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On the cohomology class of fiber-bunched cocycles on semi simple Lie groups

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Abstract. We study the cohomological equation associated to linear cocycles on semi simple Lie groups $\mathcal{G}$ over hyperbolic dynamics. We give sufficient conditions for the solution of the cohomological equation of fiber-bunched cocycles to be unique and for the Hölder conjugacy class of the cocycle to coincide with $C^\nu(M,\mathcal{G})$. In particular, we prove that there exists an open and dense subset of the set $C^\nu_b(M,\mathcal{G})$ of fiber-bunched cocycles with trivial centralizer. As a consequence we deduce that the solutions of the cohomological equation of fiber-bunched cocycles form a finite abelian subgroup of $C^\nu_b(M,\mathcal{G})$ for an open and dense subset of fiber-bunched cocycles in $C^\nu_b(M,\mathcal{G})$. Some results on the centralizer of skew-products are also given.

1. Introduction and statement of the main results

In the seminal papers [13], [14], Livšic considered the problem of obtaining solutions of the cohomological equation

$$A(x) = C(f(x))\,B(x)\,C(x)^{-1} \tag{1.1}$$

over hyperbolic dynamical systems $f\colon M \to M$ and for cocycles $A, B\colon M \to \mathcal{G}$ taking values on the group $\mathcal{G}$ that is either $\mathbb{R}$ or an abelian group. The questions raised by Livšic focus on the existence of continuous solutions C from the existence of a *a priori* measurable solution (with respect to some full supported probability measure) for (1.1), and also the evaluation of the cocycle at periodic orbits as a criterium of necessity and sufficiency for the existence of solutions for (1.1). In particular, in the case of real valued cohomological equations both questions have a positive answer and it is well known that solutions are unique up to an additive constant.

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The context of non-abelian groups $\mathcal{G}$ is more delicate and, besides the substantial contributions in the last decade, the problem of finding solutions for the cohomological equation remains not completely understood. In the abelian context, Hölder continuous cocycles over hyperbolic systems have been extensively studied, specially with the focus of establishing sufficient conditions for cohomology in terms of the periodic data and on studying the regularity of the conjugacy [14]. Some important extensions of the latter were obtained in the case of compact groups, cocycles with 'bounded distortion', cocycles taking values on semi simple Lie groups and cocycles cohomologous to the identity (see [9], [15], [17], [22] and references therein). Let us also mention [16], [25], where the authors study the regularity of the solutions of the cohomological equation and twisted cohomological equation on the setting of Anosov actions and unipotent automorphisms on a Lie group. Moreover, one should refer that in the context of matrix valued cocycles over hyperbolic dynamical systems: (i) there exists a pair of $\mathrm{GL}(d, \mathbb{R})$ valued linear cocycles that are measurable cohomologous but not Hölder cohomologous, and (ii) the existence of a conjugacy at periodic data do not always imply on the existence of a solution of the cohomological equation, even among important classes of fiber-bunched cocycles (see [1], [18], [21] and references therein). These facts reinforce that this topic remains a very active and important area of research and presenting several novelties with respect to the abelian context.

A centralizer for a dynamical system consists of the set of symmetries for the dynamics. This notion has the following formulation in the context of linear cocycles: the linear centralizer $\mathcal{Z}(A)$ of the cocycle $A\colon M \to \mathcal{G}$ is defined by

$$\mathcal{Z}(A) = \{B \in C^\nu(M, \mathcal{G})\colon A(x) = B(f(x))\, A(x)\, B(x)^{-1} \text{ for every } x \in M\}, \tag{1.2}$$

where we omit the dependence of $\mathcal{Z}(A)$ on f and the Hölder exponent ν for notational simplicity. The linear centralizer $\mathcal{Z}(A)$ is a subgroup of $C^\nu(M, \mathcal{G})$ and it is strongly related with the cohomological equation (see e.g. [21] or Lemma 3.6 below). Here we always assume that $\mathcal{G} \subset \mathrm{SL}(d, \mathbb{K})$ where $d \geq 2$ and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and we say that A has *trivial centralizer* if $\mathcal{Z}(A) = \{\lambda \mathrm{Id} : \lambda \in \mathbb{K}\} \cap C^\nu(M, \mathcal{G})$ (see also Remark 3.7).

Our goal here is to study the cohomology class of fiber-bunched cocycles over a hyperbolic homeomorphism and taking values on a semi simple Lie group $\mathcal{G}$ and address the uniqueness of solutions of the cohomological equation (1.1). In the case of real valued cocycles, since all solutions differ by an additive constant, the group of solutions is $\mathbb{R}$ [14]. As pointed out by Kalinin ([9], p. 1027), for $\mathrm{GL}(d, \mathbb{R})$ valued cocycles, topological transitivity of the dynamics f can be used to assure that any other solution C' of the cohomological equation (1.1) in the case that B is the identity is of the form $C'(x) = C(x)C$ for some $C \in \mathrm{GL}(d, \mathbb{R})$ and, consequently, the space of solutions is necessarily finite dimensional. In [24], Walkden gave conditions for twisted cohomological equations on the torus to have finitely many solutions. Here we consider fiber-bunched cocycles over a hyperbolic homeomorphism. In order to determine the number of solutions of (1.1) we use a relation between the space of solutions and the centralizer of one of the involved cocycles. First we prove that the centralizer is always finite dimensional (Corollary 3.10).

Then we prove that the centralizer is trivial for an open and dense set of fiber-bunched cocycles, which turns out to be equivalent to obtain a unique solution for the cohomological equation involving such cocycles (see Theorems 2.1 and 2.2 for the precise statements). In particular, there exists a finite number of conjugacies between a typical pair of conjugated fiber-bunched cocycles, and their number is determined by the field $\mathbb{K}$ and the dimension d.

The paper is organized as follows. The precise statements of the main results and some examples are given in Section 2. In addition, we give some examples that illustrate the need of the assumptions of non-compactness and irreducibility in the Lie groups. Section 3 is devoted to some preliminaries on fiber-bunched cocycles, holonomies, and the relation between cohomological classes and centralizers. Finally, in Section 4 we prove our main results.

