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The Pohozaev identity for the anisotropic *p*-Laplacian and estimates of the torsion function

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Abstract. In this paper we prove a Pohozaev identity for the weighted anisotropic p-Laplace operator. As an application of the identity, we deduce the nonexistence of nontrivial solutions of the Dirichlet problem for the weighted anisotropic p-Laplacian in star-shaped domains of $\mathbb{R}^n$. We also provide an upper bound estimate for the first Dirichet eigenvalue of the anisotropic p-Laplacian on bounded domains of $\mathbb{R}^n$, some sharp estimates for the torsion function of compact manifolds with boundary and a non-existence result for the solutions of the Laplace equation on closed Riemannian manifolds.

1. Introduction and main results

Let Ω be a bounded smooth domain in $\mathbb{R}^n$ and let g be a continuous function on $\mathbb{R}$. In 1965, Pohozaev [24] considered the following nonlinear elliptic problem:

$$\begin{cases} -\Delta u = g(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

and proved that if $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ is a solution of (1.1), then

$$(2-n)\int_\Omega u\, g(u)\, dx + 2n\int_\Omega G(u)\, dx = \int_{\partial\Omega} |\nabla u|^2 \langle x, \nu\rangle\, ds, \tag{1.2}$$

where $G(u) = \int_0^u g(t)\, dt$ and $\nu = \nu(x)$ is the outward unit normal vector at the point $x \in \partial\Omega$.

Based on (1.2), Pohozaev established the following well-known non-existence result.

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Theorem A (Pohozaev)**.** *Let Ω be a star-shaped domain with respect to the origin in $\mathbb{R}^n, n \geq 3$ and let $g \in C(\mathbb{R}, \mathbb{R})$ with $g(u) \geq 0$, when $u \geq 0$. If*

$$(2-n)u\,g(u) + 2n\int_0^u g(t)\,dt \leq 0, \quad \text{when } u \geq 0,$$

then the problem (1.1) has no positive solution.

Pohozaev's identity has also other important applications to the solutions of differential equations. As an example, let us assume further that

$$g(u) \equiv 1 \quad \text{and} \quad \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = c = \text{const}.$$

Then we have from (1.2) that

$$(n+2)\int_\Omega u\,dx = c^2\int_{\partial\Omega}\langle x, \nu\rangle\,ds = n\,c^2 V(\Omega). \tag{1.3}$$

Also, it is easy to see in this case that

$$\Delta\Big(|\nabla u|^2 + \frac{2}{n}u\Big) = 2\Big(|\nabla^2 u|^2 - \frac{1}{n}\Big) = 2\Big(|\nabla^2 u|^2 - \frac{(\Delta u)^2}{n}\Big) \geq 0 \tag{1.4}$$

and

$$\Big(|\nabla u|^2 + \frac{2}{n}u\Big)\Big|_{\partial\Omega} = c^2,$$

which, implies by the maximum principle that

$$|\nabla u|^2 + \frac{2}{n}u \leq c^2 \quad \text{on } \Omega.$$

On the other hand, one obtains from integration by parts and (1.3) that

$$\int_\Omega\Big(|\nabla u|^2 + \frac{2}{n}u\Big)\,dx = \frac{n+2}{n}\int_\Omega u\,dx = c^2 V(\Omega). \tag{1.5}$$

Therefore $|\nabla u|^2 + \frac{2}{n}u$ is constant in Ω and so the equality must hold in (1.4) which implies that

$$u_{ij} = -\frac{1}{n}\,\delta_{ij}$$

Hence, for a suitable choice of origin we know that u is given by

$$u = \frac{1}{2n}\,(\rho_0 - r^2),$$

where ρ_0 is a constant and r is the distance function from the origin. Since $u|_{\partial\Omega} = 0$, we conclude that $\rho_0 > 0$ and that Ω is a ball of radius $\sqrt{\rho_0}$. Also one can deduce from (1.5) that $\rho_0 = n^2c^2$. The above arguments are essentially the proof given by Weinberger [34] to the following seminal work of Serrin [31].

Theorem B (Serrin). *If $u \in C^2(\overline{\Omega})$ satisfies the overdetermined problem*

$$\begin{cases} \Delta u = -1 \quad \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \ \frac{\partial u}{\partial \nu}|_{\partial\Omega} = c, \end{cases} \tag{1.6}$$

where Ω is a bounded smooth domain in $\mathbb{R}^n$, ν is the unit outward normal of $\partial\Omega$, and c is a constant, then Ω is a ball of radius $n|c|$ and $u = (n^2c^2 - r^2)/2n$, where r is the distance from the center of the ball.

The appearance of Pohozaev's identity is a milestone in the development of differential equations. The generalizations of Pohozaev's identity have been widely used to prove the non-existence of nontrivial solutions of nonlinear elliptic equations. Here are some of the important results in this direction. Esteban–Lions [16] and Berestycki–Lions [5] considered the following problem on unbounded domain:

$$\begin{cases} -\Delta u = g(u) \quad \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

where $\Omega = \mathbb{R}^n$ or an unbounded domain of $\mathbb{R}^n$. They established the Pohozaev identity for the above problem and the existence and non-existence results. Pucci and Serrin [25] proved the Pohozaev identity satisfied by general elliptic equations on bounded domains. Guedda–Véron [20] proved the Pohozaev identity for the solutions of the quasi-linear elliptic problem

$$\begin{cases} -\text{div}(|\nabla u|^{p-2}\nabla u) = g(x,u) \quad \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

and obtained the non-existence result when Ω is a star-shaped domain. Bartsch–Peng–Zhang [3] and Kou–An [21] considered more general quasi-linear elliptic equations with weight on more general domains. Pucci and Serrin [26] studied the Pohozaev identity of polyharmonic operators and obtained non-existence of nontrivial solutions of the related equations.

Recently, Ros-Oton and Serra [29] established the Pohozaev identity for the fractional elliptic problem

$$\begin{cases} (-\Delta)^s u = g(u) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$, where $s \in (0,1)$,

$$(-\Delta)^s u(x) = c_{n,s}\, PV \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dy$$

is the fractional Laplacian, and $c_{n,s}$ is a normalization constant given by

$$c_{n,s} = \frac{s\,2^{2s}\,\Gamma((n+2s)/2)}{\pi^{n/2}\,\Gamma(1-s)}.$$

As a matter of fact, the fractional and anisotropic case has been dealt with in [30].

In this paper, we shall prove the Pohozaev identity for a weighted anisotropic p-Laplace operator. Let us fix some required notation before stating our result. Let $F\colon \mathbb{R}^n \to [0,+\infty)$ be a convex function of class $C^1(\mathbb{R}^n \setminus \{0\})$ which is even and positively homogeneous of degree 1, so that

$$F(tx) = |t|F(x), \quad \forall x \in \mathbb{R}^n,\ \forall t \in \mathbb{R}. \tag{1.7}$$

Note that there are positive constants α and β such that F satisfies

$$\alpha|\xi| \le F(\xi) \le \beta|\xi|, \quad \forall \xi \in \mathbb{R}^n.$$

Observing that F^p is positively homogeneous of degree p, we have

$$\langle Z, \nabla_\xi [F^p](Z)\rangle = pF^p(Z), \quad \forall Z \in \mathbb{R}^n. \tag{1.8}$$

For $1 < p < \infty$, the anisotropic p-Laplace operator is defined as

$$\mathcal{Q}_p(u) = \operatorname{div}\Big(\frac{1}{p}\nabla_\xi[F^p](\nabla u)\Big) = \sum_{i=1}^n \frac{\partial}{\partial x_i}\big((F(\nabla u))^{p-1}F_{\xi_i}(\nabla u)\big),$$

where ∇_ξ stands for the gradient operator with respect to the ξ variables. The first result of the present paper is as follows.

