta} - p) = A_t - p$. Therefore, by Theorem 3.1, the maximum principle, Theorem 3.3 (with respect to ℓ, S_1 and S_2) and the domain monotonicity of harmonic measure, we have that, for $t \geq 1 + t_1$,

$$\begin{aligned}
\omega(t_0, A_{t,t_1,\delta}, S_0(\alpha) \setminus A_{t,t_1,\delta}) &\leq \omega(t_0 - p, A_{t,t_1,\delta} - p, S_0(\alpha) \setminus (A_{t,t_1,\delta} - p)) \\
&\leq C_t\, \omega(t_0 - x, A_{t,t_1,\delta} - p, S_0(\alpha) \setminus (A_{t,t_1,\delta} - p)) \\
&\leq \widetilde{C}\, \omega(t_0 - x, A_{t,t_1,\delta} - p, S_0(\alpha) \setminus (A_{t,t_1,\delta} - p)) \\
&\leq \widetilde{C}\, (\omega(t_0 - x, A_{t,t_1,\delta} - p, S) + \omega(t_0 - x, A_t - p, S)) \\
&\leq 2\widetilde{C}\, \omega(t_0 - x, A_t - p, S) \\
&\leq 2\widetilde{C}\, \omega(t_0 - x, A_t - p, S_0(\alpha) \setminus (A_t - p)).
\end{aligned}$$

Note $A_t - p = A_{t-x,-y} = \{s - yi : s \geq t - x\}$. Bearing in mind Proposition 3.4 (recall $y \neq 0$), there exist $\widehat{C} = \widehat{C}(\alpha, y, t_0) > 0$ and $\widetilde{T} = \widetilde{T}(\alpha, y, t_0) > 0$ such that, for all $t \geq T := \max\{\widetilde{T} + x, 1 + t_1\}$,

$$\omega(t_0, A_{t,t_1,\delta}, S_0(\alpha) \setminus A_{t,t_1,\delta}) \leq 2\widetilde{C}\, \widehat{C}\, t^{-1/(2\alpha)}.$$

Since the numbers T and $C := 2\widetilde{C}\widehat{C}$ depend only on t_0, t_1, α, δ, the proof of this case is done.

Case (2). Either $\alpha = 1$ or $1/2 < \alpha < 1$ and $1 - \alpha \leq \delta/2$-

Consider the half-line

$$L_1 := \{re^{-i\alpha\pi} = -re^{i(1-\alpha)\pi} : r \geq 0\}$$

and the line $\ell := \{t_1 + re^{i\,\delta\pi/2} : r \in \mathbb{R}\}$. Under our two different hypothesis, L_1 does not cut ℓ. For $t \geq 1 + t_1$, denote $S := S_0(\alpha) \setminus ((A_{t,t_1,\delta} \cup A_t))$ and let $S_1 := \Pi_1 \cap S$ (resp. $S_2 := \Pi_2 \cap S$), where Π_1 (resp. Π_2) is the open upper (resp. lower) half-plane with respect to the line ℓ, that is, the half-plane which includes the points it (resp. $-it$) for t large enough. If $\mathcal{R}$ is the reflection with respect to the line ℓ, then $\mathcal{R}(S_1) \subset S_2$, $A_{t,t_1,\delta} \subset \Pi_1$ and $\mathcal{R}(A_{t,t_1,\delta}) = A_t$. Note $t_0 \in S_1$ but $t_1 + 1/2 \in S_2$ whenever $t \geq 1 + t_1$. Moreover, consider the domain

$$\widetilde{S} := S_0(\alpha) \setminus A_{1+t_1,t_1,\delta}.$$

Since $t_0, t_1 + 1/2 \in \widetilde{S}$, we can define

$$C_t := \exp\left(2\, k_{S_0(\alpha)\setminus A_{t,t_1,\delta}}(t_0, t_1 + 1/2)\right), \quad t \geq 1 + t_1,$$

and $\widetilde{C} := \exp\left(2k_{\widetilde{S}}(t_0, t_1 + 1/2)\right)$. Moreover, these constants are well-defined and positive, $C_t \leq \widetilde{C}$ and $\widetilde{C}$ only depends only on t_0, t_1, α and δ. Therefore, by Theorem 3.1, the maximum principle, Theorem 3.3 (with respect to ℓ, S_1 and S_2) and the domain monotonicity of harmonic measure, we have that, for $t \geq 1 + t_1$,

$$\begin{aligned}
\omega(t_0, A_{t,t_1,\delta}, S_0(\alpha) \setminus A_{t,t_1,\delta}) &\leq C_t, \omega(t_1 + 1/2, A_{t,t_1,\delta}, S_0(\alpha) \setminus A_{t,t_1,\delta}) \\
&\leq \widetilde{C}\, \omega(t_1 + 1/2, A_{t,t_1,\delta}, S_0(\alpha) \setminus A_{t,t_1,\delta}) \\
&\leq \widetilde{C}\, (\omega(t_1 + 1/2, A_{t,t_1,\delta}, S) + \omega(t_1 + 1/2, A_t, S)) \\
&\leq 2\widetilde{C}\omega(t_1 + 1/2, A_t, S) \\
&\leq 2\widetilde{C}\omega(t_1 + 1/2, A_t, S_0(\alpha) \setminus A_t).
\end{aligned}$$

Therefore, by Proposition 3.2 and for all $t \geq T := 1 + t_1$,

$$\omega(t_0, A_{t,t_1,\delta}, S_0(\alpha) \setminus A_{t,t_1,\delta}) \leq 2\widetilde{C}\,(t_1 + 1/2)^{1/(2\alpha)}\, t^{-1/(2\alpha)}.$$

Since the numbers T and $C := 2\widetilde{C}(t_1 + 1/2)^{1/(2\alpha)}$ depends only on t_0, t_1, α, δ, the proof of this case is done.

Case (3), $1/2 < \alpha < 1$ and $\delta/2 < 1 - \alpha$.

Consider the lines

$$L_1 := \{re^{-i\alpha\pi} = -re^{i(1-\alpha)\pi} : r \in \mathbb{R}\}$$

and $\ell_1 := \{t_1 + re^{i\delta\pi/2} : r \in \mathbb{R}\}$. Since $1/2 < \alpha < 1$ and $\delta/2 < 1-\alpha$, L_1 intersects ℓ_1 in a certain point $p = x + iy$. From this point, the proof follows the same steps given in Case (1), so that we omit the details. □

We will need an estimate of harmonic measure involving Jordan arcs in the unit disc (Theorem 9 in [15]).

Theorem 3.6. *Let $\gamma\colon [a,b] \to \overline{\mathbb{D}}$ be a Jordan arc such that $\gamma([a,b)) \subset \mathbb{D}$, $\gamma(b) \in \partial\mathbb{D}$ and $\gamma(t) \neq 0$, for every $t \in [a,b]$. Then*

$$\omega(0, \gamma, \mathbb{D} \setminus \gamma) \geq \frac{1}{\pi} \arcsin\left(\frac{|\gamma(a) - \gamma(b)|}{2}\right).$$

4. The hyperbolic case

We start this section with the rate of convergence of $\phi_t(z)$ to τ as t goes to $+\infty$. It is known (Theorem 2.8 in [6]) that given a hyperbolic semigroup, $\phi_t(z)$ goes non-tangentially to τ as t goes to $+\infty$ for all $z \in \mathbb{D}$.

Proposition 4.1. *Let (ϕ_t) be a hyperbolic semigroup in $\mathbb{D}$ with Denjoy–Wolff point $\tau \in \partial\mathbb{D}$ and let $\lambda > 0$ be its spectral value. Then*

$$\lim_{t \to +\infty} \frac{1}{t} \log\left(1 - \overline{\tau}\phi_t(z)\right) = -\lambda. \tag{4.1}$$

Proof. Let G be the infinitesimal generator of the semigroup. Given $z \in \mathbb{D}$ and $t \geq 0$,

$$\int_0^t \frac{-\overline{\tau}\, G(\phi_s(z))}{1-\overline{\tau}\,\phi_s(z)}\, ds = [\log(1-\overline{\tau}\phi_s(z))]_{s=0}^{s=t} = \log(1-\overline{\tau}\phi_t(z)) - \log(1-\overline{\tau}z).$$

Then, using L'Hôpital's rule, the non-tangential convergence, and Theorem 1 in [7] (or Theorem 1 in [12]), we obtain

$$\lim_{t\to+\infty} \frac{1}{t}\int_0^t \frac{-\overline{\tau}\, G(\phi_s(z_0))}{1-\overline{\tau}\phi_s(z_0)}\, ds = \lim_{t\to+\infty} \frac{G(\phi_t(z_0))}{\phi_t(z_0)-\tau} = -\lambda.$$

Hence, $\lim_{t\to+\infty} \frac{1}{t} \log\left(1-\overline{\tau}\,\phi_t(z)\right) = -\lambda$. □

For some hyperbolic semigroups it is possible to get a tighter rate of convergence to the Denjoy–Wolff point than the one we got in (4.1). Such a rate of convergence will be characterized in terms of the conformality of the function $g = -i\exp(-\lambda h)$ at the Denjoy–Wolff point of the semigroup where, as usual, λ is the spectral value of the semigroup and h is its Koenigs map.