2. Statement of the main results

Let (M, μ) be a probability space, let $f\colon M \to M$ be a measure-preserving map, and let $\mathbf{G}$ be a $\mathbb{K}$-algebraic subgroup of $\mathrm{SL}(d, \mathbb{C})$, i.e., a group of complex $d \times d$ matrices of determinant 1, defined by polynomial equations with coefficients in $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $d \geq 2$. In particular $\mathbf{G}$ is closed. We assume

$$\mathcal{G} := \mathbf{G} \cap \mathrm{SL}(d, \mathbb{K}) \tag{2.1}$$

satisfies:

(i) $\mathcal{G}$ contains $\mathrm{SO}(d, \mathbb{R})$;

(ii) $\mathbf{G}$ is *connected* (or equivalently, $\mathbf{G}$ is irreducible as an algebraic set);

(iii) $\mathcal{G}$ is *semisimple* (or equivalently, $\mathbf{G}$ is semisimple);

(iv) $\mathcal{G}$ is *noncompact*;

(v) $\mathcal{G}$ *acts irreducibly* on $\mathbb{K}^d$, that is, the only subspaces $V \subset \mathbb{K}^d$ invariant under the whole action of G are the trivial subspaces $V = \{0\}$ and $V = \mathbb{K}^d$.

These properties are satisfied by most classical non-compact semi simple Lie groups.

A *linear cocycle* taking values on $\mathcal{G}$ is a pair (A, f) where $A\colon M \to \mathcal{G}$ is a matrix-valued map. Throughout we assume that f is a transitive hyperbolic homeomorphism on a compact metric space M (see Subsection 3.2 for the definition). Given $\nu > 0$, denote by $C^\nu(M, \mathcal{G})$ the set of ν-Hölder continuous cocycles (in the case that $\nu = 1$ the latter denotes the space of Lipschitz continuous cocycles, also denoted by $\mathrm{Lip}(M, \mathcal{G})$). The space $C^\nu(M, \mathcal{G})$ endowed with the norm $\|A\|_\nu := \|A\|_\infty + \sup_{x \neq y} \|A(x) - A(y)\|/d(x, y)^\nu$ is a Banach space. One can associate to $A \in C^\nu(M, \mathcal{G})$ the skew-product

$$\begin{array}{rccc} F_A\colon & M \times \mathbb{K}^d & \longrightarrow & M \times \mathbb{K}^d \\ & (x, v) & \mapsto & (f(x), A(x) \cdot v). \end{array} \tag{2.2}$$

The joint base and fiber dynamics is given by $F_A^n(x, v) = (f^n(x), A^{(n)}(x)v)$, where

$$A^{(n)}(x) = \begin{cases} A(f^{n-1}(x)) \cdots A(f(x))A(x), & \text{if } n \geq 0, \\ A(f^n(x))^{-1} \ldots A(f^{-1}(x))^{-1}, & \text{otherwise.} \end{cases}$$

We say that two cocycles $A, B \in C^\nu(M,\mathcal{G})$ are: (i) *measurable cohomologous* (resp. *Hölder cohomologous*) if there exists a measurable map (resp. Hölder continuous map) $C\colon M \to \mathcal{G}$ such that $A(x) = C(f(x))B(x)C(x)^{-1}$ for all $x \in M$. The *cohomology class* $\mathcal{C}(A)$ of a cocycle $A \in C^\nu(M,\mathcal{G})$ is defined as the set of ν-Hölder continuous cocycles B that are ν-Hölder cohomologous to A. The set $C^\nu_b(M,\mathcal{G})$ of fiber-bunched cocycles over f (see Subsection 3.2 for the definition) is a C^0-open subset of $C^\nu(M,\mathcal{G})$. Our first main results are as follows:

Theorem 2.1. *Let $d \geq 2$ and let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Assume that $\mathcal{G}$ is a subgroup of* $\mathrm{SL}(d,\mathbb{K})$ *given by* (2.1) *and satisfying conditions* (i)–(v) *above. There exists an open and dense subset $\mathscr{G} \subset C^\nu_b(M,\mathcal{G})$ such that, for any $A \in \mathscr{G}$,*

(a) *there exists a finite abelian subgroup $F \subsetneq C^\nu(M,\mathcal{G})$ so that the conjugacy class $\mathcal{C}(A)$ is homeomorphic to $C^\nu(M,\mathcal{G})/F$; and*
(b) *for every $B \in \mathcal{C}(A)$, the cohomological equation $A(x) = C(f(x))^{-1}B(x)C(x)$ has exactly*
   (i) *one solution if d is odd and $\mathbb{K} = \mathbb{R}$;*
   (ii) *two solutions if d is even and $\mathbb{K} = \mathbb{R}$; and*
   (iii) *d solutions if $\mathbb{K} = \mathbb{C}$.*

The proof of the previous theorem is constructive, hence the solutions are given explicitly. The previous result relies on a relation between cohomology classes and centralizers (cf. Lemma 3.6) together with the following.

Theorem 2.2. *There exists an open and dense subset $\mathscr{G} \subset C^\nu_b(M,\mathcal{G})$ such that the centralizer $\mathcal{Z}(A)$ is trivial for every $A \in \mathscr{G}$.*

Note that the centralizer of a linear map on a finite dimensional vector space V is typically a large finite dimensional subgroup of the space of linear maps on V (see e.g. [6]). The previous result asserts that the fiber-bunching assumption and the recurrence guaranteed by the hyperbolic dynamics impose sufficient conditions for the centralizer to be typically trivial. It is still unknown if the cocycles with non-trivial centralizer can be detected using only information at periodic points. We discuss the previous results by means of some examples:

Example 2.3. Let $f\colon \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$ be the usual shift, which is a hyperbolic homeomorphism. Let $\mathrm{SO}(2,\mathbb{R}) \subset \mathrm{GL}(2,\mathbb{R})$ denote the special orthogonal group, isomorphic to $\mathbb{R}/2\pi\mathbb{Z} \simeq \mathbb{S}^1$. This is a compact group formed by rotations. Let $\alpha\colon M \to \mathbb{R}$ be Hölder continuous and consider the cocycle $A \in C^\nu(M, \mathrm{SO}(2,\mathbb{R}))$ given by

$$A(x) = \begin{pmatrix} \cos\alpha(x) & -\sin\alpha(x) \\ \sin\alpha(x) & \cos\alpha(x) \end{pmatrix}. \tag{2.3}$$

