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Regularity estimates in weighted Morrey spaces for quasilinear elliptic equations

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Abstract. We study regularity for solutions of quasilinear elliptic equations of the form $\operatorname{div} \mathbf{A}(x,u,\nabla u) = \operatorname{div} \mathbf{F}$ in bounded domains in $\mathbb{R}^n$. The vector field $\mathbf{A}$ is assumed to be continuous in u, and its growth in ∇u is like that of the p-Laplace operator. We establish interior gradient estimates in weighted Morrey spaces for weak solutions u to the equation under a small BMO condition in x for $\mathbf{A}$. As a consequence, we obtain that ∇u is in the classical Morrey space $\mathcal{M}^{q,\lambda}$ or weighted space L^q_w whenever $|\mathbf{F}|^{1/(p-1)}$ is respectively in $\mathcal{M}^{q,\lambda}$ or L^q_w, where q is any number greater than p and w is any weight in the Muckenhoupt class $A_{q/p}$. In addition, our two-weight estimate allows the possibility to acquire the regularity for ∇u in a weighted Morrey space that is different from the functional space that the data $|\mathbf{F}|^{1/(p-1)}$ belongs to.

1. Introduction

We investigate interior gradient estimates in weighted Morrey space for bounded weak solutions to the following general elliptic equations of p-Laplacian type,

$$\operatorname{div} \mathbf{A}(x,u,\nabla u) = \operatorname{div} \mathbf{F} \quad \text{in } \Omega, \tag{1.1}$$

when Ω is a bounded domain in $\mathbb{R}^n$ ($n \geq 1$) and the vector field $\mathbf{A}$ is only continuous in the u variable and possibly discontinuous in the x variable. Without loss of generality we take Ω to be the Euclidean ball $B_{10} := \{x \in \mathbb{R}^n : |x| < 10\}$, whose special radius is simply to avoid working with many fractions in our arguments later on. Let $\mathbb{K} \subset \mathbb{R}$ be an open interval and consider the general vector field

$$\mathbf{A} = \mathbf{A}(x,z,\xi) : B_{10} \times \overline{\mathbb{K}} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n,$$

which is a Carathéodory map, that is, $\mathbf{A}(x,z,\xi)$ is measurable in x for every $(z,\xi) \in \overline{\mathbb{K}} \times \mathbb{R}^n$ and continuous in (z,ξ) for a.e. x. We assume that $\xi \mapsto \mathbf{A}(x,z,\xi)$ is

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differentiable on $\mathbb{R}^n\setminus\{0\}$ for a.e. x and all $z\in\overline{\mathbb{K}}$. Also, there exist constants $\Lambda>0$, $1<p<\infty$, and a nondecreasing and right continuous function $\omega\colon[0,\infty)\to[0,\infty)$ with $\omega(0)=0$ such that the following conditions are satisfied for a.e. $x\in B_{10}$ and all $z\in\overline{\mathbb{K}}$:

$$\langle\partial_\xi \mathbf{A}(x,z,\xi)\eta,\eta\rangle \geq \Lambda^{-1}|\xi|^{p-2}|\eta|^2 \qquad \forall\xi\in\mathbb{R}^n\setminus\{0\} \text{ and } \forall\eta\in\mathbb{R}^n, \tag{1.2}$$

$$|\mathbf{A}(x,z,\xi)|+|\xi|\,|\partial_\xi\mathbf{A}(x,z,\xi)|\leq\Lambda|\xi|^{p-1} \qquad \forall\xi\in\mathbb{R}^n, \tag{1.3}$$

$$|\mathbf{A}(x,z_1,\xi)-\mathbf{A}(x,z_2,\xi)|\leq\Lambda|\xi|^{p-1}\omega(|z_1-z_2|) \qquad \forall z_1,z_2\in\overline{\mathbb{K}} \text{ and } \forall\xi\in\mathbb{R}^n. \tag{1.4}$$

The class of equations of the form (1.1) with $\mathbf{A}$ satisfying (1.2)–(1.4) contains the well known p-Laplace equations and, more generally, equations of the type

$$\operatorname{div}\mathbf{A}(x,\nabla u)=\operatorname{div}\mathbf{F} \quad \text{in } \Omega. \tag{1.5}$$

Gradient estimates for (1.5) with discontinuous coefficient have been studied extensively by several authors [3], [4], [7], [8], [9], [11], [12], [15], [20], [22], [24], [26], [27], [35], [38]. Some of these developments rely essentially on the perturbation method developed by Caffarelli and Peral [4], which allows one to deal with discontinuous coefficient and highly nonlinear structure in gradient of equation (1.5).

In this paper we study general quasilinear equation (1.1) when the principal part also depends on the z variable. In the case $\mathbf{A}$ is Lipschitz continuous in both x and z variables, the interior $C^{1,\alpha}$ regularity for locally bounded weak solutions to the corresponding homogeneous equation was established by DiBenedetto [6] and Tolksdorf [37], extending the celebrated $C^{1,\alpha}$ estimates by Uraltceva [39], Uhlenbeck [38], Evans [13], and Lewis [25] for the homogeneous p-Laplace equation. When $\mathbf{A}$ is not necessarily continuous in x but has sufficiently small BMO oscillation, it was established in [31] that: $|\mathbf{F}|^{1/(p-1)}\in L^q_{\mathrm{loc}}\Longrightarrow\nabla u\in L^q_{\mathrm{loc}}$ for any $q>p$. It is required in [31] that $\mathbf{A}$ is Lipschitz continuous in the z variable since the uniqueness property for the frozen equation is used there. However, this condition is weaken in a recent paper [2] allowing only Hölder continuity in z for $\mathbf{A}$ (i.e., $\omega(s)=s^\alpha$ for some $\alpha\in(0,1)$ in (1.4)).

Our purpose of the current work is to extend the mentioned result in [31], [2] by deriving interior estimates in weighted Morrey spaces for gradients of locally bounded weak solutions to nonhomogeneous equation (1.1). In order to state our main results, let us recall the so-called Muckenhoupt class of A_s weights. By definition, a weight is a nonnegative locally integrable function on $\mathbb{R}^n$ that is positive almost everywhere. A weight w belongs to the class A_s, $1<s<\infty$, if

$$[w]_{A_s}:=\sup\Big(\fint_B w(x)\,dx\Big)\Big(\fint_B w(x)^{-1/(s-1)}\,dx\Big)^{s-1}<\infty,$$

where the supremum is taken over all balls B in $\mathbb{R}^n$. We also say that w belongs to the class A_∞ if

$$[w]_{A_\infty}:=\sup\Big(\fint_B w(x)\,dx\Big)\exp\Big(\fint_B \log w(x)^{-1}\,dx\Big)<\infty.$$

Now let $U \subset \mathbb{R}^n$ be a bounded open set, let w be a weight, $1 \le q < \infty$, and let φ be a positive function on the set of nonempty open balls in $\mathbb{R}^n$. A function $g\colon U \to \mathbb{R}$ is said to belong to the weighted space $L^q_w(U)$ if

$$\|g\|_{L^q_w(U)} := \Big(\int_U |g(x)|^q \, w(x)\, dx \Big)^{1/q} < \infty.$$

We define the weighted Morrey space $\mathcal{M}^{q,\varphi}_w(U)$ to be the set of all functions $g \in L^q_w(U)$ satisfying

$$\|g\|_{\mathcal{M}^{q,\varphi}_w(U)} := \sup_{\bar{x}\in U,\, 0<r\le \operatorname{diam}(U)} \Big(\frac{\varphi(B_r(\bar{x}))}{w(B_r(\bar{x}))} \int_{B_r(\bar{x})\cap U} |g(x)|^q \, w(x)\, dx \Big)^{1/q} < \infty. \tag{1.6}$$

Notice that $\mathcal{M}^{q,\varphi}_w(U) = L^q_w(U)$ if $\varphi(B) = w(B)$, and we obtain the classical Morrey space $\mathcal{M}^{q,\lambda}(U)$ $(0 \le \lambda \le n)$ by taking $w = 1$ and $\varphi(B) = |B|^{\lambda/n}$. It is worth observing that $\mathcal{M}^{q,\lambda}(U) = L^q(U)$ when $\lambda = n$ and $\mathcal{M}^{q,\lambda}(U) = L^\infty(U)$ when $\lambda = 0$. For brevity, the Morrey space $\mathcal{M}^{q,\varphi}_w(U)$ with $w = 1$ will be denoted by $\mathcal{M}^{q,\varphi}(U)$. Let us next introduce some slight restrictions on the function φ.

Definition 1.1. Let φ be a positive function on the set of nonempty open balls in $\mathbb{R}^n$. We say that φ belongs to the class $\mathcal{B}_\alpha$ with $\alpha \ge 0$ if there exists $C > 0$ such that

$$\frac{\varphi(B_r(x))}{\varphi(B_s(x))} \le C\Big(\frac{r}{s}\Big)^\alpha$$

for every $x \in \mathbb{R}^n$ and every $0 < r \le s$. We define $\mathcal{B}_+ := \cup_{\alpha>0}\mathcal{B}_\alpha \subset \mathcal{B}_0$.

The function $\varphi(B) := |B|^{\alpha/n}$ with $\alpha > 0$ is an example of functions in $\mathcal{B}_+$. Also for any weight $w \in A_\infty$, the function $\varphi(B) := w(B)$ belongs to the class $\mathcal{B}_+$ due to the characterization (2.2) below. However, a member of $\mathcal{B}_+$ does not need to be even a measure. Hereafter, for a ball $B \subset \mathbb{R}^n$ we set $\bar{\mathbf{A}}_B(z,\xi) := \fint_B \mathbf{A}(x,z,\xi)\, dx$. Also, the conjugate exponent of a number $l \in (1,\infty)$ is denoted by l'. Our first main result is:

Theorem 1.2. *Let* $\mathbf{A}$ *satisfy* (1.2)–(1.4) *with* $p > 1$, *and let* w *be an* A_s *weight for some* $1 < s < \infty$. *Then for any* $q \ge p$, $M_0 > 0$, *and* $\varphi \in \mathcal{B}_+$ *with* $\sup_{x\in B_1} \varphi(B_2(x)) < \infty$, *there exists a constant* $\delta = \delta(p, q, n, \omega, \Lambda, M_0, s, [w]_{A_s}) > 0$ *such that: if*

$$\sup_{0<\rho\le 1}\ \sup_{y\in B_{15/2}}\ \sup_{z\in\overline{\mathbb{K}}\cap[-M_0,M_0]} \fint_{B_\rho(y)} \Big[\sup_{\xi\ne 0} \frac{|\mathbf{A}(x,z,\xi) - \bar{\mathbf{A}}_{B_\rho(y)}(z,\xi)|}{|\xi|^{p-1}} \Big]\, dx \le \delta, \tag{1.7}$$

and u *is a weak solution of*

$$\operatorname{div}\mathbf{A}(x,u,\nabla u) = \operatorname{div}\mathbf{F} \quad \text{in } B_{10} \tag{1.8}$$

satisfying $\|u\|_{L^\infty(B_{10})} \le M_0$, *we have*

$$\begin{aligned} \|\nabla u\|_{\mathcal{M}^{q,\varphi}_w(B_1)} \le C\big(\|\nabla u\|_{L^p(B_{10})} &+ \|\mathrm{M}_{B_{10}}(|\mathbf{F}|^{p/(p-1)})\|^{1/p}_{\mathcal{M}^{1,\varphi^{p/q}}(B_{10})} \\ &+ \|\mathrm{M}_{B_{10}}(|\mathbf{F}|^{p/(p-1)})\|^{1/p}_{\mathcal{M}^{q/p,\varphi}_w(B_{10})}\big). \end{aligned} \tag{1.9}$$

Here $\mathrm{M}_{B_{10}}$ *denotes the centered Hardy–Littlewood maximal operator (see Definition 3.1), and* $C > 0$ *is a constant depending only on* q, p, n, ω, Λ, M_0, φ, s, *and* $[w]_{A_s}$.

The above theorem holds true for any weight w in the class A_∞. When $q > p$ and certain additional information about the weights and φ and ϕ is given, we can further estimate the two quantities in (1.9) involving the maximal function of $|\mathbf{F}|^{p/(p-1)}$ to obtain:

Theorem 1.3 (Weighted Morrey space estimate). *Let* $\mathbf{A}$ *satisfy* (1.2)–(1.4) *with* $p > 1$. *Let* $q > p$, $w \in A_\infty$, $v \in A_{q/p}$, $\varphi \in \mathcal{B}_+$ *with* $\sup_{x\in B_1} \varphi(B_2(x)) < \infty$, *and* $\phi \in \mathcal{B}_0$ *satisfy*

$$[w, v^{1-(q/p)'}]_{A_{q/p}} := \sup_B \Big(\fint_B w\, dx\Big)\Big(\fint_B v^{1-(q/p)'}\, dx\Big)^{q/p-1} < \infty, \tag{1.10}$$

$$\frac{v(2B)}{w(2B)} \frac{1}{\phi(2B)} \le C_* \frac{1}{\varphi(B)} \quad \text{for all balls } B \subset \mathbb{R}^n. \tag{1.11}$$

Then for any $M_0 > 0$, *there exists a small constant* $\delta = \delta(p, q, n, \omega, \Lambda, M_0, [w]_{A_\infty}) > 0$ *such that: if* (1.7) *holds and* u *is a weak solution of* (1.8) *satisfying* $\|u\|_{L^\infty(B_{10})} \le M_0$, *we have*

$$\|\nabla u\|_{\mathcal{M}^{q,\varphi}_w(B_1)} \le C \left(\|\nabla u\|_{L^p(B_{10})} + \||\mathbf{F}|^{1/(p-1)}\|_{\mathcal{M}^{q,\phi}_v(B_{10})}\right). \tag{1.12}$$

Here $C > 0$ *is a constant depending only on* q, p, n, ω, Λ, M_0, φ, ϕ, C_*, $[w]_{A_\infty}$, $[v]_{A_{q/p}}$, *and* $[w, v^{1-(q/p)'}]_{A_{q/p}}$.

Remark 1.4. If $\phi(2B) \le C\varphi(B)$ for all balls B, then conditions (1.10)–(1.11) imply that $v(x) \le CC_* w(x)$ a.e. and $v \in A_{q/p}$ with $[v]_{A_{q/p}} \le CC_*[w, v^{1-(q/p)'}]_{A_{q/p}}$.

Estimate (1.12) is a two-weight inequality which allows the possibility to acquire the regularity for ∇u in a weighted Morrey space that is different from the functional space that the data $|\mathbf{F}|^{1/(p-1)}$ belongs to. In particular, the result in Theorem 1.3 is even new when applied to equations of special form (1.5), whose gradient estimates in the classical Morrey spaces are obtained in [35], [26], [27]. Our results also improve the L^q gradient estimates established in [31], [2] for equation (1.1) since the principal part $\mathbf{A}(x, z, \xi)$ is merely assumed to be continuous in the z variable instead of being Hölder continuous.