In [14] (see also the monograph [13], Theorem 7.2), it is proved that the limit shown in (4.2) is either zero or a univalent function with non-negative real part. In the next theorem, we characterize when this second possibility happens in terms of convergence rate. We also provide an example where the first possibility occurs. For some related results, see Theorem 7.4 in [14].

Theorem 4.2. *Let (ϕ_t) be a hyperbolic semigroup in $\mathbb{D}$ with Denjoy–Wolff point $\tau \in \partial\mathbb{D}$ and canonical model $(\mathbb{S}_{\pi/\lambda}, h, z \mapsto z+t)$. Then, for each $z \in \mathbb{D}$, the limit*

$$K(z) := \lim_{t\to+\infty} e^{\lambda t}(1-\overline{\tau}\,\phi_t(z)) \tag{4.2}$$

exists; the convergence is uniform on compact subsets of the unit disc. Moreover, the function K is either identically zero or it is a univalent function with $\operatorname{Re} K > 0$. *In addition, the following are equivalent:*

(1) $\limsup_{t\to+\infty}(1-|\phi_t(0)|)e^{\lambda t} > 0$;

(2) $\liminf_{t\to+\infty}[k_{\mathbb{D}}(0,\phi_t(z)) - k_{\mathbb{S}_{\pi/\lambda}}(h(0), h(z)+t)] < +\infty$, *for all $z \in \mathbb{D}$*;

(3) $\liminf_{t\to+\infty}[k_{\mathbb{D}}(0,\phi_t(0)) - k_{\mathbb{S}_{\pi/\lambda}}(i\frac{\pi}{2\lambda}, i\frac{\pi}{2\lambda}+t)] < +\infty$;

(4) *The function $g := ie^{-\lambda h}$ is conformal at τ, that is, $g'(\tau) := \angle\lim_{z\to\tau} \frac{g(z)}{z-\tau} \in \mathbb{C}\setminus\{0\}$*;

(5) *for each $z \in \mathbb{D}$, the limit*

$$K(z) := \lim_{t\to+\infty} e^{\lambda t}(1-\overline{\tau}\,\phi_t(z)) \tag{4.3}$$

exists and K is a univalent function with $\operatorname{Re} K > 0$;

(6) *there exist $z \in \mathbb{D}$ and (t_n) going to $+\infty$ such that the limit*

$$\lim_{n\to\infty} e^{\lambda t_n}(1-\overline{\tau}\,\phi_{t_n}(z)) \in \mathbb{C}\setminus\{0\}; \tag{4.4}$$

(7) *for each $z \in \mathbb{D}$, the limit*

$$k(z) := \lim_{t\to+\infty} e^{\lambda t}\phi_t'(z) \tag{4.5}$$

exists and belongs to $\mathbb{C}\setminus\{0\}$;

(8) *there exist $z \in \mathbb{D}$ and (t_n) going to $+\infty$ such that the limit*

$$\lim_{n\to\infty} e^{\lambda t_n}\phi_{t_n}'(z) \in \mathbb{C}\setminus\{0\}. \tag{4.6}$$

In such a case, $K(z) = \frac{g(z)}{-\tau g'(\tau)}$ and $k(z) = -\tau K'(z)$, for all $z \in \mathbb{D}$.

Proof. Throughout the proof, for each $t \geq 0$, we consider the functions $K_t(z) := e^{\lambda t}(1-\overline{\tau}\phi_t(z))$, $z \in \mathbb{D}$. Since $\operatorname{Re} K_t > 0$ for all t, the family $\{K_t :\ t \geq 0\}$ is normal. Thus, if some sequence $\{K_{t_n}\}$ converges pointwise, it does uniformly on compact subsets of the unit disc.

Also we use the following notation. The function

$$T(w) = \frac{1-ie^{-\lambda w}}{1+ie^{-\lambda w}}, \quad w \in \mathbb{S}_{\pi/\lambda},$$

sends conformally the strip $\mathbb{S}_{\pi/\lambda}$ onto the unit disc so that $f: = T\circ h$ is a self-map of the unit disc and $\lim_{r\to1} f(r\tau) = 1$ because $\lim_{r\to1}\operatorname{Re} h(r\tau) = +\infty$. As usual, we denote by $\alpha_f(\tau)$ boundary dilatation coefficient of f.

We start by showing the equivalence of (2) and (3). Notice that given $z \in \mathbb{D}$, the triangle inequality and the Schwarz–Pick lemma give

$$\begin{aligned}
&k_{\mathbb{D}}(0,\phi_t(z)) - k_{\mathbb{S}_{\pi/\lambda}}(h(0),h(z)+t)\\
&\leq k_{\mathbb{D}}(0,\phi_t(0))+k_{\mathbb{D}}(\phi_t(0),\phi_t(z))-k_{\mathbb{S}_{\pi/\lambda}}(h(0),i\tfrac{\pi}{2\lambda}+t)+k_{\mathbb{S}_{\pi/\lambda}}(i\tfrac{\pi}{2\lambda}+t,h(z)+t)\\
&\leq k_{\mathbb{D}}(0,\phi_t(0))+k_{\mathbb{D}}(0,z)-k_{\mathbb{S}_{\pi/\lambda}}(i\tfrac{\pi}{2\lambda},i\tfrac{\pi}{2\lambda}+t)+k_{\mathbb{S}_{\pi/\lambda}}(i\tfrac{\pi}{2\lambda},h(0))+k_{\mathbb{S}_{\pi/\lambda}}(i\tfrac{\pi}{2\lambda},h(z))\\
&\leq k_{\mathbb{D}}(0,\phi_t(w))-k_{\mathbb{S}_{\pi/\lambda}}(i\tfrac{\pi}{2\lambda},i\tfrac{\pi}{2\lambda}+t)+k_{\mathbb{D}}(w,z)+k_{\mathbb{S}_{\pi/\lambda}}(i\tfrac{\pi}{2\lambda},h(0))+k_{\mathbb{S}_{\pi/\lambda}}(i\tfrac{\pi}{2\lambda},h(z)).
\end{aligned}$$

Thus (3) implies (2). With a similar argument we show that (2) implies (3). Moreover, for fixed $t > 0$, we have that

$$\begin{aligned}
k_{\mathbb{D}}(0,\phi_t(0)) - k_{\mathbb{S}_{\pi/\lambda}}\Big(i\frac{\pi}{2\lambda}, i\frac{\pi}{2\lambda}+t\Big) &= \frac{1}{2}\log\Big(\frac{1+|\phi_t(0)|}{1-|\phi_t(0)|}\Big) - \frac{\lambda t}{2}\\
&= \frac{1}{2}\log\Big(e^{-\lambda t}\,\frac{1+|\phi_t(0)|}{1-|\phi_t(0)|}\Big).
\end{aligned}$$

Since $1 \leq 1+|\phi_t(0)| \leq 2$, it is clear that (1) is equivalent to (3).