Theorem 1.1. *Let Ω be a bounded domain with smooth boundary and let $g\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Let b be a real number, let $1 < p < \infty$, and let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be a solution of the problem*

$$\begin{cases} -\dfrac{1}{p}\operatorname{div}\big(|x|^{-bp}\,\nabla_\xi[F^p](\nabla u)\big) = g(x,u) & \text{in } \Omega,\\ u|_{\partial\Omega} = 0. \end{cases}$$

Then we have

$$\begin{aligned} \Big(1 + b - \frac{n}{p}\Big)\int_\Omega u\,g(x,u)\,dx + n\int_\Omega G(x,u)\,dx + \int_\Omega \langle x, \nabla_x G(x,u)\rangle\,dx \\ = \Big(1 - \frac{1}{p}\Big)\int_{\partial\Omega} F^p(\nabla u)\langle x, \nu\rangle\,ds, \end{aligned} \tag{1.9}$$

where $G(x,\rho) = \int_0^\rho g(x,\theta)d\theta$ and $\nabla_x G$ is the gradient of G with respect to the first variable x.

As an immediate application of Theorem 1.1, we have:

Corollary 1.2. *Let Ω be a bounded star-shaped domain with smooth boundary, and let $g\colon \mathbb{R} \to \mathbb{R}$ be a continuous function. Let b be a real number, $1 < p < \infty$, and suppose that*

$$\Big(1 + b - \frac{n}{p}\Big)\rho g(\rho) + n\int_0^\rho g(\sigma)\,d\sigma < 0, \quad \text{when } \rho \ne 0.$$

Then the problem

$$\begin{cases} -\frac{1}{p}\operatorname{div}\big(|x|^{-bp}\,\nabla_\xi[F^p](\nabla u)\big) = g(u) \quad \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases}$$

has no nontrivial (not identically zero) solution.

Taking

$$g(x,u) = \lambda|x|^{-\alpha}|u|^{r-2}u + \mu|x|^{-\beta}|u|^{s-2}u + \eta|x|^{-\gamma}|u|^{t-2}u$$

in Theorem 1.1, we have the following.

Corollary 1.3. *Let Ω be a bounded star-shaped domain with smooth boundary, and let $b,\alpha,\beta,\gamma,\lambda,\mu,\eta,r,s,t$ be constants such that $rst \neq 0$, and*

(1.10)
$$\lambda\Big(1+b-\frac{n}{p}+\frac{n-\alpha}{r}\Big) \le 0, \quad \mu\Big(1+b-\frac{n}{p}+\frac{n-\beta}{s}\Big) \le 0, \quad \eta\Big(1+b-\frac{n}{p}+\frac{n-\gamma}{t}\Big) < 0.$$

Then, the problem

(1.11)
$$\begin{cases} -\frac{1}{p}\operatorname{div}\big(|x|^{-bp}\,\nabla_\xi[F^p](\nabla u)\big) = \lambda|x|^{-\alpha}|u|^{r-2}u + \mu|x|^{-\beta}|u|^{s-2}u \\ \qquad\qquad\qquad\qquad\qquad\qquad +\eta|x|^{-\gamma}|u|^{t-2}u \qquad \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

has no positive solution.

Since r, s and t appear in the denominator of (1.10), hence in Corollary 1.3 it is natural to require that they do not vanish, that is, $rst \neq 0$.

In the second part of this paper, we study the first Dirichlet eigenvalue of $\mathcal{Q}_p$, which is given by

$$\lambda_{p,1}(F,\Omega) = \inf_{u\in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\int_\Omega F^p(\nabla u)\,dx}{\int_\Omega |u|^p\,dx}. \tag{1.12}$$

It is known [4] that (1.12) has a unique positive solution u_p solving the Euler–Lagrange equation

$$\begin{cases} \mathcal{Q}_p u_p + \lambda_{p,1}|u_p|^{p-2}u_p = 0 & \text{in } \Omega, \\ u_p = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.13}$$

For any bounded smooth domain Ω in a complete Riemannian manifold, there exists a unique solution u_Ω, called the torsion function of Ω, to the equation

$$\Delta u = -1 \ \text{ in } \Omega, \qquad u|_{\partial\Omega} = 0, \tag{1.14}$$

and (1.14) is the Euler equation of the minimum problem

$$\min_{u\in W_0^{1,2}(\Omega)} \int_\Omega \Big(\frac{1}{2}|\nabla u|^2 - u\Big).$$

The number

$$T(\Omega)=\int_\Omega u_\Omega=\int_\Omega |\nabla u_\Omega|^2$$

is called the *torsional rigidity* of Ω.

The torsion problem for the anisotropic p-Laplace is as follows [14]:

$$\begin{cases} -\mathcal{Q}_p v=1 & \text{in } \Omega,\\ v=0 & \text{on } \partial\Omega.\end{cases} \tag{1.15}$$

There exists a unique solution of (1.15), that will be always denoted by v_Ω, which is positive in Ω (cf. [1], [4], [14]).

The anisotropic p-torsional rigidity of Ω is defined as

$$T_{F,p}(\Omega)=\int_\Omega F^p(\nabla v_\Omega)\,dx=\int_\Omega v_\Omega\,dx. \tag{1.16}$$

The following variational characterization for $T_{F,p}(\Omega)$ holds:

$$T_{F,p}(\Omega)^{p-1}=\max_{\phi\in W_0^{1,p}(\Omega)\setminus\{0\}}\frac{\left(\int_\Omega |\phi|dx\right)^p}{\int_\Omega F^p(\nabla\phi)\,dx}, \tag{1.17}$$

and the solution v_Ω of (1.15) realizes the maximum in (1.17).

The next result is an estimate involving $\lambda_{p,1}(F,\Omega)$ and $T_{F,p}(\Omega)$ which is motivated by Theorem 1.1 in [6].

Theorem 1.4. *Let $p\geq 2$, let F be as above, and let Ω be a bounded domain in $\mathbb{R}^n$. Assume further that $F\in C^{3,\beta}(\mathbb{R}^n\setminus\{0\})$ and*

$$[F^p]_{\xi\xi}(\xi)\ \textit{is positive definite in}\ \mathbb{R}^n\setminus\{0\}. \tag{1.18}$$

Then, we have

$$\frac{\lambda_{p,1}(F,\Omega)\,T_{F,p}(\Omega)^{p-1}}{|\Omega|^{p-1}}\leq 1-\frac{p^{\frac{2p-3}{p-1}}(n\kappa_n^{1/n})^{p/(p-1)}}{n(p-1)(n(p-1)+p)}\cdot\frac{T_{F,p}(\Omega)}{|\Omega|^{1+\frac{p}{n(p-1)}}}, \tag{1.19}$$

where $|\Omega|$ and κ_n stand for the measure of Ω and K^o, respectively, being

$$K^o=\Big\{x\in\mathbb{R}^n:\sup_{z\neq 0}\frac{\langle x,z\rangle}{F(z)}\leq 1\Big\}.$$

A main tool in the proof of Theorem 1.4 is the isoperimetric inequality (Wulff's theorem) relating the perimeter of a set E with respect to F and $|E|$, the measure of E. This tool can be also used to prove the following result which is motivated by [23].

Theorem 1.5. *Let Ω be a bounded domain with smooth boundary in $\mathbb{R}^n$, and let $g\in C(\mathbb{R},\mathbb{R})$ with $g(\sigma)\geq 0$, when $\sigma\geq 0$. Let u be a smooth positive solution of the Dirichlet problem for the anisotropic n-Laplace operator*:

$$\begin{cases} -\frac{1}{n}\operatorname{div}(\nabla_\xi[F^n](\nabla u)) = g(u) & \textit{in } \Omega,\\ u=0 & \textit{on } \partial\Omega.\end{cases}$$

Then we have

$$\Big(\int_\Omega g(u)\,dx\Big)^{n/(n-1)} \geq \frac{n^{(2n-1)/(n-1)}\,\kappa_n^{1/(n-1)}}{n-1}\int_\Omega G(u)\,dx, \tag{1.20}$$

where $G(u)=\int_0^u g(s)\,ds$.