One of the main difficulties in proving the above main results is that equations of the form (1.8) are not invariant with respect to dilations and rescaling of domains due to the dependence of $\mathbf{A}(x, z, \xi)$ on the z variable. To handle this issue, we use the key idea introduced in [19], [31] by enlarging the class of equations under consideration in a suitable way. Precisely, we consider the associated quasilinear elliptic equations with two parameters, i.e., equation (2.4) below. The class of these equations is the smallest one that is invariant with respect to dilations and rescaling of domains and that contains equations of the form (1.8). Given the invariant structure, we employ the direct argument in [1], [2] to show that the gradient of the solution u can be approximated by a bounded gradient in L^p norm (see Proposition 4.1). It is essential for us that this approximation is uniform with respect to the two parameters. With this and through a standard procedure, we derive some density estimates and obtain Theorem 2.4 about gradient estimates in weighted L^q spaces (see Subsections 5.1 and 5.2). This estimate plays a central role in proving our main results. Indeed, by using the trick in [10], [27] we show in Subsection 5.3 that Theorem 1.2 can be derived as a consequence of Theorem 2.4. In order to prove Theorem 1.3, we need to further estimate the last two terms in inequality (1.9) involving the maximal function of $|\mathbf{F}|^{p'}$. This is another difficulty that needs to overcome and it can be solved if one has a suitable estimate for the maximal function in weighted Morrey spaces. This type of estimate is proved by Chiarenza and Frasca [5] in the unweighted setting, and there are some recent works [17], [18], [23], [29] establishing the estimate in the weighted setting under certain conditions. However, they are inadequate for our purpose and we need to extend these results by attaining in Theorem 3.4 a weighted Morrey space estimate for the maximal function. This two-weight inequality holds true for quite general weights and we believe that it is of independent interest.

We end the introduction by pointing out that our method of establishing interior gradient estimates in weighted Morrey spaces could be combined with the boundary techniques used in [3], [2], [26], [27] to derive global estimates for Reifenberg flat domains and homogeneous Dirichlet boundary condition. Furthermore, by following [34] it might be possible to weaken the assumption $u \in L^\infty$ in Theorem 1.3 by assuming only that $u \in \mathrm{BMO}$.

The organization of the paper is as follows. We recall some basic properties of A_s weights in Subsection 2.1 and state a key result (Theorem 2.4) in Subsection 2.2 about gradient estimates in weighted L^q spaces. In Section 3, we derive a weighted Morrey space estimate for the maximal function (see Theorem 3.4 and its corollaries). Section 4 is devoted to proving Proposition 4.1 which shows that gradients of weak solutions to two-parameter equation (2.4) can be approximated by bounded gradients under some smallness conditions. Using this crucial result, we establish in Section 5 some density estimates for gradients and then prove Theorem 2.4. In this same section, the main results stated in Theorem 1.2 and Theorem 1.3 are derived as consequences of Theorem 2.4.

2. Preliminaries and a key result

2.1. Some basic properties of A_s weights

Let us recall some properties of weights which can be found in Chapter 9 of [16]. Given a weight w and a measurable set $E \subset \mathbb{R}^n$, we use the notation $dw(x) = w(x)\,dx$ and $w(E) = \int_E w(x)\,dx$.

Lemma 2.1. *Let $w \in A_s$ for some $1 < s < \infty$. Then:*

1) *There exist $0 < \beta \le 1$ and $K > 0$, depending only on n and $[w]_{A_s}$, such that*

$$[w]_{A_s}^{-1} \left(\frac{|E|}{|B|}\right)^s \le \frac{w(E)}{w(B)} \le K \left(\frac{|E|}{|B|}\right)^\beta \tag{2.1}$$

for all balls B and all measurable sets $E \subset B$. In particular, w is doubling with $w(2B) \le 2^{ns}[w]_{A_s} w(B)$.

2) *The function $w^{1-s'} \in A_{s'}$ with characteristic constant $[w^{1-s'}]_{A_{s'}} = [w]_{A_s}^{1/(s-1)}$.*
3) *The classes A_s are increasing as s increases; precisely, for $1 < s < q \le \infty$ we have $[w]_{A_q} \le [w]_{A_s}$.*
4) *For any $1 < q \le \infty$, we have $A_q = \bigcup_{1<s<q} A_s$.*

Lemma 2.2 (Characterizations of A_∞ weights)**.** *Suppose that w is a weight. Then w is in A_∞ if and only if there exist $A > 0$ and $\nu < \infty$ such that for all balls B and all measurable sets $E \subset B$ we have*

$$\frac{w(E)}{w(B)} \le A\Big(\frac{|E|}{|B|}\Big)^{\nu}. \tag{2.2}$$

When $w \in A_\infty$, the above constants A and ν depend only on n and $[w]_{A_\infty}$. Conversely, given constants A and ν satisfying (2.2), we have $[w]_{A_\infty} \le C(n, A, \nu)$.

2.2. Quasilinear equations with two parameters

Our goal is to derive interior gradient estimates for weak solutions to (1.8). Let us consider a function $u \in W^{1,p}_{\mathrm{loc}}(B_{rR})$ such that $u(y) \in \overline{\mathbb{K}}$ for a.e. $y \in B_{rR}$ and u satisfies $\operatorname{div} \mathbf{A}(y, u, \nabla u) = \operatorname{div} \mathbf{F}$ in B_{rR} in the sense of distribution. Then the rescaled function

$$v(x) := \frac{u(rx)}{\mu r} \quad \text{for } r, \mu > 0 \tag{2.3}$$

has the properties: $v(x) \in \frac{1}{\mu r}\overline{\mathbb{K}}$ for a.e. $x \in B_R$, and v solves the equation

$$\operatorname{div} \mathbf{A}_{\mu,r}(x, \mu r v, \nabla v) = \operatorname{div} \mathbf{F}_{\mu,r} \quad \text{in } B_R$$

in the distributional sense. Here,

$$\mathbf{A}_{\mu,r}(x, z, \xi) := \frac{\mathbf{A}(rx, z, \mu\xi)}{\mu^{p-1}} \quad \text{and} \quad \mathbf{F}_{\mu,r}(x) := \frac{\mathbf{F}(rx)}{\mu^{p-1}}.$$

It is clear that if $\mathbf{A}\colon B_{rR} \times \overline{\mathbb{K}} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ satisfies conditions (1.2)–(1.4), then the rescaled vector field $\mathbf{A}_{\mu,r}\colon B_R \times \overline{\mathbb{K}} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ also satisfies these conditions with the same constants.

The above observation shows that equations of type (1.8) are not invariant with respect to the standard scalings (2.3). This presents a serious obstacle in obtaining weighted L^q estimates for their solutions by using the methods in [3], [26], [27]. Here we follow the idea in [19], [31] by considering associated quasilinear equations with two parameters

$$\operatorname{div}\Big[\frac{\mathbf{A}(x, \lambda\theta u, \lambda\nabla u)}{\lambda^{p-1}}\Big] = \operatorname{div} \mathbf{F} \quad \text{in } B_{10}, \tag{2.4}$$

with λ, $\theta > 0$. The class of these equations is the smallest one that is invariant with respect to the transformations (2.3) and that contains equations of type (1.8).

Indeed, if u solves (2.4) and v is given by (2.3), then v satisfies an equation of similar form, namely, $\operatorname{div}[\mathbf{A}'(y,\lambda'\theta' v,\lambda'\nabla v)/(\lambda')^{p-1}] = \operatorname{div}\mathbf{F}'$ in $B_{10/r}$ with $\mathbf{A}'(y,z,\xi) := \mathbf{A}(ry,z,\xi)$, $\mathbf{F}'(y) := \mathbf{F}(ry)/\mu^{p-1}$, $\lambda' := \mu\lambda$, and $\theta' := r\theta$.

Let us give the precise definition of weak solutions that is used throughout the paper.

Definition 2.3. Let $\mathbf{F} \in L^{p'}(B_{10};\mathbb{R}^n)$. A function $u \in W^{1,p}_{\text{loc}}(B_{10})$ is called a weak solution of (2.4) if $u(x) \in \frac{1}{\lambda\theta}\overline{\mathbb{K}}$ for a.e. $x \in B_{10}$ and

$$\int_{B_{10}} \Big\langle \frac{\mathbf{A}(x,\lambda\theta u,\lambda\nabla u)}{\lambda^{p-1}}, \nabla\varphi\Big\rangle\, dx = \int_{B_{10}} \langle \mathbf{F},\nabla\varphi\rangle\, dx \quad \forall\varphi \in W^{1,p}_0(B_{10}).$$

Fix a number $M_0 > 0$. Then for a ball $B \subset \mathbb{R}^n$, we define

$$\Theta_B(\mathbf{A}) := \sup_{z\in\overline{\mathbb{K}}\cap[-M_0,M_0]} \fint_B \Big[\sup_{\xi\neq 0} \frac{|\mathbf{A}(x,z,\xi) - \bar{\mathbf{A}}_B(z,\xi)|}{|\xi|^{p-1}}\Big]\, dx. \tag{2.5}$$

In order to achieve the main results stated in Section 1, we will prove the following gradient estimate in weighted L^q spaces:

Theorem 2.4. *Let $\mathbf{A}$ satisfy (1.2)–(1.4) with $p > 1$, and let w be an A_s weight for some $1 < s < \infty$. For any $q \geq p$ and $M_0 > 0$, there exists a constant $\delta = \delta(p,q,n,\omega,\Lambda,M_0,s,[w]_{A_s}) > 0$ such that: if $\lambda > 0$, $\theta > 0$,*

$$\sup_{0<\rho\leq 1/5}\ \sup_{y\in B_{3/2}} \Theta_{B_\rho(y)}(\mathbf{A}) \leq \delta, \tag{2.6}$$

and u is a weak solution of (2.4) satisfying $\|u\|_{L^\infty(B_2)} \leq M_0/(\lambda\theta)$, then

$$\fint_{B_1} |\nabla u|^q\, dw \leq C\Big(\|\nabla u\|^q_{L^p(B_2)} + \fint_{B_{3/2}} \mathrm{M}_{B_2}(|\mathbf{F}|^{p/(p-1)})^{q/p}\, dw\Big). \tag{2.7}$$

Here $C > 0$ is a constant depending only on q, p, n, ω, Λ, M_0, s, and $[w]_{A_s}$.

The proof of this key result will be given in Subsection 5.2, and we will demonstrate in Subsection 5.3 that the gradient estimates in weighted Morrey spaces (Theorem 1.2) can be derived as a direct consequence of Theorem 2.4. Notice that by combining Theorem 2.4 with Muckenhoupt's strong type weighted estimate for the maximal function (see [28] and Theorem 9.1.9 in [16]), we immediately get:

Corollary 2.5 (Weighted L^q space estimate)**.** *Let $\mathbf{A}$ satisfy (1.2)–(1.4) with $p > 1$. Then for any $q > p$, $M_0 > 0$, and any weight $w \in A_{q/p}$, there exists a constant $\delta > 0$ such that: if $\lambda > 0$, $\theta > 0$, (2.6) holds, and u is a weak solution of (2.4) satisfying $\|u\|_{L^\infty(B_2)} \leq M_0/(\lambda\theta)$, we have*

$$\fint_{B_1} |\nabla u|^q\, dw \leq C\Big(\|\nabla u\|^q_{L^p(B_2)} + \fint_{B_2} |\mathbf{F}|^{q/(p-1)}\, dw\Big).$$

Here C, δ are constants depending only on q, p, n, ω, Λ, M_0, and $[w]_{A_{q/p}}$.

3. Weighted Morrey space estimates for the maximal function

This section is devoted to deriving a weighted Morrey space estimate for the maximal function. Let us first recall standard maximal operators used in this paper.

Definition 3.1. Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. The centered Hardy–Littlewood maximal function of f is defined by

$$\mathrm{M}(f)(x) = \sup_{\rho>0} \fint_{B_\rho(x)} |f(y)|\, dy.$$

The uncentered Hardy–Littlewood maximal operator $\tilde{\mathrm{M}}$ is defined over balls by

$$\tilde{\mathrm{M}}(f)(x) = \sup_{B\ni x} \fint_B |f(y)|\, dy.$$

If $U \subset \mathbb{R}^n$ is an open set and $f \in L^1(U)$, then we denote $\mathrm{M}_U(f) = \mathrm{M}(f\chi_U)$ and $\tilde{\mathrm{M}}_U(f) = \tilde{\mathrm{M}}(f\chi_U)$.

The following characterization about the two-weight norm inequality

$$\|\tilde{\mathrm{M}}(f)\|_{L^p_w(\mathbb{R}^n)} \le C\|f\|_{L^p_v(\mathbb{R}^n)} \tag{3.1}$$

is due to Sawyer [36] and Kairema [21].

Theorem 3.2 (Two-weight norm inequality)**.** *Let w and v be two weights, and let $1 < q < \infty$. Then we have*

$$\|\tilde{\mathrm{M}}(f)\|_{L^q_w(\mathbb{R}^n)} \le C_n\, q'[w, v^{1-q'}]_{S_q}\, \|f\|_{L^q_v(\mathbb{R}^n)} \quad \forall f \in L^q_v(\mathbb{R}^n),$$

where

$$[w,\sigma]_{S_q} := \sup_B \Big(\frac{1}{\sigma(B)} \int_B [\tilde{\mathrm{M}}(\sigma\chi_B)]^q\, w\, dx\Big)^{1/q}.$$

Moreover, if there exists $C > 0$ such that (3.1) holds for all $f \in L^q_v(\mathbb{R}^n)$, then $[w, v^{1-q'}]_{S_q} < \infty$.

Remark 3.3. From the definition of $\tilde{\mathrm{M}}$, it is obvious that

$$[w, v^{1-q'}]_{A_q} := \sup_B \Big(\fint_B w(x)\, dx\Big)\Big(\fint_B v(x)^{1-q'}\, dx\Big)^{q-1} \le [w, v^{1-q'}]^q_{S_q}.$$

Our main result in this subsection is:

Theorem 3.4. *Let w, v be two weights and $1 < q < \infty$. Assume that φ and ϕ are two positive functions on the set of nonempty open balls in $\mathbb{R}^n$. Then for any bounded open set $U \subset \mathbb{R}^n$, we have*

$$\|\tilde{\mathrm{M}}_U(f)\|_{\mathcal{M}^{q,\varphi}_w(U)} \le C(n,q)\, [w, v^{1-q'}]_{S_q}\, [w\varphi, v\phi]_q\, \|f\|_{\mathcal{M}^{q,\phi}_v(U)} \tag{3.2}$$

for any function $f \in L^1(U)$*. Here,*

$$[w\varphi, v\phi]_q := \sup_{y\in U,\, 0<r\le \mathrm{diam}(U)} \Big[\frac{v(B_{2r}(y))}{w(B_r(y))}\,\frac{\phi(B_{2r}(y))^{-1}}{\varphi(B_r(y))^{-1}} + \sup_{2r\le s\le 2\mathrm{diam}(U)} \frac{v(B_s(y))}{w(B_s(y))}\,\frac{\phi(B_s(y))^{-1}}{\varphi(B_r(y))^{-1}}\Big]^{1/q}.$$

Proof. Without loss of generality, we can assume that $f \in L^1(U)$ and $f \ge 0$.