Assume (3) holds. Notice that

$$\begin{aligned}
k_{\mathbb{D}}(0,\phi_t(0)) - k_{\mathbb{D}}(0,f(\phi_t(0))) &\leq k_{\mathbb{D}}(0,\phi_t(0)) - k_{\mathbb{D}}(f(0),f(\phi_t(0))) + k_{\mathbb{D}}(0,f(0))\\
&= k_{\mathbb{D}}(0,\phi_t(0)) - k_{\mathbb{S}_{\pi/\lambda}}(h(0),h(\phi_t(0))) + k_{\mathbb{D}}(0,f(0))\\
&= k_{\mathbb{D}}(0,\phi_t(0)) - k_{\mathbb{S}_{\pi/\lambda}}(h(0),h(0)+t) + k_{\mathbb{D}}(0,f(0)).
\end{aligned}$$

Since

$$k_{\mathbb{S}_{\pi/\lambda}}\left(i\frac{\pi}{2\lambda}, i\frac{\pi}{2\lambda}+t\right) \leq k_{\mathbb{S}_{\pi/\lambda}}(h(0), h(0)+t) + 2k_{\mathbb{S}_{\pi/\lambda}}\left(i\frac{\pi}{2\lambda}, h(0)\right),$$

calling $C = 2k_{\mathbb{S}_{\pi/\lambda}}(i\frac{\pi}{2\lambda}, h(0)) + k_{\mathbb{D}}(0, f(0))$, statement (2) and (2.5) imply

$$\begin{aligned}\frac{1}{2}\log\alpha_f(\tau) &= \liminf_{z\to\tau}\,[k_{\mathbb{D}}(0,z) - k_{\mathbb{D}}(0,f(z))]\\ &\leq \liminf_{t\to+\infty}\,[k_{\mathbb{D}}(0,\phi_t(0)) - k_{\mathbb{D}}(0,f(\phi_t(0)))]\\ &\leq \liminf_{t\to+\infty}\left[k_{\mathbb{D}}(0,\phi_t(0)) - k_{\mathbb{S}_{\pi/\lambda}}\left(i\frac{\pi}{2\lambda}, i\frac{\pi}{2\lambda}+t\right)\right] + C < +\infty.\end{aligned}$$

By the Julia–Wolff–Carathéodory theorem and the non-tangential convergence to the Denjoy–Wolff point in the hyperbolic case, we have that

$$\overline{\tau}\alpha_f(\tau) = \angle\lim_{z\to\tau}\frac{1-f(z)}{\tau - z} = \angle\lim_{z\to\tau}\frac{1}{\tau-z}\,\frac{2ie^{-\lambda h(z)}}{ie^{-\lambda h(z)}+1}.$$

Hence

$$\angle\lim_{z\to\tau}\frac{e^{-\lambda h(z)}}{z-\tau} = \frac{i}{2}\overline{\tau}\alpha_f(\tau) \in \mathbb{C}\setminus\{0\}. \tag{4.7}$$

That is, (4) holds. Notice that, in fact, the above argument shows that the conformality of g is equivalent to the fact that $\alpha_f(\tau) < +\infty$. Thus, following a similar idea using again the triangle inequality, we see that (4) implies (3). That is, so far we have obtained that the first four statements are equivalent.

Assume (4). Using again the non-tangential convergence to the Denjoy–Wolff point in the hyperbolic case, we directly obtain that

$$\frac{i}{2}\overline{\tau}\alpha_f(\tau) = \lim_{t\to+\infty}\frac{e^{-\lambda h(\phi_t(z))}}{\phi_t(z)-\tau} = \lim_{t\to+\infty}\frac{e^{-\lambda h(z)-\lambda t}}{\phi_t(z)-\tau}.$$

Therefore,

$$\lim_{t\to+\infty} e^{\lambda t}(1-\overline{\tau}\phi_t(z)) = \frac{2ie^{-\lambda h(z)}}{\alpha_f(\tau)}. \tag{4.8}$$

Since $\mathsf{Re}\, e^{\lambda t}(1-\overline{\tau}\phi_t(z)) > 0$ for all z and t and the limit is not constant, we deduce that the function $K(z) = 2ie^{-\lambda h(z)}/\alpha_f(\tau)$ has non-negative real part, it is not constant and then univalent, getting (5).

Clearly (5) implies (6). Assume now that (6) holds. Write $B = \lim_{n\to\infty} e^{\lambda t_n}(1-\overline{\tau}\phi_{t_n}(z))$. By Proposition 4.13 in [19], the following limit does exist:

$$A := \angle\lim_{z\to\tau}\frac{1-f(z)}{\tau-z} \in \mathbb{C}_\infty\setminus\{0\}. \tag{4.9}$$

Moreover,

$$\frac{1-f(\phi_{t_n}(z))}{\tau-\phi_{t_n}(z)} = \frac{1}{e^{\lambda t_n}(1-\overline{\tau}\phi_{t_n}(z))}\,\frac{2ie^{-\lambda h(z)}}{1+ie^{-\lambda h(z)}e^{-\lambda t_n}}. \tag{4.10}$$

Taking limits, we deduce that $A = \frac{2i}{B}e^{-\lambda h(z)} \in \mathbb{C}$. That is, $\alpha_f(\tau) < +\infty$. Now, following the argument used to prove that (3) implies (4), we conclude again (4).

Using the normality of the family $\{K_t\}$ and the Weierstrass theorem, we see that (5) implies (7). Since, clearly, (8) is a particular case of (7), we are left to prove that (8) implies (6) to obtain that the eight statements are equivalent. Take $z \in \mathbb{D}$ and (t_n) going to $+\infty$ such that the limit (4.6) is not zero. Denote by G the infinitesimal generator of the semigroup. Then

$$K_{t_n}(z) = e^{\lambda t_n}(1 - \overline{\tau}\phi_{t_n}(z)) = \overline{\tau}e^{\lambda t_n}\, G(\phi_{t_n}(z))\, \frac{\tau - \phi_{t_n}(z)}{G(\phi_{t_n}(z))}$$
$$= \overline{\tau}\, e^{\lambda t_n}\phi'_{t_n}(z)\, G(z)\, \frac{\tau - \phi_{t_n}(z)}{G(\phi_{t_n}(z))}.$$

Therefore, in view of the non-tangential convergence of $(\phi_t(z))$ to τ, we conclude that

$$\lim_{n\to\infty} K_{t_n}(z) = \lambda\,\overline{\tau}\, G(z) \lim_{n\to\infty} e^{\lambda t_n}\phi'_{t_n}(z) \in \mathbb{C}\setminus\{0\}.$$

Summing up, we have proved that statements (1) to (8) are equivalent.

Suppose now that none of the statements (1) to (8) holds. Looking at the proof of (6) implies (4), the number A in (4.9) must be ∞. Therefore, using (4.10), we see that for every sequence (t_n) converging to $+\infty$, we have $\lim_{n\to\infty} K_{t_n}(z) = 0$.

In the case K is a univalent function, the equality $K(z) = -g(z)/(\tau g'(\tau))$ follows from (4.7) and (4.8). Also, the formula $k(z) = -\tau K'(z)$, $z \in \mathbb{D}$, comes from (4.3) and (4.5). □

Remark 4.3. Let (ϕ_t) be a hyperbolic semigroup in $\mathbb{D}$ with Denjoy–Wolff point $\tau \in \partial\mathbb{D}$ and canonical model $(\mathbb{S}_{\pi/\lambda}, h, z \mapsto z + t)$. We know that $g(z) = i\,e^{-\lambda h(z)}$ has non-negative real part and $\angle\lim_{z\to\tau} g(z) = 0$. Writing C the Cayley map that maps the right half-plane onto the unit disc sending the point 0 to τ, by Proposition 4.13 in [19], $C \circ g$ has a boundary fixed point at τ and there exists its angular derivative at τ. Such angular derivative is either a positive real number or ∞. Thus $\angle\lim_{z\to\tau} \frac{g(z)}{z-\tau} \in \mathbb{C}_\infty \setminus \{0\}$. Therefore the following are equivalent:

(1) g is conformal at τ;

(2) $\angle\lim_{z\to\tau} g'(z) \in \mathbb{C}$;

(3) $\lim_{r\to 1} \frac{\operatorname{Re} g(r\tau)}{1-r} \in \mathbb{R}$;

(4) $\liminf_{r\to 1} \frac{\operatorname{Re} g(r\tau)}{1-r} < +\infty$.

Some elementary computations show that conformality of g at τ can also be characterized in terms of the Koenigs function as follows.

Proposition 4.4. *Let (ϕ_t) be a hyperbolic semigroup in $\mathbb{D}$ with Denjoy–Wolff point $\tau \in \partial\mathbb{D}$ and canonical model $(S_{\pi/\lambda}, h, z \mapsto z + t)$. Write $g(z) = i\,e^{-\lambda h(z)}$. The following are equivalent:*

(1) *g is conformal at τ, i.e., there exists $g'(\tau) := \angle\lim_{z\to\tau} g(z)/(z-\tau) \in \mathbb{C}\setminus\{0\}$;*

(2) $$A_1 := \angle\lim_{z\to\tau}(\lambda h(z) + \log(1 - \overline{\tau}z)) \in \mathbb{C};$$

(3) $$A_2 := \lim_{r\to 1}(\lambda h(r\tau) + \log(1-r)) \in \mathbb{C};$$

(4) $$A_3 := \lim_{r\to 1}(\lambda \operatorname{Re} h(r\tau) + \log(1-r)) \in \mathbb{R}.$$

In such a case, $A_1 = A_2$, $ig'(\tau)\tau = e^{-A_1}$ *and* $A_3 = \operatorname{Re} A_1$.

