Rev. Mat. Iberoam. (online first)
DOI 10.4171/RMI/1165

On differential polynomial rings over nil algebras

Mikhail Chebotar and Wen-Fong Ke

Abstract. Let R be a nil algebra over a field of characteristic 0, and let δ be a derivation of R. Then the differential polynomial ring $R[X,\delta]$ cannot be mapped onto a unital simple ring homomorphically.

1. Introduction

Let $\delta\colon R \to R$ be a derivation of a ring R. By $R[X;\delta]$ we denote the differential polynomial ring, that is the set of left polynomials $a_0 + a_1X + \cdots + a_{n-1}X^{n-1} + a_nX^n$, where $a_i \in R$, $i = 0,\ldots,n$. Addition of such polynomials is defined in the usual way. Multiplication is defined by the associative rule, except that the right polynomials should be converted into left polynomials, which can be done by repeatedly applying the rule $Xa = aX + a^\delta$ for all $a \in R$. Here, a^δ denotes the image of a by δ.

Recently, there has been a significant interest to the radical properties of differential polynomial rings [3], [4], [8], [9], [13], [14]. It is quite interesting to compare polynomial rings over nil rings with the differential polynomial rings over nil rings. A famous result due to Smoktunowicz [12] asserts that there exists a nil ring R such that the polynomial ring $R[x]$ is not nil. Obviously, this result easily extends to differential polynomial rings.

Recall that the Koethe conjecture, one of the most challenging open questions in ring theory, has an equivalent statement: whether a polynomial ring $R[X]$ is Jacobson radical if the ring R is nil. Along the long stream of attempts to solve this problem, Puczyłowski and Smoktunowicz [11] laid a cornerstone by showing that a polynomial ring $R[X]$ over a nil ring R is Brown–McCoy radical, i.e., it cannot be mapped homomorphically onto a unital simple ring.

Now, it is natural to verify similar statements for differential polynomial rings over nil rings. First of all, in their important paper [14], Smoktunowicz and Ziembowski showed that there exists a locally nilpotent ring R and a derivation δ of R such that $R[X;\delta]$ is not Jacobson radical, thus solving an open problem by

Mathematics Subject Classification (2010): 16N40.
Keywords: Differential polynomial ring, nil ring.

Shestakov. Recently, Smoktunowicz ([9], Proposition 3.1) proved that if R is locally nilpotent, then $R[X;\delta]$ is Brown–McCoy radical. Hence, $R[X;\delta]$ cannot be mapped onto a unital simple ring homomorphically. Still, one has to ponder on the following question.

Question A. *Let $\delta\colon R \to R$ be a derivation of a nil ring R. Can $R[X;\delta]$ be mapped homomorphically onto a unital simple ring?*

In this short paper we will show that the answer is negative if the target simple ring is of characteristic 0. Namely, we will prove the following theorem.

Theorem 1. *Let R be a nil ring and let δ be a derivation of R. If φ is a homomorphism from the differential polynomial ring $R[X,\delta]$ to a unital simple ring S, where S is of characteristic 0, then φ cannot be surjective.*

As an obvious corollary, we state:

Corollary 2. *Let R be a nil algebra over a field of characteristic 0, and let δ be a derivation of R. Then the differential polynomial ring $R[X,\delta]$ cannot be mapped homomorphically onto a unital simple ring.*

We remark that, to answer Question A in full generality, it is sufficient to consider nil algebras over $\mathbb{Z}_p$. From this perspective, it is interesting to compare this question with the one about polynomial rings in several variables over nil rings posted by Puczyłowski and Smoktunowicz. In [11], Question 1a, it was asked whether polynomial rings in several variables over nil rings are Brown–McCoy radical. For this problem, it was sufficient to consider nil algebras instead of nil rings. For polynomial rings in two variables over nil algebras of prime characteristic, the answer was obtained in [5]; for general situation, the answer was given only very recently in [6] by means of convex geometry.

2. Results

One of the main tools in our proof is Kharchenko's theory of differential identities. For a general introduction to Kharchenko's technique as well as the properties of symmetric ring of quotients and extended centroid, one is referred to the book [2]. We will include a short and nice presentation from [7] on differential polynomial identities before we put down the Kharchenko's theorem, namely Theorem 4, that we need for our proof.

Let us start with the following useful result on nil prime rings.