It is not hard to check that every $Q \in \mathcal{Z}(A)$ is a rotation cocycle of the form

$$B(x) = \begin{pmatrix} \cos\beta(x) & -\sin\beta(x) \\ \sin\beta(x) & \cos\beta(x) \end{pmatrix} \tag{2.4}$$

for some $\beta\colon M \to \mathbb{R}$ such that $\beta\circ f-\beta=0$. Since f is a hyperbolic homeomorphism the classical Livsic theorem holds and, consequently, all possible solutions of the equation $u\circ f-u=0$ are of the form $u=\beta+c$ for some $c\in\mathbb{R}$. Finally, since all angles that differ by an integer multiple of 2π determine the same cocycle in (2.4) then

$$\mathcal{Z}(A)\simeq \mathbb{R}/2\pi\mathbb{Z}\simeq \mathbb{S}^1\simeq \mathrm{SO}(2,\mathbb{R}).$$

In particular, this example shows that the non-compactness assumption is crucial in the statements of Theorems 2.1 and 2.2.

Example 2.4. Let M be a compact Riemannian manifold, $f\colon M\to M$ be an Anosov diffeomorphism and $A\in C^\nu(M,\mathrm{SL}(2,\mathbb{R}))$ be given by

$$A(x)=\begin{pmatrix} a(x) & 0\\ 0 & b(x)\end{pmatrix}$$

with $a,b\colon M\to\mathbb{R}$ Hölder continuous. It is easily checkable that every $B\in\mathcal{Z}(A)$ is formed by diagonal matrices

$$B(x)=\begin{pmatrix} c(x) & 0\\ 0 & d(x)\end{pmatrix}$$

such that the functions c,d satisfy $\log c\circ f-\log c=\log d\circ f-\log d=0$. Thus the centralizer of A is non-trivial and that $\mathcal{C}(A)\simeq C^\nu(M,\mathrm{Diag}(2))\neq C^\nu(M,\mathrm{SL}(2,\mathbb{R}))$, where $\mathrm{Diag}(2)\subset \mathrm{SL}(2,\mathbb{R})$ denote the subgroup of diagonal matrices with determinant constant to 1. In particular, although one can expect typical fiber-bunched $\mathrm{SL}(2,\mathbb{R})$-cocycles to have simple Lyapunov spectrum (see e.g. [7], [23]). As the latter cocycle needs not be fiber-bunched, there exists no relation between simplicity of the Lyapunov spectrum and the bunching condition with the triviality of the centralizer. The cocycle A can be assumed to be fiber-bunched provided that a,b are taken close to 1. If, in addition, $a(x)b(x)^{-1}<1$ for all $x\in M$, then every $B\in C^\nu(M,\mathrm{SL}(2,\mathbb{R}))$ that is C^0-close to A admits a dominated splitting $M\times\mathbb{R}^2=E\oplus F$, where the subbundles E and F can be defined by the existence of invariant cone fields for B. Then Theorem 2.2 implies that there exists a C^0-open neighborhood $\mathcal{U}$ of A so that the centralizer is trivial for a C^ν-open and dense set of cocycles in $\mathcal{U}$.

Remark 2.5. The subgroup $\mathrm{Diag}(2)\subsetneq \mathrm{SL}(2,\mathbb{R})$ is not irreducible. Moreover, if the cocycle A in Example 2.4 is taken as an element of $C^\nu(M,\mathrm{Diag}(2))$, then there is an open neighborhood of A in $C^\nu(M,\mathrm{Diag}(2))$ formed by cocycles formed by diagonal elements, which have non-trivial centralizer. This shows that the irreducibility assumption on the group $\mathcal{G}$ is necessary.

The centralizer of a cocycle depends intrinsically on the underlying dynamics. In the case of cocycles over Anosov diffeomorphisms, one can prove the abundance of trivial centralizer on the space $\mathrm{Diff}^{\,1}(M)\times \mathrm{Lip}(M,\mathcal{G})$, endowed with the product topology. To state this result, we first set some notation. Given $g\in\mathcal{W}$, $A\in \mathrm{Lip}(M,\mathcal{G})$, denote $A_g^{(n)}(x):=A(g^{n-1}(x))\,\dots\,A(g(x))\,A(x)$, for every $x\in M$, $n\ge 1$.

Theorem 2.6. *Given an Anosov diffeomorphism $f\in \mathrm{Diff}^{\,1}(M)$ and a cocycle $A\in \mathrm{Lip}(M,\mathcal{G})$ that is fiber-bunched with respect to f, there exists a open neighborhood $\mathcal{W}\times\mathcal{V}$ of (f,A) such that $\mathcal{Z}_g(A)$ is trivial for an open and dense subset of pairs (g,A) in $\mathcal{W}\times\mathcal{V}$.*

Finally we turn our attention to the skew-product dynamics of the form (2.2). Given integers $n,r\ge 1$ and $\nu>0$, let $\mathcal{A}^r(M)$ denote the space of C^r-transitive Anosov diffeomorphisms on a compact manifold M, and let $\mathcal{S}=\mathcal{S}^{r,\nu}$ denote the space of skew-products $F_A(x,v)=(f(x),A(x)v)$ so that $f\in\mathcal{A}^r(M)$ and $A\in C^\nu(M,\mathcal{G})$ is fiber-bunched with respect to f. While it is an open question to determine if all Anosov diffeomorphisms are transitive, it is well known that all Anosov diffeomorphisms on tori are transitive. We endow $\mathcal{S}$ with the topology induced by the product topology on $\mathrm{Diff}^r(M)\times C^\nu(M,\mathcal{G})$. Note that $\mathcal{S}$ is a open subset in the space of the skew-products with linear fiber action. Given $F_A\in\mathcal{S}$, the centralizer $\mathcal{Z}(F_A)$ is defined as follows:

$$\mathcal{Z}(F_A):=\{G_B\in\mathcal{S}\colon F_A\circ G_B=G_B\circ F_A\}.$$

We say that the centralizer $\mathcal{Z}(F_A)$ is *trivial* if for every $G_B\in\mathcal{Z}(F_A)$ there exists $\ell\in\mathbb{Z}$ and $\lambda\in\mathbb{K}$ so that $\lambda^d=1$ and $G_B(x,v)=(f^\ell(x),\lambda A^{(\ell)}(x)v)$ for every $x\in M$ and $v\in\mathbb{K}^d$. This means that all elements of $\mathcal{Z}(F_A)$ coincide with a power of F_A up to a rational rotation on the fiber. The centralizer of partially hyperbolic skew-products having $\mathbb{S}^1$ as one-dimensional compact fiber was studied in [5]. Here we consider skew-products with a d-dimensional vector space fiber and a linear constraint in the fiber direction. Recall that a subset $\mathcal{R}\subset\mathcal{S}$ is *Baire residual* if it contains a countable intersection of open and dense subsets of $\mathcal{S}$. We prove the following.