Next example is built using techniques developed in [5]. A related example appears in [11], Example 6.1.

Example 4.5. In this example we provide a hyperbolic semigroup with Denjoy–Wolff point 1 such that, for all $z \in \mathbb{D}$,

(4.11) $$\lim_{t\to+\infty} e^{\lambda t}(1-\phi_t(z)) = 0.$$

Consider the strip $\mathbb{S}_2 = \{w \in \mathbb{C} : 0 < \operatorname{Im} w < 2\}$. Given $0 < y < 1$ and $M > 0$, we write $S_y = \{w \in \mathbb{C} : y < \operatorname{Im} w < 2-y\}$ and $S_y^M = \{w \in S_y : \operatorname{Re} w > M\}$. Take $c_k := \frac{(k+1)^2}{(k+1)^2-1}$ and $y_k := 1/(k+1)$ for all $k \in \mathbb{N}$.

Fix k and assume we have chosen $x_1, x_2, ..., x_{k-1}$. For each $b \in \mathbb{R}$, denote

$$\Omega_{k,b} = \mathbb{C} \setminus \{z \in \mathbb{C} : \operatorname{Im} z \in \{y_k, 2-y_k\}, \operatorname{Re} z \le b\}.$$

By Proposition 5.3 in [5], there is a constant $D_k = D(c_k)$ (notice that D_k does not depend on b) such that

(4.12) $$k_{S_{y_k}}(z,w) \le c_k\, k_{\Omega_{k,b}}(z,w)$$

for every $z, w \in S_{y_k}$ with $\operatorname{Re} z, \operatorname{Re} w \le b - 2(1-y_k)D_k$. Choose $x_k > k+1+2(1-y_k)D_k$.

Now, consider the domain

$$\Omega = \mathbb{S}_2 \setminus \cup_{k\in\mathbb{N}} \left(L_-^k \cup L_+^k\right),$$

where

$$L_+^k := \{w \in \mathbb{C} : \operatorname{Im} w = y_k, \operatorname{Re} w \le x_k\}, \quad L_-^k := \{w \in \mathbb{C} : \operatorname{Im} w = 2-y_k, \operatorname{Re} w \le x_k\}.$$

Clearly, we have that $\Omega \subset \Omega_{k,x_k}$ so that $k_{\Omega_{k,y_k}} \le k_\Omega$.

Take h a Riemann map of Ω such that $h(0) = i$, $h(-1,1) = i + \mathbb{R}$, and $\lim_{r\to 1} \operatorname{Re} h(r) = +\infty$. Consider the semigroup given by $\phi_t(z) := h^{-1}(h(z)+t)$, for all $z \in \mathbb{D}$ and $t \ge 0$. Then $\tau = 1$ is its Denjoy–Wolff point.

Let us prove that

(4.13) $$\lim_{t\to+\infty} [k_{\mathbb{D}}(0, \phi_t(0)) - k_{\mathbb{S}_2}(i, i+t)] = +\infty,$$

so that (4.11) follows from Theorem 4.2.

Since the curve $i+\mathbb{R}$ is a geodesic for the hyperbolic metric in $\mathbb{S}_2$, using that $k_{\mathbb{S}_2} \leq k_\Omega$ and (4.12), we have that, for all $N \in \mathbb{N}$,

$$
\begin{aligned}
&k_{\mathbb{D}}(0,\phi_N(0)) - k_{\mathbb{S}_2}(i,i+N) = k_\Omega(i,i+N) - k_{\mathbb{S}_2}(i,i+N)\\
&\geq \sum_{k=0}^{N-1}[k_\Omega(i+k,i+(k+1)) - k_{\mathbb{S}_2}(i+k,i+(k+1))] = \sum_{k=0}^{N-1}\left[k_\Omega(i+k,i+(k+1)) - \frac{\pi}{4}\right]\\
&\geq \sum_{k=0}^{N-1}\left[k_{\Omega_{k,x_k}}(i+k,i+(k+1)) - \frac{\pi}{4}\right] \geq \sum_{k=0}^{N-1}\left[\frac{1}{c_k}k_{S_{y_k}}(i+k,i+(k+1)) - \frac{\pi}{4}\right]\\
&= \sum_{k=0}^{N-1}\left[\frac{1}{c_k}\,\frac{\pi}{4(1-y_k)} - \frac{\pi}{4}\right] = \sum_{k=0}^{N-1}\frac{\pi}{4(k+1)}.
\end{aligned}
$$

Thus, (4.13) holds.

Remark 4.6. The problem of conformality at a boundary point for conformal mappings of strip domains was deeply studied with the help of the method of extremal lengths. In particular, examples of the same kind as above can be obtained by means of Theorem 1 in [21].

5. The parabolic case

Recall that for a holomorphic self-map ϕ of $\mathbb{D}$, we denote by (ϕ_n) the sequence of iterates of ϕ.

Theorem 5.1. *Let ϕ be a parabolic holomorphic self-map in $\mathbb{D}$ with Denjoy–Wolff point $\tau \in \partial\mathbb{D}$. Then, for every pair $z_0, z_1 \in \mathbb{D}$, it holds*

$$\lim_{n\to+\infty}\frac{\phi_n(z_0)-\tau}{\phi_n(z_1)-\tau} = 1. \tag{5.1}$$

Proof. We may assume that $\tau = 1$ and $z_0 = 0$. Consider the Cayley map C from the unit disc onto the right half-plane Ω given by $C(z) = (1+z)/(1-z)$ and define $\psi = C\circ\phi\circ C^{-1}$. Notice that ψ is a self-map of Ω with Denjoy–Wolff point ∞ and angular derivative at such point equal to 1. Write $w_j = C(z_j)$ for $j = 0, 1$. Notice that $w_0 = 1$.

Since

$$1-\phi_n(z) = \frac{2}{1+\psi_n(C(z))}$$

for all $z \in \mathbb{D}$, (5.1) is equivalent to

$$\lim_{n\to+\infty}\frac{1+\psi_n(w_1)}{1+\psi_n(w_0)} = 1. \tag{5.2}$$

In addition, since $\lim_{n\to\infty}\psi_n(w) = \infty$ for all $w \in \Omega$, we get that (5.1) is equivalent to

$$\lim_{n\to+\infty}\frac{\psi_n(w_1)}{\psi_n(1)} = 1. \tag{5.3}$$

Write $x_n = \operatorname{Re}\psi_n(1)$ and $y_n = \operatorname{Im}\psi_n(1)$.

We consider separately the two parabolic subcases. Assume firstly that the function ϕ is of positive hyperbolic step. Then, by Theorem 1 in [18] (see also Lemma 2.1 in [8]), there is a univalent function in Ω, let us say σ, such that $\lim_{n\to+\infty}(\psi_n(w) - iy_n)/x_n = \sigma(w)$ for all $w \in \Omega$. Moreover, by Remark 1 in [18], $\lim_{n\to+\infty}|y_n|/x_n = \infty$ and then $\lim_{n\to+\infty} x_n/\psi_n(1) = 0$. Therefore,

(5.4)
$$\lim_{n\to+\infty}\frac{\psi_n(w_1)}{\psi_n(1)} = \lim_{n\to+\infty}\left[\frac{\psi_n(w_1)-iy_n}{x_n}\frac{x_n}{\psi_n(1)} + 1 - \frac{x_n}{\psi_n(1)}\right] = \sigma(w)\cdot 0+1-0=1,$$

getting (5.3) when ϕ has positive hyperbolic step.

Assume now that ϕ is of zero hyperbolic step. Using again Theorem 1 in [18] (see also Lemma 2.1 in [8]), we have that $\lim_{n\to+\infty}(\psi_n(w) - iy_n)/x_n = 1$ for all $w \in \Omega$. Since $|x_n/\psi_n(1)| \leq 1$, we deduce

$$\lim_{n\to+\infty}\frac{\psi_n(w_1)}{\psi_n(1)} = \lim_{n\to+\infty}\left[\left(\frac{\psi_n(w_1)-iy_n}{x_n} - 1\right)\frac{x_n}{\psi_n(1)} + 1\right] = 1, \tag{5.5}$$

getting (5.3) when the function is of zero hyperbolic step. □

In the next corollary we pass from the above discrete iteration result to the corresponding result for parabolic semigroups. Such a result was proved in Proposition 3.3 of [24] under the additional assumption that $\phi_t(z)$ converges non-tangentially to τ, as $t \to +\infty$. As usual to pass from the non-tangential case to the general case requires deeper techniques.