The study of anisotropic operators is quite active in recent years. One can find some of the interesting results about this topic, e. g. in [7], [8], [9], [11], [12], [13], [17], [33], etc.

Serrin's theorem above says that if the torsion function of a bounded smooth domain Ω in a Euclidean space has constant derivative in the direction of the outward unit normal of $\partial\Omega$, then Ω is a ball. In the third part of this paper, we give some sharp estimates for the torsion function of a compact manifold with boundary.

Theorem 1.6. *Let M be an n-dimensional compact Riemannian manifold with boundary. Denote by $T(M)$, ρ and V the torsion, the torsion function and the volume of M, respectively. Let ν be the outward unit normal of ∂M and assume that the Ricci curvature of M is bounded below by $(n-1)\kappa$.*

i) *We have*

$$\min_{x\in\partial M}\frac{\partial^2\rho}{\partial\nu^2}(x)\leq -\frac{1}{n}-\frac{(n-1)\kappa T(M)}{V}, \tag{1.21}$$

with equality holding if and only if M is isometric to a ball in $\mathbb{R}^n$, $\kappa=0$ and

$$\frac{\partial^2\rho}{\partial\nu^2}=-\frac{1}{n}\quad \text{on } \partial M.$$

ii) *Let A and H be the area and the mean curvature of ∂M, respectively. If $H\geq 0$ on ∂M, then*

$$\int_{\partial M}\frac{\partial^2\rho}{\partial\nu^2}\leq (n-1)\Big(\frac{V}{n}-\kappa T(M)\Big)^{1/2}\Big(\int_{\partial M}H\Big)^{1/2}-A, \tag{1.22}$$

with equality holding if and only if $\kappa=0$ and M is isometric to a ball in $\mathbb{R}^n$.

Theorem 1.7. *Let M be an n-dimensional compact Riemannian manifold with boundary and Ricci curvature bounded below by $(n-1)\kappa$. Let u be the solution of the Dirichlet problem*

$$\Delta u=-1 \quad \text{in } M,\qquad u|_{\partial M}=0.$$

Then

$$\max_{\partial M}|\nabla u|^2\geq\frac{(n+2)T(M)}{nV}+\frac{2(n-1)\kappa}{V}\int_M u|\nabla u|^2, \tag{1.23}$$

with equality holding if and only if $\kappa=0$ and M is isometric to a ball in $\mathbb{R}^n$.

Remark. One can obtain Theorem B from Theorem 1.7. In fact, when Ω is a bounded smooth domain in $\mathbb{R}^n$, if u is a solution to the equation (1.6), then we have from (1.3) that

$$c^2 = \frac{(n+2)T(\Omega)}{nV}.$$

Thus, the equality sign in (1.23) is attained since the Ricci curvature of Ω is zero. Theorem B follows.

The next result is a Pohozaev-type inequality on compact Riemannian manifolds.

Theorem 1.8. *Let M be an n-dimensional compact Riemannian manifold with or without boundary. Assume that the Ricci curvature of M is bounded below by $(n-1)\kappa$ and let $g\colon \mathbb{R}\to\mathbb{R}$ be a continuous function. If $u \in C^3(M)\cap C^1(\partial M)$ is a non-negative solution of the problem*

$$-\Delta u = g(u) \ \text{ in } M, \quad u|_{\partial M} = 0,$$

then we have

$$\int_M g(u)\Big(\frac{2(n-1)u\,g(u)}{n} - 3G(u) - (n-1)\kappa u^2\Big) \geq \begin{cases} \int_{\partial M}\big(\frac{\partial u}{\partial \nu}\big)^3, & \text{when } \partial M \neq \emptyset, \\ 0, & \text{when } \partial M = \emptyset, \end{cases} \tag{1.24}$$

where $G(u) = \int_0^u g(\sigma)\,d\sigma$.

From Theorem 1.8, we have the following non-existence result.

Corollary 1.9. *Let M be an n-dimensional closed Riemannian manifold with Ricci curvature bounded below by $(n-1)\kappa$. Assume that $g\colon\mathbb{R}\to\mathbb{R}$ is a continuous function and there exists a discrete subset S of $[0,+\infty)$ such that*

$$g(t)\Big(\frac{2(n-1)t\,g(t)}{n} - 3\int_0^t g(\sigma)\,d\sigma - (n-1)\kappa t^2\Big) \begin{cases} = 0, & \text{if } t \in S, \\ < 0, & \text{if } t\in[0,+\infty)\setminus S. \end{cases}$$

Then any non-negative solution of the equation

$$\Delta u = -g(u) \quad \text{on } M$$

is a zero of g.

2. Proof of Theorem 1.1 and Corollary 1.3

In this section, we shall prove Theorem 1.1 and Corollary 1.3.

Proof of Theorem 1.1. Multiplying the equation

$$-\frac{1}{p}\operatorname{div}\big(|x|^{-bp}\nabla_\xi[F^p](\nabla u)\big) = g(x,u)$$

by $-p\langle x, \nabla u\rangle$ and integrating on Ω, one deduces from the divergence theorem that

$$\begin{aligned}
-p\int_\Omega g(x,u)\langle x,\nabla u\rangle\,dx &= \int_\Omega \operatorname{div}\big(|x|^{-bp}\,\nabla_\xi[F^p](\nabla u)\big)\langle x,\nabla u\rangle\,dx\\
&= \int_\Omega \Big(\operatorname{div}\big(|x|^{-bp}\,\nabla_\xi[F^p](\nabla u)\langle x,\nabla u\rangle\big)\,\big\langle |x|^{-bp}\,\nabla_\xi[F^p](\nabla u),\nabla\langle x,\nabla u\rangle\big\rangle\Big)\,dx\\
&= \int_{\partial\Omega}\big\langle |x|^{-bp}\,\nabla_\xi[F^p](\nabla u),\nu\big\rangle\,\langle x,\nabla u\rangle\,d\mathcal{H}^{n-1} \qquad (2.1)\\
&\quad -\int_\Omega \big\langle |x|^{-ap}\nabla_\xi[F^p](\nabla u),\nabla u+\nabla^2u(x)\big\rangle\,dx.
\end{aligned}$$

Here $\nabla^2 u\colon \mathfrak{X}(\Omega)\to\mathfrak{X}(\Omega)$ denotes the self-adjoint linear operator metrically equivalent to the Hessian of u, and is given by [15]

$$\langle\nabla^2u(Z),W\rangle = \nabla^2u(Z,W) = \langle\nabla_Z\nabla u,W\rangle$$

for all $Z,W\in\mathfrak{X}(\Omega)$. It follows from (1.8) and $u|_{\partial\Omega}=0$ that

$$\begin{aligned}
&\int_{\partial\Omega}\big\langle |x|^{-bp}\,\nabla_\xi[F^p](\nabla u),\nu\big\rangle\,\langle x,\nabla u\rangle\,d\mathcal{H}^{n-1} \qquad (2.2)\\
&\qquad= \int_{\partial\Omega}\big\langle |x|^{-bp}\,\nabla_\xi[F^p](\nabla u),\nu\big\rangle\,\langle x,u_\nu\nu\rangle\,d\mathcal{H}^{n-1}\\
&\qquad= \int_{\partial\Omega}\big\langle |x|^{-bp}\,\nabla_\xi[F^p](\nabla u),u_\nu\nu\big\rangle\,\langle x,\nu\rangle\,d\mathcal{H}^{n-1}\\
&\qquad= \int_{\partial\Omega}\big\langle |x|^{-bp}\,\nabla_\xi[F^p](\nabla u),\nabla u\big\rangle\,\langle x,\nu\rangle\,d\mathcal{H}^{n-1}.\\
&\qquad= p\int_{\partial\Omega}|x|^{-bp}F^p(\nabla u)\langle x,\nu\rangle\,d\mathcal{H}^{n-1}.
\end{aligned}$$