For each $y \in U$ and $0 < r \le \mathrm{diam}(U)$, let us write $f = f\chi_{B_{2r}(y)} + f\chi_{\mathbb{R}^n\setminus B_{2r}(y)} =: f_1 + f_2$. Then $\tilde{\mathrm{M}}_U(f) \le \tilde{\mathrm{M}}_U(f_1) + \tilde{\mathrm{M}}_U(f_2)$, and hence

$$\begin{aligned}\|\tilde{\mathrm{M}}_U(f)\|_{L^q_w(B_r(y)\cap U)} &\le \|\tilde{\mathrm{M}}(f_1\chi_U)\|_{L^q_w(\mathbb{R}^n)} + \|\tilde{\mathrm{M}}(f_2\chi_U)\|_{L^q_w(B_r(y)\cap U)}\\ &\le C_n\, q'[w, v^{1-q'}]_{S_q}\|f_1\chi_U\|_{L^q_v(\mathbb{R}^n)} + \|\tilde{\mathrm{M}}(f_2\chi_U)\|_{L^q_w(B_r(y)\cap U)},\end{aligned} \tag{3.3}$$

where we have used Theorem 3.2 to obtain the last inequality. Now for any $x \in B_r(y)\cap U$ and any ball $B_t(z) \ni x$, we clearly have $B_t(z) \subset B_{2t+r}(y)$. Therefore, by considering separately the case $t \le r/2$ and the case $t > r/2$, we see that

$$\tilde{\mathrm{M}}(\chi_U f_2)(x) \le 4^n \sup_{s>2r} \frac{1}{|B_s(y)|}\int_{B_s(y)\setminus B_{2r}(y)} f\chi_U\, dz =: 4^n I_r(y) \quad \forall x \in B_r(y)\cap U.$$

This together with (3.3) yields

$$\|\tilde{\mathrm{M}}_U f\|_{L^q_w(B_r(y)\cap U)} \le C_n\, q'[w, v^{1-q'}]_{S_q}\, \|f\|_{L^q_v(B_{2r}(y)\cap U)} + 4^n\, w(B_r(y))^{1/q}\, I_r(y) \tag{3.4}$$

for every $y \in U$ and $0 < r \le \mathrm{diam}(U)$. Using definition (1.6) and (3.4), we get

$$\begin{aligned}\|\tilde{\mathrm{M}}_U(f)\|_{\mathcal{M}^{q,\varphi}_w(U)} &= \sup_{y\in U,\, r\le \mathrm{diam}(U)} \Big(\frac{\varphi(B_r(y))}{w(B_r(y))}\int_{B_r(y)\cap U} |\tilde{\mathrm{M}}_U(f)|^q\, dw\Big)^{1/q}\\ &\le C_n\, q'[w, v^{1-q'}]_{S_q} \sup_{y\in U,\, r\le \mathrm{diam}(U)} \Big(\frac{v(B_{2r}(y))}{w(B_r(y))}\,\frac{\varphi(B_r(y))}{\phi(B_{2r}(y))}\Big)^{1/q} J_{2r}(y)\\ &\quad + 4^n \sup_{y\in U,\, r\le \mathrm{diam}(U)} \varphi(B_r(y))^{1/q}\, I_r(y),\end{aligned}$$

with

$$J_t(y) := \Big(\frac{\phi(B_t(y))}{v(B_t(y))}\int_{B_t(y)\cap U} |f|^q\, dv\Big)^{1/q} \le \|f\|_{\mathcal{M}^{q,\phi}_v(U)}.$$

When $r \ge \mathrm{diam}(U)/2$, we have $U \subset B_{2r}(y)$ and hence $I_r(y) = 0$. Therefore, we deduce from the above inequality that

$$\begin{aligned}&\|\tilde{\mathrm{M}}_U(f)\|_{\mathcal{M}^{q,\varphi}_w(U)}\\ &\le C_n\, q'[w, v^{1-q'}]_{S_q}\|f\|_{\mathcal{M}^{q,\phi}_v(U)} \sup_{y\in U,\, r\le \mathrm{diam}(U)} \Big(\frac{v(B_{2r}(y))}{w(B_r(y))}\,\frac{\varphi(B_r(y))}{\phi(B_{2r}(y))}\Big)^{1/q}\\ &\quad + 4^n \sup_{y\in U,\, r<\frac{1}{2}\mathrm{diam}(U)} \varphi(B_r(y))^{1/q}\, I_r(y).\end{aligned} \tag{3.5}$$

We use Hölder inequality to estimate $I_r(y)$ for $r < \frac{1}{2}\mathrm{diam}(U)$ as follows:

$$\begin{aligned} I_r(y) &\leq \sup_{2r<s} \fint_{B_s(y)} f\chi_U\,dz = \sup_{2r<s\leq \mathrm{diam}(U)} \fint_{B_s(y)} f\chi_U\,dz \\ &\leq \sup_{2r<s\leq \mathrm{diam}(U)} \frac{1}{|B_s(y)|}\Big(\int_{B_s(y)} v^{-1/(q-1)}\,dz\Big)^{(q-1)/q}\Big(\int_{B_s(y)} |f\chi_U|^q\,dv(z)\Big)^{1/q} \\ &= \sup_{2r<s\leq \mathrm{diam}(U)} \Big[\Big(\fint_{B_s(y)} v\,dz\Big)\Big(\fint_{B_s(y)} v^{-1/(q-1)}\,dz\Big)^{q-1}\Big]^{1/q}\Big(\fint_{B_s(y)} |f\chi_U|^q\,dv(z)\Big)^{1/q}. \end{aligned}$$

It then follows from Remark 3.3 that

$$I_r(y) \leq [w, v^{1-q'}]_{S_q} \sup_{2r<s\leq \mathrm{diam}(U)} \Big(\frac{v(B_s(y))}{w(B_s(y))}\Big)^{1/q}\Big(\fint_{B_s(y)} |f\chi_U|^q\,dv(z)\Big)^{1/q}.$$

Plugging this into (3.5), we conclude that

$$\begin{aligned} &\|\tilde{\mathrm{M}}_U(f)\|_{\mathcal{M}^{q,\varphi}_w(U)} \\ &\leq C_n\, q'\,[w, v^{1-q'}]_{S_q}\,\|f\|_{\mathcal{M}^{q,\phi}_v(U)} \sup_{y\in U,\, r\leq \mathrm{diam}(U)} \Big(\frac{v(B_{2r}(y))}{w(B_r(y))}\,\frac{\varphi(B_r(y))}{\phi(B_{2r}(y))}\Big)^{1/q} \\ &\quad + 4^n\,[w, v^{1-q'}]_{S_q} \sup_{y\in U,\, r<\frac{1}{2}\mathrm{diam}(U)}\ \sup_{2r<s\leq \mathrm{diam}(U)} \Big(\frac{v(B_s(y))}{w(B_s(y))}\,\frac{\varphi(B_r(y))}{\phi(B_s(y))}\Big)^{1/q} J_s(y) \\ &\leq C(n,q)\,[w, v^{1-q'}]_{S_q}\,[w\varphi, v\phi]_q\,\|f\|_{\mathcal{M}^{q,\phi}_v(U)}. \end{aligned}$$

This gives estimate (3.2) as desired. □

Remark 3.5. By inspecting the proof we see that inequality (3.2) also holds true if U is replaced by $\mathbb{R}^n$.

Given Theorem 3.4, it is desirable to have concrete conditions on w, v, φ, and ϕ ensuring that $[w, v^{1-q'}]_{S_q} < \infty$ and $[w\varphi, v\phi]_q < \infty$. The finiteness of the constant $[w, v^{1-q'}]_{S_q}$ has been investigated extensively (see [14], [30], [32], [33], [40]) and for this to be true it is necessary that $[w, v^{1-q'}]_{A_q} < \infty$ (see Remark 3.3). In particular, we have from Corollary 1.3 in [32] and Corollary 1.4 in [33] that $[w, v^{1-q'}]_{S_q} < \infty$ if any of the following two conditions is satisfied:

(**A**) There exists $r > 1$ such that $\displaystyle\sup_B \Big(\fint_B w\,dx\Big)\Big(\fint_B v^{r(1-q')}\,dx\Big)^{(q-1)/r} < \infty.$

(**B**) $[w, v^{1-q'}]_{A_q} < \infty$ and $v^{1-q'} \in A_\infty$.

We refer readers to [32], [33] for more general conditions guaranteeing the finiteness of $[w, v^{1-q'}]_{S_q}$. On the other hand, it is clear that $[w\varphi, v\phi]_q < \infty$ if w is doubling and there exists a constant $C_* > 0$ such that

$$\sup_{2r\leq s\leq 2\mathrm{diam}(U)} \frac{v(B_s(y))}{w(B_s(y))}\,\frac{1}{\phi(B_s(y))} \leq C_*\,\frac{1}{\varphi(B_r(y))} \tag{3.6}$$

for all $y \in U$ and $0 < r \leq \mathrm{diam}(U)$. For convenience, we summarize the above discussions into two separate results:

Corollary 3.6 (Two-weight case)**.** *Let $U \subset \mathbb{R}^n$ be a bounded open set. Assume that the hypotheses of Theorem 3.4 and condition (3.6) hold, and w is doubling. Assume in addition that either* (**A**) *or* (**B**) *is satisfied. Then $[w, v^{1-q'}]_{S_q} < \infty$ and there exists a constant $C > 0$ depending only on n, q, C_*, the doubling constant for w, and $[w, v^{1-q'}]_{S_q}$, such that*

$$\|\tilde{\mathrm{M}}_U(f)\|_{\mathcal{M}^{q,\varphi}_w(U)} \le C\,\|f\|_{\mathcal{M}^{q,\phi}_v(U)} \quad \forall f \in L^1(U).$$

Corollary 3.7 (One-weight case)**.** *Let $1 < q < \infty$, $w \in A_q$, and let $U \subset \mathbb{R}^n$ be a bounded open set. Assume that φ and ϕ are two positive functions on the set of open balls in $\mathbb{R}^n$ such that there exists $C_* > 0$ satisfying*

$$\sup_{2r \le s \le 2\,\mathrm{diam}(U)} \phi(B_s(y))^{-1} \le C_*\, \varphi(B_r(y))^{-1} \tag{3.7}$$

for all $y \in U$ and $0 < r \le \mathrm{diam}(U)$. Then there exists a constant $C > 0$ depending only on n, q, C_*, and $[w]_{A_q}$ such that

$$\|\tilde{\mathrm{M}}_U(f)\|_{\mathcal{M}^{q,\varphi}_w(U)} \le C\,\|f\|_{\mathcal{M}^{q,\phi}_w(U)} \quad \textit{for any } f \in L^1(U).$$

Proof. Since $w \in A_q$, we have from Lemma 2.1 that $w^{1-q'} \in A_{q'} \subset A_\infty$. Moreover, $[w, w^{1-q'}]_{A_q} = [w]_{A_q}$. Therefore, the conclusion is a consequence of Corollary 3.6 and the first inequality in (2.1). □

Notice that if $\phi = \varphi$, then condition (3.7) trivially holds if $\varphi \in \mathcal{B}_0$ defined in Definition 1.1. In particular, we recover the estimate by Chiarenza and Frasca in [5] by taking $\varphi(B) := |B|^{\lambda/n}$ with $0 \le \lambda \le n$. Furthermore, Corollary 3.7 extends the weighted Morrey space estimates obtained recently in [17], [18], [23], and [29].

4. Approximating gradients of solutions

The purpose of this section is to show that gradients of weak solutions to (2.4) can be approximated by bounded gradients in an invariant way. Proposition 4.1 below is an extension of Lemma 5.3 in [31] and Lemma 4.2 in [2] where the modulus ω is respectively required to be $\omega(s) = s$ (Lipschitz continuity) and $\omega(s) = s^\alpha$ with $\alpha \in (0,1)$ (Hölder continuity). Here by a simple modification of the arguments in [2] we prove that the approximation holds true for any modulus of continuity ω. This result plays a crucial role in our derivation of gradient estimates for solutions of (1.8).

Proposition 4.1. *Let* **A** *satisfy* (1.2)–(1.4) *with $p > 1$, and let $M_0 > 0$. For any $\varepsilon \in (0,1]$, there exist small positive constants δ and σ depending only on ε, p, n, ω, Λ, and M_0 such that: if $\lambda > 0$, $\theta > 0$, $r > 0$,*

$$\Theta_{B_{3\sigma r}}(\mathbf{A}) \le \delta \quad \textit{and} \quad \fint_{B_{4r}} |\mathbf{F}|^{p'}\, dx \le \delta,$$

and $u \in W^{1,p}_{\mathrm{loc}}(B_{4r})$ *is a weak solution of* $\operatorname{div}[\mathbf{A}(x, \lambda\theta u, \lambda\nabla u)/\lambda^{p-1}] = \operatorname{div}\mathbf{F}$ *in* B_{4r} *satisfying*

$$\|u\|_{L^\infty(B_{4r})} \leq \frac{M_0}{\lambda\theta}, \quad \fint_{B_{4r}} |\nabla u|^p\,dx \leq 1, \quad \text{and} \quad \fint_{B_{4\sigma r}} |\nabla u|^p dx \leq 1,$$

then

$$\fint_{B_{3\sigma r}} |\nabla u - \nabla v|^p\,dx \leq \varepsilon^p$$

for some function $v \in W^{1,p}(B_{3\sigma r})$ *with*

$$\|\nabla v\|^p_{L^\infty(B_{2\sigma r})} \leq C(p,n,\Lambda) \fint_{B_{3\sigma r}} |\nabla v|^p\,dx.$$

Remark 4.2. Since the class of our equations is invariant under the transformation $x \mapsto x + y$, Proposition 4.1 still holds true if B_r is replaced by $B_r(y)$.

We will use the fact (see the proof of Lemma 1 in [37]) that condition (1.2) implies that

$$\langle \mathbf{A}(x,z,\xi) - \mathbf{A}(x,z,\eta), \xi - \eta\rangle \geq \begin{cases} 4^{1-p}\Lambda^{-1}|\xi-\eta|^p & \text{if } p \geq 2,\\ (4\Lambda)^{-1}\big(|\xi| + |\eta|\big)^{p-2}|\xi-\eta|^2 & \text{if } 1 < p < 2, \end{cases} \tag{4.1}$$

for a.e. $x \in B_{10}$, all $z \in \overline{\mathbb{K}}$, and all ξ, $\eta \in \mathbb{R}^n$. The next result is the first step of proving Proposition 4.1.