Lemma 3. *Let R be a nil prime ring with symmetric ring of quotients Q and let $q \in Q$ be an element such that $[q,r] = qr - rq \in R$ for all $r \in R$. Then the ring S generated by R and q cannot be a simple ring with 1.*

Proof. Suppose that the ring S generated by R and q is a simple ring with 1. We claim that q is invertible in S. As $1 \in S$, there are $a_0, a_1, \ldots, a_k \in R$ such that $1 = a_0 + a_1q + \cdots + a_kq^k$. Write $a_0 = 1 - a_1q - \cdots - a_kq^k$, which is nilpotent, and so $(1 - a_1q - \cdots - a_kq^k)^\ell = 0$ for some $\ell \geq 1$. Expanding, we see that

$$1 = b_1q + \cdots + b_tq^t = (b_1 + \cdots + b_tq^{t-1})q$$

for some $t \in \mathbb{N}$ and $b_1, \ldots, b_t \in R$. Since $b_1 + \cdots + b_tq^{t-1} \in S$, q has a left inverse in S. Similarly, changing the side on which the element q appears, we see that q is has a right inverse in S. Hence q is invertible in S.

Now, for some positive integer N and $u_{i,0}, v_{j,0} \in R$ ($i, j \in \{0, 1, \ldots, N\}$), we can write

$$q^{-1} = \sum_{i=0}^{N} u_{i,0}\, q^i = \sum_{j=0}^{N} q^j\, v_{j,0}.$$

Let $k \geq 1$, and assume that

$$q^{-k} = \sum_{i=0}^{N} u_{i,k}\, q^i = \sum_{j=0}^{N} q^j\, v_{j,k},$$

where $u_{i,k}, v_{j,k} \in R$ ($i, j \in \{0, 1, \ldots, N\}$). Then

$$\begin{aligned} q^{-(k+1)} &= \Big(\sum_{i=0}^{N} u_{i,k}\, q^i\Big) \cdot q^{-1} = u_{0,k} \cdot q^{-1} + \Big(\sum_{i=1}^{N} u_{i,k}\, q^i\Big) \cdot q^{-1} \\ &= u_{0,k}\Big(\sum_{i=0}^{N} u_{i,0}\, q^i\Big) + \sum_{i=1}^{N} u_{i,k}\, q^{i-1} = u_{0,k}\, u_{0,0} + \sum_{i=1}^{N} (u_{0,k}\, u_{i,0} + u_{i,k+1})\, q^i \\ &= \sum_{i=0}^{N} u_{i,k+1}\, q^i, \end{aligned}$$

and similarly,

$$q^{-(k+1)} = \sum_{j=0}^{N} q^j\, v_{j,k+1},$$

where $u_{i,k+1}, v_{j,k+1} \in R$ ($i, j \in \{0, 1, \ldots, N\}$). Thus, by induction, we have for any positive integer t, there are $u_{i,t}, v_{j,t} \in R$ $i, j \in \{0, 1, \ldots, N\}$, such that

$$q^{-t} = \sum_{i=0}^{N} u_{i,t}\, q^i = \sum_{j=0}^{N} q^j\, v_{j,t}.$$

Take a nonzero ideal I of R with the property that $q^i I \subseteq R$ and $Iq^j \subseteq R$ for $i, j \in \{1, \ldots N\}$. Such an ideal exists by Proposition 2.2.3 in [2]. Let $J = I^2$. Then we have

$$q^{-s} J\, q^{-t} \subseteq R \quad \text{for all positive integers } s \text{ and } t. \tag{2.1}$$

We note that the ideal generated by J and q is S due to the simplicity of S. Hence, the unity 1 is a sum $\sum_{(i,j)\in\Lambda} q^i w_{ij} q^j$, where Λ is a finite subset of $\mathbb{N}_0 \times \mathbb{N}_0$, and $w_{ij} \in J$ for all $(i,j) \in \Lambda$. Let K and L be positive integers such that for every $(i,j) \in \Lambda$, $K > i$ and $L > j$. With (2.1), for each $(i,j) \in \Lambda$, there is an $r_{ij} \in R$ with

$$q^i\, w_{ij}\, q^j = q^K (q^{i-K}\, w_{ij}\, q^{j-L})\, q^L = q^K\, r_{ij}\, q^L.$$

Thus,

$$1 = \sum_{(i,j)\in\Lambda} q^i w_{ij}\, q^j = \sum_{(i,j)\in\Lambda} q^K\, r_{ij}\, q^L = q^K \Big(\sum_{(i,j)\in\Lambda} r_{ij} \Big)\, q^L = q^K r\, q^L,$$

where $r = \sum_{(i,j)\in\Lambda} r_{ij} \in R$. But then, $r = q^{-K-L}$, which is impossible since r is nilpotent. Therefore, S cannot be a simple ring with 1. □