Using the divergence theorem again, we infer

$$\begin{aligned}
-\int_\Omega\big\langle |x|^{-bp}\,\nabla_\xi[F^p](\nabla u),\nabla u\big\rangle\,dx &= \int_\Omega u\,\operatorname{div}(|x|^{-bp}\,\nabla_\xi[F^p](\nabla u))\,dx \qquad (2.3)\\
&= -p\int_\Omega u\,g(x,u)\,dx.
\end{aligned}$$

Let $\{e_1=(1,0,\dots,0),\dots,e_n=(0,\dots,0,1)\}$ be the canonical base of $\mathbb{R}^n$ and set $u_i=\partial u/\partial x_i$, $u_{ij}=\partial^2u/\partial x_ix_j$, $i,j=1,\dots,n$. We calculate

$$\begin{aligned}
\big\langle\nabla_\xi[F^p](\nabla u),\nabla^2u(x)\big\rangle &= \big\langle\nabla_{\nabla_\xi[F^p](\nabla u)}\nabla u,x\big\rangle = \big\langle\nabla_{\sum_{i=1}^n\frac{\partial[F^p]}{\partial\xi_i}(\nabla u)e_i}\nabla u,x\big\rangle\\
&= \sum_{i=1}^n\frac{\partial[F^p]}{\partial\xi_i}(\nabla u)\,\langle\nabla_{e_i}\nabla u,x\rangle = \sum_{i=1}^n\frac{\partial[F^p]}{\partial\xi_i}(\nabla u)\nabla^2u(e_i,x)\\
&= \sum_{i=1}^n\frac{\partial[F^p]}{\partial\xi_i}(\nabla u)\sum_{j=1}^n\langle x,e_j\rangle\nabla^2u(e_i,e_j) = \sum_{i,j=1}^n\frac{\partial[F^p]}{\partial\xi_i}(\nabla u)\langle x,e_j\rangle u_{ij}\\
&= \Big\langle x,\sum_{i,j=1}^n\frac{\partial[F^p]}{\partial\xi_i}(\nabla u)u_{ij}\,e_j\Big\rangle = \Big\langle x,\sum_{j=1}^n\frac{\partial(F^p(\nabla u))}{\partial x_j}e_j\Big\rangle = \langle x,\nabla(F^p(\nabla u))\rangle.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}\big\langle |x|^{-bp}\nabla_\xi[F^p](\nabla u),\nabla^2u(x)\big\rangle &= |x|^{-bp}\langle\nabla(F^p(\nabla u)),x\rangle = \langle\nabla(F^p(\nabla u)),|x|^{-bp}x\rangle\\ &= \mathrm{div}(F^p(\nabla u)|x|^{-bp}x) - F^p(\nabla u)\mathrm{div}(|x|^{-bp}x)\\ &= \mathrm{div}(F^p(\nabla u)|x|^{-bp}x) - (n-bp)|x|^{-ap}F^p(\nabla u).\end{aligned}$$

Thus

$$\begin{aligned}&\int_\Omega \big\langle |x|^{-bp}\nabla_\xi F^p(\nabla u),\nabla^2u(x)\big\rangle\,dx\\ &= \int_{\partial\Omega}|x|^{-bp}F^p(\nabla u)\langle x,\nu\rangle\,d\mathcal{H}^{n-1} - (n-bp)\int_\Omega |x|^{-bp}F^p(\nabla u)\,dx\\ &= \int_{\partial\Omega}|x|^{-bp}F^p(\nabla u)\langle x,\nu\rangle\,d\mathcal{H}^{n-1} - (n-bp)\int_\Omega |x|^{-bp}\cdot\frac{1}{p}\langle\nabla_\xi[F^p](\nabla u),\nabla u\rangle\,dx\\ &= \int_{\partial\Omega}|x|^{-bp}F^p(\nabla u)\langle x,\nu\rangle\,d\mathcal{H}^{n-1} + \frac{n-bp}{p}\int_\Omega u\,\mathrm{div}\big(|x|^{-bp}\nabla_\xi[F^p](\nabla u)\big)dx\\ &= \int_{\partial\Omega}|x|^{-bp}F^p(\nabla u)\langle x,\nu\rangle\,d\mathcal{H}^{n-1} - (n-bp)\int_\Omega ug(x,u)\,dx. \qquad (2.4)\end{aligned}$$

Substituting (2.2), (2.3) and (2.4) into (2.1), we get

$$\begin{aligned}&-\int_\Omega g(x,u)\langle x,\nabla u\rangle\,dx\\ &= \Big(1-\frac{1}{p}\Big)\int_{\partial\Omega}|x|^{-bp}F^p(\nabla u)\langle x,\nu\rangle\,d\mathcal{H}^{n-1} + \Big(\frac{n}{p}-1-b\Big)\int_\Omega u\,g(x,u)\,dx. \qquad (2.5)\end{aligned}$$

On the other hand, we have

$$g(x,u)\,\langle x,\nabla u\rangle = \langle x,\nabla(G(x,u))\rangle - \langle x,\nabla_x G(x,u)\rangle$$

and so

$$\begin{aligned}-\int_\Omega g(x,u)\,\langle x,\nabla u\rangle\,dx &= -\int_\Omega\big(\langle x,\nabla(G(x,u))\rangle + \langle x,\nabla_x G(x,u)\rangle\big)dx \qquad (2.6)\\ &= n\int_\Omega G(x,u)\,dx + \int_\Omega\langle x,\nabla_x G(x,u)\rangle\,dx.\end{aligned}$$

Combining (2.5) and (2.6), we get (1.9). □

Proof of Corollary 1.3. Suppose that u is a positive solution of (1.11). Then the equality (1.9) holds. Thus, we have

$$\begin{aligned}&\Big(1+b-\frac{n}{p}\Big)\int_\Omega u(\lambda|x|^{-\alpha}u^{r-1}+\mu|x|^{-\beta}u^{s-1}+\eta|x|^{-\gamma}u^{t-1})\,dx \qquad (2.7)\\ &\quad + n\int_\Omega G(x,u)\,dx + \int_\Omega\langle x,\nabla_x G(x,u)\rangle\,dx\\ &= \Big(1-\frac{1}{p}\Big)\int_{\partial\Omega}F^p(\nabla u)\langle x,\nu\rangle\,ds \geq 0,\end{aligned}$$

where

$$G(x,u) = \int_0^u (\lambda |x|^{-\alpha}|\sigma|^{r-2}\sigma + \mu |x|^{-\beta}|\sigma|^{s-2}\sigma + \eta |x|^{-\gamma}|\sigma|^{t-2}\sigma)\, d\sigma = \frac{\lambda}{r}|x|^{-\alpha}u^r + \frac{\mu}{s}|x|^{-\beta}u^s + \frac{\eta}{t}|x|^{-\beta}u^t, \tag{2.8}$$

$$\langle x, \nabla_x G(x,u)\rangle = \sum_{i=1}^n x_i \frac{\partial G}{\partial x_i}(x,u) = -\frac{\lambda\alpha}{r}\alpha\, |x|^{-\alpha}\, u^r - \frac{\mu\beta}{s}|x|^{-\beta}u^s - \frac{\eta\gamma}{t}|x|^{-\gamma}u^t. \tag{2.9}$$

Substituting (2.8) and (2.9) into (2.7), we have

$$\lambda\Big(1+b-\frac{n}{p}+\frac{n-\alpha}{r}\Big)\int_\Omega |x|^{-\alpha}u^r\,dx + \mu\Big(1+b-\frac{n}{p}+\frac{n-\beta}{s}\Big)\int_\Omega |x|^{-\beta}u^s\,dx + \eta\Big(1+b-\frac{n}{p}+\frac{n-\gamma}{t}\Big)\int_\Omega |x|^{-\gamma}u^t\,dx \ge 0.$$

This is a contradiction if (1.10) holds. □

Using Theorem 1.1 we can also prove the following non-existence result.