Theorem 2.7. *There exists a Baire residual subset $\mathcal{R}\subset\mathcal{S}$ such that $\mathcal{Z}(F_A)$ is trivial for every $F_A\in\mathcal{R}$.*

3. Preliminaries

3.1. Relation between conjugacies and centralizers of linear cocycles

Given an integer $d\ge 1$, we define the *linear centralizer* $C(A)$ of a matrix $A\in \mathrm{GL}(d,\mathbb{K})$ as the set of matrices $B\in\mathrm{GL}(d,\mathbb{K})$ that commute with A.[1] Using Jordan canonical form, it is not hard to see that the linear centralizer of matrices in $\mathrm{GL}(d,\mathbb{K})$ is typically non-trivial. In some cases, such centralizers can be characterized in terms of the spectrum of the matrix and can be used to describe the centralizer of Anosov diffeomorphisms on the torus (see e.g. [20] and references

[1] We use the nomenclature of linear centralizer as there are non-linear maps commuting with linear transformations (see e.g. [12]).

therein). Let $\mathbb{P}^{d-1}\mathbb{K}$ denote the projective space of $\mathbb{K}^d$, which is a compact manifold. Given a linear transformation $A\colon \mathbb{K}^d \to \mathbb{K}^d$, let $P_A\colon \mathbb{P}^{d-1}\mathbb{K} \to \mathbb{P}^{d-1}\mathbb{K}$ denote its projectivization.

Example 3.1. The invariant subspaces of the matrix $A \in \mathrm{SL}(3,\mathbb{R})$ given by

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

are: $E_1 = \mathrm{Ker}(A-I)^2 = \mathbb{R}^3$, $E_2 = \{(x,0,0) \in \mathbb{R}^3 \colon x \in \mathbb{R}\}$, $E_3 = \{(0,0,z) \in \mathbb{R}^3 \colon z \in \mathbb{R}\}$ and $E_4 = \{(x,y,0) \in \mathbb{R}^3 \colon x, y \in \mathbb{R}\}$.

For any pair of commuting matrices, eigenspaces and generalized eigenspaces are preserved, and the space of projectively invariant measures is non-empty. More precisely:

Lemma 3.2. *If $A, B \in \mathcal{G}$ commute, then the set of common invariant measures for the projectivizations P_A and P_B on $\mathbb{P}^{d-1}\mathbb{K}$ is a non-empty convex set.*

Proof. Let $\sigma(A) \subset \mathbb{C}$ denote the generalized eigenvalues of A. Given $\lambda \in \sigma(A)$, let E^A_λ denote the generalized eigenspace of A associated to λ. We first claim that if $AB = BA$ then the generalized eigenspaces of A are preserved by B, and vice-versa. Indeed, if λ is an eigenvalue for A then there exists $k \geq 1$ so that $E^A_\lambda = \mathrm{Ker}((A-\lambda I)^k)$. Hence, if $v \in E^A_\lambda$ then $(A-\lambda I)^k(B(v)) = B(A-\lambda I)^k(v) = B(0) = 0$, which proves that $B(E^A_\lambda) \subset E^A_\lambda$. Since there exists a basis of $\mathbb{K}^d$ formed by basis of the generalized subspaces for A, we conclude that $B(E^A_\lambda) = E^A_\lambda$ for all $\lambda \in \sigma(A)$. Replacing the roles of A and B above we conclude $A(E^B_\lambda) = E^B_\lambda$ for all $\lambda \in \sigma(B)$, proving the claim. Moreover, since the previous computations hold for $k = 1$, we also have that commuting matrices preserve the eigenspaces of each other.

We now translate the previous information on the existence of common invariant measures for the projective dynamics. Let

$$\mathbb{K}^d = E^A_{\lambda_1} \oplus E^A_{\lambda_2} \oplus \cdots \oplus E^A_{\lambda_\kappa} \tag{3.1}$$

be the decomposition of $\mathbb{K}^d$ on generalized eigenspaces for A and set $d_i = \dim E^A_{\lambda_i}$. Assume first that some of the $E^A_{\lambda_i}$ is an eigenspace and $A|_{E^A_{\lambda_i}}$ is conformal (i.e., $A|_{E^A_{\lambda_i}} = \lambda_i \,\mathrm{Id}|_{E^A_{\lambda_i}}$). Since B preserves (3.1), one can write

$$E^A_{\lambda_i} = E^B_{\lambda_i,1} \oplus E^B_{\lambda_i,2} \oplus \cdots \oplus E^B_{\lambda_i,\ell} \tag{3.2}$$

as the splitting of $E^A_{\lambda_i}$ on generalized eigenspaces for $B|_{E^A_{\lambda_i}}$. Since $A(E^A_{\lambda_i}) = B(E^A_{\lambda_i}) = E^A_{\lambda_i}$ and A, B commute, then $A|_{E^A_{\lambda_i}}$ preserve the generalized eigenspaces of $B|_{E^A_{\lambda_i}}$. Hence all subspaces $E^B_{\lambda_i,j}$ are A-invariant and B-invariant, that is,

$$A(E^B_{\lambda_i,j}) = B(E^B_{\lambda_i,j}) = E^B_{\lambda_i,j} \quad \text{for every } 1 \leq j \leq \ell.$$