Corollary 5.2. *Let (ϕ_t) be a parabolic semigroup in $\mathbb{D}$ with Denjoy–Wolff point $\tau \in \partial\mathbb{D}$. Then, for every pair $z_0, z_1 \in \mathbb{D}$, it holds*

$$\lim_{t\to+\infty}\frac{\phi_t(z_0)-\tau}{\phi_t(z_1)-\tau} = 1. \tag{5.6}$$

Proof. Fix $z_0, z_1 \in \mathbb{D}$. It is enough to check that every sequence $(t_n) \subset (1, +\infty)$ converging to $+\infty$ has a subsequence (t_{n_k}) such that

$$\lim_{k\to+\infty}\frac{\phi_{t_{n_k}}(z_0)-\tau}{\phi_{t_{n_k}}(z_1)-\tau} = 1.$$

Thus, consider such a sequence $(t_n) \subset (1, +\infty)$ with $\lim_{n\to+\infty} t_n = +\infty$. Write $t_n = N_n + r_n$ with $N_n \in \mathbb{N}$ and $r_n \in (0, 1]$ and take a subsequence of the natural numbers (n_k) such that $r := \lim_{k\to+\infty} r_{n_k} \in [0, 1]$. Clearly, $\lim_{k\to+\infty} N_{n_k} = +\infty$.

Note that, for $k \in \mathbb{N}$,

$$L_k := \frac{\phi_{t_{n_k}}(z_0)-\tau}{\phi_{t_{n_k}}(z_1)-\tau} = \frac{\phi_{N_{n_k}}(\phi_{r_{n_k}}(z_0))-\tau}{\phi_{N_{n_k}}(\phi_{r_{n_k}}(z_1))-\tau}.$$

Define $w_0 := \phi_r(z_0) \in \mathbb{D}$ and $w_1 := \phi_r(z_1) \in \mathbb{D}$. By Theorem 5.1, we have

$$\lim_{k\to+\infty}\frac{\phi_{N_{n_k}}(w_0)-\tau}{\phi_{N_{n_k}}(w_1)-\tau} = 1.$$

For each k, the functions

$$z \mapsto p_k(z) := \frac{\phi_{N_{n_k}}(z) - \tau}{\phi_{N_{n_k}}(w_0) - \tau} \quad \text{and} \quad z \mapsto q_k(z) := \frac{\phi_{N_{n_k}}(w_1) - \tau}{\phi_{N_{n_k}}(z) - \tau}$$

map $\mathbb{D}$ into $\mathbb{C} \setminus (-\infty, 0]$. Therefore, by Montel's theorem, the families $\{p_k\}_{k\in\mathbb{N}}$ and $\{q_k\}_{k\in\mathbb{N}}$ are normal. Moreover, by Theorem 5.1 and Vitali's theorem, we also have that the following two limits,

$$\lim_{k\to+\infty} \frac{\phi_{N_{n_k}}(z) - \tau}{\phi_{N_{n_k}}(w_0) - \tau} = \lim_{k\to+\infty} \frac{\phi_{N_{n_k}}(w_1) - \tau}{\phi_{N_{n_k}}(z) - \tau} = 1,$$

hold when z runs on compact subsets of $\mathbb{D}$. Hence

$$L_k = \frac{\phi_{N_{n_k}}(\phi_{r_{n_k}}(z_0)) - \tau}{\phi_{N_{n_k}}(w_0) - \tau} \frac{\phi_{N_{n_k}}(w_0) - \tau}{\phi_{N_{n_k}}(w_1) - \tau} \frac{\phi_{N_{n_k}}(w_1) - \tau}{\phi_{N_{n_k}}(\phi_{r_{n_k}}(z_1)) - \tau}$$

converges to 1, when k tends to $+\infty$, as wanted. □

Theorem 5.3. *Let (ϕ_t) be a parabolic semigroup in $\mathbb{D}$ with Denjoy–Wolff point $\tau \in \partial\mathbb{D}$ and let h be a Koenigs function of the semigroup. Assume that $h(\mathbb{D})$ is contained in a sector $S_p(\alpha, \beta)$ where $0 \leq \alpha, \beta \leq 1$ with $\alpha + \beta > 0$. Then, for every $z \in \mathbb{D}$, there exists a constant $C = C(\alpha, \beta, z) > 0$ such that, for all $t > 0$,*

$$|\phi_t(z) - \tau| \leq \frac{C}{t^{1/(\alpha+\beta)}}. \tag{5.7}$$

Proof. Note that $\phi_t(z) = (h-p)^{-1}((h-p)(z) + t)$, for every $z \in \mathbb{D}$ and $t \geq 0$. Therefore, we may and do assume that $p = 0$.

Let $a := e^{i\pi(\beta-\alpha)/2}$, thus $E := \{ra : r > 0\}$ is the axis of symmetry of the sector $S_0(\alpha, \beta)$. Note $\overline{a}S_0(\alpha, \beta) = S_0((\alpha + \beta)/2)$. Without lose of generality, we may also assume that E cuts $h(\mathbb{D})$. Otherwise, consider the number $\theta_0 := \inf\{\alpha' + \beta' : h(\mathbb{D}) \subset S_0(\alpha', \beta')\}$, which is well-defined by hypothesis and positive since $h(\mathbb{D})$ is a domain. Then, take any sector $S_0(\alpha', \beta')$ containing $h(\mathbb{D})$ such that $\alpha' + \beta' < 2\theta_0$ and $\alpha' + \beta' \leq \alpha + \beta$. By geometrical reasons, the axis of symmetry of $S_0(\alpha', \beta')$ necessarily cuts $h(\mathbb{D})$. Finally, note that, for all $t > 0$, we have $1/t^{1/(\alpha'+\beta')} \leq 1/t^{1/(\alpha+\beta)}$.

Hence, we may suppose there is $0 < r_0 < r_1$ such that $w_0 := r_0 a$ and $w_1 := r_1 a$ belong to $E \cap h(\mathbb{D})$. Define $z_0 := h^{-1}(w_0) \in \mathbb{D}$ and $z_1 := h^{-1}(w_1) \in \mathbb{D}$. Since $\lim_{t\to+\infty} \phi_t(z_1) = \tau$, we can choose $T > 0$ such that, for all $t \geq T$,

$$|\phi_t(z_1)| > \frac{1 + |z_0|}{2}. \tag{5.8}$$

For every $t \geq T$, consider the sets

$$A_t := \{\phi_s(z_1) : s \geq t\}, \qquad B_t := h(A_t) = \{w_1 + s : s \geq t\}.$$

Bearing in mind (5.8), we see that $0, z_0 \in \mathbb{D} \setminus A_t$, for every $t \geq T$. Therefore, by Harnack's inequality (see Theorem 3.1),

$$\omega(0, A_t, \mathbb{D}\setminus A_t) \leq e^{2\,k_{\mathbb{D}\setminus A_t}(0,z_0)}\omega(z_0, A_t, \mathbb{D}\setminus A_t) \leq e^{2\,k_{\mathbb{D}\setminus A_T}(0,z_0)}\omega(z_0, A_t, \mathbb{D}\setminus A_t). \tag{5.9}$$

Bearing in mind again that $\lim_{t\to+\infty}\phi_t(z_1) = \tau$ and (5.8), we can apply Theorem 3.6 to the curve $s \in [t, +\infty) \mapsto \phi_s(z_1)$ and deduce that, for every $t \geq T$,

$$|\phi_t(z_1) - \tau| \leq 2\sin\left(\pi\,\omega(0, A_t, \mathbb{D}\setminus A_t)\right).$$

Therefore, using (5.9), the conformal invariance and the domain monotonicity of harmonic measure, we obtain that for every $t \geq T$,

$$\begin{aligned}
|\phi_t(z_1) - \tau| &\leq 2\pi\,\omega(0, A_t, \mathbb{D}\setminus A_t) \leq 2\pi\, e^{2\,k_{\mathbb{D}\setminus A_T}(0,z_0)}\,\omega(z_0, A_t, \mathbb{D}\setminus A_t) \\
&= 2\pi\, e^{2\,k_{\mathbb{D}\setminus A_T}(0,z_0)}\,\omega(w_0, B_t, h(\mathbb{D})\setminus B_t) \\
&\leq 2\pi\, e^{2\,k_{\mathbb{D}\setminus A_T}(0,z_0)}\,\omega(w_0, B_t, S_0(\alpha,\beta)\setminus B_t) \\
&= 2\pi\, e^{k_{\mathbb{D}\setminus A_T}(0,z_0)}\,\omega(r_0, \overline{a}B_t, S_0((\alpha+\beta)/2)\setminus \overline{a}B_t).
\end{aligned}$$

Note that $\overline{a}\,B_t = \{r_1 + s\overline{a} = r_1 + se^{i\pi(\alpha-\beta)/2} : s \geq t\}$ and $|(\alpha-\beta)/2| \leq (\alpha+\beta)/2 \leq 1$. Therefore, we can apply Proposition 3.5 to obtain two constants $C_1 = C_1(r_0, r_1, \alpha, \beta) > 0$ and $T_1 = T_1(r_0, r_1, \alpha, \beta)$ such that, for all $t \geq \max\{T, T_1\}$, we have

$$|\phi_t(z_1) - \tau| \leq 2\pi\, e^{2\,k_{\mathbb{D}\setminus A_T}(0,z_0)}\, C_1\, t^{-1/(\alpha+\beta)}.$$

Note that the quantity $2\pi e^{2\,k_{\mathbb{D}\setminus A_T}(0,z_0)}C_1$ does not depend on t; indeed and in the end, it depends only on α and β. Since it suffices to prove the inequality of the theorem for every sufficiently large t, the proof is done for the point z_1.