Lemma 4.3. *For any* $\varepsilon > 0$*, there exist small positive constants* δ *and* σ *depending only on* ε, p, n, ω, Λ, *and* M_0 *such that: if* $\lambda > 0$, $\theta > 0$,

$$\fint_{B_4} |\mathbf{F}|^{p'}\,dx \leq \delta,$$

and $u \in W^{1,p}_{\mathrm{loc}}(B_4)$ *is a weak solution of* $\operatorname{div}[\mathbf{A}(x, \lambda\theta u, \lambda\nabla u)/\lambda^{p-1}] = \operatorname{div}\mathbf{F}$ *in* B_4 *satisfying*

$$\|u\|_{L^\infty(B_4)} \leq \frac{M_0}{\lambda\theta} \quad \text{and} \quad \fint_{B_4} |\nabla u|^p\,dx \leq 1,$$

then

$$\fint_{B_{4\sigma}} |\nabla u - \nabla f|^p\,dx \leq \varepsilon^p,$$

where f *is a weak solution of*

$$\begin{cases} \operatorname{div}\Big[\dfrac{\mathbf{A}(x, \lambda\theta\bar{h}_{B_{4\sigma}}, \lambda\nabla f)}{\lambda^{p-1}}\Big] = 0 & \text{in } B_{4\sigma},\\ f = h & \text{on } \partial B_{4\sigma}, \end{cases} \tag{4.2}$$

with $\bar{h}_{B_{4\sigma}} := \fint_{B_{4\sigma}} h(x)\,dx$, *and* h *being a weak solution of*

$$\begin{cases} \operatorname{div}\Big[\dfrac{\mathbf{A}(x, \lambda\theta u, \lambda\nabla h)}{\lambda^{p-1}}\Big] = 0 & \text{in } B_4,\\ h = u & \text{on } \partial B_4. \end{cases} \tag{4.3}$$

The proof of this result relies on the following classical regularity result for weak solutions of homogeneous equation (4.3) (see for instance estimates (6. 64) and (7.45) in [15]).

Lemma 4.4 (Classical interior regularity)**.** *Assume that* $\mathbf{a}(x,\xi)$ *satisfies* (1.2)–(1.3) *with* $p>1$. *Let* $R\in(0,10)$ *and* $w\in W^{1,p}(B_R)$ *be a weak solution of* $\operatorname{div}\mathbf{a}(x,\nabla w)=0$ *in* B_R. *Then there exist constants* $p_0\in(p,\infty)$, $\beta\in(0,1)$, *and* $C>0$ *depending only on* p, n, *and* Λ, *such that for every* $r\in(0,R)$ *we have*

$$\Big(\fint_{B_r}|\nabla w|^{p_0}\,dx\Big)^{1/p_0}\le C\Big(\fint_{B_R}|\nabla w|^{p}\,dx\Big)^{1/p}$$

and

$$\sup_{x\in B_r}|w(x)-\bar{w}_{B_r}|\le C\left(\frac{r}{R}\right)^{\beta}\|w\|_{L^\infty(B_R)}.$$

Proof of Lemma 4.3. We present the complete proof only for the case $p\ge 2$ by using an idea in [2], and will indicate at the end how the argument can be modified for the case $1<p<2$. Consider $p\ge 2$ and, for convenience, let $\tilde{\mathbf{A}}(x,z,\xi):=\mathbf{A}(x,\lambda\theta z,\lambda\xi)/\lambda^{p-1}$. We write

$$\nabla u-\nabla f=\nabla(u-h)+\nabla(h-f)$$

and will estimate $\|\nabla(u-h)\|_{L^p(B_{4\sigma})}$ and $\|\nabla(h-f)\|_{L^p(B_{4\sigma})}$. By using $u-h$ as a test function in the equations for u and h, we have

$$\int_{B_4}\langle\tilde{\mathbf{A}}(x,u,\nabla u)-\tilde{\mathbf{A}}(x,u,\nabla h),\nabla(u-h)\rangle\,dx=\int_{B_4}\langle\mathbf{F},\nabla(u-h)\rangle\,dx.$$

We then use (4.1) to bound the above left-hand side from below. As a consequence, we obtain

$$\int_{B_4}|\nabla(u-h)|^p\,dx\le 4^{p-1}\Lambda\int_{B_4}\langle\mathbf{F},\nabla(u-h)\rangle\,dx.$$

Then we infer from Young's inequality that

$$\int_{B_4}|\nabla(u-h)|^p\,dx\le C\int_{B_4}|\mathbf{F}|^{p'}\,dx, \tag{4.4}$$

yielding

$$\fint_{B_{4\sigma}}|\nabla(u-h)|^p\,dx\le\frac{C}{\sigma^n}\fint_{B_4}|\mathbf{F}|^{p'}dx. \tag{4.5}$$

By using $h-f$ as a test function in the equations for h and f, we have

$$\int_{B_{4\sigma}}\langle\tilde{\mathbf{A}}(x,\bar{h}_{B_{4\sigma}},\nabla f),\nabla(h-f)\rangle\,dx=\int_{B_{4\sigma}}\langle\tilde{\mathbf{A}}(x,u,\nabla h),\nabla(h-f)\rangle\,dx.$$

This together with (4.1) gives

$$\begin{aligned}\int_{B_{4\sigma}} |\nabla(h-f)|^p\,dx &\leq 4^{p-1}\Lambda \int_{B_{4\sigma}} \langle \tilde{\mathbf{A}}(x,\bar{h}_{B_{4\sigma}},\nabla h) - \tilde{\mathbf{A}}(x,\bar{h}_{B_{4\sigma}},\nabla f), \nabla(h-f)\rangle\,dx\\ &= 4^{p-1}\Lambda \int_{B_{4\sigma}} \langle \tilde{\mathbf{A}}(x,\bar{h}_{B_{4\sigma}},\nabla h) - \tilde{\mathbf{A}}(x,u,\nabla h), \nabla(h-f)\rangle\,dx\\ &\leq 4^{p-1}\Lambda \int_{B_{4\sigma}} \min\{2\Lambda, \omega(\lambda\theta|u-\bar{h}_{B_{4\sigma}}|)\}\,|\nabla h|^{p-1}|\nabla(h-f)|\,dx. \end{aligned} \tag{4.6}$$

As a consequence of (4.6) and Young's inequality, we obtain

$$\int_{B_{4\sigma}} |\nabla(h-f)|^p\,dx \leq C(p,\Lambda)\int_{B_{4\sigma}} |\nabla h|^p\,dx.$$

On the other hand, it follows from equation (4.3) for h that $\int_{B_{4\sigma}} |\nabla h|^p\,dx \leq C\int_{B_4} |h|^p\,dx$. Thus we conclude that

$$⨍_{B_{4\sigma}} |\nabla(h-f)|^p\,dx \leq \frac{C}{\sigma^n}\int_{B_4} |h|^p\,dx \leq \frac{C_*}{\sigma^n}\|h\|^p_{L^\infty(B_4)} \leq \frac{C_*}{\sigma^n}\|u\|^p_{L^\infty(B_4)} \leq \frac{C_*}{\sigma^n}\Big(\frac{M_0}{\lambda\theta}\Big)^p.$$

This together with (4.5) gives the desired conclusion if $\frac{C_*}{\sigma^n}(M_0/(\lambda\theta))^p \leq \varepsilon^p/2$. We hence only need to consider the case

$$\frac{C_*}{\sigma^n}\Big(\frac{M_0}{\lambda\theta}\Big)^p > \frac{\varepsilon^p}{2}. \tag{4.7}$$

For this, note first that (4.4) and the assumption yield

$$\begin{aligned}\|\nabla h\|_{L^p(B_4)} &\leq \|\nabla(h-u)\|_{L^p(B_4)} + \|\nabla u\|_{L^p(B_4)}\\ &\leq C\Big[\Big(\int_{B_4} |\mathbf{F}|^{p'}\,dx\Big)^{1/p} + 1\Big] \leq C.\end{aligned} \tag{4.8}$$

We deduce from (4.6), Young's inequality, the higher integrability for ∇h given by Lemma 4.4, and (4.8) that

$$\begin{aligned}⨍_{B_{4\sigma}} |\nabla(h-f)|^p\,dx &\leq C ⨍_{B_{4\sigma}} \omega(\lambda\theta|u-\bar{h}_{B_{4\sigma}}|)^{p'}\,|\nabla h|^p\,dx\\ &\leq C\Big[⨍_{B_{4\sigma}} \omega(\lambda\theta|u-\bar{h}_{B_{4\sigma}}|)^{p'p_0/(p_0-p)}\,dx\Big]^{(p_0-p)/p_0}\Big[⨍_{B_{4\sigma}} |\nabla h|^{p_0}\,dx\Big]^{p/p_0}\\ &\leq C\Big[⨍_{B_{4\sigma}} \omega(\lambda\theta|u-\bar{h}_{B_{4\sigma}}|)^{p'p_0/(p_0-p)}\,dx\Big]^{(p_0-p)/p_0} ⨍_{B_4} |\nabla h|^p\,dx\\ &\leq C\Big[⨍_{B_{4\sigma}} \omega(\lambda\theta|u-\bar{h}_{B_{4\sigma}}|)^{p'p_0/(p_0-p)}\,dx\Big]^{(p_0-p)/p_0}\end{aligned}$$

with $p_0 > p$ and $C > 0$ depending only on p, n, and Λ. As for any $\gamma > 0$, we have

$$\begin{aligned}
&\int_{B_{4\sigma}} \omega(\lambda\theta|u-\bar{h}_{B_{4\sigma}}|)^{p'p_0/(p_0-p)}\,dx \\
&\qquad = \int_{\{B_{4\sigma}:\lambda\theta|u-\bar{h}_{B_{4\sigma}}|\le\gamma\}} \omega(\lambda\theta|u-\bar{h}_{B_{4\sigma}}|)^{p'p_0/(p_0-p)}\,dx \\
&\qquad\quad + \int_{\{B_{4\sigma}:\lambda\theta|u-\bar{h}_{B_{4\sigma}}|>\gamma\}} \omega(\lambda\theta|u-\bar{h}_{B_{4\sigma}}|)^{p'p_0/(p_0-p)}\,dx \\
&\qquad \le |B_{4\sigma}|\,\omega(\gamma)^{p'p_0/(p_0-p)} + \frac{\omega(2M_0)^{p'p_0/(p_0-p)}}{\gamma^p}\int_{B_{4\sigma}}\big(\lambda\theta|u-\bar{h}_{B_{4\sigma}}|\big)^p\,dx,
\end{aligned}$$

we infer that

$$\fint_{B_{4\sigma}}|\nabla(h-f)|^p\,dx \le C\omega(\gamma)^{p'}+C\frac{\omega(2M_0)^{p'}}{\gamma^{p(p_0-p)/p_0}}\Big[(\lambda\theta)^p\fint_{B_{4\sigma}}|u-\bar{h}_{B_{4\sigma}}|^p\,dx\Big]^{(p_0-p)/p_0} \tag{4.9}$$

for all $\gamma > 0$. Let us now estimate the last integral in (4.9). Using Sobolev's inequality and the oscillation estimate in Lemma 4.4, we have

$$\begin{aligned}
\fint_{B_{4\sigma}}|u-\bar{h}_{B_{4\sigma}}|^p\,dx &\le 2^{p-1}\Big[\frac{1}{|B_{4\sigma}|}\int_{B_4}|u-h|^p\,dx+\fint_{B_{4\sigma}}|h-\bar{h}_{B_{4\sigma}}|^p\,dx\Big] \\
&\le C\Big[\sigma^{-n}\int_{B_4}|\nabla(u-h)|^p\,dx+\big(\sigma^\beta\|h\|_{L^\infty(B_4)}\big)^p\Big].
\end{aligned}$$

This, together with (4.7) and the fact $\|h\|_{L^\infty(B_4)} \le M_0/\lambda\theta$, yields

$$(\lambda\theta)^p\fint_{B_{4\sigma}}|u-\bar{h}_{B_{4\sigma}}|^p\,dx \le C\Big[\sigma^{-2n}\big(\frac{M_0}{\varepsilon}\big)^p\int_{B_4}|\nabla(u-h)|^p\,dx+\Big(M_0\sigma^\beta\Big)^p\Big].$$

Plugging this estimate into (4.9) gives

$$\begin{aligned}
&\fint_{B_{4\sigma}}|\nabla(h-f)|^p\,dx \\
&\le C\omega(\gamma)^{p'}+C\,M_0^{p(p_0-p)/p_0}\omega(2M_0)^{p'}\Big[\frac{\sigma^{-2n}}{(\gamma\varepsilon)^p}\int_{B_4}|\nabla(u-h)|^p\,dx+\Big(\frac{\sigma^\beta}{\gamma}\Big)^p\Big]^{(p_0-p)/p_0}.
\end{aligned}$$

By combining this with (4.5) and using (4.4) we obtain

$$\begin{aligned}
\fint_{B_{4\sigma}}|\nabla(u-f)|^p\,dx &\le \frac{C}{\sigma^n}\fint_{B_4}|\mathbf{F}|^{p'}\,dx+C\omega(\gamma)^{p'} \\
&\quad +C\,M_0^{p(p_0-p)/p_0}\,\omega(2M_0)^{p'}\Big[\frac{\sigma^{-2n}}{(\gamma\varepsilon)^p}\fint_{B_4}|\mathbf{F}|^{p'}dx+\Big(\frac{\sigma^\beta}{\gamma}\Big)^p\Big]^{(p_0-p)/p_0}
\end{aligned} \tag{4.10}$$

for every $\gamma > 0$. From this, we get the desired conclusion by choosing γ small first, then σ, and δ last. For the case $1 < p < 2$, the proof is similar with some slight adjustments as follows, which can also be found in the proofs of Lemma 4.3 and

Lemma 4.6 in [2]. By arguing as above but using (4.1) together with Lemma 3.1 in [31], then in place of (4.5) and (4.6) we now have for every τ_1, $\tau_2 > 0$ small that

$$\begin{aligned}\fint_{B_{4\sigma}} |\nabla(u-h)|^p\,dx &\le \frac{C}{\sigma^n}\Big[\tau_1 \fint_{B_4} |\nabla u|^p\,dx + \tau_1^{(p-2)/(p-1)} \fint_{B_4} |\mathbf{F}|^{p'}\,dx\Big]\\ &\le \frac{C}{\sigma^n}\Big[\tau_1 + \tau_1^{(p-2)/(p-1)} \fint_{B_4} |\mathbf{F}|^{p'}\,dx\Big]\end{aligned}$$

and

$$\begin{aligned}&\int_{B_{4\sigma}} |\nabla(h-f)|^p\,dx\\ &\le \tau_2 \int_{B_{4\sigma}} |\nabla h|^p\,dx + C_p\Lambda\tau_2^{1-2/p} \int_{B_{4\sigma}} \langle \tilde{\mathbf{A}}(x,\bar h_{B_{4\sigma}},\nabla h) - \tilde{\mathbf{A}}(x,\bar h_{B_{4\sigma}},\nabla f), \nabla(h-f)\rangle\,dx\\ &\le \tau_2 \int_{B_{4\sigma}} |\nabla h|^p\,dx + C_p\Lambda\tau_2^{1-2/p} \int_{B_{4\sigma}} \min\{2\Lambda, \omega(\lambda\theta|u-\bar h_{B_{4\sigma}}|)\}\,|\nabla h|^{p-1}|\nabla(h-f)|\,dx.\end{aligned}$$