If $B|_{E^B_{\lambda_i,j}}$ is conformal, then both restrictions of the projective maps P_A and P_B to $\mathbb{P}E^B_{\lambda_i,j}$ coincide with the identity (recall $A|_{E^B_{\lambda_i}}$ is also conformal), and so P_A and P_B have common invariant measures. If $B|_{E^B_{\lambda_i,j}}$ is not conformal, then write $E^B_{\lambda_i,j} = E^{B,0}_{\lambda_i,j} \oplus E^{B,N}_{\lambda_i,j}$, where $E^{B,0}_{\lambda_i,j}$ has dimension s_i and admits a basis of eigenvectors associated to the generalized eigenvalue λ_i, the subspace $E^{B,N}_{\lambda_i,j}$ has dimension $d_i - s_i$, and $(B-\lambda_i I)|_{E^{B,N}_{\lambda_i,j}}$ is a nilpotent matrix. Since $E^{B,0}_{\lambda_i,j}$ is an eigenspace for $B|_{E^A_{\lambda_i}}$ then it is preserved by A and consequently A and B are conformal in $E^{B,0}_{\lambda_i,j}$. In consequence, P_A and P_B have some common invariant measure.

Assume now that all $E^A_{\lambda_i}$ are generalized eigenspaces. Given $1 \le i \le k$, let $\hat{E}^A_{\lambda_i} \subset E^A_{\lambda_i}$ denote the eigenspace associated to the eigenvalue λ_i. Note that $A|_{\hat{E}^A_{\lambda_i}}$ is conformal and that $B(\hat{E}^A_{\lambda_i}) = \hat{E}^A_{\lambda_i}$ because B preserves the eigenspaces of A. The proof follows exactly as above using $A|_{\hat{E}^A_{\lambda_i}}$ instead of $A|_{E^A_{\lambda_i}}$. Since the convexity of the space of common invariant measures is immediate then the lemma follows. □

Remark 3.3. The previous lemma relies on the fact that commuting matrices preserve both eigenspaces and generalized eigenspaces. Observe also that if A, B are commuting matrices and $F \subset \mathbb{K}^d$ is an A-invariant subspace, then $A(B(F)) = B(A(F)) = B(F)$, which means that $B(F)$ is an A-invariant subspace. It often occurs that invariant subspaces are not preserved by commuting matrices. For instance, the matrices

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

commute and, if $\alpha \notin \mathbb{Q}$, the A-invariant subspace $\{(x,y) \in \mathbb{R}^2 \colon x = 0\}$ (not an eigenspace) is not preserved by B.

Remark 3.4. The projectivization P_A of a linear map A whose all eigenvalues have different norms acts as a Morse–Smale dynamics on $\mathbb{PK}^{d-1}$ and, in particular, the space of P_A-invariant measures is a finite dimensional simplex.

Remark 3.5. Assume that $A \in C^\nu(M, \mathcal{G})$, $B \in \mathcal{Z}(A)$ and p is a periodic point of period $n \ge 1$ for f. Since (1.2) holds, then $A^{(n)}(p) = B(p)\, A^{(n)}(p) B(p)^{-1}$ or, equivalently, $B(p) \in C(A^{(n)}(p))$.

The relation between conjugacy classes and centralizers is well known in different settings (see e.g. [6], [19], [21]). We need the following simple formulation for linear cocycles.

Lemma 3.6 (Proposition 4.9 in [21])**.** *Assume that $A, B, C_1, C_2 \in C^\nu(M, \mathcal{G})$ are so that $B(x) = C_i(f(x))\, A(x) C_i(x)^{-1}$ for every $x \in M$ and $i = 1, 2$. Then $B \in \mathcal{C}(A)$ and $C_2^{-1} C_1 \in \mathcal{Z}(A)$. Moreover, the space of solutions of the cohomological equation involving A and B coincides with $C_1 \mathcal{Z}(A)$.*

Remark 3.7. We note that the triviality of the centralizer means it is the smallest possible and not necessarily the identity cocycle. As $\mathcal{G} \subset \mathrm{SL}(d, \mathbb{K})$, if $B \in \mathcal{Z}(A)$ and the centralizer $\mathcal{Z}(A)$ is trivial, the following holds:

1. if $\mathbb{K} = \mathbb{R}$ and $d \geq 2$ is odd, then $B = \mathrm{Id}$;
2. if $\mathbb{K} = \mathbb{R}$ and $d \geq 2$ is even, then $B = \pm\mathrm{Id}$; and
3. if $\mathbb{K} = \mathbb{C}$, then $B = \lambda\mathrm{Id}$ for some $\lambda \in \mathbb{C}$ so that $\lambda^d = 1$.

Hence, either $\mathcal{Z}(A) = \{\mathrm{Id}\}$ (in the first case above) or $\mathcal{Z}(A)$ is isomorphic to the cyclic group $\mathbb{Z}_2$ or $\mathbb{Z}_d$ (in the second and third cases above, respectively). If we assumed $\mathcal{G} \subset \mathrm{GL}(d, \mathbb{K})$ instead, triviality would mean the centralizer to coincide with $\{\lambda\mathrm{Id}\colon \lambda \in \mathbb{K}\}$.

3.2. Fiber-bunched cocycles and holonomies

In this subsection we recall the notion of stable and unstable holonomies, referring e.g. to [3] for the proofs. Let (M, d) be a compact metric space and $f\colon M \to M$ be a continuous and transitive map. Given $x \in M$, the stable set of x with respect to f is $W^s(x) = \{y \in M;\, d(f^n(x), f^n(y)) \to 0 \text{ as } n \to +\infty\}$, and the stable set of size $\varepsilon > 0$ is $W^s_\varepsilon(x) = \{y \in M;\, d(f^n(x), f^n(y)) \leq \varepsilon \text{ for all } n \geq 0\}$. If f is invertible, unstable and local unstable sets of x are defined just replacing f by f^{-1} above. A homeomorphism $f\colon M \to M$ is *hyperbolic* if there are constants $C, \lambda, \varepsilon, \delta > 0$ so that

- $d(f^n(y), f^n(z)) \leq C e^{-\lambda n} d(y, z)$ for all $y, z \in W^s_\varepsilon(x)$ and $n \geq 0$;
- $d(f^{-n}(y), f^{-n}(z)) \leq C e^{-\lambda n} d(y, z)$ for all $y, z \in W^u_\varepsilon(x)$ and $n \geq 0$; and
- if $d(x, y) < \delta$, then $W^s_\varepsilon(x) \cap W^u_\varepsilon(y)$ consists of an unique point which varies continuously with x and y.