Finally, for an arbitrary point $z \in \mathbb{D}$, we apply Corollary 5.2 to the points z and z_1 and we obtain a constant $C = C(z, \alpha, \beta) > 0$ such that for all $t > 0$

$$|\phi_t(z) - \tau| \leq C t^{-1/(\alpha+\beta)}. \qquad \square$$

The following corollary follows from the fact that Koenigs functions of parabolic semigroups are starlike at infinite.

Corollary 5.4 (Theorem 1 in [4]). *Let (ϕ_t) be a parabolic semigroup in $\mathbb{D}$ with Denjoy–Wolff point $\tau \in \partial\mathbb{D}$. Then for each $z \in \mathbb{D}$, there is a constant $C = C(z)$ such that, for every $t \geq 0$,*

$$|\phi_t(z) - \tau| \leq \frac{C}{\sqrt{t}}. \tag{5.10}$$

Since the range of Koenigs functions of parabolic semigroups of positive hyperbolic step are contained in half-planes, we also have the following.

Corollary 5.5 (Theorem 1 in [4]). *Let (ϕ_t) be a parabolic semigroup in $\mathbb{D}$ of positive hyperbolic step with Denjoy–Wolff point $\tau \in \partial\mathbb{D}$. Then for each $z \in \mathbb{D}$, there is a constant $C = C(z)$ such that, for every $t \geq 0$,*

$$|\phi_t(z) - \tau| \leq \frac{C}{t}. \tag{5.11}$$

In the appendix, we show it is possible to deduce the previous corollary without using arguments from harmonic measures. Indeed, in that alternative proof, the core are some estimates of hyperbolic geometry.

Theorem 5.6. *Let (ϕ_t) be a parabolic semigroup in $\mathbb{D}$ with Denjoy–Wolff point $\tau \in \partial\mathbb{D}$ and let h be a Koenigs function of the semigroup. Assume that $h(\mathbb{D})$ contains a sector $S_p(\alpha,\beta)$.*

(1) *If α and β are positive, then, for every $z \in \mathbb{D}$, there exist $C = C(\alpha,\beta,z) > 0$ and $T = T(z) > 0$ such that, for all $t > T$,*

$$|\phi_t(z) - \tau| \geq \frac{C}{t^{1/(\alpha+\beta)}}.$$

(2) *If α or β is zero, then, for every $z \in \mathbb{D}$, there exist $C = C(\alpha,\beta,z) > 0$ and $T = T(\alpha,\beta,z) > 0$ such that, for all $t > T$,*

$$|\phi_t(z) - \tau| \geq \frac{1}{t}\,\frac{C}{t^{1/(\alpha+\beta)}}.$$

Proof. Assume initially $0 \leq \alpha,\beta \leq 1$, let $a := e^{i\pi(\beta-\alpha)/2}$, and fix any point $p + r_0\, a$ ($r_0 > 0$) of the axis of symmetry of the sector $S_p(\alpha,\beta)$. If $z_0 := h^{-1}(p + r_0\, a) \in \mathbb{D}$, then

$$S_{h(z_0)}(\alpha,\beta) \subset S_p(\alpha,\beta) \subset h(\mathbb{D}).$$

Note that, by Corollary 5.2, it is enough to prove the theorem for this point z_0. Moreover, if $\theta := \alpha+\beta > 0$, $\overline{a}S_0(\alpha,\beta) = S_0(\theta/2,\theta/2)$ and, using the principal branch of the logarithm, the function $g(\zeta) := (\overline{a}\zeta)^{1/\theta}$ maps $S_0(\alpha,\beta)$ conformally onto $\mathbb{H}$.

Using the continuous Denjoy–Wolff theorem, we can choose $T = T(z_0) > 1$ such that, for all $t > T$,

$$0 < \delta_t := |\phi_t(z_0) - \tau| < 1/2.$$

Then, for $t > T$,

$$k_{\mathbb{D}}(0, 1-\delta_t) = \frac{1}{2}\log\frac{2-\delta_t}{\delta_t},$$

and, therefore,

$$\delta_t > \frac{\delta_t}{2-\delta_t} = \exp\left(-2\, k_{\mathbb{D}}(0,1-\delta_t)\right).$$

Moreover, by an elementary inequality for hyperbolic distances in $\mathbb{D}$ and conformal invariance,

$$k_{\mathbb{D}}(0,1-\delta_t) \leq k_{\mathbb{D}}(0,\phi_t(z_0)) = k_{h(\mathbb{D})}(h(0),h(\phi_t(z_0))) = k_{h(\mathbb{D})}(h(0),h(z_0)+t). \tag{5.12}$$

Let us prove the first statement, that is, assume now α and β are positive. In this case, $h(z_0) + s \in S_{h(z_0)}(\alpha,\beta)$, for every $s > 0$. Then, for $t > T$, applying (5.12) and the domain monotonicity of the hyperbolic metric

$$\begin{aligned}k_{\mathbb{D}}(0,1-\delta_t) &\leq k_{h(\mathbb{D})}(h(0),h(z_0)+1)+k_{h(\mathbb{D})}(h(z_0)+1,h(z_0)+t)\\ &\leq k_{\mathbb{D}}(0,\phi_1(z_0))+k_{S_{h(z_0)}(\alpha,\beta)}(h(z_0)+1,h(z_0)+t)\\ &= k_{\mathbb{D}}(0,\phi_1(z_0))+k_{S_0(\alpha,\beta)}(1,t).\end{aligned}$$

Define

$$b := g(1) = \overline{a}^{1/\theta} = e^{i\frac{\pi}{2}\frac{\alpha-\beta}{\alpha+\beta}} \in \mathbb{H},$$

thus $g(t) = bt^{1/\theta}$. Note that $b^2 \in \partial\mathbb{D}\setminus\{-1\}$. Therefore, and since $t > 1$,

$$\begin{aligned}k_{S_0(\alpha,\beta)}(1,t) = k_{\mathbb{H}}(b,bt^{1/\theta}) &= \frac{1}{2}\log\Big(\frac{|b+\overline{b}\,t^{1/\theta}|+t^{1/\theta}-1}{|b+\overline{b}\,t^{1/\theta}|-t^{1/\theta}+1}\Big)\\ &\leq \frac{1}{2}\log\Big(\frac{(2t^{1/\theta})^2}{2t^{1/\theta}(1+\operatorname{Re}(b^2))}\Big) \leq \frac{1}{2}\log\Big(\frac{2}{1+\operatorname{Re}(b^2)}\,t^{1/\theta}\Big).\end{aligned}$$

Hence $2\,k_{\mathbb{D}}(0,1-\delta_t) \leq C + \log(t^{1/\theta})$, where

$$C := 2\,k_{\mathbb{D}}(0,\phi_1(z_0)) + \log\Big(\frac{2}{1+\operatorname{Re}(b^2)}\Big) > 0.$$

Therefore, for every $t > T$,

$$|\phi_t(z_0)-\tau| \geq \frac{e^{-C}}{t^{1/\theta}}.$$

Let us prove the second statement. We assume that $0 < \beta \leq 1$ and $\alpha = 0$. The other possibility ($0 < \alpha \leq 1$ and $\beta = 0$) can be dealt in a very similar way. Note that, in this case, $a = e^{i\beta\pi/2}$ and $\theta = \beta$.