Let R be a prime ring with symmetric ring of quotients Q and extended centroid C, which is a field. Let $\mathrm{Der}(R)$ and $\mathrm{Der}(Q)$ denote the set of all derivations of R and Q, respectively. Since every derivation of R can be uniquely extended to a derivation of Q (cf. [2], Proposition 2.5.1), we regard $\mathrm{Der}(R) \subseteq \mathrm{Der}(Q)$. For $\alpha \in C$, $\sigma \in \mathrm{Der}(Q)$ and $x \in Q$, define $x^{\sigma\alpha} = x^\sigma \alpha$. This makes $\sigma\alpha$ a derivation on Q, and turns $\mathrm{Der}(Q)$ into a right vector space over C. Let $D = \mathrm{Der}(R)C$ and let D_{int} be a subspace of $\mathrm{Der}(Q)$ consisting of all inner derivations of Q.

Now, a differential polynomial is a generalized polynomial involving noncommutative indeterminates Y_i which are acted by derivations of R as unary operations. The coefficients of a differential polynomial are allowed to lie in Q. Every differential polynomial can be transformed and written in the form $\chi(Y_i^{\Delta_j})$, where $\chi(Z_{ij})$ is an ordinary generalized polynomial in Z_{ij} and Δ_j's are words of derivations of R. We say that $\chi = 0$ is a differential identity for R if $\chi(r_i^{\Delta_j}) = 0$ for all $r_i \in R$.

Choose a basis M_0 for D_{int}, and augment it to a basis M of $D_{\mathrm{int}} + D$. Fix a total order $<$ in the set M such that $\mu < \delta$ for $\delta \in M_0$ and $\mu \in M \setminus M_0$, and extend it to the set of all derivation words in M: a shorter word is smaller than a longer one, while words of the same length are ordered lexicographically. A differential identity can always be transformed into one in the form $\chi(Y_i^{\Delta_j}) = 0$, where (1) $\chi(Z_{ij})$ is a generalized polynomial in distinct indeterminates Z_{ij}, and (2) each Δ_j is a regular word. Here a word $\Delta = \delta_1^{s_1}\delta_2^{s_2}\ldots\delta_m^{s_m}$, is regular if

1. $\delta_j \in M \setminus M_0$ for all $j = 1, \ldots, m$;
2. $\delta_1 < \delta_2 < \cdots < \delta_m$; and
3. if R has characteristic $p > 0$, then $s_k < p$ for all $k = 1, \ldots, m$.

We now state Kharchenko's theorem as it was done in [7], p. 254.

Theorem 4. *If $\chi(Y_i^{\Delta_j}) = 0$ is a differential identity on a prime ring R, where Δ_j are distinct regular derivation words in M and where $\chi(Z_{ij})$ is a generalized polynomial in indeterminates Z_{ij}, then $\chi(Z_{ij}) = 0$ is a generalized polynomial identity on R.*

Applying the theorem, we obtain a result on linear generalized differential polynomials, which will be used for Lemma 6. It also illustrates well how Kharchenko's theorem works.

In our case, we will have just one derivation d on R, so the linear generalized differential polynomial in our consideration is of the form $\sum_{i=0}^{\ell}\sum_{j=0}^{s_i} a_{ij}Y^{d^i}b_{ij}$, $a_{ij}, b_{ij} \in Q$, where Y is an indeterminate. When $\sum_{j=0}^{s_\ell} a_{\ell j}Y^{d^\ell}b_{\ell j} \neq 0$, we say that the leading exponent is d^ℓ. If there is some $q \in Q$ such that $r^d = [q, r]$ for all $r \in R$, we say that d is X-inner.

Corollary 5. *Let R be a prime ring of characteristic 0 and let d be a derivation of R which is not X-inner. Let*

$$\chi(Y^{d^i}) = \sum_{i=0}^{\ell}\sum_{j=0}^{s_i} a_{ij}Y^{d^i}b_{ij}$$

be a generalized differential polynomial, where $a_{ij}, b_{ij} \in Q$ and Y is an indeterminate. Then there exists $r \in R$ such that $\chi(r^{d^i}) \neq 0$.