It is a standard assumption that hyperbolic homeomorphisms are bi-Lipschitz and that homoclinic points are dense in M. Examples of hyperbolic homeomorphisms include Anosov diffeomorphisms, hyperbolic basic sets and bilateral subshifts of finite type. Let $f\colon M \to M$ be a hyperbolic homeomorphism. We say that a cocycle $A \in C^\nu(M, \mathcal{G})$ over f is *fiber-bunched* (and write $A \in C^\nu_b(M, \mathcal{G})$) if there are constants $N \geq 1$ and $\theta > 0$, with $\theta < \lambda\nu$, so that

$$\|A^{(N)}(x)\|\, \|A^{(N)}(x)^{-1}\| < e^{\theta N} \quad \text{for all } x \in M. \tag{3.3}$$

It is clear from the definition that the set $C^\nu_b(M, \mathcal{G})$ of fiber-bunched cocycles over f form a C^0-open subset of $C^\nu(M, \mathcal{G})$. Moreover, if $A \in C^\nu_b(M, \mathcal{G})$ then there exists $K > 0$ so that $\|A^{(n)}(x)\|\ \|A^{(n)}(x)^{-1}\| \leq K e^{\theta n}$ for all $x \in M$ and $n \geq 0$. If the cocycle $A \in C^\nu(M, \mathcal{G})$ is fiber-bunched, then there are well defined stable holonomies

$$H^{s,A}_{x,y} := \lim_{n \to +\infty} [A^{(n)}(y)]^{-1} A^{(n)}(x) \in \mathcal{G} \quad \text{for all points } y \in W^s_\varepsilon(x). \tag{3.4}$$

For points x, y in the same global stable manifold, define

$$H^{s,A}_{x,y} := [A^{(k)}(y)]^{-1} H^{s,A}_{f^k(x), f^k(y)} A^{(k)}(x),$$

where $k \geq 1$ is large so that $f^k(x), f^k(y)$ belong to the same local stable manifold (this expression does not depend on k, cf. Proposition 3.8 (2) below). Unstable holonomies $H^u_{x,y}$ are defined analogously using unstable manifolds and f^{-1} instead of f.

Proposition 3.8 ([3]). *Let f be a hyperbolic homeomorphism and A be a fiber-bunched cocycle. Then*

(1) $H^{s,A}_{x,x} = \mathrm{Id}$ *and* $H^{s,A}_{x,z} = H^{s,A}_{y,z} \circ H^{s,A}_{x,y}$;

(2) $A(y)^{-1} \circ H^{s,A}_{f(x),f(y)} \circ A(x) = H^{s,A}_{x,y}$; *and*

(3) $\|H^{s,A}_{x,y} - \mathrm{Id}\| \leq C_1\, d(x,y)^\nu$

for every x, y, z in the same stable manifold.

3.3. Centralizers and solutions of the cohomological equation

After Livšic's pioneer work [14], the solutions of the cohomological equation for real valued cocycles over hyperbolic dynamics can be characterized in terms of its evaluation at periodic points. As mentioned at the introduction, such a characterization of the solutions of the cohomological equation does not hold even for fiber-bunched matrix cocycles: there are fiber-bunched linear cocycles A, B such that the matrices at periodic points are conjugated but A and B fail to be cohomologous [18]. Nevertheless, if f is a Anosov diffeomorphism, A, B are $\mathrm{GL}(d,\mathbb{R})$-cocycles and there exists a Hölder continuous map $C\colon \mathrm{Per}(f) \to \mathrm{GL}(d,\mathbb{R})$ such that $A(p) = C(f(p))B(p)C(p)^{-1}$ for every $p \in \mathrm{Per}(f)$, then A and B are Hölder cohomologous [1], [21]. We will use the following.

Proposition 3.9 (Proposition 4.7 in [21]). *Let f be a hyperbolic homeomorphism, let $p \in M$ be a fixed point for f, and let $A, B \in C^\nu_b(M, \mathcal{G})$. Assume that there exists a matrix $C_0 \in \mathcal{G}$ satisfying*

(a) $A(p) = C_0 \circ B(p) \circ C_0^{-1}$, *and*

(b) $H^{s,A}_{x,p} \circ H^{u,A}_{p,x} = C_0 \circ H^{s,B}_{x,p} \circ H^{u,B}_{p,x} \circ C_0^{-1}$ *for every $x \in W^s(p) \cap W^u(p)$.*

Then there exists a unique ν-Hölder continuous conjugacy $C \in C^\nu(M, \mathcal{G})$ between A and B so that $C(p) = C_0$.

Sketch of the proof. Proposition 4.7 in [21] was originally stated in the case that f is an Anosov diffeomorphism and $\mathrm{GL}(d,\mathbb{R})$-cocycles, but the argument in the proof extends straightforward to our context. For completeness reasons we will give a sketch of the key ideas involved in the proof. Let p be a fixed point for f, let $A \in C^\nu_b(M, \mathcal{G})$ and let $C_0 \in \mathcal{G}$ satisfy (a) and (b). Define $C^s\colon W^s(p) \to \mathcal{G}$ and $C^u\colon W^u(p) \to \mathcal{G}$ by

$$\begin{cases} C^s(x) := H^{s,A}_{p,x}\, C_0\, H^{s,B}_{x,p}, & \text{for } x \in W^s(p), \\ C^u(x) := H^{u,A}_{p,x}\, C_0\, H^{u,B}_{x,p}, & \text{for } x \in W^u(p). \end{cases}$$