With these changes and by repeating the above lines of argument, we obtain the following version of (4.10) for the case $1 < p < 2$:

$$\begin{aligned}\fint_{B_{4\sigma}} |\nabla(u-f)|^p\,dx &\le \frac{C}{\sigma^n}\Big[\tau_1 + \tau_1^{(p-2)/(p-1)} \fint_{B_4} |\mathbf{F}|^{p'}\,dx\Big] + C\tau_2 + C\tau_2^{(p-2)/(p-1)}\\ &\quad\times\Big\{\omega(\gamma)^{p'} + M_0^{p(p_0-p)/p_0}\omega(2M_0)^{p'}\Big[\frac{\sigma^{-2n}}{(\gamma\varepsilon)^p}\fint_{B_4}|\mathbf{F}|^{p'}\,dx + \Big(\frac{\sigma^\beta}{\gamma}\Big)^p\Big]^{(p_0-p)/p_0}\Big\}.\end{aligned}$$

We then get the conclusion by choosing τ_2 small first, γ second, σ third, then τ_1, and δ last. □

To obtain Proposition 4.1, our second step is to show that the gradient of the solution f to (4.2) can be approximated by a bounded gradient. Precisely, we have:

Lemma 4.5. *Let $\varepsilon \in (0,1]$, and let σ be its corresponding constant given by Lemma 4.3. Let $\mathbf{F}$, u, and h be as in Lemma 4.3, and assume in addition that $\fint_{B_{4\sigma}} |\nabla u|^p\,dx \le 1$. Suppose that f is a weak solution of $\operatorname{div}[\mathbf{A}(x,\lambda\theta\bar h_{B_{4\sigma}},\lambda\nabla f)/\lambda^{p-1}] = 0$ in $B_{4\sigma}$ and v is a weak solution of*

$$\begin{cases}\operatorname{div}\Big[\dfrac{\bar{\mathbf{A}}_{B_{3\sigma}}(\lambda\theta\bar h_{B_{4\sigma}},\lambda\nabla v)}{\lambda^{p-1}}\Big] = 0 & \text{in } B_{3\sigma},\\ v = f & \text{on } \partial B_{3\sigma}.\end{cases} \tag{4.11}$$

There exist constants $p_0 \in (p,\infty)$ and $C > 0$ depending only on p, n, and Λ such that: if $p \ge 2$, then

$$\fint_{B_{3\sigma}} |\nabla f - \nabla v|^p\,dx \le C\,\Theta_{B_{3\sigma}}(\mathbf{A})^{(p_0-p)/p_0}, \tag{4.12}$$

and if $1 < p < 2$, then

$$\fint_{B_{3\sigma}} |\nabla f - \nabla v|^p\,dx \le \tau + C\tau^{(1-2/p)p'}\Theta_{B_{3\sigma}}(\mathbf{A})^{(p_0-p)/p_0} \quad \text{for every } \tau\in(0,1).$$

Proof. For convenience, let us define $\mathbf{a}(x,\xi) := \mathbf{A}(x, \lambda\theta\bar{h}_{B_{4\sigma}}, \lambda\xi)/\lambda^{p-1}$ and $\bar{\mathbf{a}}_{B_{3\sigma}}(\xi) := \fint_{B_{3\sigma}} \mathbf{a}(x,\xi)\,dx$. We first consider the case $p \geq 2$. Then using (4.1) we get

$$\int_{B_{3\sigma}} |\nabla(f-v)|^p\,dx \leq \Lambda \int_{B_{3\sigma}} \langle \bar{\mathbf{a}}_{B_{3\sigma}}(\nabla f) - \bar{\mathbf{a}}_{B_{3\sigma}}(\nabla v), \nabla(f-v)\rangle\,dx := I. \tag{4.13}$$

To estimate the term I, we use $f-v$ as a test function in equation (4.11) and the equation for f to obtain

$$\int_{B_{3\sigma}} \langle \bar{\mathbf{a}}_{B_{3\sigma}}(\nabla v), \nabla(f-v)\rangle\,dx = \int_{B_{3\sigma}} \langle \mathbf{a}(x,\nabla f), \nabla(f-v)\rangle\,dx.$$

Therefore,

$$\begin{aligned} I &= \Lambda \int_{B_{3\sigma}} \langle \bar{\mathbf{a}}_{B_{3\sigma}}(\nabla f) - \mathbf{a}(x,\nabla f), \nabla(f-v)\rangle\,dx \\ &\leq \Lambda \int_{B_{3\sigma}} \sup_{\xi\neq 0} \frac{|\mathbf{a}(x,\xi) - \bar{\mathbf{a}}_{B_{3\sigma}}(\xi)|}{|\xi|^{p-1}}\, |\nabla f|^{p-1}|\nabla(f-v)|\,dx. \end{aligned} \tag{4.14}$$

From (4.13)–(4.14), Young's inequality, the interior higher integrability for ∇f given by Lemma 4.4, and the fact $|\mathbf{a}(x,\xi)| \leq \Lambda|\xi|^{p-1}$, it follows that

$$\begin{aligned} &\fint_{B_{3\sigma}} |\nabla(f-v)|^p\,dx \leq C \fint_{B_{3\sigma}} \left[\sup_{\xi\neq 0} \frac{|\mathbf{a}(x,\xi) - \bar{\mathbf{a}}_{B_{3\sigma}}(\xi)|}{|\xi|^{p-1}}\right]^{p'} |\nabla f|^p dx \\ &\leq C\Big(\fint_{B_{3\sigma}} \left[\sup_{\xi\neq 0} \frac{|\mathbf{a}(x,\xi) - \bar{\mathbf{a}}_{B_{3\sigma}}(\xi)|}{|\xi|^{p-1}}\right]^{p'p_0/(p_0-p)} dx\Big)^{(p_0-p)/p_0} \Big(\fint_{B_{3\sigma}} |\nabla f|^{p_0} dx\Big)^{p/p_0} \\ &\leq C\Big(\fint_{B_{3\sigma}} \sup_{\xi\neq 0} \frac{|\mathbf{a}(x,\xi) - \bar{\mathbf{a}}_{B_{3\sigma}}(\xi)|}{|\xi|^{p-1}}\,dx\Big)^{(p_0-p)/p_0} \fint_{B_{4\sigma}} |\nabla f|^p\,dx, \end{aligned}$$

where $C>0$ depends only on p, n, and Λ. But we have from the definition of $\mathbf{a}$ that

$$\sup_{\xi\neq 0} \frac{|\mathbf{a}(x,\xi) - \bar{\mathbf{a}}_{B_{3\sigma}}(\xi)|}{|\xi|^{p-1}} = \sup_{\eta\neq 0} \frac{|\mathbf{A}(x,\lambda\theta\bar{h}_{B_{4\sigma}},\eta) - \bar{\mathbf{A}}_{B_{3\sigma}}(\lambda\theta\bar{h}_{B_{4\sigma}},\eta)|}{|\eta|^{p-1}},$$

and Lemma 4.3 yields

$$\begin{aligned} \fint_{B_{4\sigma}} |\nabla f|^p dx &\leq 2^{p-1}\Big[\fint_{B_{4\sigma}} |\nabla f - \nabla u|^p\,dx + \fint_{B_{4\sigma}} |\nabla u|^p\,dx\Big] \\ &\leq 2^{p-1}[\varepsilon^p + 1] \leq 2^p. \end{aligned} \tag{4.15}$$

Therefore we conclude that

$$\fint_{B_{3\sigma}} |\nabla(f-v)|^p\,dx \leq C\Big(\fint_{B_{3\sigma}} \sup_{\xi\neq 0} \frac{|\mathbf{A}(x,\lambda\theta\bar{h}_{B_{4\sigma}},\xi) - \bar{\mathbf{A}}_{B_{3\sigma}}(\lambda\theta\bar{h}_{B_{4\sigma}},\xi)|}{|\xi|^{p-1}}\,dx\Big)^{(p_0-p)/p_0}.$$

This together with the fact $\lambda\theta\bar{h}_{B_{4\sigma}} \in \overline{\mathbb{K}} \cap [-M_0, M_0]$ and the definition of $\Theta_{B_{3\sigma}}(\mathbf{A})$ given by (2.5) yields estimate (4.12). We next consider the case $1 < p < 2$. Then condition (1.2) in [31] is satisfied thanks to (4.1). Therefore, instead of (4.13) we now have from Lemma 3.1 in [31] that

$$\int_{B_{3\sigma}} |\nabla(f-v)|^p\,dx \le \tau \int_{B_{3\sigma}} |\nabla f|^p\,dx + C_p\,\tau^{1-2/p} I \quad \text{for all } \tau \in (0,1/2).$$

Then we deduce from estimate (4.14) for I and Young's inequality that

$$\fint_{B_{3\sigma}} |\nabla(f-v)|^p\,dx \le 2\tau \int_{B_{3\sigma}} |\nabla f|^p\,dx \\ + C(p,\Lambda)\tau^{(1-2/p)p'} \fint_{B_{3\sigma}} \Big[\sup_{\xi\neq 0} \frac{|\mathbf{a}(x,\xi)-\bar{\mathbf{a}}_{B_{3\sigma}}(\xi)|}{|\xi|^{p-1}}\Big]^{p'} |\nabla f|^p\,dx.$$

The first integral is estimated by (4.15) and the last integral can be estimated exactly as above. As a consequence, we obtain

$$\fint_{B_{3\sigma}} |\nabla(f-v)|^p\,dx \le 2^{p+1}\Big(\frac{4}{3}\Big)^n \tau + C(p,n,\Lambda)\,\tau^{(1-2/p)p'}\,\Theta_{B_{3\sigma}}(\mathbf{A})^{(p_0-p)/p_0}$$

for all $\tau \in (0,1/2)$. □

We are now ready to prove Proposition 4.1.

Proof of Proposition 4.1. The stated result follows by scaling and then applying Lemma 4.3 and Lemma 4.5. Precisely, let us define

$$\mathbf{A}'(x,z,\xi) = \mathbf{A}(rx,z,\xi), \quad \mathbf{F}'(x) = \mathbf{F}(rx), \quad u'(x) = r^{-1}u(rx), \quad \text{and} \quad \theta' = \theta r.$$

Then u' is a weak solution of $\operatorname{div}[\mathbf{A}'(x,\lambda\theta' u',\lambda\nabla u')/\lambda^{p-1}] = \operatorname{div}\mathbf{F}'$ in B_4 satisfying

$$\|u'\|_{L^\infty(B_4)} \le \frac{M_0}{\lambda\theta'}, \quad \fint_{B_4} |\nabla u'|^p\,dx = \fint_{B_{4r}} |\nabla u|^p\,dy \le 1,$$
$$\fint_{B_{4\sigma}} |\nabla u'|^p\,dx = \fint_{B_{4\sigma r}} |\nabla u|^p\,dy \le 1.$$

We also have

$$\Theta_{B_{3\sigma}}(\mathbf{A}') = \Theta_{B_{3\sigma r}}(\mathbf{A}) \le \delta \quad \text{and} \quad \fint_{B_4} |\mathbf{F}'|^{p'}\,dx = \fint_{B_{4r}} |\mathbf{F}|^{p'}\,dy \le \delta.$$

Therefore, we can apply Lemma 4.3 and Lemma 4.5 to obtain that

$$\fint_{B_{3\sigma}} |\nabla u' - \nabla v'|^p\,dx \le \varepsilon^p, \tag{4.16}$$

where v' is a weak solution of

$$\operatorname{div}\Big[\frac{\bar{\mathbf{A}}'_{B_\sigma}(\lambda\theta'\,\bar{h}_{B_{4\sigma}},\lambda\nabla v')}{\lambda^{p-1}}\Big] = 0 \quad \text{in } B_{3\sigma}.$$

From the well-known interior Lipschitz estimate for this constant coefficient equation we have

$$\|\nabla v'\|^p_{L^\infty(B_{2\sigma})} \le C(p,n,\Lambda)\fint_{B_{3\sigma}} |\nabla v'|^p\,dx. \tag{4.17}$$

Let $v(y) := rv'(r^{-1}y)$. Then by rescaling we obtain from (4.16) and (4.17) that

$$\fint_{B_{3\sigma r}} |\nabla u - \nabla v|^p \, dy \leq \varepsilon^p \quad \text{and} \quad \|\nabla v\|^p_{L^\infty(B_{2\sigma r})} \leq C(p,n,\Lambda) \fint_{B_{3\sigma r}} |\nabla v|^p \, dy. \quad \square$$

5. Density and gradient estimates

We derive interior gradient estimates for weak solution u of (2.4) by estimating the distribution functions of the maximal function of $|\nabla u|^p$. This is carried out in the next two subsections, while the last subsection (Subsection 5.3) is devoted to proving the main results stated in Section 1.

5.1. Density estimates

The next result gives a density estimate for the distribution of $\mathrm{M}_{B_1}(|\nabla u|^p)$. It roughly says that if the maximal function $\mathrm{M}_{B_1}(|\nabla u|^p)$ is bounded at one point in $B_{\sigma r}(y)$, then this property can be propagated for all points in $B_{\sigma r}(y)$ except on a set of small measure w.

Lemma 5.1. *Assume that* $\mathbf{A}$ *satisfies* (1.2)–(1.4) *and that* $\mathbf{F} \in L^{p'}(B_{10};\mathbb{R}^n)$. *Let* $M_0 > 0$ *and let* w *be an* A_∞ *weight. There exists a constant* $N = N(p,n,\Lambda) > 1$ *satisfying that, for any* $\varepsilon > 0$, *we can find small positive constants* δ *and* σ *depending only on* ε, p, n, ω, Λ, M_0, *and* $[w]_{A_\infty}$ *such that: if* $\lambda > 0$, $\theta > 0$,

$$\sup_{0<\rho\leq \frac{3}{5}} \sup_{y \in B_{\sigma/10}} \Theta_{B_{\sigma\rho}(y)}(\mathbf{A}) \leq \delta,$$

then for any weak solution u *of* (2.4) *with* $\|u\|_{L^\infty(B_1)} \leq M_0/(\lambda\theta)$, *and for any* $y \in B_{\sigma/10}$, $0 < r \leq 1/5$ *with*

$$B_{\sigma r}(y) \cap B_{\sigma/10} \cap \big\{B_1 : \mathrm{M}_{B_1}(|\nabla u|^p) \leq 1\big\} \cap \{B_1 : \mathrm{M}_{B_1}(|\mathbf{F}|^{p'}) \leq \delta\} \neq \emptyset, \tag{5.1}$$

we have

$$w\big(\{B_{\sigma/10} : \mathrm{M}_{B_1}(|\nabla u|^p) > N\} \cap B_{\sigma r}(y)\big) < \varepsilon\, w(B_{\sigma r}(y)).$$

Proof. By (5.1), there exists $x_0 \in B_{\sigma r}(y) \cap B_{\sigma/10}$ such that

$$\mathrm{M}_{B_1}(|\nabla u|^p)(x_0) \leq 1 \quad \text{and} \quad \mathrm{M}_{B_1}(|\mathbf{F}|^{p'})(x_0) \leq \delta. \tag{5.2}$$