Proof. Since d is not X-inner, we may assume that $d \in M\backslash M_0$ and all d^i are regular words. Let d^ℓ be the leading exponent of $\chi(Y^{d^i})$, that is $\sum_{j=0}^{s_\ell} a_{\ell j}Y^{d^\ell}b_{\ell j} \neq 0$.

Suppose that $\chi(r^{d^i}) = 0$ for all $r \in R$, that is $\chi(Y^{d^i})$ is a differential identity on R. According to Theorem 4 we get

$$\chi(Z_i) = \sum_{i=0}^{\ell}\sum_{j=0}^{s_i} a_{ij}\,Z_i\,b_{ij} = 0$$

for all $Z_i \in R$, $i = 0, \ldots, \ell$. Taking $Z_0 = \cdots = Z_{\ell-1} = 0$, we get

$$\sum_{j=0}^{s_\ell} a_{\ell j}\,Z_\ell\,b_{\ell j} = 0$$

for all $Z_\ell \in R$, but this contradicts Corollary 6.1.3 in [2], which asserts that there are no nonzero linear generalized polynomial identities in one variable. □

Lemma 6. *Let S be a simple ring of characteristic 0. Let R be a prime ring with extended centroid C, and $x \in S$ an element such that $[x, r] = xr - rx \in R$ for all $r \in R$. Assume that S is generated by R and x, and that the derivation d on R given by $r^d = [x, r]$ for all $r \in R$ is not X-inner. Then for elements $a_0, a_1, \ldots, a_n \in RC$, $a_0 + a_1x + \cdots + a_nx^n = 0$ only if $a_i = 0$ for all i.*

Proof. We proceed by induction on n. The case $n = 0$ is obviously true. Assume that the statement holds for $n = k > 0$, and that $a_0, a_1, \ldots, a_{k+1} \in RC$ are such that $a_0 + a_1x + \cdots + a_{k+1}x^{k+1} = 0$. For convenience, write

$$s = a_0 + a_1x + \cdots + a_{k+1}\,x^{k+1} = 0.$$

If $a_{k+1} = 0$, there is nothing to prove. So we assume $a_{k+1} \neq 0$.

By Proposition 2.5.1 in [2], we can extend the derivation d uniquely to a derivation on RC, and by the assumption, we have $[x, b] = b^d$ for all $b \in RC$. Hence,

$$xb = bx + b^d \quad \text{for all } b \in RC. \tag{2.2}$$

For $r \in R$, set $t(r) = sr\,a_{k+1} - a_{k+1}rs = 0$. After using (2.2) to rearrange the terms of $t(r)$ in the powers of x, we get

$$t(r) = b_0(r) + b_1(r)x + \cdots + b_k(r)x^k + b_{k+1}(r)x^{k+1} = 0,$$

where each $b_m(Y)$, $0 \le m \le k+1$, is a linear generalized differential polynomial in Y. In particular,

$$b_{k+1}(r) = a_{k+1}\,r\,a_{k+1} - a_{k+1}\,r\,a_{k+1} = 0,$$

while

$$b_0(r) = a_{k+1}\,r^{d^{k+1}}a_{k+1} + b(r),$$

with $b(Y)$ a linear generalized differential polynomial in Y whose leading exponent is smaller than d^{k+1}. Therefore, $t(r)$ is in the form

$$t(r) = b_0(r) + b_1(r)x + \cdots + b_k(r)x^k = 0,$$

and the induction hypothesis says that $b_m(r) = 0$ for $m = 0, \ldots, k$. Moreover, this is true for arbitrary $r \in R$. However, as a_{k+1} is not zero, $b_0(Y)$ is a nonzero linear generalized differential polynomial in Y, and so, according to Corollary 5, there exists an element $r' \in R$ such that $b_0(r')$ is not zero. This is a contradiction, and the lemma is proved. □

Proof of Theorem 1. Suppose that the skew polynomial ring $R[X, \delta]$ is homomorphically mapped onto a unital simple ring S of characteristic 0. Let φ be such an epimorphism, and A the image of R. Let $p(X) \in R[X, \delta]$ be a preimage of 1 and let x be the image of $Xp(X)$ under this homomorphism.