Take $0 < \varepsilon$ small enough such that $h(z_0) - \varepsilon a$ belongs to the axis of symmetry of the sector $S_p(0,\beta)$. Since $\operatorname{Im} a = \sin(\beta\pi/2) > 0$, we have

$$h(z_0)+s \in S_{h(z_0)-\varepsilon a}(0,\beta) \subset S_p(0,\beta) \subset h(\mathbb{D}),\ \text{ for } s \geq 0.$$

As above, for $t > T$, applying (5.12) and the domain monotonicity of the hyperbolic metric

$$\begin{aligned}k_{\mathbb{D}}(0,1-\delta_t) &\leq k_{h(\mathbb{D})}(h(0),h(z_0)+1)+k_{h(\mathbb{D})}(h(z_0)+1,h(z_0)+t)\\ &\leq k_{\mathbb{D}}(0,\phi_1(z_0))+k_{S_{h(z_0)-\varepsilon a}(0,\beta)}(h(z_0)+1,h(z_0)+t)\\ &= k_{\mathbb{D}}(0,\phi_1(z_0))+k_{S_0(0,\beta)}(1+\varepsilon a,t+\varepsilon a).\end{aligned}$$

Denote

$$\delta(s) := \frac{1}{\beta}\arctan\Big(\frac{s\sin(\beta\pi/2)}{\varepsilon+s\cos(\beta\pi/2)}\Big), \quad \text{for } s \geq 1.$$

Hence, $(\varepsilon + sa)^{1/\beta} = |\varepsilon + sa|^{1/\beta} e^{i\delta(s)}$. Therefore,

$$\begin{aligned} k_{S_0(0,\beta)}(1+\varepsilon a, t+\varepsilon a) &= k_{\mathbb{H}}(g(1+\varepsilon a), g(t+\varepsilon a)) = k_{\mathbb{H}}((\varepsilon+\overline{a})^{1/\beta}, (\varepsilon + t\overline{a})^{1/\beta}) \\ &= \frac{1}{2}\log\Big(\frac{|(\varepsilon+t\overline{a})^{1/\beta} + (\varepsilon+a)^{1/\beta}| + |(\varepsilon+t\overline{a})^{1/\beta} - (\varepsilon+\overline{a})^{1/\beta}|}{|(\varepsilon+t\overline{a})^{1/\beta} + (\varepsilon+a)^{1/\beta}| - |(\varepsilon+t\overline{a})^{1/\beta} - (\varepsilon+\overline{a})^{1/\beta}|}\Big) \\ &\le \frac{1}{2}\log\Big(\frac{4\left(|\varepsilon+ta|^{1/\beta} + |\varepsilon+a|^{1/\beta}\right)^2}{2|\varepsilon+ta|^{1/\beta}|\varepsilon+a|^{1/\beta}\mathrm{Re}\left(e^{-i\delta(t)-i\delta(1)} + e^{-i\delta(t)+i\delta(1)}\right)}\Big) \\ &= \frac{1}{2}\log\Big(\frac{4\left(|\varepsilon+ta|^{1/\beta} + |\varepsilon+a|^{1/\beta}\right)^2}{2|\varepsilon+ta|^{1/\beta}|\varepsilon+a|^{1/\beta} 2\cos(\delta(1))\cos(\delta(t))}\Big). \end{aligned}$$

Notice that

$$\lim_{t\to+\infty} \frac{\left(|\varepsilon+ta|^{1/\beta} + |\varepsilon+a|^{1/\beta}\right)^2}{t^{1/\beta}|\varepsilon+ta|^{1/\beta}} = 1.$$

Moreover, $\lim_{s\to+\infty}\delta(s) = \pi/2$ and some computations show

$$\lim_{s\to+\infty} s\cos(\delta(s)) = \frac{\sin(\beta\pi/2)}{\beta}\,\varepsilon > 0.$$

Therefore, there exist $C = C(\beta,\varepsilon) > 0$ and $T_1 = T_1(\beta,\varepsilon) > T$, such that for every $t > T_1$

$$k_{S_0(0,\beta)}(1+\varepsilon a, t+\varepsilon a) \le \frac{1}{2}\log(Ctt^{1/\beta}).$$

Hence $2k_{\mathbb{D}}(0, 1-\delta_t) \le C_1 + \log(tt^{1/\beta})$, where

$$C_1 := 2k_{\mathbb{D}}(0, \phi_1(z_0)) + \log(C).$$

Therefore, for every $t > T_1$,

$$|\phi_t(z_0) - \tau| \ge \frac{e^{-C_1}}{tt^{1/\beta}}. \qquad \square$$

Next example shows that the estimates obtained in Theorem 5.3 and Theorem 5.6 (1) are sharp.

Example 5.7. Fix $0 < \alpha \le 1$ and consider the holomorphic function $h(z) = (\frac{1+z}{1-z})^{2\alpha}$, $z \in \mathbb{D}$. Consider the semigroup (ϕ_t) given by $\phi_t(z) = h^{-1}(h(z) + t)$. Its Denjoy–Wolff point is 1 and

$$\angle\lim_{z\to 1} h(z)(1-z)^{2\alpha} = \angle\lim_{z\to 1}(1+z)^{2\alpha} = 2^{2\alpha} \in \mathbb{C}\setminus\{0\}.$$

Therefore, bearing in mind that

$$t(1-\overline{\tau}\phi_t(z))^{2\alpha} = \frac{t}{h(z)+t}\, h(\phi_t(z))(1-\overline{\tau}\phi_t(z))^{2\alpha}$$

and that $\phi_t(z)$ converges to 1 non-tangentially for all z, we conclude that

$$\lim_{t\to+\infty} t(1-\phi_t(z))^{2\alpha} = 2^{2\alpha}.$$

6. Rate of convergence at the boundary

Suppose that (ϕ_t) is a semigroup in $\mathbb{D}$. By a theorem of P. Gumenyuk (Theorem 2.3), for every $t \geq 0$ and every $\sigma \in \partial\mathbb{D}$, the angular limit

$$\phi_t(\sigma) := \angle \lim_{z \to \sigma} \phi_t(z)$$

exists. Moreover (Proposition 3.2 in [16]), for every $z \in \overline{\mathbb{D}}$, the function

$$[0, +\infty) \ni t \mapsto \phi_t(z) \in \mathbb{C}$$

is continuous. So we can consider the trajectory

$$\gamma_z : [0, +\infty) \to \overline{\mathbb{D}}, \quad \gamma_z(t) := \phi_t(z)$$

for every $z \in \overline{\mathbb{D}}$. If $z \in \partial\mathbb{D}$ and the trajectory γ_z enters the unit disk (namely, for some $t > 0$, $\gamma_z(t) \in \mathbb{D}$), then the study of the rate of convergence is reduced to the trajectories starting from an interior point. The following theorem is about the case where the whole trajectory lies on $\partial\mathbb{D}$.

Theorem 6.1. *Suppose that (ϕ_t) is a semigroup in $\mathbb{D}$ with Denjoy–Wolff point $\tau \in \partial\mathbb{D}$. Let $(\Omega, h, z \mapsto z + t)$ be its holomorphic model given in Theorem 2.5. Let $z \in \partial\mathbb{D}$, $z \neq \tau$, and assume that $\phi_t(z) \in \partial\mathbb{D}$ for every $t \geq 0$ and $\lim_{t \to +\infty} \phi_t(z) = \tau$. Then the semigroup is of positive hyperbolic step and one of the following two cases occurs:*

(a) *The semigroup (ϕ_t) is hyperbolic and there exists a constant $C_1 = C_1(z)$ such that*

$$|\phi_t(z) - \tau| \leq C_1 \, e^{-\lambda t}, \quad t \in (0, +\infty), \tag{6.1}$$

where λ is the spectral value of the semigroup.

(b) *The semigroup (ϕ_t) is parabolic of positive hyperbolic step and there exists a constant $C_2 = C_2(z)$ such that*

$$|\phi_t(z) - \tau| \leq \frac{C_2}{t}, \quad t \in (0, +\infty). \tag{6.2}$$

In addition, if (b) *is satisfied with $\Omega = \Pi^+$ (resp. $\Omega = \Pi^-$) and $h(\mathbb{D}) \subset S_p(0, \alpha)$ (resp. $h(\mathbb{D}) \subset S_p(\alpha, 0)$) for some $p \in \mathbb{C}$ and $0 < \alpha < 1$, then there is a constant C_3 such that*

$$|\phi_t(z) - \tau| \leq \frac{C_3}{t^{1/\alpha}}, \quad t \in (0, +\infty). \tag{6.3}$$

Proof. For $t > 0$, let ℓ_t be the length of the circular arc Γ_t on the unit circle with end points τ and $\phi_t(z)$. Then

$$|\phi_t(z) - \tau| \leq \ell_t = 2\pi \, \omega(0, \Gamma_t, \mathbb{D}) = 2\pi \, \omega(h(0), h(\Gamma_t), h(\mathbb{D})).$$

Note that h, being a univalent function, extends almost everywhere on the unit circle by taking angular limits. The set $h(\Gamma_t)$ is a horizontal half-line emanating from the point $h(\phi_t(z))$ and lies on the boundary of $h(\mathbb{D})$. Therefore, the whole horizontal line passing from the point $h(\phi_t(z))$ is contained in the complement of $h(\mathbb{D})$. It follows that either $h(\mathbb{D})$ is contained in a horizontal strip, or it is contained in a horizontal half-plane and it is not contained in any horizontal strip (see Theorem 2.5 and the very definition of holomorphic model). Thus the hyperbolic step is positive.