Condition (b) ensures that $C(x) := C^s(x) = C^u(x)$ coincide at the homoclinic points $W^s(p) \pitchfork W^u(p)$, which are dense in M. Moreover, since $\mathcal{G}$ is a closed subgroup and stable and unstable holonomies have uniform Hölder constants, then the arguments in Proposition 4.7 of [21] show that $C\colon W^s(p) \pitchfork W^u(p) \to \mathcal{G}$ is ν-Hölder continuous with bounded Hölder constant, thus admits a unique extension to a ν-Hölder continuous cocycle which we still denote by $C\colon M \to \mathcal{G}$, by some abuse of notation. Using assumption (b) and that $H^{u,A}_{f(x),f(y)} \circ A(x) = A(f(y)) \circ H^{u,A}_{x,y}$ and $(H^{u,A}_{p,x})^{-1} = H^{u,A}_{x,p}$ for every $x, y \in W^u(p)$, we conclude that

$$\begin{aligned} C(f(x))B(x)C(x)^{-1} &= H^{u,A}_{p,f(x)}\ C_0\ H^{u,B}_{f(x),p} B(x)\ H^{u,B}_{p,x}\ C_0^{-1}\ H^{u,A}_{x,p} \\ &= H^{u,A}_{p,f(x)}\ C_0\ B(p)\ C_0^{-1}\ H^{u,A}_{x,p} = H^{u,A}_{p,f(x)}\ A(p)\ H^{u,A}_{x,p} = A(x) \end{aligned}$$

for every point $x \in W^s(p) \pitchfork W^u(p)$. Finally, since homoclinic points are dense in M, C is the unique solution of the cohomological equation involving A and B. □

Now we relate the centralizer of a fiber-bunched cocycle with the centralizer of matrices at periodic points. Given a fixed point p for f and $B \in \mathcal{Z}(A)$, the evaluation matrix $B(p)$ belongs to $C(A(p))$ (recall Remark 3.5). By Proposition 3.9 (taking $B = A$, which are fiber-bunched), every element in $\mathcal{Z}(A)$ is determined by the matrices C_0 subject to the conditions (a) and (b) determined at a fixed point p. As the converse of Proposition 3.9 is a simple algebraic manipulation using Proposition 3.8 (see e.g. [21]), we obtain the following.

Corollary 3.10. *Under the assumptions of Proposition 3.9, if $\Psi_p\colon C^\nu(M,\mathcal{G}) \to \mathcal{G}$ is the evaluation map $B \mapsto B(p)$, then for every $A \in C^\nu_b(M,\mathcal{G})$, the map*

$$\Psi_{p,A} := \Psi_p\,|_{\mathcal{Z}(A)}\colon \mathcal{Z}(A) \to C(A(p)) \cap \Big[\bigcap_{x \in W^s(p) \cap W^u(p)} C(H^{s,A}_{x,p} \circ H^{u,A}_{p,x}) \Big] \subset \mathcal{G} \tag{3.5}$$

is a bijection. In particular, $\mathcal{Z}(A)$ is a finite dimensional subgroup of $C^\nu(M,\mathcal{G})$.

4. Proof of the theorems

4.1. Proof of Theorem 2.1

Let $\mathscr{G} \subset C^\nu_b(M,\mathcal{G})$ be given by Theorem 2.2 and take $A \in \mathscr{G}$. If $B \in \mathcal{C}(A)$ and there are $C_1, C_2 \in C^\nu(M,\mathcal{G})$ so that

$$B(x) = C_1(f(x))^{-1}\,A(x)\,C_1(x) = C_2(f(x))^{-1}\,A(x)\,C_2(x)$$

for every $x \in M$, then $C_1C_2^{-1} \in \mathcal{Z}(A)$. Since A has trivial centralizer and $\mathcal{G} \subset \mathrm{SL}(d,\mathbb{K})$, then $C_1 = \lambda^d C_2$ for some $\lambda \in \mathbb{K}$ such that $\lambda^d = 1$. This, together with Remark 3.7, proves item (b) in the theorem.

In order to prove item (a), first note that given $B \in \mathcal{C}(A)$ there exists $C \in C^\nu(M,\mathcal{G})$ so that $B(x) = C(f(x))^{-1}A(x)C(x)$ for every $x \in M$. We proceed to prove that $C^\nu(M,\mathcal{G})/\!\sim$ is homeomorphic to $\mathcal{C}(A)$, where $C_1 \sim C_2$ if and only if $C_1(f(x))^{-1}\,A(x)\,C_1(x) = C_2(f(x))^{-1}\,A(x)\,C_2(x)$ for every $x \in M$. Every $C \in$

$C^\nu(M,\mathcal{G})$ induces $B = B_C \in \mathcal{C}(A)$ by $B(x) := C(f(x))^{-1}A(x)C(x)$ for every $x \in M$. The map

$$F_A\colon \quad \begin{array}{ccc} C^\nu(M,\mathcal{G}) & \to & \mathcal{C}(A) \\ C & \mapsto & B_C \end{array}$$

is clearly continuous and surjective. Since A has trivial centralizer then $F_A(C_1) = F_A(C_2)$ if and only if $C_1 = \lambda^d C_2$ for some $\lambda \in \mathbb{K}$ such that $\lambda^d = 1$. In particular, taking the finite abelian subgroup $F = \{\lambda \mathrm{Id}\colon \lambda^d = 1\}$ of $C^\nu(M,\mathcal{G})$, the quotient space $C^\nu(M,\mathcal{G})/F$ is a Banach space (the norm $\|[A]\| := \|A\|$ in the quotient space is well defined as every element in the same class differ by a multiplicative constant of norm one). Moreover, the map

$$\tilde F_A\colon \quad \begin{array}{ccc} C^\nu(M,\mathcal{G})/F & \to & \mathcal{C}(A) \\ [C] & \mapsto & B_C \end{array}$$

is a continuous bijection. The inverse map $\tilde F_A^{-1}\colon \mathcal{C}(A) \to C^\nu(M,\mathcal{G})/F$ is continuous because

$$C_B := \tilde F_A^{-1}(B) \in C^\nu(M,\mathcal{G})/F$$

is determined up to a multiplicative constant by the continuous map $B \mapsto C_B(p)$ (uniquely defined by Proposition 3.9 items (a) and (b)), by the (uniform) continuity of the map $B \mapsto H^{u,B}_{x,p}$ on local unstable manifolds (see e.g. [3], Lemme 1.14) and the subsequent uniform continuity of the map $B \mapsto H^{u,A}_{p,x} C_B(p) H^{u,B}_{x,p}$ on $W^u(p)$. This shows that $\tilde F_A$ is a homeomorphism and completes the proof of Theorem 2.1. □

Remark 4.1. L. Backes informed us that Theorem 2.1 extends to the setting of accessible partially hyperbolic diffeomorphisms considered by Kalinin and Sadovskaya [10], replacing Proposition 3.9 by Proposition 4.6 and Theorem 4.7 in [10].