Suppose that $r \in R$ and $\varphi(r) = a$. Then

$$\varphi(rX) = \varphi(rX)\,\varphi(p(X)) = \varphi(rXp(X)) = \varphi(r)\,\varphi(Xp(X)) = ax,$$

and for $i > 1$,

$$\varphi(r(Xp(X))^i) = \varphi(rXp(X))\,\varphi(Xp(X))^{i-1} = ax \cdot x^{i-1} = ax^i.$$

Consequently, if $h = \sum_{i=0}^{k} r_i X^i$, then $\varphi(h) = \sum_{i=0}^{k} \varphi(r_i X^i) = \sum_{i}^{k} \varphi(r_i)x^i$. Therefore, every element $u \in S$ can be written in the form $u = a_0 + a_1 x + \cdots + a_k x^k$ for some $k \ge 0$ and each $a_i \in A$ for $i = 0, \ldots, k$.

Since R is nil, we have that A is nil as well. Further, with S being of zero characteristic, we will show that A is prime. Our approach is based on the idea of the proof of Theorem 4 in [10] with appropriate modifications.

First, we observe that if I is a nonzero differential ideal of A, then I cannot not be locally nilpotent. In particular, A itself is not locally nilpotent. Indeed, for a nonzero differential ideal I of A, consider the differential polynomial ring $I[Y;d]$, where $a^d = Ya - aY$ for all $a \in I$. Take the homomorphism ψ from $I[Y;d]$ into S defined by $a_0 + a_1Y + \cdots + a_kY^k \mapsto a_0 + a_1x + \cdots + a_kx^k$ for all $a_0, a_1, \ldots, a_k \in I$. Then $\psi(I[Y;d])$ is a nonzero ideal of S, and so $\psi(I[Y;d]) = S$ since S is simple. By Proposition 3.1 in [9], this cannot happen if I is locally nilpotent.

Next, we claim that A is primary in the sense of [10], p. 338, i.e., every ideal divisor of zero of A is nilpotent. Let I be a nonzero ideal of A such that $I\beta = 0$ for some nonzero $\beta \in A$. Assume that I is not nilpotent, and set

$$J = \{\gamma \in A \mid \exists M(\gamma) > 0 \text{ such that } I^{M(\gamma)}\gamma = 0\}.$$

Then J is a nonzero ideal of A. For any $\gamma \in J$, as $I^{M(\gamma)}\gamma = 0$, we have $I^{2M(\gamma)}\gamma^d = 0$, and so $\gamma^d \in J$ as well. Therefore, J is a nonzero differential ideal of A. We will derive a contradiction that J is locally nilpotent.

Let $\Gamma = \{\gamma_1, \ldots, \gamma_l\}$ be a finite subset of J and, for any $k > 0$, denote $\Gamma^k = \{\gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_k} \mid 1 \le i_j \le l\}$. It is sufficient to show that $\Gamma^k = 0$ for some k. Let

$$K = \{\alpha \in A \mid \exists N(\alpha) > 0 \text{ such that } \alpha A\Gamma^{N(\alpha)} = 0\}.$$

Then K is an ideal of A. From $I^{M(\gamma_{i_1})}A\gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_k} \subseteq I^{M(\gamma_{i_1})}\gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_k} = 0$, we have $I^{M(\gamma_{i_1})} \subseteq K$. Since I is assumed to be not nilpontent, K is nonzero. Now, as shown in the proof of Theorem 4 in [10], for $\alpha \in K$, it holds that

$$\alpha^d A\Gamma^{2N(\alpha)} = 0,$$

thus $d(\alpha) \in K$. This shows that K is a differential ideal of A. Therefore S is generated by K and x, and we have

$$\alpha_0 + x\alpha_1 + \cdots + x^t\alpha_t = 1 \tag{2.3}$$

for some $\alpha_0, \ldots, \alpha_t \in K$. Let $N = \max\{N(\alpha_0), \ldots, N(\alpha_t)\}$. Then $\alpha_i\Gamma^{N+1} \subseteq \alpha_iA\Gamma^N = 0$ for all i, $0 \le i \le t$. Multiplying Γ^{N+1} from the right on the identity (2.3), we get $0 = \Gamma^{N+1}$. Therefore, J is locally nilpotent. This contradiction assures us that I is nilpotent. Hence, A is primary.

Let us continue to show that A is prime. This will be the case if we could show that A has no nonzero nilpotent ideal. Let T be the sum of all nilpotent ideals, and we set to show that T is a differential ideal and conclude that $T = 0$. To this goal, it is sufficient to show that if an element $\alpha \in A$ is contained in a nilpotent ideal, then α^d is sitting is some nilpotent ideal as well.

