



Almost everywhere divergence of spherical harmonic expansions and equivalence of summation methods

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Abstract. We show that there exists an integrable function on the n -sphere ($n \geq 2$), whose Cesàro $(C, (n-1)/2)$ means with respect to the spherical harmonic expansion diverge unboundedly almost everywhere. This extends results of Stein (1961) for flat tori and complements the work of Taibleson (1985) for spheres. We also study relations among Cesàro, Riesz, and Bochner–Riesz means.

In memoriam: Elias M. Stein

1. Introduction

Let \mathbb{S}^n ($n \geq 2$) be the n -dimensional sphere equipped with the uniform measure. We consider for $f \in L^1(\mathbb{S}^n)$ the spherical harmonic expansion

$$(1.1) \quad f \sim \sum_{k=0}^{\infty} \text{proj}_k f,$$

where proj_k denotes the orthogonal projection operator from $L^2(\mathbb{S}^n)$ to the space of spherical harmonics of degree k (see Chapter IV, Section 2 in [25]). It is well known that the series in (1.1) diverges for general $f \in L^1(\mathbb{S}^n)$. So it is natural to consider summation methods that guarantee the convergence of (1.1). In this regard, we consider the Cesàro (C, δ) means, $S_N^\delta f$, $N = 0, 1, \dots$, for the series in (1.1). Bonami and Clerc [2] showed that $S_N^\delta f$ converges almost everywhere to f provided $\delta > \delta_0 := (n-1)/2$. For $\delta < \delta_0$, Meaney [19] showed that there exists a zonal function $f \in L^1(\mathbb{S}^n)$ such that $S_N^\delta f$ diverges almost everywhere. At the critical index $\delta = \delta_0$, a general result of Christ and Sogge [5] implies that $S_N^{\delta_0} f$

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always converges in measure to f . Chanillo and Muckenhoupt [3] showed that, if moreover f is zonal, then $S_N^{\delta_0} f$ converges almost everywhere to f .

In view of Kolmogoroff's counterexample [16] on \mathbb{S}^1 and Stein's counterexample [23] on the tori \mathbb{T}^n ($n \geq 2$), it can be expected that there exists an $f \in L^1(\mathbb{S}^n)$ such that $S_N^{\delta_0} f$ diverges almost everywhere on \mathbb{S}^n . Indeed, such a result is claimed in a paper by Taibleson [27] with f belonging to the Hardy space $H^1(\mathbb{S}^n)$ (see also remarks in [6], [30]; note that $H^1 \subset L^1$). However, Taibleson addresses only how to obtain such a result from the corresponding result with $f \in L^1(\mathbb{S}^n)$, by adapting an idea of Stein in [24] for \mathbb{T}^n . The main purpose of this paper is to give a complete proof of Taibleson's claim by constructing an $f \in L^1(\mathbb{S}^n)$ such that $S_N^{\delta_0} f$ diverges almost everywhere (Theorem 3.1 in Section 3). Although our proof bears some similarity to those for \mathbb{S}^1 and \mathbb{T}^n , some aspects are new and may provide motivation for the study of more general settings and of more refined questions.

The construction of f relies on precise estimates of the summation kernel. In the case of tori [23] and compact semisimple Lie groups [4], such estimates were obtained using the Poisson summation formula. However, except for some special cases, such an approach does not seem to carry over to \mathbb{S}^n . Based on detailed study of the Jacobi polynomials (cf. [26]), Bonami and Clerc [2] were able to obtain rather precise estimates of the Cesàro kernels on \mathbb{S}^n . They applied the estimates mainly with $\delta > \delta_0$ in [2]. For the sake of self-containedness, in Section 2 we give a detailed presentation for the case $\delta = \delta_0$ following their approach.

With the kernel estimates, to obtain the almost everywhere divergence result, we combine ideas of Stein [23] in his treatments of \mathbb{S}^1 and resp. \mathbb{T}^n , as well as the treatment of compact semisimple Lie groups by the first two authors in [4]. More precisely, as in [4] we first use Young's inequality and the weak (1,1) boundedness of the Hardy–Littlewood maximal function to remove the influence of the global part of the kernel. We then need to find an appropriate $f \in L^1(\mathbb{S}^n)$ to blow up the local part of the kernel. For this we use an idea of Stein in his treatment of \mathbb{S}^1 , that is to replace f by an appropriate probability measure μ whose mass is equally distributed on finitely many points. The points in the support of μ need to be suitably equidistributed, and the distance functions generated by them need to be rationally independent almost everywhere. In the case of compact semisimple Lie groups, such points were constructed in [4] using a probabilistic approach similar to that of Kahane [15] for \mathbb{S}^1 . However, the probabilistic approach is somewhat limited and provides limited information about details of the divergence. Therefore we opt for a deterministic approach for \mathbb{S}^n . As the proofs will show, this approach provides much more flexibility in choosing the points. In particular, it will be shown that the equidistribution property is satisfied for any sufficiently dense packing of \mathbb{S}^n (or modifications thereof; see Lemma 3.5 below). It will also be shown that the rational independence property holds whenever the points are distinct and contain no antipodal pairs (Lemma 3.6). It is worth mentioning that the latter requires knowledge about the analyticity of the distance functions on \mathbb{S}^n (see [1], [23] for the case of \mathbb{R}^n); in particular, it is used in the proof that the cut loci of \mathbb{S}^n are singletons. Interestingly, similar considerations are not needed in the case of \mathbb{T}^n . This is because the global part of the kernel on \mathbb{T}^n already diverges almost

everywhere, so that one can simply take μ to be a point mass, see [23] (see also Chapter VII, Section 4 in [25]).