Corollary 2.1. *Let Ω be a bounded star-shaped domain with smooth boundary, and let b, α, β, λ, μ and r be constants such that $r \neq 0$ and*

$$\lambda\Big(n+1+b-\frac{n}{p}-\alpha\Big) \le 0, \quad \mu\Big(1+b-\frac{n}{p}+\frac{n-\beta}{r}\Big) < 0. \tag{2.10}$$

Then, the problem

$$\begin{cases} -\frac{1}{p}\operatorname{div}\big(|x|^{-bp}\,\nabla_\xi[F^p](\nabla u)\big) = \lambda|x|^{-\alpha} + \mu|x|^{-\beta}|u|^{r-2}u & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{2.11}$$

has no positive solution.

Proof of Corollary 2.1. If u is a positive solution of (2.11), then we have from (1.9) that

$$\Big(1+b-\frac{n}{p}\Big)\int_\Omega u(\lambda|x|^{-\alpha} + \mu|x|^{-\beta}u^{r-1})\,dx + n\int_\Omega G(x,u)\,dx + \int_\Omega \langle x, \nabla_x G(x,u)\rangle\,dx = \Big(1-\frac{1}{p}\Big)\int_{\partial\Omega} F^p(\nabla u)\langle x,\nu\rangle\,ds \ge 0. \tag{2.12}$$

Here

$$G(x,u) = \int_0^u (\lambda|x|^{-\alpha} + \mu|x|^{-\beta}|\sigma|^{r-2}\sigma)\,d\sigma = \lambda|x|^{-\alpha}u + \frac{\mu}{r}|x|^{-\beta}u^r, \tag{2.13}$$

and

$$\langle x, \nabla_x G(x,u)\rangle = \sum_{i=1}^{n} x_i \frac{\partial G}{\partial x_i}(x,u) = -\lambda\alpha|x|^{-\alpha}u - \frac{\mu\beta}{r}|x|^{-\beta}u^r. \tag{2.14}$$

Substituting (2.13) and (2.14) into (2.12), we have

$$\lambda\Big(n+1+b-\frac{n}{p}-\alpha\Big)\int_\Omega |x|^{-\alpha}u\,dx + \mu\Big(1+b-\frac{n}{p}+\frac{n-\beta}{r}\Big)\int_\Omega |x|^{-\beta}u^r dx \geq 0,$$

contradicting (2.10). □

3. Proof of Theorems 1.4 and 1.5

In this section we prove Theorems 1.4 and 1.5. Firstly we recall some facts needed about the function F introduced in section 1. Because of (1.7) we can assume, without loss of generality, that the convex closed set

$$K = \{x \in \mathbb{R}^n : F(x) \leq 1\}$$

has measure $|K|$ equal to the measure ω_n of the unit sphere in $\mathbb{R}^n$. We say that F is the gauge of K. The support function of K is defined as [28]

$$F^o(x) = \sup_{\xi\in K}\langle x, \xi\rangle.$$

It is easy to see that $F^o : \mathbb{R}^n \to [0,+\infty)$ is a convex, homogeneous function and that F, F^o are polar each other in the sense that

$$F^o(x) = \sup_{\xi\neq 0}\frac{\langle x,\xi\rangle}{F(\xi)}, \quad \text{and} \quad F(x) = \sup_{\xi\neq 0}\frac{\langle x,\xi\rangle}{F^o(\xi)}.$$

We set

$$K^o = \{x \in \mathbb{R}^n : F^o(x) \leq 1\}$$

and denote by κ_n the measure of K^o.

Let Ω be an open subset of $\mathbb{R}^n$. The total variation of a function $u \in BV(\Omega)$ with respect to a guage fuction F is defined by ([2]):

$$\int_\Omega |\nabla u|_F = \sup\Big\{\int_\Omega u \text{ div } \sigma\; dx : \sigma \in C_0^1(\Omega;\mathbb{R}^n), F^o(\sigma) \leq 1\Big\}.$$

The perimeter of a set E with respect to F is then defined as

$$P_F(E;\Omega) = \int_\Omega |\nabla \chi_E|_F = \sup\Big\{\int_\Omega \text{div } \sigma\; dx : \sigma \in C_0^1(\Omega;\mathbb{R}^n), F^o(\sigma) \leq 1\Big\}.$$

The following co-area formula,

$$\int_\Omega |\nabla u|_F = \int_0^\infty P_F(\{u > s\};\Omega)\,ds, \quad \forall u \in \mathrm{BV}(\Omega),$$

and the equality

$$P_F(E;\Omega) = \int_{\Omega\cap\partial^* E} F(\nu^E)\,d\mathcal{H}^{n-1}$$

hold, where $\partial^* E$ is the reduced boundary of E and ν^E is the outer normal to E (see [2]). The following result can be found in [1], [10], [18].

Lemma 3.1 (Wulff's theorem). *If E is a set of finite perimeter in $\mathbb{R}^n$, then*

$$P_F(E;\mathbb{R}^n) \geq n\kappa_n^{1/n}\,|E|^{1-1/n}, \tag{3.1}$$

and equality holds if and only if E has Wulff shape, i.e., E is a sub-level set of F^o, modulo translations.

Now we are ready to give a:

Proof of Theorem 1.4. Let v_Ω be the unique solution of the equation

$$-\mathcal{Q}_p v = 1, \text{ in}\Omega, \quad v = 0 \text{ on } \partial\Omega. \tag{3.2}$$

Then v_Ω is positive in Ω. By (1.18) and since $F \in C^3(\mathbb{R}^n \setminus \{0\})$, we know that $v_\Omega \in C^{1,\alpha}(\Omega) \cap C^3((\{\nabla v_\Omega \neq 0\})$ (see [22], [32]).

It follows from (1.12) that

$$\lambda_{p,1}(F,\Omega) \leq \frac{\int_\Omega (F(\nabla v_\Omega))^p\,dx}{\int_\Omega v_\Omega^p\,dx} = \frac{\int_\Omega v_\Omega\,dx}{\int_\Omega v_\Omega^p\,dx}. \tag{3.3}$$

Combining (1.16) and (3.3), we infer

$$\lambda_{p,1}(F,\Omega)\,T_{F,p}(\Omega)^{p-1} \leq \frac{\left(\int_\Omega v_\Omega\,dx\right)^p}{\int_\Omega v_\Omega^p\,dx}.$$

Let $M = \sup_\Omega v_\Omega$. For $s \in [0,M]$, we denote by $\mu(s) = |\{x \in \Omega : v_\Omega > s\}|$the distribution function of v_Ω. Then

$$\int_\Omega v_\Omega = \int_0^M \mu(s)\,ds \quad \text{and} \quad \int_\Omega v_\Omega^p\,dx = \int_0^M p\,s^{p-1}\mu(s)\,ds.$$

Observe that the boundary of

$$\{x \in \Omega : v_\Omega > s\} \tag{3.4}$$

is

$$\{x \in \Omega : v_\Omega = s\}$$

for almost every $s > 0$ and the inner normal to this boundary at a point x is exactly $\nabla v_\Omega(x)/|\nabla v_\Omega(x)|$. These facts and others will be used to establish a differential inequality involving $\mu'(s)$ and $\mu(s)$ for almost every s so as to get an upper bound for $\mu(s)$ by integration which is not affected by "almost everywhere". Integrating $-\mathcal{Q}_p v_\Omega = 1$ over (3.4) gives

$$\begin{aligned}
\mu(s) &= -\frac{1}{p}\int_{v_\Omega(x)>s} \operatorname{div}\big(\nabla_\xi[F^p](\nabla v_\Omega)\big)\,dx \\
&= \frac{1}{p}\int_{v_\Omega(x)=s} \big\langle \nabla_\xi[F^p](\nabla v_\Omega), \tfrac{\nabla v_\Omega}{|\nabla v_\Omega|}\big\rangle\,d\mathcal{H}^{n-1} = \int_{v_\Omega(x)=s} \frac{F^p(\nabla v_\Omega)}{|\nabla v_\Omega|}\,d\mathcal{H}^{n-1} \qquad (3.5)
\end{aligned}$$

and we have

$$-\mu'(s)=\int_{v_\Omega(x)=s}\frac{1}{|\nabla v_\Omega|}\,d\mathcal{H}^{n-1} \tag{3.6}$$

for almost every $s\in[0,M)$.