This, together with the facts $B_{4r}(y) \subset B_{5r}(x_0) \cap B_1$ and $B_{4\sigma r}(y) \subset B_{5\sigma r}(x_0) \cap B_1$, implies that

$$\fint_{B_{4r}(y)} |\nabla u|^p \, dx \leq \frac{|B_{5r}(x_0)|}{|B_{4r}(y)|} \frac{1}{|B_{5r}(x_0)|} \int_{B_{5r}(x_0)\cap B_1} |\nabla u|^p \, dx \leq \left(\frac{5}{4}\right)^n,$$

$$\fint_{B_{4\sigma r}(y)} |\nabla u|^p \, dx \leq \frac{|B_{5\sigma r}(x_0)|}{|B_{4\sigma r}(y)|} \frac{1}{|B_{5\sigma r}(x_0)|} \int_{B_{5\sigma r}(x_0)\cap B_1} |\nabla u|^p \, dx \leq \left(\frac{5}{4}\right)^n,$$

$$\fint_{B_{4r}(y)} |\mathbf{F}|^{p'} \, dx \leq \frac{|B_{5r}(x_0)|}{|B_{4r}(y)|} \frac{1}{|B_{5r}(x_0)|} \int_{B_{5r}(x_0)\cap B_1} |\mathbf{F}|^{p'} \, dx \leq \left(\frac{5}{4}\right)^n \delta.$$

Therefore, we can apply Proposition 4.1 and Remark 4.2 for $\tilde{\varepsilon} \in (0,1]$ that will be determined later. As a consequence, we obtain

$$\fint_{B_{3\sigma r}(y)} |\nabla u - \nabla v|^p \, dx \leq \tilde{\varepsilon}^p \tag{5.3}$$

for some function $v \in W^{1,p}(B_{3\sigma r}(y))$ with

$$\|\nabla v\|^p_{L^\infty(B_{2\sigma r}(y))} \leq C(p,n,\Lambda) \fint_{B_{3\sigma r}(y)} |\nabla v|^p \, dx.$$

These, combined with the first inequality in (5.2), give

$$\begin{aligned} \|\nabla v\|^p_{L^\infty(B_{2\sigma r}(y))} &\leq 2^{p-1} C \Big[\fint_{B_{3\sigma r}(y)} |\nabla u - \nabla v|^p \, dx + \fint_{B_{3\sigma r}(y)} |\nabla u|^p \, dx \Big] \\ &\leq 2^{p-1} C \Big[\tilde{\varepsilon}^p + \Big(\frac{4}{3}\Big)^n \Big] \leq C_*, \end{aligned} \tag{5.4}$$

where $C^* > 0$ depends only on p, n, and Λ. We claim that (5.2)–(5.4) yield

$$\big\{B_{\sigma r}(y) : \mathrm{M}_{B_{3\sigma r}(y)}(|\nabla u - \nabla v|^p) \leq C_*\big\} \subset \big\{B_{\sigma r}(y) : \mathrm{M}_{B_1}(|\nabla u|^p) \leq N\big\} \tag{5.5}$$

with $N := \max\{2^p C_*, 3^n\}$. Indeed, let x be a point in the set on the left-hand side of (5.5), and consider $B_\rho(x)$. If $\rho \leq \sigma r$, then $B_\rho(x) \subset B_{2\sigma r}(y) \subset B_1$, and hence

$$\begin{aligned} &\frac{1}{|B_\rho(x)|} \int_{B_\rho(x) \cap B_1} |\nabla u|^p \, dy \\ &\qquad \leq \frac{2^{p-1}}{|B_\rho(x)|} \Big[\int_{B_\rho(x)\cap B_1} |\nabla u - \nabla v|^p \, dy + \int_{B_\rho(x)\cap B_1} |\nabla v|^p \, dy \Big] \\ &\qquad \leq 2^{p-1} \big[\mathrm{M}_{B_{3\sigma r}(y)}(|\nabla u - \nabla v|^p)(x) + \|\nabla v\|^p_{L^\infty(B_{2\sigma r}(y))} \big] \leq 2^p C_*. \end{aligned}$$

On the other hand, if $\rho > \sigma r$, then $B_\rho(x) \subset B_{3\rho}(x_0)$. This and the first inequality in (5.2) give

$$\frac{1}{|B_\rho(x)|} \int_{B_\rho(x)\cap B_1} |\nabla u|^p \, dy \leq \frac{3^n}{|B_{3\rho}(x_0)|} \int_{B_{3\rho}(x_0)\cap B_1} |\nabla u|^p \, dy \leq 3^n.$$

Therefore, $\mathrm{M}_{B_1}(|\nabla u|^p)(x) \leq N$ and claim (5.5) is proved. Notice that (5.5) is equivalent to

$$\big\{B_{\sigma r}(y) : \mathrm{M}_{B_1}(|\nabla u|^p) > N\big\} \subset \big\{B_{\sigma r}(y) : \mathrm{M}_{B_{3\sigma r}(y)}(|\nabla u - \nabla v|^p) > C_*\big\}.$$

It follows from this, the weak type 1-1 estimate, and (5.3) that

$$\begin{aligned} \big|\big\{B_{\sigma r}(y) : \mathrm{M}_{B_1}(|\nabla u|^p) > N\big\}\big| &\leq \big|\big\{B_{\sigma r}(y) : \mathrm{M}_{B_{3\sigma r}(y)}(|\nabla u - \nabla v|^p) > C_*\big\}\big| \\ &\leq C \int_{B_{3\sigma r}(y)} |\nabla u - \nabla v|^p \, dx \leq C_1 \, \tilde{\varepsilon}^p \, |B_{\sigma r}(y)|. \end{aligned}$$

We then infer from property (2.2) that

$$\begin{aligned} w\big(\{B_{\sigma r}(y): \mathrm{M}_{B_1}(|\nabla u|^p) > N\}\big) &\le A\Big(\frac{|\{B_{\sigma r}(y): \mathrm{M}_{B_1}(|\nabla u|^p) > N\}|}{|B_{\sigma r}(y)|}\Big)^{\nu} w(B_{\sigma r}(y)) \\ &\le A(C_1\,\tilde{\varepsilon}^{\,p})^{\nu}\, w(B_{\sigma r}(y)) \end{aligned}$$

with A and ν being the constants given by characterization (2.2) for w. We choose $\tilde{\varepsilon}^{\,p} := \min\{C_1^{-1}(\varepsilon A^{-1})^{1/\nu}, 1\}$ to complete the proof. □

In view of Lemma 5.1 and similar to Lemma 3.8 in [27], we can apply a variation of the Vitali covering lemma to obtain:

Lemma 5.2. *Assume that* $\mathbf{A}$ *satisfies* (1.2)–(1.4) *and* $\mathbf{F} \in L^{p'}(B_{10};\mathbb{R}^n)$. *Let* $M_0 > 0$ *and let* w *be an* A_s *weight for some* $1 < s < \infty$. *There exists a constant* $N = N(p,n,\Lambda) > 1$ *satisfying for any* $\varepsilon > 0$, *we can find small positive constants* δ *and* σ *depending only on* ε, p, n, ω, Λ, M_0, *and* $[w]_{A_s}$ *such that: if* $\lambda > 0$, $\theta > 0$,

$$\sup_{0<\rho\le 3/5}\ \sup_{y\in B_{\sigma/10}} \Theta_{B_{\sigma\rho}(y)}(\mathbf{A}) \le \delta,$$

then for any weak solution u *of* (2.4) *satisfying* $\|u\|_{L^\infty(B_1)} \le M_0/(\lambda\theta)$ *and*

$$w\big(\{B_{\sigma/10} : \mathrm{M}_{B_1}(|\nabla u|^p) > N\}\big) < \varepsilon\, w(B_{\sigma/10}), \tag{5.6}$$

we have

$$\begin{aligned} &w\big(\{B_{\sigma/10} : \mathrm{M}_{B_1}(|\nabla u|^p) > N\}\big) \\ &\quad \le 10^{ns}\,[w]_{A_s}^2\,\varepsilon\,\big[w\big(\{B_{\sigma/10} : \mathrm{M}_{B_1}(|\nabla u|^p) > 1\}\big) + w\big(\{B_{\sigma/10} : \mathrm{M}_{B_1}(|\mathbf{F}|^{p'}) > \delta\}\big)\big]. \end{aligned}$$

Proof. Set $C = \{B_{\sigma/10} : \mathrm{M}_{B_1}(|\nabla u|^p) > N\}$ and

$$D = \{B_{\sigma/10} : \mathrm{M}_{B_1}(|\nabla u|^p) > 1\} \cup \{B_{\sigma/10} : \mathrm{M}_{B_1}(|\mathbf{F}|^{p'}) > \delta\}.$$

Let y be any point in C, and define

$$m(r) := \frac{w(C\cap B_{\sigma r}(y))}{w(B_{\sigma r}(y))} \quad \text{for } r > 0.$$

Due to the lower semicontinuity of $\mathrm{M}_{B_1}(|\nabla u|^p)$, we have $\lim_{r\to 0^+} m(r) = 1$ as C is open. Moreover, condition (5.6) implies that $m(r) \le w(C)/w(B_{\sigma/10}) < \varepsilon$ when $r \ge 1/5$. Hence, there exists $r_y \in (0,1/5)$ such that $m(r_y) = \varepsilon$ and $m(r) < \varepsilon$ for all $r > r_y$. That is,

$$w(C\cap B_{\sigma r_y}(y)) = \varepsilon\, w(B_{\sigma r_y}(y)) \quad \text{and} \quad w(C\cap B_{\sigma r}(y)) < \varepsilon\, w(B_{\sigma r}(y)) \quad \forall r > r_y. \tag{5.7}$$

Therefore, by Vitali's covering lemma we can select a countable sequence $\{y_i\}_{i=1}^{\infty}$ such that $\{B_{\sigma r_i}(y_i)\}$ is a sequence of disjoint balls and

$$C \subset \bigcup_{i=1}^{\infty} B_{5\sigma r_i}(y_i), \tag{5.8}$$

where $r_i := r_{y_i}$. Since $w(C \cap B_{\sigma r_i}(y_i)) = \varepsilon\, w(B_{\sigma r_i}(y_i))$ by (5.7) and $[w]_{A_\infty} \le [w]_{A_s}$ by Lemma 2.1, it follows from Lemma 5.1 that

$$B_{\sigma r_i}(y_i) \cap B_{\sigma/10} \subset D. \tag{5.9}$$

Using (5.7)–(5.8) and the first inequality in (2.1), we have

$$\begin{aligned} w(C) &\le w\Big(\bigcup_{i=1}^{\infty} B_{5\sigma r_i}(y_i) \cap C\Big) \le \sum_{i=1}^{\infty} w(B_{5\sigma r_i}(y_i) \cap C) \\ &\le \varepsilon \sum_{i=1}^{\infty} w(B_{5\sigma r_i}(y_i)) \le \varepsilon\, [w]_{A_s}\, 5^{ns} \sum_{i=1}^{\infty} w(B_{\sigma r_i}(y_i)). \end{aligned} \tag{5.10}$$

We claim that

$$\sup_{y \in B_{\sigma/10},\, 0<r<1/5} \frac{|B_{\sigma r}(y)|}{|B_{\sigma r}(y) \cap B_{\sigma/10}|} \le 2^n, \tag{5.11}$$

which together with (2.1) gives $w(B_{\sigma r_i}(y_i)) \le [w]_{A_s} 2^{ns}\, w(B_{\sigma r_i}(y_i) \cap B_{\sigma/10})$. It follows from this and (5.9)–(5.10) that

$$\begin{aligned} w(C) &\le \varepsilon\, [w]_{A_s}^2\, 10^{ns} \sum_{i=1}^{\infty} w(B_{\sigma r_i}(y_i) \cap B_{\sigma/10}) = \varepsilon\, [w]_{A_s}^2\, 10^{ns}\, w\Big(\bigcup_{i=1}^{\infty} B_{\sigma r_i}(y_i) \cap B_{\sigma/10}\Big) \\ &\le \varepsilon\, [w]_{A_s}^2\, 10^{ns}\, w(D), \end{aligned}$$

which implies the desired estimate. Thus it remains to show (5.11). For this, let $y \in B_{\sigma/10}$, $r \in (0, 1/5)$, and it is enough to consider only the case $y \ne 0$. Then the line passing through y and 0 intersects $\partial B_{\sigma r}(y)$ at two distinct points, say a_1 and a_2 with a_1 being the one on the same side as the origin 0 with respect to the point y. If $a_1 \notin B_{\sigma/10}$, then $B_{\sigma/10} \subset B_{\sigma r}(y)$ since for any $x \in B_{\sigma/10}$ we have $|x-y| \le |x| + |y| = |x| + |y - a_1| - |a_1| < \sigma/10 + \sigma r - \sigma/10 = \sigma r$. It follows that

$$\frac{|B_{\sigma r}(y)|}{|B_{\sigma r}(y) \cap B_{\sigma/10}|} = \frac{|B_{\sigma r}(y)|}{|B_{\sigma/10}|} = (10r)^n < 2^n.$$

On the other hand, if $a_1 \in B_{\sigma/10}$, then by letting z be the midpoint of a_1 and y, we obviously have $B_{\sigma r/2}(z) \subset B_{\sigma r}(y)$. If 0 belongs to the line segment $[z, y]$ connecting z and y, then $|z| = |a_1| - |a_1 - z| < \sigma/10 - \sigma r/2$. In case $0 \notin [z, y]$, then $|z| = |y| - |y - z| < \sigma/10 - \sigma r/2$. Thus we always have $|z| < \sigma/10 - \sigma r/2$, and hence $B_{\sigma r/2}(z) \subset B_{\sigma/10}$ as $x \in B_{\sigma r/2}(z)$ implies that $|x| \le |x - z| + |z| < \sigma r/2 + \sigma/10 - \sigma r/2 = \sigma/10$. Therefore, we deduce that $B_{\sigma r/2}(z) \subset B_{\sigma r}(y) \cap B_{\sigma/10}$, giving

$$\frac{|B_{\sigma r}(y)|}{|B_{\sigma r}(y) \cap B_{\sigma/10}|} \le \frac{|B_{\sigma r}(y)|}{|B_{\sigma r/2}(z)|} = 2^n.$$

We then conclude that (5.11) holds, and the proof is complete. □

5.2. Gradient estimates in weighted L^q spaces

We are now ready to prove Theorem 2.4.