As in the proof of Theorem 4 in [10], we assume that $(A, +)$ is torsion-free, and let $\alpha \in A$ be an element of some nilpotent ideal. Let us show that α^d belongs to some nilpotent ideal as well. If α^d belongs to an ideal divisor of zero, then it is in a nilpotent ideal and there is nothing to do. So we assume that α^d is not in any ideal divisor of zero. As some power of αA is zero, there is a smallest integer m such that $(\alpha A)^m = 0$. If $m = 1$, then $\alpha = 0$ and $\alpha^d = 0$, and we are done. Assume $m > 1$, and let $a_i, b_i \in A$, $1 \le i \le m-1$. For all $a \in A$, we have

$\alpha a_1 \alpha a_2 \ldots \alpha a_{m-1} \alpha a = 0$. Differentiating this equation m times and multiplying it from the left with $\alpha b_1 \alpha b_2 \ldots \alpha b_{m-1}$, we get

$$(m+1)\alpha b_1 \alpha b_2 \ldots \alpha b_{m-1} \alpha^d a_1 \alpha^d a_2 \ldots \alpha^d a_{m-1} \alpha^d a = 0 \quad \text{for all } a \in A.$$

Since A is torsion-free,

$$\alpha b_1 \alpha b_2 \ldots \alpha b_{m-1} \alpha^d a_1 \alpha^d a_2 \ldots \alpha^d a_{m-1} \alpha^d a = 0 \quad \text{for all } a \in A.$$

As A is primary, we have

$$\alpha b_1 \alpha b_2 \ldots \alpha b_{m-1} \alpha^d a_1 \alpha^d a_2 \ldots \alpha^d a_{m-1} \alpha^d = 0.$$

The elements $a_i, b_j \in A$, $1 \le i \le m-1$, $1 \le j \le m-1$, are arbitrary, and since α^d is not in any ideal divisor of zero, we have

$$\alpha b_1 \alpha b_2 \ldots \alpha b_{m-1} \alpha^d a_1 \alpha^d a_2 \ldots \alpha^d a_{m-1} = 0$$

for all $a_i, b_j \in A$, $1 \le i \le m-1$, $1 \le j \le m-1$. Continuing in this fashion, we shall reach the conclusion that

$$\alpha b_1 \alpha b_2 \ldots \alpha b_{m-1} \alpha^d = 0$$

for all $b_j \in A$, $1 \le j \le m-1$, and so

$$\alpha b_1 \alpha b_2 \ldots \alpha b_{m-1} = 0$$

for all $b_j \in A$, $1 \le j \le m-1$. This contradicts the minimality of m. Thus, α^d is contained in some nilpotent ideal of A as required. Therefore, as said, A has no nonzero nilpotent ideals, and so A is prime.

To finish the proof of the theorem, we consider two cases.

If the derivation d is X-inner, i.e., x is contained in the symmetric ring of quotients Q of A, then we are done by Lemma 3. Hence, we assume that d is not X-inner. Suppose that for some $a_0, a_1, \ldots, a_n \in A$,

$$a_0 + a_1 x + \cdots + a_n x^n = 1.$$

Then

$$(a_0 - 1) + a_1 x + \cdots + a_n x^n = 0,$$

and it follows from Lemma 6 that $a_0 - 1 = a_1 = \cdots = a_n = 0$. This yields $a_0 = 1$ which is impossible since a_0 is nilpotent. The proof is complete. □

After we have proved the main result, we ask the following natural question.

Question B. *Let $\delta\colon R \to R$ be a derivation of a nil ring R. Can $R[X;\delta]$ be mapped homomorphically onto a ring with a nonzero idempotent?*

We remark that the questions A and B are "differential analogies" of well-known solved problems. After showing that $R[x]$ cannot be mapped homomorphically onto a simple ring with unity if R is nil, Puczyłowski and Smoktunowicz asked if $R[x]$ can be mapped homomorphically onto a simple ring with a nonzero idempotent (see Question 1b in [11]). The answer is "no", as shown by Beidar, Fong and Puczyłowski in [1].

To conclude the paper, we note that in case of finite characteristic, a differentially simple ring may not be prime, or semiprime. This prevents us from applying Kharchenko's theory directly as we have done above.

Acknowledgement. We would like to thank the referee for her/his useful suggestions.

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Received July 11, 2018; revised October 25, 2018. Published online January 7, 2020.

Mikhail Chebotar: Department of Mathematical Sciences, Kent State University, Kent, USA.
E-mail: chebotar@math.kent.edu

Wen-Fong Ke: Department of Mathematics and Research Center for Theoretical Sciences, National Cheng Kung University, Tainan, Taiwan.
E-mail: wfke@mail.ncku.edu.tw