In Section 4 we study relations among the Cesàro, Riesz, and Bochner–Riesz means. These summation methods are known to give rise to the same order of summability (i.e., the critical δ) at every point on \mathbb{S}^n (cf. Hardy–Riesz [10]). In particular, the almost everywhere divergence of Cesàro (C, δ_0) means implies that of the Riesz and Bochner–Riesz means of the same order. On the other hand, it is not clear if the divergence behaviors are exactly the same for these summation methods (cf. Kuttner [17]). Using a result of Ingham [14], we show that, if appropriately defined, the differences among the three summation processes are bounded almost everywhere when $\delta = \delta_0$ and $f \in L^1(\mathbb{S}^n)$.

Throughout the paper, unless otherwise stated, C denotes a positive, dimensional constant whose value may change from line to line, and $A = O(B)$ means that $|A| \leq \tilde{C}B$ holds for a constant $\tilde{C} > 0$ independent of the testing inputs (which will usually be clear from the context).

2. Notation and preliminaries

In what follows, we denote by \mathbb{S}^n the unit sphere in \mathbb{R}^{n+1} ($n \geq 2$) equipped with the standard round metric. Denote by Δ the Laplace–Beltrami operator on \mathbb{S}^n . For $k = 0, 1, \dots$, denote by \mathcal{H}_k^n the space of spherical harmonics of degree k (for background on spherical harmonics, we refer to [29], [7], and Chapter IV in [25]). It is well known that we have the orthogonal decomposition

$$L^2(\mathbb{S}^n) = \bigoplus_{k=0}^{\infty} \mathcal{H}_k^n;$$

moreover,

$$(2.1) \quad \Delta Y_k = -k(k+n-1)Y_k, \quad \forall Y_k \in \mathcal{H}_k^n,$$

$$\dim \mathcal{H}_k^n = \binom{k+n}{k} - \binom{k-2+n}{k-2}.$$

Denote by

$$\text{proj}_k : L^2(\mathbb{S}^n) \rightarrow \mathcal{H}_k^n$$

the orthogonal projection from $L^2(\mathbb{S}^n)$ to \mathcal{H}_k^n . Set

$$\lambda = \frac{n-1}{2}.$$

Let $C_k^\lambda(t)$ be the Gegenbauer polynomial of degree k and index λ . Equivalently,

$$(2.2) \quad C_k^\lambda(t) = \frac{\Gamma(\lambda+1/2)}{\Gamma(2\lambda)} \frac{\Gamma(k+2\lambda)}{\Gamma(k+\lambda+1/2)} P_k^{(\lambda-1/2, \lambda-1/2)}(t),$$

where $P_k^{(\alpha, \beta)}$ is the Jacobi polynomial of degree k (cf. Szegö [26], p. 80). Denote by

$$|x - y| \in [0, \pi]$$

the great-circle distance between x and y on \mathbb{S}^n . It is a standard fact that

$$(2.3) \quad \text{proj}_k f = \int_{\mathbb{S}^n} Z_k(\cdot, y) f(y) dy,$$

where

$$Z_k(x, y) = \frac{k + \lambda}{\lambda} C_k^\lambda(\cos |x - y|).$$

Note that

$$\cos |x - y| = \langle x, y \rangle$$

represents the inner product in \mathbb{R}^{n+1} . Note also that, by (2.2) and Theorem 7.32.2 in Szegö [26], we have

$$(2.4) \quad |Z_k(x, y)| \leq C(k + 1)^{(n-1)/2} |x - y|^{-(n-1)/2} |x - \hat{y}|^{-(n-1)/2}, \quad k = 0, 1, \dots,$$

where \hat{y} denotes the antipodal point of y on \mathbb{S}^n . (See also Bonami–Clerc [2], p. 231.)