The co-area formula gives that

$$-\frac{d}{dt}\int_{v_\Omega>s}F(\nabla v_\Omega)\,dx=P_F(\{v_\Omega>s\};\Omega),$$

for almost all s. Also, since v_Ω is smooth with compact support, it is known [1] that for almost every $s\in[0,M)$,

$$-\frac{d}{dt}\int_{u>s}F(\nabla v_\Omega)\,dx=\int_{v_\Omega=t}\frac{F(\nabla v_\Omega)}{|\nabla v_\Omega|}\,d\mathcal{H}^{n-1}.$$

Hence,

$$P_F(\{v_\Omega>s\};\Omega)=\int_{v_\Omega=s}\frac{F(\nabla v_\Omega)}{|\nabla v_\Omega|}\,d\mathcal{H}^{n-1}.$$

From Hölder's inequality, (3.5) and (3.6), we obtain

$$\begin{aligned}\big(P_F(\{v_\Omega>s\};\Omega)\big)^p&=\Big(\int_{v_\Omega=s}\frac{F(\nabla v_\Omega)}{|\nabla v_\Omega|}\,d\mathcal{H}^{n-1}\Big)^p\\ &\le\int_{v_\Omega=s}\frac{F(\nabla v_\Omega)^p}{|\nabla v_\Omega|}\,d\mathcal{H}^{n-1}\Big(\int_{v_\Omega=s}\frac{1}{|\nabla v_\Omega|}\,d\mathcal{H}^{n-1}\Big)^{p-1}=\mu(s)(-\mu'(s))^{p-1}.\end{aligned} \tag{3.7}$$

The isoperimetric inequality (3.1) tells us that

$$P(\{v_\Omega>s\})\ge n\kappa_n^{1/n}\,\mu(s)^{(n-1)/n}. \tag{3.8}$$

One can then use the same arguments as in the proof of Theorem 1 in [35] to finish the proof of Theorem 1.4. For the sake of completeness, we include it. It follows from (3.7) and (3.8) that

$$\big(n\kappa_n^{1/n}\big)^{p/(p-1)}\le-\mu(s)^{\frac{n+p-np}{n(p-1)}}\,\mu'(s) \tag{3.9}$$

Integrating (3.9) gives

$$\mu(s)\le\Big(|\Omega|^{\frac{p}{n(p-1)}}-\big(n\kappa_n^{1/n}\big)^{p/(p-1)}\cdot\frac{p}{n(p-1)}s\Big)^{n(p-1)/p}=|\Omega|\,(1-bs)^a, \tag{3.10}$$

where

$$a=\frac{n(p-1)}{p},\qquad b=\big(n\kappa_n^{1/n}\big)^{p/(p-1)}\cdot\frac{p}{n(p-1)}\cdot|\Omega|^{-1/a}.$$

Define $F\colon[0,M]\to\mathbb{R}$ by

$$F(t)=\Big(\int_0^t\mu(s)\,ds\Big)^p-p\Big(\int_0^t s^{p-1}\mu(s)\,ds\Big)|\Omega|^{p-1}.$$

It is easy to see from (3.10) that

$$F'(t) = p\Big(\Big(\int_0^t \mu(s)\,ds\Big)^{p-1} - t^{p-1}|\Omega|^{p-1}\Big)\mu(t) \le p\,|\Omega|^{p-1}\Big(\Big(\frac{1}{b(a+1)}\Big(1-(1-bt)^{a+1}\Big)\Big)^{p-1} - t^{p-1}\Big)\mu(t) \tag{3.11}$$

Since $p \ge 2 \ge n/(n-1)$, we have $a+1 \ge 2$. Using

$$(1+x)^\alpha \ge 1 + \alpha x + x^2, \quad \alpha \ge 2,\ x \ge -1$$

in (3.11), we obtain

$$F'(t) \le p\,|\Omega|^{p-1}\Big(\Big(t - \frac{bt^2}{a+1}\Big)^{p-1} - t^{p-1}\Big)\,\mu(t) = p\,|\Omega|^{p-1}\,t^{p-1}\Big(\Big(1-\frac{bt}{a+1}\Big)^{p-1} - 1\Big)\,\mu(t) \le -\frac{p\,b\,|\Omega|^{p-1}\,t^p\mu(t)}{a+1} \tag{3.12}$$

Integrating (3.12) over $[0, M]$ and using Hölder's inequality we have

$$\begin{aligned} F(M) &\le -\frac{p\,b\,|\Omega|^{p-1}}{a+1}\int_0^M t^p\mu(t)\,dt \\ &\le -\frac{p\,b\,|\Omega|^{p-1}}{a+1}\cdot\frac{\big(\int_0^M t^{p-1}\mu(t)\,dt\big)^{p/(p-1)}}{\big(\int_0^M \mu(t)\,dt\big)^{1/(p-1)}} \\ &= -\frac{b\,|\Omega|^{p-1}}{a+1}\cdot\frac{\big(\int_\Omega v_\Omega^p\,dx\big)^{p/(p-1)}}{p^{1/(p-1)}\,\big(\int_\Omega v_\Omega\,dx\big)^{1/(p-1)}}, \end{aligned}$$

that is,

$$\Big(\int_\Omega v_\Omega\,dx\Big)^p - |\Omega|^{p-1}\int_\Omega v_\Omega^p\,dx \le -\frac{b\,|\Omega|^{p-1}}{a+1}\,\frac{\big(\int_\Omega v_\Omega^p\,dx\big)^{p/(p-1)}}{p^{1/(p-1)}\big(\int_\Omega v_\Omega\,dx\big)^{1/(p-1)}}.$$

Dividing by $|\Omega|^{p-1}\int_\Omega v_\Omega^p dx$ and using Hölder's inequality, we infer

$$\begin{aligned} &\frac{\lambda_{p,1}(F,\Omega)\,T_{F,p}(\Omega)^{p-1}}{|\Omega|^{p-1}} - 1 \le \frac{\big(\int_\Omega v_\Omega\,dx\big)^p}{\big(\int_\Omega v_\Omega^p\,dx\big)\,|\Omega|^{p-1}} - 1 \\ &\quad\le -\frac{b}{(a+1)\,p^{1/(p-1)}}\cdot\Big(\frac{\int_\Omega v_\Omega^p\,dx}{\int_\Omega v_\Omega\,dx}\Big)^{1/(p-1)} \le -\frac{b}{(a+1)\,p^{1/(p-1)}}\cdot\frac{\int_\Omega v_\Omega\,dx}{|\Omega|} \\ &\quad= -\frac{p^{\frac{2p-3}{p-1}}(n\,\kappa_n^{1/n})^{p/(p-1)}}{n(p-1)(n(p-1)+p)}\cdot\frac{T_{F,p}(\Omega)}{|\Omega|^{1+\frac{p}{n(p-1)}}}. \end{aligned}$$