Proof of Theorem 2.4. Let $N = N(p,n,\Lambda) > 1$ be as in Lemma 5.2, and let $l = q/p \geq 1$. We choose $\varepsilon = \varepsilon(p,q,n,\Lambda,s,[w]_{A_s}) > 0$ be such that

$$\varepsilon_1 := 10^{ns}\,[w]_{A_s}^2\,\varepsilon = \frac{1}{2N^l},$$

and let δ and σ (depending only on p, q, n, ω, Λ, M_0, s, and $[w]_{A_s}$) be the corresponding positive constants given by Lemma 5.2. Assuming for a moment that u satisfies

$$w\big(\{B_{\sigma/10} : \mathrm{M}_{B_1}(|\nabla u|^p) > N\}\big) < \varepsilon\, w(B_{\sigma/10}), \tag{5.12}$$

then it follows from Lemma 5.2 that

$$\begin{aligned} &w\big(\{B_{\sigma/10} : \mathrm{M}_{B_1}(|\nabla u|^p) > N\}\big)\\ &\quad\leq \varepsilon_1\big[w\big(\{B_{\sigma/10} : \mathrm{M}_{B_1}(|\nabla u|^p) > 1\}\big) + w\big(\{B_{\sigma/10} : \mathrm{M}_{B_1}(|\mathbf{F}|^{p'}) > \delta\}\big)\big]. \end{aligned}\tag{5.13}$$

Let us iterate this estimate by considering

$$u_1(x) = \frac{u(x)}{N^{1/p}}, \quad \mathbf{F}_1(x) = \frac{\mathbf{F}(x)}{N^{(p-1)/p}} \quad \text{and} \quad \lambda_1 = N^{1/p}\lambda.$$

It is clear that $\|u_1\|_{L^\infty(B_1)} \leq M_0/(\lambda_1\theta)$, and that u_1 is a weak solution of $\operatorname{div}[\mathbf{A}(x,\lambda_1\theta u_1,\lambda_1\nabla u_1)/\lambda_1^{p-1}] = \operatorname{div}\mathbf{F}_1$ in B_{10}. Moreover, thanks to (5.12) we have

$$w\big(\{B_{\sigma/10} : \mathrm{M}_{B_1}(|\nabla u_1|^p) > N\}\big) = w\big(\{B_{\sigma/10} : \mathrm{M}_{B_1}(|\nabla u|^p) > N^2\}\big) < \varepsilon\, w(B_{\sigma/10}).$$

Therefore, by applying Lemma 5.2 to u_1 we obtain

$$\begin{aligned} &w\big(\{B_{\sigma/10} : \mathrm{M}_{B_1}(|\nabla u_1|^p) > N\}\big)\\ &\quad\leq \varepsilon_1\big[w\big(\{B_{\sigma/10} : \mathrm{M}_{B_1}(|\nabla u_1|^p) > 1\}\big) + w\big(\{B_{\sigma/10} : \mathrm{M}_{B_1}(|\mathbf{F}_1|^{p'}) > \delta\}\big)\big]\\ &\quad= \varepsilon_1\big[w\big(\{B_{\sigma/10} : \mathrm{M}_{B_1}(|\nabla u|^p) > N\}\big) + w\big(\{B_{\sigma/10} : \mathrm{M}_{B_1}(|\mathbf{F}|^{p'}) > \delta N\}\big)\big]. \end{aligned}$$

We infer from this and (5.13) that

$$\begin{aligned} &w\big(\{B_{\sigma/10} : \mathrm{M}_{B_1}(|\nabla u|^p) > N^2\}\big) \leq \varepsilon_1^2\, w\big(\{B_{\sigma/10} : \mathrm{M}_{B_1}(|\nabla u|^p) > 1\}\big)\\ &\quad+ \varepsilon_1^2\, w\big(\{B_{\sigma/10} : \mathrm{M}_{B_1}(|\mathbf{F}|^{p'}) > \delta\}\big) + \varepsilon_1\, w\big(\{B_{\sigma/10} : \mathrm{M}_{B_1}(|\mathbf{F}|^{p'}) > \delta N\}\big). \end{aligned}\tag{5.14}$$

Next, let

$$u_2(x) = \frac{u(x)}{N^{2/p}}, \quad \mathbf{F}_2(x) = \frac{\mathbf{F}(x)}{N^{2(p-1)/p}} \quad \text{and} \quad \lambda_2 = N^{2/p}\,\lambda.$$

Then u_2 is a weak solution of $\operatorname{div}[\mathbf{A}(x,\lambda_2\theta u_2,\lambda_2\nabla u_2)/\lambda_2^{p-1}] = \operatorname{div}\mathbf{F}_2$ in B_{10} satisfying

$$\|u_2\|_{L^\infty(B_1)} \le \frac{M_0}{\lambda_2\theta}$$

and

$$w\big(\{B_{\sigma/10} : \mathrm{M}_{B_1}(|\nabla u_2|^p) > N\}\big) = w\big(\{B_{\sigma/10} : \mathrm{M}_{B_1}(|\nabla u|^p) > N^3\}\big) < \varepsilon\, w(B_{\sigma/10}).$$

Hence, by applying Lemma 5.2 to u_2, we get

$$\begin{aligned} &w\big(\{B_{\sigma/10} : \mathrm{M}_{B_1}(|\nabla u_2|^p) > N\}\big)\\ &\quad\le \varepsilon_1\big[w\big(\{B_{\sigma/10} : \mathrm{M}_{B_1}(|\nabla u_2|^p) > 1\}\big) + w\big(\{B_{\sigma/10} : \mathrm{M}_{B_1}(|\mathbf{F}_2|^{p'}) > \delta\}\big)\big]\\ &\quad= \varepsilon_1\big[w\big(\{B_{\sigma/10} : \mathrm{M}_{B_1}(|\nabla u|^p) > N^2\}\big) + w\big(\{B_{\sigma/10} : \mathrm{M}_{B_1}(|\mathbf{F}|^{p'}) > \delta N^2\}\big)\big]. \end{aligned}$$

This together with (5.14) gives

$$\begin{aligned} &w\big(\{B_{\sigma/10} : \mathrm{M}_{B_1}(|\nabla u|^p) > N^3\}\big)\\ &\quad\le \varepsilon_1^3\, w\big(\{B_{\sigma/10} : \mathrm{M}_{B_1}(|\nabla u|^p) > 1\}\big) + \sum_{i=1}^{3} \varepsilon_1^i\, w\big(\{B_{\sigma/10} : \mathrm{M}_{B_1}(|\mathbf{F}|^{p'}) > \delta N^{3-i}\}\big). \end{aligned}$$

By repeating the iteration, we then conclude that

$$\begin{aligned} &w\big(\{B_{\sigma/10} : \mathrm{M}_{B_1}(|\nabla u|^p) > N^k\}\big)\\ &\quad\le \varepsilon_1^k\, w\big(\{B_{\sigma/10} : \mathrm{M}_{B_1}(|\nabla u|^p) > 1\}\big) + \sum_{i=1}^{k} \varepsilon_1^i\, w\big(\{B_{\sigma/10} : \mathrm{M}_{B_1}(|\mathbf{F}|^{p'}) > \delta N^{k-i}\}\big) \end{aligned}$$

for all $k = 1, 2, \ldots$ This together with

$$\begin{aligned} \int_{B_{\sigma/10}} \mathrm{M}_{B_1}(|\nabla u|^p)^l\, dw &= l\int_0^\infty t^{l-1} w\big(\{B_{\sigma/10} : \mathrm{M}_{B_1}(|\nabla u|^p) > t\}\big)\, dt\\ &= l\int_0^N t^{l-1} w\big(\{B_{\sigma/10} : \mathrm{M}_{B_1}(|\nabla u|^p) > t\}\big)\, dt\\ &\quad + l\sum_{k=1}^\infty \int_{N^k}^{N^{k+1}} t^{l-1} w\big(\{B_{\sigma/10} : \mathrm{M}_{B_1}(|\nabla u|^p) > t\}\big)\, dt\\ &\le N^l\, w(B_{\sigma/10}) + (N^l - 1)\sum_{k=1}^\infty N^{lk}\, w\big(\{B_{\sigma/10} : \mathrm{M}_{B_1}(|\nabla u|^p) > N^k\}\big) \end{aligned}$$

gives

$$\begin{aligned} \int_{B_{\sigma/10}} \mathrm{M}_{B_1}(|\nabla u|^p)^l\, dw &\le N^l w(B_{\sigma/10}) + (N^l-1) w(B_{\sigma/10}) \sum_{k=1}^\infty (\varepsilon_1 N^l)^k\\ &\quad + \sum_{k=1}^\infty \sum_{i=1}^k (N^l - 1)\, N^{lk}\, \varepsilon_1^i\, w\big(\{B_{\sigma/10} : \mathrm{M}_{B_1}(|\mathbf{F}|^{p'}) > \delta N^{k-i}\}\big). \end{aligned}$$

But we have

$$\begin{aligned}
&\sum_{k=1}^{\infty}\sum_{i=1}^{k}(N^l-1)\,N^{lk}\,\varepsilon_1^i\,w\big(\{B_{\sigma/10}:\mathrm{M}_{B_1}(|\mathbf{F}|^{p'})>\delta N^{k-i}\}\big)\\
&=\Big(\frac{N}{\delta}\Big)^l\sum_{i=1}^{\infty}(\varepsilon_1N^l)^i\Big[\sum_{k=i}^{\infty}(N^l-1)\,\delta^l\,N^{l(k-i-1)}\,w\big(\{B_{\sigma/10}:\mathrm{M}_{B_1}(|\mathbf{F}|^{p'})>\delta N^{k-i}\}\big)\Big]\\
&=\Big(\frac{N}{\delta}\Big)^l\sum_{i=1}^{\infty}(\varepsilon_1N^l)^i\Big[\sum_{j=0}^{\infty}(N^l-1)\,\delta^l\,N^{l(j-1)}\,w\big(\{B_{\sigma/10}:\mathrm{M}_{B_1}(|\mathbf{F}|^{p'})>\delta N^{j}\}\big)\Big]\\
&\le\Big(\frac{N}{\delta}\Big)^l\Big[\int_{B_{\sigma/10}}\mathrm{M}_{B_1}(|\mathbf{F}|^{p'})^l\,dw\Big]\sum_{i=1}^{\infty}(\varepsilon_1N^l)^i.
\end{aligned}$$

Thus we infer

$$\begin{aligned}
&\int_{B_{\sigma/10}}\mathrm{M}_{B_1}(|\nabla u|^p)^l\,dw\\
&\le N^lw(B_{\sigma/10})+\Big[(N^l-1)\,w(B_{\sigma/10})+\Big(\frac{N}{\delta}\Big)^l\int_{B_{\sigma/10}}\mathrm{M}_{B_1}(|\mathbf{F}|^{p'})^l\,dw\Big]\sum_{k=1}^{\infty}(\varepsilon_1N^l)^k\\
&=N^lw(B_{\sigma/10})+\Big[(N^l-1)\,w(B_{\sigma/10})+\Big(\frac{N}{\delta}\Big)^l\int_{B_{\sigma/10}}\mathrm{M}_{B_1}(|\mathbf{F}|^{p'})^l\,dw\Big]\sum_{k=1}^{\infty}2^{-k}\\
&\le C\Big(w(B_{\sigma/10})+\int_{B_{\sigma/10}}\mathrm{M}_{B_1}(|\mathbf{F}|^{p'})^l\,dw\Big),
\end{aligned}$$

with the constant C depending only on p, q, n, ω, Λ, M_0, s, and $[w]_{A_s}$. This, together with the facts $l=q/p$ and $|\nabla u(x)|^p\le\mathrm{M}_{B_1}(|\nabla u|^p)(x)$ for a.e. $x\in B_{\sigma/10}$, yields

$$\fint_{B_{\sigma/10}}|\nabla u|^q\,dw\le C\Big(1+\fint_{B_{\sigma/10}}\mathrm{M}_{B_1}(|\mathbf{F}|^{p'})^{q/p}\,dw\Big). \tag{5.15}$$

We next remove the extra assumption (5.12) for u. Notice that for any $M>0$, by using the weak type 1-1 estimate for the maximal function, we get

$$\big|\{B_{\sigma/10}:\mathrm{M}_{B_1}(|\nabla u|^p)>NM^p\}\big|\le\frac{C}{NM^p}\int_{B_1}|\nabla u|^p\,dx. \tag{5.16}$$

Let $\bar{u}(x,t)=u(x,t)/M$, where

$$M^p:=\frac{2C\,\|\nabla u\|^p_{L^p(B_1)}}{N(\varepsilon K^{-1})^{1/\beta}\,|B_{\sigma/10}|},$$

with K and β being the constants given by (2.1) and depending only on n and $[w]_{A_s}$. Then it follows from (5.16) that

$$\big|\{B_{\sigma/10}:\mathrm{M}_{B_1}(|\nabla\bar{u}|^p)>N\}\big|\le\frac{1}{2}\,(\varepsilon K^{-1})^{17\beta}\,|B_{\sigma/10}|.$$

This, together with the second inequality in (2.1), gives

$$w\big(\{B_{\sigma/10} : \mathrm{M}_{B_1}(|\nabla \bar{u}|^p) > N\}\big) < \varepsilon\, w(B_{\sigma/10}).$$

Hence we can apply (5.15) to $\bar{u}$ with $\mathbf{F}$ and λ being replaced by $\bar{\mathbf{F}} = \mathbf{F}/M^{p-1}$ and $\bar{\lambda} = \lambda M$. By reversing back to the functions u and $\mathbf{F}$, we then obtain

$$\begin{aligned}\fint_{B_{\sigma/10}} |\nabla u|^q\, dw &\leq C\Big(M^q + \fint_{B_{\sigma/10}} \mathrm{M}_{B_1}(|\mathbf{F}|^{p'})^{q/p}\, dw\Big)\\ &\leq C\Big(\|\nabla u\|^q_{L^p(B_1)} + \fint_{B_{\sigma/10}} \mathrm{M}_{B_1}(|\mathbf{F}|^{p'})^{q/p}\, dw\Big).\end{aligned}$$

Due to the translation invariance of our equation, the above estimate holds true if $B_{\sigma/10}$ and B_1 are respectively replaced by $B_{\sigma/10}(z) \subset B_{3/2}$ and $B_1(z) \subset B_2$ for any $z \in B_1$. Therefore, desired estimate (2.7) follows by covering B_1 by a finite number of balls $B_{\sigma/10}(z)$ with $z \in B_1$. □

5.3. Gradient estimates in weighted Morrey spaces

Proof of Theorem 1.2. Let $\bar{x} \in B_1$ and $0 < r \leq 2$. Then by rescaling and applying Theorem 2.4 we obtain

$$\fint_{B_r(\bar{x})} |\nabla u|^q\, dw \leq C\Big[\Big(\fint_{B_{2r}(\bar{x})} |\nabla u|^p dx\Big)^{q/p} + \fint_{B_{\frac{3}{2}r}(\bar{x})} \mathrm{M}_{B_{2r}(\bar{x})}(|\mathbf{F}|^{p'})^{q/p}\, dw\Big].$$

Since $B_{\frac{3}{2}r}(\bar{x}) \subset B_4$ and $B_{2r}(\bar{x}) \subset B_5$, it then follows that

$$\begin{aligned}\varphi(B_r(\bar{x})) \fint_{B_r(\bar{x})} |\nabla u|^q\, dw \leq & C\Big[\varphi(B_r(\bar{x}))\Big(\fint_{B_{2r}(\bar{x})} |\nabla u|^p dx\Big)^{q/p}\\ &+ \frac{\varphi(B_r(\bar{x}))}{w(B_{\frac{3}{2}r}(\bar{x}))} \int_{B_{\frac{3}{2}r}(\bar{x})\cap B_4} \mathrm{M}_{B_5}(|\mathbf{F}|^{p'})^{q/p}\, dw\Big].\end{aligned} \tag{5.17}$$