For $f \in L^1(\mathbb{S}^n)$, we can consider the formal spherical harmonic expansion

$$f \sim \sum_{k=0}^{\infty} \text{proj}_k f,$$

with $\text{proj}_k f \in \mathcal{H}_k^n$ given by (2.3). Let $\delta > -1$. The corresponding Cesàro (C, δ) means are defined by

$$S_N^\delta f = \frac{1}{A_N^\delta} \sum_{k=0}^N A_{N-k}^\delta \text{proj}_k f, \quad N = 0, 1, \dots,$$

where

$$A_k^\delta = \binom{k + \delta}{k} = \frac{(\delta + k)(\delta + k - 1) \cdots (\delta + 1)}{k(k - 1) \cdots 1} = \frac{\Gamma(k + \delta + 1)}{\Gamma(k + 1)\Gamma(\delta + 1)}.$$

By (2.3), we can also write

$$S_N^\delta f = \int_{\mathbb{S}^n} K_N^\delta(\cdot, y) f(y) dy$$

where

$$K_N^\delta(x, y) = \frac{1}{A_N^\delta} \sum_{k=0}^N A_{N-k}^\delta Z_k(x, y)$$

is called the $(N$ -th) Cesàro kernel of order δ . Note that $K_N^\delta(x, y)$ depends only on $|x - y|$. In general, for kernels $K(x, y)$ satisfying this property, we will write

$$K * f := \int_{\mathbb{S}^n} K(\cdot, y) f(y) dy.$$

In particular, we will write

$$S_N^\delta f = K_N^\delta * f.$$

In what follows, we will focus on the case

$$\delta = \delta_0 = \frac{n-1}{2}.$$

For simplicity, we will write

$$K_N(x, y) = K_N^{\delta_0}(x, y).$$

Following Bonami–Clerc [2], we first decompose $K_N(x, y)$ as

$$K_N(x, y) = K_N^{(0)}(x, y) + K_N^{(\pi)}(x, y)$$

where

$$\begin{aligned} K_N^{(0)}(x, y) &= K_N(x, y) \mathbf{1}_{\{|x-y| \leq \pi/2\}}, \\ K_N^{(\pi)}(x, y) &= K_N(x, y) \mathbf{1}_{\{|x-y| > \pi/2\}}. \end{aligned}$$

We will need the following estimate for $K_N^{(\pi)}(x, y)$. See Corollary 2.5 in [2].

Lemma 2.1. *There exists a constant $C > 0$ such that*

$$|K_N^{(\pi)}(x, y)| \leq C |x - y|^{-(n-1)/2}, \quad N = 0, 1, \dots$$

To estimate $K_N^{(0)}(x, y)$, following [2], we will use the following formula. See Szegő [26], p. 261.

Lemma 2.2. *For $N = 1, 2, \dots$, we have*

$$K_N(x, y) = C_N P_N^{(n-1/2, (n-2)/2)}(\cos |x - y|) + E_N(x, y),$$

where

$$\begin{aligned} C_N &= \frac{1}{A_N^{\delta_0}} \left\{ 2^{n-1} \Gamma\left(\frac{n}{2}\right) \frac{\Gamma(N + n/2)}{\Gamma(N + 3n/2 - 1)} \frac{\Gamma(2N + 2n - 1)}{\Gamma(2N + 3n/2)} \right\}^{-1}, \\ E_N(x, y) &= - \sum_{\ell=1}^{\infty} (-1)^\ell \binom{\delta_0}{\ell} \frac{(N + \frac{n+1}{2}) \cdots (N + \frac{n+1}{2} + \ell - 1)}{(2N + \frac{3n}{2}) \cdots (2N + \frac{3n}{2} + \ell - 1)} K_N^{\delta_0 + \ell}(x, y). \end{aligned}$$

Regarding the coefficients in Lemma 2.2, we have

$$(2.5) \quad \lim_{N \rightarrow \infty} \frac{C_N}{N^{1/2}} = 2^{-(3n-4)/2} \frac{\Gamma((n+1)/2)}{\Gamma(n/2)},$$

and

$$(2.6) \quad \sum_{\ell=1}^{\infty} \left| \binom{\delta_0}{\ell} \frac{(N + \frac{n+1}{2}) \cdots (N + \frac{n+1}{2} + \ell - 1)}{(2N + \frac{3n}{2}) \cdots (2N + \frac{3n}{2} + \ell - 1)} \right| \leq \sum_{\ell=1}^{\infty} \left| \binom{\delta_0}{\ell} \right| < \infty,$$

where the last series converges because

$$\left| \binom{\delta_0}{\ell} \right| = \left| \frac{\delta_0(\delta_0 - 1) \cdots (\delta_0 - \ell + 1)}{1 \cdot 2 \cdots \ell} \right| \leq C \ell^{-1-\delta_0}.$$

We will need the following asymptotic formula for the Jacobi polynomial. See Theorem 8.21.8 in Szegő [26].

Lemma 2.3. For every $t = \cos |x - y| \in (-1, 1)$,

$$P_N^{(n-1/2, (n-2)/2)}(t) = N^{-1/2} k(x, y) \cos \left(\left(N + \frac{3n-1}{4} \right) |x - y| - \frac{n\pi}{2} \right) + O(N^{-3/2}),$$

where

$$(2.7) \quad k(x, y) = \pi^{-1/2} \left(\sin \frac{|x - y|}{2} \right)^{-n} \left(\sin \frac{|x - \hat{y}|}{2} \right)^{-(n-1)/2}.$$

To estimate $E_N(x, y)$ in Lemma 2.2, we will use the following lemma. See Hardy [9], p. 100–101.