Thus (1.19) holds. □

Proof of Theorem 1.5. Let $u_M = \sup_\Omega u$. Consider two functions $\eta, V\colon [0, u_M] \to \mathbb{R}$ given by

$$\eta(t) = \int_{\{x\in\Omega: u(x)>t\}} g(u)\, dx, \quad V(t) = |\{x \in \Omega : u(x) > t\}|.$$

Integrating the equation $-\frac{1}{n}\mathrm{div}(\nabla_\xi[F^n](\nabla u)) = g(u)$ on $\{x \in \Omega : u(x) > t\}$, we have

$$\eta(t) = \int_{\Gamma(t)} \frac{F^n(\nabla u)}{|\nabla u|}\, d\mathcal{H}^{n-1}, \tag{3.13}$$

where $\Gamma(t) = \{x \in \Omega : u(x) = t\}$. The co-area formula gives

$$-\frac{dV}{dt} = \int_{\{x\in\Omega: u(x)=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u|}. \tag{3.14}$$

From (3.13), (3.14) and Hölder's inequality, we get

$$\begin{aligned} \eta(-V')^{n-1} &= \int_{\Gamma(t)} \frac{F^n(\nabla u)}{|\nabla u|}\, d\mathcal{H}^{n-1} \Big(\int_{\{x\in\Omega: u(x)=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u|} \Big)^{n-1} \\ &\geq \Big(\int_{\Gamma(t)} \frac{F(\nabla u)}{|\nabla u|}\, d\mathcal{H}^{n-1} \Big)^n = \big(P_F(\{u > t\}; \Omega)\big)^n \geq \big(n \kappa_n^{1/n} V(t)^{(n-1)/n}\big)^n. \end{aligned}$$

Hence

$$-\eta^{1/(n-1)}\, V'(t) \geq \big(n^n \kappa_n\big)^{1/(n-1)}\, V(t),$$

which, combining with

$$\frac{d\eta}{dt} = g(t)\, \frac{dV}{dt},$$

gives

$$-\,\eta^{1/(n-1)}\, \eta'(t) \geq \big(n^n \kappa_n\big)^{1/(n-1)}\, g(t) V(t). \tag{3.15}$$

Integration of (3.15) on $[0, u_M]$ yields

$$\begin{aligned} \frac{n-1}{n} \eta(0)^{n/(n-1)} &\geq \big(n^n \kappa_n\big)^{1/(n-1)} \int_0^{u_M} g(t) V(t)\, dt \\ &= -\big(n^n \kappa_n\big)^{1/(n-1)} \int_0^{u_M} G(t) V'(t)\, dt = \big(n^n \kappa_n\big)^{\frac{1}{n-1}} \int_\Omega G(u)\, dx, \end{aligned}$$

that is

$$\frac{n-1}{n} \Big(\int_\Omega g(u)\, dx \Big)^{n/(n-1)} \geq \big(n^n \kappa_n\big)^{1/(n-1)} \int_\Omega G(u)\, dx.$$

Thus, (1.20) follows. □

4. Proof of Theorems 1.6–1.8

In this section, we prove Theorems 1.6 and 1.7. Before doing this, we first recall Reilly's formula which will be used later. Let M be an n-dimensional compact manifold with boundary. We will often write $\langle\,,\rangle$ the Riemannian metric on M as well as that induced on ∂M. Let ∇ and Δ be the connection and the Laplacian on M, respectively. Let ν be the unit outward normal vector of ∂M. The shape operator of ∂M is given by $S(X) = \nabla_X \nu$ and the second fundamental form of ∂M is defined as $II(X,Y) = \langle S(X), Y\rangle$, here $X, Y \in T(\partial M)$. The eigenvalues of S are called the principal curvatures of ∂M and the mean curvature H of ∂M is given by $H = \frac{1}{n-1}\mathrm{tr}\ S$, here tr S denotes the trace of S. For a smooth function f defined on M, the following identity holds [27] if $h = \frac{\partial}{\partial\nu} f\big|_{\partial M}$, $z = f|_{\partial M}$ and Ric denotes the Ricci tensor of M:

$$\int_M \big((\Delta f)^2 - |\nabla^2 f|^2 - \mathrm{Ric}(\nabla f, \nabla f)\big)$$
$$= \int_{\partial M} \big(((n-1)Hh + 2\overline{\Delta} z)h + II(\overline{\nabla} z, \overline{\nabla} z)\big).$$

Here $\nabla^2 f$ is the Hessian of f; $\overline{\Delta}$ and $\overline{\nabla}$ represent the Laplacian and the gradient on ∂M with respect to the induced metric on ∂M, respectively.

Proof of Theorem 1.6. i) Since ρ satisfies the equation

$$\Delta\rho = -1 \text{ in } M, \quad \rho|_{\partial M} = 0,$$

we know from the strong maximum principle and Hopf's lemma [19] that ρ is positive in the interior of M and

$$\frac{\partial\rho}{\partial\nu}(x) < 0, \quad \forall x \in \partial M. \tag{4.1}$$

It follows from Bochner's formula that

$$\begin{aligned} \frac{1}{2}\Delta|\nabla\rho|^2 &= |\nabla^2\rho|^2 + \langle\nabla\rho, \nabla(\Delta\rho)\rangle + \mathrm{Ric}(\nabla\rho,\nabla\rho) \\ &= |\nabla^2\rho|^2 + \mathrm{Ric}(\nabla\rho,\nabla\rho) \ge |\nabla^2\rho|^2 + (n-1)\kappa|\nabla\rho|^2. \end{aligned} \tag{4.2}$$

Integrating (4.2) on M and using the divergence theorem, we get

$$\begin{aligned} \int_M |\nabla^2\rho|^2 + (n-1)\kappa\cdot T(M) &= \int_M (|\nabla^2\rho|^2 + (n-1)\kappa|\nabla\rho|^2) \\ &\le \frac{1}{2}\int_{\partial M} \nu|\nabla\rho|^2 = \int_{\partial M} \nabla^2\rho(\nabla\rho,\nu). \end{aligned} \tag{4.3}$$

Since $\rho|_{\partial M} = 0$, we have

$$\nabla\rho|_{\partial M} = \Big(\frac{\partial\rho}{\partial\nu}\Big)\nu, \quad \nabla^2\rho(\nu,\nu) = \frac{\partial^2\rho}{\partial\nu^2}.$$

Hence

$$\int_{\partial M} \nabla^2\rho(\nabla\rho,\nu) = \int_{\partial M} \Big(\frac{\partial\rho}{\partial\nu}\Big)\Big(\frac{\partial^2\rho}{\partial\nu^2}\Big).$$

Setting

$$l = \min_{x\in\partial M} \frac{\partial^2 \rho}{\partial \nu^2}(x);$$

we have from (4.1) that

$$\left(\frac{\partial \rho}{\partial \nu}\right)\left(\frac{\partial^2 \rho}{\partial \nu^2}\right) \le \left(\frac{\partial \rho}{\partial \nu}\right) l. \tag{4.4}$$

Hence

$$\int_{\partial M} \nabla^2 \rho(\nabla \rho, \nu) \le l \int_{\partial M} \frac{\partial \rho}{\partial \nu} = l \int_M \Delta \rho = -lV. \tag{4.5}$$

The Schwarz inequality implies that

$$|\nabla^2 \rho|^2 \ge \frac{1}{n}(\Delta \rho)^2 = \frac{1}{n} \tag{4.6}$$

with equality holding if and only if

$$\nabla^2 \rho = \frac{\Delta \rho}{n}\langle\, ,\rangle = -\frac{1}{n}\langle\, ,\rangle. \tag{4.7}$$