(a) If $h(\mathbb{D})$ is contained in a horizontal strip, then the semigroup is hyperbolic with $\Omega = \mathbb{S}_{\frac{\pi}{\lambda}}$ where λ is the spectral value of the semigroup. In fact, $h(\Gamma_t) \subset \mathbb{R} \cup (\mathbb{R} + i\pi/\lambda)$. Thus

$$h(\Gamma_t) = [\mathsf{Re}\, h(\phi_t(z)), +\infty) + i\,\mathsf{Im}\, h(\phi_t(z)) = [\mathsf{Re}\, h(z) + t, +\infty) + i\,\mathsf{Im}\, h(z),$$

being $\mathsf{Im}\, h(z) \in \{0, \pi/\lambda\}$. By well-known estimates for harmonic measures on strips (see e.g. [4]),

$$\omega(h(0), h(\Gamma_t), h(\mathbb{D})) \leq \omega(h(0), h(\Gamma_t), \mathbb{S}_{\pi/\lambda}) \leq C_1\, e^{-\lambda t},$$

where C_1 is a constant that depends on $\mathsf{Re}\, h(z)$.

(b) If $h(\mathbb{D})$ is not contained in a horizontal strip, then the semigroup is parabolic with positive hyperbolic step. Assume $\Omega = \Pi^+$ (the argument is similar if $\Omega = \Pi^-$). In this case, $h(\phi_t(z)) \in \mathbb{R}$ and $\Gamma_t = [h(\phi_t(z)), +\infty) = [h(z) + t, +\infty)$. By Theorem 4.3.13 in [20], there is a constant C_2, that depends on $\mathsf{Re}\, h(z)$, such that

$$\omega(h(0), h(\Gamma_t), h(\mathbb{D})) \leq \frac{C_2}{t}.$$

Finally, assume $h(\mathbb{D}) \subset S_p(0, \alpha)$ for some $p \in \mathbb{C}$ and $0 < \alpha < 1$. Up to a translation, we may assume that $p = 0$. Arguing as above, and taking t large enough such that $h(z) + t > 0$, we deduce

$$\begin{aligned}\omega(h(0), h(\Gamma_t), h(\mathbb{D})) &\leq \omega(h(0), [h(z) + t, +\infty), S(0, \alpha)) \\ &= \omega(h(0)^{1/\alpha}, [(h(z) + t)^{1/\alpha}, +\infty), \Pi^+),\end{aligned}$$

ending the proof applying again Theorem 4.3.13 in [20]. □

Remark 6.2. Concerning the hypothesis of the above theorem, it is worth mentioning that in Remark 5.1 of [16] it was proved that given $z \in \partial\mathbb{D}$ then either z is a boundary fixed point of the semigroup different from the Denjoy–Wolff point or $\lim_{t\to+\infty} \phi_t(z) = \tau$.

7. Appendix

Lemma 7.1. *Let (ϕ_t) be a non-elliptic semigroup in $\mathbb{D}$ with Denjoy–Wolff point $\tau \in \partial\mathbb{D}$. Then, for each $z \in \mathbb{D}$, there is a positive constant $C(z)$, such that*

$$k_{\mathbb{D}}(z, \phi_t(z)) \leq \frac{1}{2} \log \frac{C(z)}{|1 - \overline{\tau}\phi_t(z)|^2}.$$

Proof. By Julia's lemma, we have that

$$\frac{|1-\overline{\tau}\,\phi_t(z)|^2}{1-|\phi_t(z)|^2} \leq \alpha_{\phi_t}(\tau)\,\frac{|\tau-z|^2}{1-|z|^2} \leq \frac{|\tau-z|^2}{1-|z|^2}, \tag{7.1}$$

for all $z\in\mathbb{D}$ and $t\geq 0$. Denote by T_z the automorphism of the unit disc given by $T_z(w):=(z-w)/(1-\overline{z}w)$. Therefore,

$$\begin{aligned}
k_{\mathbb{D}}(z,\phi_t(z)) &= \frac{1}{2}\log\frac{1+|T_z(\phi_t(z))|}{1-|T_z(\phi_t(z))|} = \frac{1}{2}\log\frac{(1+|T_z(\phi_t(z))|)^2}{1-|T_z(\phi_t(z))|^2}\\
&= \frac{1}{2}\log\frac{(1+|T_z(\phi_t(z))|)^2\,|1-\overline{z}\,\phi_t(z)|^2}{(1-|z|^2)\,(1-|\phi_t(z)|^2)}\\
&\leq \frac{1}{2}\log\frac{(1+|T_z(\phi_t(z))|)^2\,|1-\overline{z}\,\phi_t(z)|^2|\tau-z|^2}{(1-|z|^2)^2\,|1-\overline{\tau}\,\phi_t(z)|^2}\\
&\leq \frac{1}{2}\log\frac{16\,|\tau-z|^2}{(1-|z|^2)^2\,|1-\overline{\tau}\,\phi_t(z)|^2}.
\end{aligned}$$

□

Theorem 7.2 (Theorem 1 (b) in [4])**.** *Let (ϕ_t) be a parabolic semigroup in $\mathbb{D}$ of positive hyperbolic step with Denjoy–Wolff point $\tau\in\partial\mathbb{D}$ and canonical model $(\Omega,h,z\mapsto z+t)$. Then for each $z\in\mathbb{D}$, there is a constant $C=C(z)$ such that, for every $t\geq 0$,*

$$t\,|1-\overline{\tau}\,\phi_t(z)|\leq C. \tag{7.2}$$

Proof. We may assume that $\Omega=\Pi^+$. In case $\Omega=\Pi^-$, the proof follows the same steps. Let us get a lower estimation of $k_{\mathbb{D}}(z,\phi_t(z))$:

$$\begin{aligned}
k_{\mathbb{D}}(z,\phi_t(z)) &= k_\Omega(h(z),h(z)+t)\geq k_{\Pi^+}(h(z),h(z)+t) = k_{\Pi^+}(\mathsf{Im}\,h(z),\mathsf{Im}\,h(z)+t)\\
&= \frac{1}{2}\log\frac{(2\,\mathsf{Im}\,h(z))^2+2t^2+2t\sqrt{(2\,\mathsf{Im}\,h(z))^2+t^2}}{(2\,\mathsf{Im}\,h(z))^2}. \qquad (7.3)
\end{aligned}$$

Thus, Lemma 7.1 shows that

$$\left[(2\,\mathsf{Im}\,h(z))^2+2t^2+2t\sqrt{(2\,\mathsf{Im}\,h(z))^2+t^2}\right]|1-\overline{\tau}\,\phi_t(z)|^2\leq (2\,\mathsf{Im}\,h(z))^2\,C(z)$$

for some positive constant $C(z)$. That is,

$$t^2|1-\overline{\tau}\,\phi_t(z)|^2\leq (2\,\mathsf{Im}\,h(z))^2\,C(z)\,\frac{t^2}{(2\,\mathsf{Im}\,h(z))^2+2t^2+2t\sqrt{(2\,\mathsf{Im}\,h(z))^2+t^2}}.$$

Noticing that, for fixed $z\in\mathbb{D}$, the right hand term of this inequality is bounded, the proof is complete. □

Remark 7.3. The proof of the above theorem also works replacing a parabolic semigroup of positive hyperbolic step by a univalent parabolic function of positive hyperbolic step and the real number t by the natural number n.

Remark 7.4. If the semigroup is of zero hyperbolic step, the above proof does not give the best result. We can replace in (7.3) the role of the domain $\mathbb{H}$ by $\mathbb{C} \setminus B$ where B is an appropriate half-line. But the point is that Lemma 7.1 is not a good enough bound in the zero hyperbolic step case. The *reason* is that when we use Julia's lemma to get

$$\frac{|1-\overline{\tau}\phi_t(z)|^2}{1-|\phi_t(z)|^2} \leq \frac{|\tau - z|^2}{1-|z|^2},$$

the inequality is *close* to be an equality if the semigroup is of positive hyperbolic step. But in the zero case, this is far from being true.

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