Lemma 2.4. Let $\{a_k\}_{k=0}^\infty$ be a sequence of numbers. For $\delta > -1$, let

$$S_N^\delta = \frac{1}{A_N^\delta} \sum_{k=0}^N A_{N-k}^\delta a_k, \quad N = 0, 1, \dots$$

Then for any $\rho > 0$, we have

$$(2.8) \quad |S_N^{\delta+\rho}| \leq \max_{0 \leq k \leq N} |S_k^\delta|, \quad N = 0, 1, \dots$$

We will also need the following estimates for $K_N^{\delta_0+1}$. See Corollary 2.5 in [2].

Lemma 2.5. (i) When $|x - y| \leq \pi/2$,

$$|K_N^{\delta_0+1}(x, y)| \leq C N^n, \quad N = 1, 2, \dots$$

(ii) When $2/N \leq |x - y| \leq \pi/2$,

$$|K_N^{\delta_0+1}(x, y)| \leq C N^{-1} |x - y|^{-n-1}.$$

Let ν be a finite Borel measure on \mathbb{S}^n . Denote by

$$(2.9) \quad M(\nu)(x) = \sup_{r>0} \frac{|\nu|(B(x, r))}{|B(x, r)|}, \quad x \in \mathbb{S}^n,$$

the Hardy–Littlewood maximal function of ν on \mathbb{S}^n , where

$$B(x, r) = \{y \in \mathbb{S}^n : |x - y| < r\}.$$

Note that, for a suitable constant $C > 1$, we have

$$(2.10) \quad r^n/C \leq |B(x, r)| \leq C r^n, \quad 0 < r \leq \pi.$$

Using a standard argument (see Theorem 2.2 in [12]), we have the weak-type (1, 1) inequality

$$(2.11) \quad |\{x \in \mathbb{S}^n : M(\nu)(x) > t\}| \leq C \frac{\|\nu\|}{t}, \quad t > 0.$$

In particular, by taking $t \rightarrow \infty$, we have

$$M(\nu)(x) < \infty, \quad \text{a.e. } x \in \mathbb{S}^n.$$

Using Lemma 2.5 and a dyadic decomposition, we also have

$$(2.12) \quad \sup_{N \geq 0} \left| \int_{|x-y| \leq \pi/2} K_N^{\delta_0+1}(x, y) d\nu(y) \right| \leq CM(\nu)(x), \quad x \in \mathbb{S}^n.$$

Now, let us denote (following the notations in Lemma 2.2)

$$(2.13) \quad \tilde{K}_N^{(0)}(x, y) = C_N P_N^{(n-1/2, (n-2)/2)}(\cos |x-y|) \mathbf{1}_{\{|x-y| \leq \pi/2\}},$$

$$(2.14) \quad E_N^{(0)}(x, y) = E_N(x, y) \mathbf{1}_{\{|x-y| \leq \pi/2\}}.$$

Then we can further decompose

$$K_N(x, y) = \tilde{K}_N^{(0)}(x, y) + E_N^{(0)}(x, y) + K_N^{(\pi)}(x, y), \quad N = 1, 2, \dots$$

Applying Lemma 2.4 with $\delta = \delta_0 + 1$,

$$a_k = \int_{|x-y| \leq \pi/2} Z_k(x, y) d\nu(y),$$

and $\rho = 1, 2, \dots$, we see that for $\ell = 1, 2, 3, \dots$,

$$\left| \int_{|x-y| \leq \pi/2} K_N^{\delta_0+\ell}(x, y) d\nu(y) \right| \leq \sup_{N \geq 0} \left| \int_{|x-y| \leq \pi/2} K_N^{\delta_0+1}(x, y) d\nu(y) \right|.$$

Thus, it follows from (2.6) that

$$|E_N^{(0)} * \nu(x)| \leq C \sup_{N \geq 0} \left| \int_{|x-y| \leq \pi/2} K_N^{\delta_0+1}(x, y) d\nu(y) \right|, \quad N = 1, 2, \dots$$

Combining this with Lemma 2.1 and (2.12), we obtain the desired estimate:

Lemma 2.6. *There exists a constant $C > 0$ such that, for any finite Borel measure ν on \mathbb{S}^n ,*

$$|(K_N - \tilde{K}_N^{(0)}) * \nu| \leq CM(\nu) + Ck^{(\pi)} * |\nu|, \quad N = 1, 2, \dots,$$

where

$$k^{(\pi)}(x, y) = |x - \hat{y}|^{-(n-1)/2}.$$

3. Almost everywhere divergence of Cesàro means

With Lemma 2.6, we can now prove the main theorem.