Combining (4.3), (4.5) and (4.6), we get (1.21). On the other hand, if (1.21) take equality sign, then the inequalities (4.2)–(4.6) should be equalities. Thus, (4.7) holds on M. Taking the covariant derivative of (4.7), we get $\nabla^3 \rho = 0$ and from the Ricci identity,

$$R(X, Y)\nabla \rho = 0, \tag{4.8}$$

for any tangent vectors X, Y on M, where R is the curvature tensor of M. By the the maximum principle ρ attains its maximum at some point x_0 in the interior of M. Let r be the distance function to x_0; then from (4.7) it follows that

$$\nabla \rho = -\frac{1}{n} r \frac{\partial}{\partial r}. \tag{4.9}$$

Using (4.8), (4.9), Cartan's theorem (cf. [15]) and $\rho|_{\partial M} = 0$, we conclude that M is a ball in $\mathbb{R}^n$ whose center is x_0, and

$$\rho(x) = \frac{1}{2n}(r_0^2 - |x - x_0|)$$

in M, here r_0 is the radius of the ball. This in turn implies that $\kappa = 0$.

ii) Restricting $\Delta \rho = -1$ on ∂M and noticing $\rho|_{\partial M} = 0$, we infer

$$\frac{\partial^2 \rho}{\partial \nu^2} + (n-1) H \frac{\partial \rho}{\partial \nu} = -1 \quad \text{on } \partial M. \tag{4.10}$$

Integrating (4.10) on ∂M yields

$$A + \int_{\partial M} \frac{\partial^2 \rho}{\partial \nu^2} = -(n-1)\int_{\partial M} H \partial_\nu \rho. \tag{4.11}$$

Substituting ρ into Reilly's formula, we get

$$(4.12)\quad (n-1)\int_{\partial M} H(\partial_\nu \rho)^2 = \int_M ((\Delta\rho)^2 - |\nabla^2\rho|^2 - \mathrm{Ric}(\nabla\rho, \nabla\rho))$$
$$\leq \int_M \Big((\Delta\rho)^2 - \frac{1}{n}(\Delta\rho)^2 - (n-1)\kappa|\nabla\rho|^2\Big) = \frac{(n-1)V}{n} - (n-1)\kappa T(M),$$

with equality holding if and only if

$$(4.13)\quad |\nabla^2\rho|^2 = \frac{1}{n} \quad \text{on } M,$$

and

$$\mathrm{Ric}(\nabla\rho, \nabla\rho) = (n-1)\kappa|\nabla\rho|^2 \quad \text{on } M.$$

Since $H \geq 0$ on ∂M, one obtains from Hölder's inequality that

$$(4.14)\quad -\int_{\partial M} H\partial_\nu\rho \leq \Big(\int_{\partial M} H(\partial_\nu\rho)^2\Big)^{1/2}\Big(\int_{\partial M} H\Big)^{1/2}.$$

Combining (4.11), (4.12) and (4.14), one gets ((1.22). Also, if the equality in (1.22) holds, recalling (4.12), one obtains (4.13). Using the same arguments as in the proof of item i), we conclude that M is isometric to a ball in $\mathbb{R}^n$. □

Proof of Theorem 1.7. As stated in the proof of Theorem 1.6, the function u is positive in the interior of M and $-\partial u/\partial\nu|_{\partial M} > 0$. Multiplying the equation

$$-\Delta u = 1$$

by $\big(|\nabla u|^2 + \frac{2}{n}u\big)$ and integrating on M, we have from the divergence theorem and $u|_{\partial M} = 0$ that

$$\frac{n+2}{n}T(M) = \frac{n+2}{n}\int_M u = \int_M \Big(|\nabla u|^2 + \frac{2}{n}u\Big) = \int_M \Big(|\nabla u|^2 + \frac{2}{n}u\Big)(-\Delta u)$$
$$= \int_M \Big\langle \nabla\Big(|\nabla u|^2 + \frac{2}{n}u\Big), \nabla u\Big\rangle - \int_{\partial M}\Big(|\nabla u|^2 + \frac{2}{n}u\Big)\frac{\partial u}{\partial\nu}$$
$$= -\int_M u\Delta\Big(|\nabla u|^2 + \frac{2}{n}u\Big) - \int_{\partial M}\Big(|\nabla u|^2 + \frac{2}{n}u\Big)\frac{\partial u}{\partial\nu}$$
$$= -2\int_M u\Big(|\nabla^2 u|^2 + \mathrm{Ric}(\nabla u, \nabla u) - \frac{1}{n}\Big) - \int_{\partial M}\Big(\frac{\partial u}{\partial\nu}\Big)^3$$
$$(4.15)\quad \leq -2\int_M u\,\mathrm{Ric}(\nabla u, \nabla u) - \int_{\partial M}\Big(\frac{\partial u}{\partial\nu}\Big)^3,$$

with equality holding if and only if

$$(4.16)\quad |\nabla^2 u|^2 = \frac{1}{n} \quad \text{on } M.$$

Setting $\max_{x\in\partial M}|\nabla u| = m$ and using

$$u\,\mathrm{Ric}(\nabla u, \nabla u) \geq (n-1)\kappa u|\nabla u|^2,$$

we conclude from (4.15) that

$$\frac{n+2}{n}T(M)+2(n-1)\kappa\int_M u|\nabla u|^2\le m^2\int_{\partial M}\Big(-\frac{\partial u}{\partial\nu}\Big)=m^2\int_M(-\Delta u)=m^2V.$$

Thus (1.23) holds. It is clear from the above proof that if the equality in (1.23) holds then (4.16) holds and so $\kappa=0$ and M is isometric to a ball in $\mathbb{R}^n$. □

Proof of Theorem 1.8. We shall only consider the case that $\partial M\neq\emptyset$, since the case $\partial M=\emptyset$ is similar. Multiplying the equation $\Delta u=-g(u)$ by $|\nabla u|^2$, integrating on M and using the divergence theorem, we get

$$\begin{aligned}\int_M g(u)|\nabla u|^2&=-\int_M|\nabla u|^2\Delta u=\int_M\langle\nabla|\nabla u|^2,\nabla u\rangle-\int_{\partial M}|\nabla u|^2\frac{\partial u}{\partial\nu}\\&=-\int_M u\Delta|\nabla u|^2-\int_{\partial M}\Big(\frac{\partial u}{\partial\nu}\Big)^3\\&=-\int_M 2u(|\nabla^2u|^2+\mathrm{Ric}(\nabla u,\nabla u)+\langle\nabla u,\nabla(\Delta u)\rangle)-\int_{\partial M}\Big(\frac{\partial u}{\partial\nu}\Big)^3\\&\le-\int_M 2u\Big(\frac{(\Delta u)^2}{n}+(n-1)\kappa|\nabla u|^2+\langle\nabla u,\nabla(\Delta u)\rangle\Big)-\int_{\partial M}\Big(\frac{\partial u}{\partial\nu}\Big)^3\\&=-\int_M\Big(\frac{2ug(u)^2}{n}+(n-1)\kappa\langle\nabla u,\nabla u^2\rangle+\langle\nabla u^2,\nabla(\Delta u)\rangle\Big)-\int_{\partial M}\Big(\frac{\partial u}{\partial\nu}\Big)^3\\&=\int_M\Big(-\frac{2ug(u)^2}{n}+(n-1)\kappa u^2\Delta u+\Delta u\Delta u^2\Big)-\int_{\partial M}\Big(\frac{\partial u}{\partial\nu}\Big)^3\\&=\int_M\Big(-\frac{2ug(u)^2}{n}-(n-1)\kappa u^2g(u)+2ug(u)^2-2g(u)|\nabla u|^2\Big)-\int_{\partial M}\Big(\frac{\partial u}{\partial\nu}\Big)^3.\end{aligned}\tag{4.17}$$

On the other hand, it is easy to see that

$$\int_M g(u)|\nabla u|^2=\int_M\langle\nabla G(u),\nabla u\rangle=-\int_M G(u)\Delta u=\int_M G(u)g(u).\tag{4.18}$$

Substituting (4.18) into (4.17), one gets (1.24). □

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