We next estimate the first term in the above right-hand side. For this, let $\varepsilon \in (0, n)$ to be determined later and use the trick in page 2506 of [27] (see also the proof of Lemma 3 in [10]) to write

$$\fint_{B_{2r}(\bar{x})} |\nabla u|^p\, dx = \omega_n^{-1}(2r)^{-\varepsilon} \int_{B_{2r}(\bar{x})} |\nabla u|^p \bar{w}\, dx \leq \omega_n^{-1}(2r)^{-\varepsilon} \int_{B_5} |\nabla u|^p \bar{w}\, dx$$

with $\omega_n := |B_1|$, and $\bar{w}$ being the weight defined by

$$\bar{w}(x) := \min\{|x - \bar{x}|^{-n+\varepsilon}, (2r)^{-n+\varepsilon}\}.$$

As $\bar{w} \in A_t$ with $[\bar{w}]_{A_t} \leq C(t, \varepsilon, n)$ for any $1 < t < \infty$ (see Lemma 3.2 in [27]), we can apply Theorem 2.4 with $q = p$ to estimate the above last integral. As a consequence, we obtain

$$
\begin{aligned}
\fint_{B_{2r}(\bar{x})} |\nabla u|^p dx &\le C\,(2r)^{-\varepsilon}\Big(\bar{w}(B_5)\,\|\nabla u\|^p_{L^p(B_{10})} + \int_{B_{15/2}} \mathrm{M}_{B_{10}}(|\mathbf{F}|^{p'})\,\bar{w}\,dx\Big)\\
&\le C\,(2r)^{-\varepsilon}\Big(\|\nabla u\|^p_{L^p(B_{10})} + \int_{B_{15/2}} \mathrm{M}_{B_{10}}(|\mathbf{F}|^{p'})\,\bar{w}\,dx\Big) \qquad (5.18)
\end{aligned}
$$

with $C > 0$ depending only on p, n, ω, Λ, M_0, and ε. Notice that to obtain the last inequality we have used the fact

$$
\bar{w}(B_5) \le \int_{B_6(\bar{x})} |x-\bar{x}|^{-n+\varepsilon}\,dx = \omega_n \int_0^6 t^{\varepsilon-1}dt = \frac{\omega_n}{\varepsilon}\,6^{\varepsilon}.
$$

To bound the last integral in (5.18), we employ Fubini's theorem to get

$$
\begin{aligned}
&\int_{B_{15/2}} \mathrm{M}_{B_{10}}(|\mathbf{F}|^{p'})\bar{w}\,dx = \int_0^{\infty}\int_{\{B_{15/2}:\bar{w}(x)>t\}} \mathrm{M}_{B_{10}}(|\mathbf{F}|^{p'})\,dxdt\\
&\le \int_0^{(2r)^{-n+\varepsilon}} \int_{B_{t^{(-n+\varepsilon)^{-1}}}(\bar{x})\cap B_{15/2}} \mathrm{M}_{B_{10}}(|\mathbf{F}|^{p'})\,dx\,dt\\
&\le \int_0^{15^{-n+\varepsilon}}\int_{B_{15/2}} \mathrm{M}_{B_{10}}(|\mathbf{F}|^{p'})\,dx\,dt + \int_{15^{-n+\varepsilon}}^{(2r)^{-n+\varepsilon}}\int_{B_{t^{(-n+\varepsilon)^{-1}}}(\bar{x})\cap B_{15/2}} \mathrm{M}_{B_{10}}(|\mathbf{F}|^{p'})\,dx\,dt.
\end{aligned}
$$

Since $B_{15/2} \subset B_{17/2}(\bar{x})$, we have $B_{15/2} = B_{17/2}(\bar{x}) \cap B_{15/2}$, and then deduce that

$$
\begin{aligned}
\int_{B_{15/2}} \mathrm{M}_{B_{10}}(|\mathbf{F}|^{p'})\,\bar{w}\,dx \le C\,\|\mathrm{M}_{B_{10}}(|\mathbf{F}|^{p'})\|_{\mathcal{M}^{1,\varphi^{p/q}}(B_{15/2})}\Big[\varphi(B_{17/2}(\bar{x}))^{-p/q}\\
+ \int_{15^{-n+\varepsilon}}^{(2r)^{-n+\varepsilon}} t^{n/(-n+\varepsilon)}\,\varphi(B_{t^{(-n+\varepsilon)^{-1}}}(\bar{x}))^{-p/q}\,dt\Big],
\end{aligned}
$$

where we recall that $\mathcal{M}^{1,\varphi}(U)$ denotes the Morrey space $\mathcal{M}^{1,\varphi}_w(U)$ with $w = 1$. As $\varphi \in \mathcal{B}_+$ by the assumption, and $\{\mathcal{B}_\alpha\}$ is decreasing in α, there exists $\alpha \in (0, n)$ such that $\varphi \in \mathcal{B}_\alpha$. It then follows if $\varepsilon < \alpha p/q$ that

$$
\begin{aligned}
&\int_{B_{15/2}} \mathrm{M}_{B_{10}}(|\mathbf{F}|^{p'})\bar{w}\,dx\\
&\le C\,(2r)^{\alpha p/q}\,\varphi(B_{2r}(\bar{x}))^{-p/q}\,\|\mathrm{M}_{B_{10}}(|\mathbf{F}|^{p'})\|_{\mathcal{M}^{1,\varphi^{p/q}}(B_{15/2})}\Big[1+\int_0^{(2r)^{-n+\varepsilon}} t^{\frac{n-\alpha p/q}{-n+\varepsilon}}\,dt\Big]\\
&\le C\,(2r)^{\varepsilon}\,\varphi(B_r(\bar{x}))^{-p/q}\,\|\mathrm{M}_{B_{10}}(|\mathbf{F}|^{p'})\|_{\mathcal{M}^{1,\varphi^{p/q}}(B_{15/2})}.
\end{aligned}
$$

Combining this with (5.18), we arrive at

$$
\fint_{B_{2r}(\bar{x})} |\nabla u|^p\,dx \le C\big(r^{-\varepsilon}\|\nabla u\|^p_{L^p(B_{10})} + \varphi(B_r(\bar{x}))^{-p/q}\|\mathrm{M}_{B_{10}}(|\mathbf{F}|^{p'})\|_{\mathcal{M}^{1,\varphi^{p/q}}(B_{15/2})}\big).
$$

Therefore, we infer from (5.17) and the fact $\varphi \in \mathcal{B}_\alpha$ that

$$
\begin{aligned}
&\varphi(B_r(\bar{x})) ⨍_{B_r(\bar{x})} |\nabla u|^q \, dw \\
&\quad \leq C_\varepsilon \Big[\varphi(B_r(\bar{x}))\, r^{-\varepsilon q/p} \, \|\nabla u\|^q_{L^p(B_{10})} \\
&\qquad\qquad + \|\mathrm{M}_{B_{10}}(|\mathbf{F}|^{p'})\|^{q/p}_{\mathcal{M}^{1,\varphi^{p/q}}(B_{15/2})} + \|\mathrm{M}_{B_5}(|\mathbf{F}|^{p'})\|^{q/p}_{\mathcal{M}^{q/p,\varphi}_w(B_4)} \Big] \\
&\quad \leq C_\varepsilon \Big[\varphi(B_2(\bar{x}))\, r^{\alpha-\varepsilon q/p} \, \|\nabla u\|^q_{L^p(B_{10})} \\
&\qquad\qquad + \|\mathrm{M}_{B_{10}}(|\mathbf{F}|^{p'})\|^{q/p}_{\mathcal{M}^{1,\varphi^{p/q}}(B_{15/2})} + \|\mathrm{M}_{B_5}(|\mathbf{F}|^{p'})\|^{q/p}_{\mathcal{M}^{q/p,\varphi}_w(B_4)} \Big]
\end{aligned}
$$

for all $\bar{x} \in B_1$ and $0 < r \leq 2$. By taking $\varepsilon = \frac{\alpha}{2} p/q$ and as $\sup_{\bar{x}\in B_1} \varphi(B_2(\bar{x})) < \infty$, this gives estimate (1.9). □

Proof of Theorem 1.3. Since $v \in A_{q/p}$, Lemma 2.1 gives $v^{1-(q/p)'} \in A_{(q/p)'} \subset A_\infty$. Thus our assumptions imply that condition (**B**) in Corollary 3.6 is satisfied. Also as $\varphi \in \mathcal{B}_0$, it is clear that (1.11) implies (3.6). Indeed, for any $y \in \mathbb{R}^n$ and any $s \geq 2r > 0$ we have from (1.11) and $\varphi \in \mathcal{B}_0$ that

$$
\frac{v(B_s(y))}{w(B_s(y))} \frac{1}{\phi(B_s(y))} \leq C_* \frac{1}{\varphi(B_{s/2}(y))} \leq C_* C \frac{1}{\varphi(B_r(y))},
$$

yielding (3.6). Moreover, by Theorem 9.3.3 in [16] there exist $s \in (1,\infty)$ and $C > 0$ depending only on n and $[w]_{A_\infty}$ such that $[w]_{A_s} \leq C$. Therefore, it follows from Theorem 1.2 and Corollary 3.6 that

$$
\begin{aligned}
&\|\nabla u\|_{\mathcal{M}^{q,\varphi}_w(B_1)} \\
&\quad \leq C\big(\|\nabla u\|_{L^p(B_{10})} + \|\mathrm{M}_{B_{10}}(|\mathbf{F}|^{p'})\|^{1/p}_{\mathcal{M}^{1,\varphi^{p/q}}(B_{10})} + \||\mathbf{F}|^{1/(p-1)}\|_{\mathcal{M}^{q,\phi}_v(B_{10})} \big)
\end{aligned} \tag{5.19}
$$

with $C > 0$ depending only on q, p, n, ω, Λ, M_0, φ, C_*, $[w]_{A_\infty}$, $[v]_{A_{q/p}}$, and $[w, v^{1-(q/p)'}]_{A_{q/p}}$. Thus it remains to estimate the middle term on the right-hand side of (5.19). Let $l := q/p > 1$. Then for any nonnegative function $g \in L^1(B_{10})$, we obtain from Hölder inequality and assumption (1.10) that

$$
\begin{aligned}
\frac{\varphi(B_R(\bar{x}))}{|B_R(\bar{x})|^l} \Big(\int_{B_R(\bar{x})\cap B_{10}} g \, dx \Big)^l &\leq \frac{\varphi(B_R(\bar{x}))}{|B_R(\bar{x})|^l} \Big(\int_{B_R(\bar{x})\cap B_{10}} g^l v \, dx \Big) \Big(\int_{B_R(\bar{x})} v^{1-l'} \Big)^{l-1} \\
&\leq [w, v^{1-l'}]_{A_l} \frac{\varphi(B_R(\bar{x}))}{w(B_R(\bar{x}))} \int_{B_R(\bar{x})\cap B_{10}} g^l v \, dx \\
&\leq [w, v^{1-l'}]_{A_l} \, \|g\|^l_{\mathcal{M}^{l,\hat{\varphi}}_v(B_{10})}
\end{aligned}
$$

for all $\bar{x} \in B_{10}$ and all $0 < R \leq 20$, where

$$
\hat{\varphi}(B) := \frac{v(B)}{w(B)} \varphi(B).
$$

Hence we infer that

$$\|\mathrm{M}_{B_{10}}(|\mathbf{F}|^{p'})\|_{\mathcal{M}^{1,\varphi^{1/l}}(B_{10})} \le [w, v^{1-l'}]_{A_l}^{1/l} \, \|\mathrm{M}_{B_{10}}(|\mathbf{F}|^{p'})\|_{\mathcal{M}_v^{l,\hat{\varphi}}(B_{10})}. \tag{5.20}$$

Using $\phi \in \mathcal{B}_0$, condition (1.11), and the doubling property of w due to Lemma 2.1, we have

$$\begin{aligned}\sup_{s\ge 2r} \frac{1}{\phi(B_s(y))} \le C\,\frac{1}{\phi(B_{2r}(y))} &\le C\,C_*\,\frac{w(B_{2r}(y))}{v(B_{2r}(y))}\,\frac{1}{\varphi(B_r(y))}\\ &\le C'\,\frac{w(B_r(y))}{v(B_r(y))}\,\frac{1}{\varphi(B_r(y))} = C'\,\frac{1}{\hat{\varphi}(B_r(y))}\end{aligned}$$

for all $y \in \mathbb{R}^n$ and $r > 0$. Thus as $v \in A_l$ we can use the strong type estimate for the Hardy–Littlewood maximal function given by Corollary 3.7 to estimate the right-hand side of (5.20). As a result, we get

$$\|\mathrm{M}_{B_{10}}(|\mathbf{F}|^{p'})\|_{\mathcal{M}^{1,\varphi^{1/l}}(B_{10})} \le C\,\||\mathbf{F}|^{p'}\|_{\mathcal{M}_v^{l,\phi}(B_{10})} = C\,\||\mathbf{F}|^{1/(p-1)}\|^p_{\mathcal{M}_v^{q,\phi}(B_{10})}. \tag{5.21}$$

This and (5.19) yield desired estimate (1.12). □

We close the paper by noting that if $w \in A_{q/p}$, then the assumption $v \in A_{q/p}$ in Theorem 1.3 can be disposed if condition (1.10) is strengthened by assuming that there exists $r > 1$ such that

$$\sup_B \Big(\fint_B w\,dx\Big)\Big(\fint_B v^{r[1-(q/p)']}\,dx\Big)^{(q-p)/(pr)} < \infty.$$

This follows from the above proof of Theorem 1.3 and the fact that condition (**A**) in Corollary 3.6 is satisfied in this case. Indeed, having (**A**) allows us to use Corollary 3.6 to obtain (5.19) and it remains to show (5.21). But for any nonnegative function $g \in L^1(B_{10})$, we have from Hölder inequality and $w \in A_l$ that

$$\begin{aligned}\frac{\varphi(B_R(\bar{x}))}{|B_R(\bar{x})|^l}\Big(\int_{B_R(\bar{x})\cap B_{10}} g\,dx\Big)^l &\le \frac{\varphi(B_R(\bar{x}))}{|B_R(\bar{x})|^l}\Big(\int_{B_R(\bar{x})\cap B_{10}} g^l w\,dx\Big)\Big(\int_{B_R(\bar{x})} w^{1-l'}\Big)^{l-1}\\ &\le [w]_{A_l}\,\frac{\varphi(B_R(\bar{x}))}{w\big(B_R(\bar{x})\big)}\int_{B_R(\bar{x})\cap B_{10}} g^l\,w\,dx \le [w]_{A_l}\,\|g\|^l_{\mathcal{M}_w^{l,\varphi}(B_{10})}\end{aligned}$$

for all $\bar{x} \in B_{10}$ and all $0 < R \le 20$. This observation together with Corollary 3.6 gives

$$\begin{aligned}\|\mathrm{M}_{B_{10}}(|\mathbf{F}|^{p'})\|_{\mathcal{M}^{1,\varphi^{1/l}}(B_{10})} &\le [w]_{A_l}^{1/l}\,\|\mathrm{M}_{B_{10}}(|\mathbf{F}|^{p'})\|_{\mathcal{M}_w^{l,\varphi}(B_{10})}\\ &\le C\,\||\mathbf{F}|^{p'}\|_{\mathcal{M}_v^{l,\phi}(B_{10})} = C\,\||\mathbf{F}|^{1/(p-1)}\|^p_{\mathcal{M}_v^{q,\phi}(B_{10})}\end{aligned}$$

which is precisely (5.21). As a consequence, we still arrive at conclusion (1.12).

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