Theorem 3.1. *There exists a function $f \in L^1(\mathbb{S}^n)$ such that*

$$\limsup_{N \rightarrow \infty} |S_N^{(n-1)/2} f(x)| = \infty, \quad \text{a.e. } x \in \mathbb{S}^n.$$

In a similar spirit as in [23], we will deduce Theorem 3.1 from the following lemma. Denote by $\mathcal{P}(\mathbb{S}^n)$ the set of Borel probability measures on \mathbb{S}^n .

Lemma 3.2. *Given $L > 1$, there exists a finitely supported measure $\mu \in \mathcal{P}(\mathbb{S}^n)$ such that*

$$\limsup_{N \rightarrow \infty} |\tilde{K}_N^{(0)} * \mu(x)| > L, \quad a.e. \ x \in \mathbb{S}^n.$$

The proof of Lemma 3.2 is postponed to the end of this section. By using Lemma 2.6 and (2.11), the next lemma follows easily from Lemma 3.2.

Lemma 3.3. *Given $L > 1$ and $\varepsilon > 0$, there exists a finitely supported measure $\mu \in \mathcal{P}(\mathbb{S}^n)$ such that*

$$\limsup_{N \rightarrow \infty} |K_N * \mu(x)| > L$$

holds on a set $E \subset \mathbb{S}^n$ with $|\mathbb{S}^n \setminus E| < \varepsilon$.

By a standard limiting argument, the supremum above may be truncated at a large enough N_0 . Furthermore, by convolving with $K_{N_1}^{\delta_0+1}$ for a large enough N_1 , the measure μ above can be replaced by a function $f \in L^1(\mathbb{S}^n)$. We call f a polynomial of degree N if

$$\text{proj}_N f \neq 0 \quad \text{and} \quad \text{proj}_k f = 0, \quad \forall k \geq N + 1.$$

Lemma 3.4. *Given $L > 1$ and $\varepsilon > 0$, there exist a polynomial f with $\|f\|_{L^1(\mathbb{S}^n)} \leq 1$, and an integer N_0 , such that*

$$\max_{0 \leq N \leq N_0} |K_N * f(x)| > L$$

holds on a set $E \subset \mathbb{S}^n$ with $|\mathbb{S}^n \setminus E| < \varepsilon$.

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. The proof is an adaptation of the argument in p. 273–274 of [25]. The function f will be taken to be of the form

$$f = \sum_{j=0}^{\infty} \eta_j f_j,$$

where $\{\eta_j\}_{j=0}^{\infty} \in \ell^1$ is a sequence of positive numbers, and each f_j is a polynomial satisfying $\|f_j\|_{L^1(\mathbb{S}^n)} \leq 1$. Obviously,

$$\|f\|_{L^1(\mathbb{S}^n)} \leq \sum_{j=0}^{\infty} \eta_j < \infty.$$

Each f_j will have an associated integer, denoted by N_j . The choice of η_j, f_j, N_j is based on induction. More precisely, set $\eta_0 = 1, f_0 = 0, N_0 = 0$. Assuming that $\eta_{j-1}, f_{j-1}, N_{j-1}$ have been chosen, we now describe how we choose η_j, f_j, N_j .

First, choose $\eta_j > 0$ so small that

$$(3.1) \quad \eta_j \leq \eta_{j-1}/2$$

and so that

$$(3.2) \quad \eta_j \max_{0 \leq N \leq N_{j-1}} \|K_N\|_\infty \leq 1.$$

With η_j chosen, by Lemma 3.4 we can find a polynomial f_j with $\|f_j\|_{L^1(\mathbb{S}^n)} \leq 1$ and an integer N_j , such that

$$(3.3) \quad \eta_j \max_{0 \leq N \leq N_j} |K_N * f_j(x)| > \sup_{N \geq 0} \|K_N * (\eta_0 f_0 + \cdots + \eta_{j-1} f_{j-1})\|_\infty + j$$

holds on a set $E_j \subset \mathbb{S}^n$ with $|\mathbb{S}^n \setminus E_j| < j^{-1}$ (note that the right-hand side of (3.3) is finite because $\eta_0 f_0 + \cdots + \eta_{j-1} f_{j-1}$ is a polynomial). By induction, this completes the choice of η_j, f_j, N_j for all j .

Now set

$$E = \bigcap_{i=1}^{\infty} \bigcup_{j \geq i} E_j.$$

It is easy to see that $|\mathbb{S}^n \setminus E| = 0$. To complete the proof, we will show that

$$\limsup_{N \rightarrow \infty} |K_N * f(x)| = \infty, \quad \forall x \in E.$$

Fix $x \in E$. By definition, there are infinitely many j 's for which $x \in E_j$. Fix such an index j_0 . We can write

$$\begin{aligned} K_N * f(x) &= K_N * \left(\sum_{j < j_0} \eta_j f_j \right)(x) + \eta_{j_0} K_N * f_{j_0}(x) + K_N * \left(\sum_{j > j_0} \eta_j f_j \right)(x) \\ &= I + II + III. \end{aligned}$$

By (3.1) and (3.2), for $N \leq N_{j_0}$ we have

$$\begin{aligned} |III| &\leq \|K_N\|_\infty \sum_{j > j_0} \eta_j \|f_j\|_{L^1(\mathbb{S}^n)} \leq \left(\sum_{j > j_0} \eta_j \right) \max_{0 \leq N \leq N_{j_0}} \|K_N\|_\infty \\ &\leq 2\eta_{j_0+1} \max_{0 \leq N \leq N_{j_0}} \|K_N\|_\infty \leq 2. \end{aligned}$$

Combining this with (3.3), we see that

$$\max_{0 \leq N \leq N_{j_0}} |K_N * f(x)| \geq \max_{0 \leq N \leq N_{j_0}} |II| - \max_{0 \leq N \leq N_{j_0}} |I| - \max_{0 \leq N \leq N_{j_0}} |III| \geq j_0 - 2.$$

Since j_0 can be chosen arbitrarily large, we obtain

$$\sup_{N \geq 0} |K_N * f(x)| = \infty.$$

Since $|K_N * f(x)| < \infty$ for all N , this completes the proof of Theorem 3.1. \square

It remains to prove Lemma 3.2. Let $r > 0$. We will call $\{y_j\}_{j=1}^m \subset \mathbb{S}^n$ an r -separated set if

$$|y_j - y_{j'}| \geq r \quad \text{whenever } j \neq j'.$$

We call $\{y_j\}_{j=1}^m$ a maximal r -separated set if it is not strictly contained in another r -separated set. By successively adding points to a singleton, it is easy to see that maximal r -separated sets exist for all $r > 0$; moreover, when $0 < r \leq \pi$, by (2.10), the corresponding cardinality m must satisfy

$$r^{-n}/C \leq m \leq Cr^{-n}$$

for a suitable constant $C > 1$.

Lemma 3.5. *Let $\{y_j\}_{j=1}^m \subset \mathbb{S}^n$ be a maximal r -separated set with $0 < r \leq \pi$. Then*

$$(3.4) \quad \frac{1}{m} \sum_{j=1}^m \frac{1}{|x - y_j|^n} \geq C \log(\pi/r), \quad \forall x \in \mathbb{S}^n.$$

Proof. The proof follows easily from a C -adic (with $C > 1$ sufficiently large) decomposition around x , and the mass distribution principle (for the latter, see Theorem 5.7 in [18]). \square

Let $\{y_j\}_{j=1}^m \subset \mathbb{S}^n$ satisfy (3.4). By a small perturbation of $\{y_j\}_{j=1}^m$, we may assume that $\{y_j\}_{j=1}^m$ contains no antipodal pairs.

Lemma 3.6. *Let $\{y_j\}_{j=1}^m \subset \mathbb{S}^n$ be a set of m distinct points which contains no antipodal pairs. Then the following hold.*

(i) *For almost every $x \in \mathbb{S}^n$, the numbers $\pi, |x - y_1|, \dots, |x - y_m|$ are rationally independent.*

(ii) *If moreover $\{y_j\}_{j=1}^m$ satisfies (3.4), then, letting*

$$\mu = \frac{1}{m} \sum_{j=1}^m \delta_{y_j}$$

(where δ_y is the Dirac delta at y), we have

$$\limsup_{N \rightarrow \infty} |\tilde{K}_N^{(0)} * \mu(x)| \geq C \log(\pi/r) - \tilde{C}, \quad \text{a.e. } x \in \mathbb{S}^n,$$

where $\tilde{C} > 0$ is a constant depending only on n .

Proof. (i) It suffices to show that, for any $q_0, q_1, \dots, q_m \in \mathbb{Q}$ not all of which are zero, letting

$$F(x) = q_0 \pi + \sum_{j=1}^m q_j |x - y_j|,$$

the zero set

$$Z = \{x \in \mathbb{S}^n : F(x) = 0\}$$

has measure zero. Without loss of generality, we may assume that q_1, \dots, q_m are not all equal to zero, since otherwise $F(x) \equiv q_0\pi \neq 0$. Notice that $F(x)$ is real-analytic on

$$U = \mathbb{S}^n \setminus (\{y_j\}_{j=1}^m \cup \{\hat{y}_j\}_{j=1}^m),$$

and is not constantly zero on U (since $F(x)$ is non differentiable at the y_j 's for which $q_j \neq 0$). It follows that Z must have measure zero (see Chapter VII, Lemma 4.17 in [25]; see also [20]).

(ii) Note that

$$\tilde{K}_N^{(0)} * \mu(x) = \frac{1}{m} \sum_{j=1}^m \tilde{K}_N^{(0)}(x, y_j).$$

By (2.13), (2.5), and Lemma 2.3, for $x \in U$ we have

$$\tilde{K}_N^{(0)} * \mu(x) = \frac{\theta_N}{m} \sum_{j=1}^m k^{(0)}(x, y_j) \cos\left(\left(N + \frac{3n-1}{4}\right)|x - y_j| - \frac{n\pi}{2}\right) + O(N^{-1}),$$

where $\theta_N = C_N N^{-1/2}$ satisfies (2.5),

$$k^{(0)}(x, y) = \mathbf{1}_{\{|x-y| \leq \pi/2\}} k(x, y)$$

(with $k(x, y)$ given by (2.7)). By (i), for almost every $x \in U$, the numbers $2\pi, |x - y_1|, \dots, |x - y_m|$ are rationally independent. It follows from Kronecker's theorem (see Theorem 442 in Chapter 23 of [11]) that, for such x ,

$$\limsup_{N \rightarrow \infty} |\tilde{K}_N^{(0)} * \mu(x)| = \frac{C}{m} \sum_{j=1}^m k^{(0)}(x, y_j).$$

On the other hand, by (2.7),

$$\begin{aligned} \frac{C}{m} \sum_{j=1}^m k^{(0)}(x, y_j) &\geq \frac{C}{m} \sum_{j=1}^m \mathbf{1}_{\{|x-y_j| \leq \pi/2\}} \frac{1}{|x - y_j|^n} \\ &= \frac{C}{m} \sum_{j=1}^m \frac{1}{|x - y_j|^n} - \frac{C}{m} \sum_{j=1}^m \mathbf{1}_{\{|x-y_j| > \pi/2\}} \frac{1}{|x - y_j|^n} \geq \frac{C}{m} \sum_{j=1}^m \frac{1}{|x - y_j|^n} - \tilde{C}. \end{aligned}$$

Combining this with the assumption that $\{y_j\}_{j=1}^m$ satisfies (3.4), the proof of (ii) is complete. \square

Proof of Lemma 3.2. Lemma 3.2 now follows immediately by taking

$$\mu = \frac{1}{m} \sum_{j=1}^m \delta_{y_j},$$

where $\{y_j\}_{j=1}^m$ is as in Lemma 3.6 (ii), with $r > 0$ chosen small enough that $C \log(\pi/r) - \tilde{C} > L$. \square

4. Equivalence of summation methods

It is known that the Cesàro means can be written as linear transformations of the Riesz means of the same order, and vice versa (cf. Hardy–Riesz [10], Gergen [8], Kuttner [17]). Consequently, the corresponding maximal functions are comparable up to multiplicative constants. In particular, one obtains from Theorem 3.1 almost everywhere divergence of the Riesz and Bochner–Riesz means of the same order, with the same set of divergence.

In order to obtain more precise information about the latter summation processes, instead of comparing the maximal functions, it seems desirable to also make a term-by-term comparison. In what follows, we sketch an approach which shows that, if appropriately defined, the differences among the Cesàro, Riesz, and Bochner–Riesz means are bounded almost everywhere when $\delta = (n - 1)/2$ and $f \in L^1(\mathbb{S}^n)$. This alternate approach originates in the work of Hardy [9], p. 113–114, Ingham [14], and Trebels [28], p. 47–48 (see also Riesz [21], and Section 58 in Hobson [13] for related treatments).

4.1. Cesàro means vs. Riesz means

Let $\delta > 0$. The Riesz (R, δ) means of $f \in L^1(\mathbb{S}^n)$ are defined by

$$(4.1) \quad \sigma_R^\delta f = \sum_{k < R} \left(1 - \frac{k}{R}\right)^\delta \text{proj}_k f, \quad R > 0.$$

To compare (4.1) with the Cesàro means, we invoke the following special case of a theorem of Ingham [14].

Lemma 4.1 (Theorem B of [14]). *Let $\delta > 0$ and let $m = \lceil \delta \rceil + 2$. There exist constants c_1, \dots, c_m , such that*

$$(4.2) \quad \binom{k + \delta}{k} = \sum_{j=1}^m c_j \left(k + \frac{j}{m}\right)^\delta + O((k + 1)^{-2}), \quad k = 0, 1, \dots$$

Note that, by taking $k \rightarrow \infty$ in (4.2), it is implied that

$$(4.3) \quad \sum_{j=1}^m c_j = \frac{1}{\Gamma(\delta + 1)}.$$

Lemma 4.1 immediately leads to the following.

Lemma 4.2. *Let δ , m , and c_j ($j = 1, \dots, m$) be as in Lemma 4.1. Then*

$$S_N^\delta f = \Sigma_N^\delta f + \eta_N^\delta f,$$

where

$$(4.4) \quad \Sigma_N^\delta f = \sum_{j=1}^m c_j \frac{(N + j/m)^\delta}{A_N^\delta} \sigma_{N+j/m}^\delta f,$$

$$(4.5) \quad \eta_N^\delta f = O(1) \max_{k \leq N} \frac{|\text{proj}_k f|}{(k + 1)^\delta}, \quad N = 0, 1, \dots$$

Since

$$\lim_{N \rightarrow \infty} \sum_{j=1}^m c_j \frac{(N + j/m)^\delta}{A_N^\delta} = 1,$$

$\Sigma_N^\delta f$ can be regarded as an average of $\sigma_R^\delta f$ as R ranges over $(N, N+1]$.¹ Thus $\eta_N^\delta f$ may be regarded as measuring the difference between the Cesàro (C, δ) and the Riesz (R, δ) means.

In the case $\delta = (n-1)/2$, it can be shown that $\eta_N^\delta f$ converges to zero almost everywhere. This is an easy consequence of (4.5) and the following lemma.

Lemma 4.3. *There exists a constant $C > 0$, such that for any $f \in L^1(\mathbb{S}^n)$,*

$$(4.6) \quad \left\| \sup_{k \geq 0} \frac{|\text{proj}_k f|}{(k+1)^{(n-1)/2}} \right\|_{L^1(\mathbb{S}^n)} \leq C \|f\|_{L^1(\mathbb{S}^n)}.$$

Proof. This follows easily from (2.4) and Fubini's theorem. \square

4.2. Riesz means vs. Bochner–Riesz means

The comparison between the Riesz (R, δ) and the Bochner–Riesz means will be done by comparing their appropriately normalized and ‘shifted’ versions.

Let $c \in \mathbb{R}$. Define the *shifted* Riesz (R, δ) means by

$$(4.7) \quad \sigma_R^{\delta, c} f = \sum_{k+c < R} \left(1 - \frac{k+c}{R}\right)^\delta \text{proj}_k f, \quad R > 0.$$

Note that when $R > c$, we have

$$(4.8) \quad \sigma_R^{\delta, c} f = \left(1 - \frac{c}{R}\right)^\delta \sigma_{R-c}^\delta f.$$

Thus $\sigma_R^{\delta, c} f$ is essentially the same as $\sigma_{R-c}^\delta f$, as $R \rightarrow \infty$. Note also that, when $\delta = \delta_0 + 1$, it follows from Theorem 3.3 in [2] and (4.8) that, for any $f \in L^1(\mathbb{S}^n)$ and $c \in \mathbb{R}$,

$$(4.9) \quad \sup_{R > 0} |\sigma_R^{\delta_0+1, c} f(x)| < \infty, \quad \text{a.e. } x \in \mathbb{S}^n.$$

Now let $c \geq 0$. Define the shifted Riesz (R, δ) means of *quadratic type* by

$$(4.10) \quad B_R^{\delta, c} f = \sum_{k+c < R} \left(1 - \frac{(k+c)^2}{R^2}\right)^\delta \text{proj}_k f, \quad R > 0.$$

Lemma 4.4. *For any $\delta > 0$ and $c \geq 0$, we have*

$$(4.11) \quad B_R^{\delta, c} f = 2^\delta \sigma_R^{\delta, c} f + O\left(\sup_{r \leq R} |\sigma_r^{\delta+1, c} f|\right), \quad R > 0,$$

where the implied constant depends only on δ .

¹Note that allowing R to take noninteger values is crucial for the equivalence of Cesàro and Riesz means, see Riesz [22] (see also Hardy [9], p. 114).

Proof. Observe that, by a Taylor expansion of $(1+t)^\delta$ about $t=1$, we have

$$(1-t^2)^\delta = 2^\delta(1-t)^\delta + 2^\delta \sum_{\ell=1}^{\infty} \binom{\delta}{\ell} \frac{(-1)^\ell}{2^\ell} (1-t)^{\delta+\ell}, \quad t \in [0, 1].$$

This leads to the identity

$$B_R^{\delta,c} f = 2^\delta \sigma_R^{\delta,c} f + 2^\delta \sum_{\ell=1}^{\infty} \binom{\delta}{\ell} \frac{(-1)^\ell}{2^\ell} \sigma_R^{\delta+\ell,c} f.$$

The desired bound (4.11) then follows from the fact that, for any $\ell \geq 1$,

$$|\sigma_R^{\delta+\ell,c} f| \leq \sup_{r \leq R} |\sigma_r^{\delta+1,c} f|, \quad R > 0.$$

See Theorem 16 in [10] (see also [25], p. 269–270). \square

Finally, recall that the Bochner–Riesz means of order δ are defined by

$$B_R^\delta f = \sum_{k(k+n-1) < R^2} \left(1 - \frac{k(k+n-1)}{R^2}\right)^\delta \text{proj}_k f, \quad R > 0,$$

where $-k(k+n-1)$ is the eigenvalue of the Laplace–Beltrami operator shown in (2.1). Observe that, with $c = (n-1)/2$, we have

$$B_R^\delta f = \left(1 + \frac{c^2}{R^2}\right)^\delta B_{\sqrt{R^2+c^2}}^{\delta,c} f.$$

Combining this with Lemma 4.4, (4.8), and (4.9), we see that, essentially, the difference between $B_R^\delta f$ and $2^\delta \sigma_R^\delta f$ is bounded almost everywhere when $\delta = (n-1)/2$ and $f \in L^1(\mathbb{S}^n)$ (it also follows that the difference between $B_R^\delta f$ and $\sigma_R^\delta f$ can diverge unboundedly almost everywhere).

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