4.2. Proof of Theorem 2.2

Let $A \in C^\nu_b(M,\mathcal{G})$ be a fiber-bunched cocycle over a transitive hyperbolic homeomorphism f. If p is a fixed point for f, Corollary 3.10 ensures that every $B \in \mathcal{Z}(A)$ is completely determined by a matrix $B(p) \in \mathcal{G}$ that belongs to the centralizer of the matrices $A(p)$ and $H^{s,A}_{x,p} \circ H^{u,A}_{p,x}$ for every $x \in W^u(p) \cap W^s(p)$. Recall that $A \in \mathcal{G}$ has non-trivial spectrum if it has more than one eigenvalue. We need the following.

Lemma 4.2. *The set $\{A \in C^\nu_b(M,\mathcal{G})\colon A$ has trivial centralizer$\}$ contains the set of cocycles $A \in C^\nu_b(M,\mathcal{G})$ so that there exists $x \in W^s(p) \cap W^u(p)$ satisfying:*

(1) *the matrices $A(p)$ and $H(p) := H^{s,A}_{x,p} \circ H^{u,A}_{p,x}$ have non-trivial spectrum; and*

(2) *if F_1, F_2 are proper invariant subspaces for $A(p)$ and $H^{s,A}_{x,p} \circ H^{u,A}_{p,x}$, respectively, then $F_1 \cap F_2 = \{0\}$.*[2]

[2] Assumptions (1) and (2) define C^0-open conditions in the space $C^\nu_b(M,\mathcal{G})$. Moreover, assumption (2) ensures that the supports of probability measures invariant by the projective map $P_{A(p)}$ are disjoint from the corresponding ones for the projective map $P_{H(p)}$, which is an obstruction to the presence of common invariant measures.

Proof. Let $A \in C_b^\nu(M, \mathcal{G})$ be such that the matrices $A(p)$ and $H(p)$ do not preserve a common invariant subspace. In what follows, by some abuse of notation, we omit the dependence of p on $A(p), B(p)$ and $H(p)$ for notational simplicity. Since commuting matrices preserve the same generalized eigenspaces (recall the proof of Lemma 3.2), if $B \in \mathcal{Z}(A)$ then the matrix $B(p)$ preserves the sum of generalized eigenspaces of both $A(p)$ and $H(p)$. Since A has non-trivial spectrum, there exists an A-invariant splitting

$$\mathbb{K}^d = \oplus_{\lambda\in\sigma(A)} E_\lambda^A, \tag{4.1}$$

where $\sigma(A)$ denotes the spectrum of A, $\#\sigma(A) \geq 2$, and E_λ^A is the generalized eigenspace associated to $\lambda \in \sigma(A)$. Since A and B commute, the proof of Lemma 3.2 ensures that (4.1) is a B-invariant splitting. In particular, for every $\lambda \in \sigma(A)$, one can write

$$E_\lambda^A = \oplus_{\rho\in\sigma(B|_{E_\lambda^A})} E_{\rho,\lambda}^B,$$

where $E_{\rho,\lambda}^B$ is the generalized eigenspace associated to ρ for $B\mid_{E_\lambda^A}$.

We claim that B has only one eigenvalue. Fix $\rho \in \sigma(B)$ and set $E := E_\rho^B$. Since E is a generalized eigenspace for B, by commutativity of A and B, it follows that

$$A(E) = E \quad \text{and} \quad E = \oplus_{\lambda\in\sigma(A)} F_\lambda, \tag{4.2}$$

where E is written as sum of generalized eigenspaces for $A|_E$, where $F_\lambda \subset E_\lambda^A$, and the sum is taken among elements $\lambda \in \sigma(A)$ so that $E_\lambda^A \cap E \neq \emptyset$. Using (4.2) and the fact that $A(E) = E$ and $A(E_\lambda^A) = E_\lambda^A$ for every $\lambda \in \sigma(A)$, we have

$$A(\oplus_{\lambda\in\sigma(A)} F_\lambda) = \oplus_{\lambda\in\sigma(A)} F_\lambda$$

and $A(F_\lambda) = F_\lambda$ for every $\lambda \in \sigma(A)$ such that $E_\lambda^A \cap E \neq \emptyset$. In particular, F_λ is an A-invariant subspace. This proves that E_ρ^B is the sum of A-invariant subspaces. The same reasoning applied to H instead of A implies that E_ρ^B is also the sum of H-invariant subspaces. Assumption (2) implies that E cannot be a proper subspace, hence $E = \mathbb{K}^d$.

We now prove that B is conformal, i.e., it is a multiple of the identity. Since $E_\rho^B = \mathbb{K}^d$ and B preserves (4.1), the Jordan canonical form theorem (see e.g. [8]) applied to each map $B|_{E_\lambda^A}$ implies that there exists a suitable basis of $\mathbb{K}^d$ so that one can write $B = \rho \mathrm{Id} + \sum_{\lambda\in\sigma(A)} N_\lambda$, where each linear map $N_\lambda \colon E_\lambda^A \to E_\lambda^A$ is nilpotent. We claim that all nilpotent linear maps N_λ are trivial. In fact, since H has non-trivial spectrum then there exists a H-invariant splitting $\mathbb{K}^d = \oplus_{\eta\in\sigma(H)} E_\eta^H$ by generalized eigenspaces for H. By commutativity, $B(E_\eta^H) = E_\eta^H$ for all $\eta \in \sigma(H)$. The latter holds if and only if $N_\lambda(E_\eta^H) = E_\eta^H$ for every $\eta \in \sigma(H)$ or, equivalently,

$$N_\lambda(E_\eta^H \cap E_\lambda^A) = E_\eta^H \cap E_\lambda^A \quad \text{for every } \lambda \in \sigma(A),\, \eta \in \sigma(H).$$

By the previous expression, we conclude that if there exists $\lambda \in \sigma(A)$ so that $N_\lambda \neq 0$, then there exists $\eta \in \sigma(H)$ for which

$$E_\eta^H \cap E_\lambda^A = N_\lambda(E_\eta^H \cap E_\lambda^A) \neq \{0\}$$

leading to a contradiction with assumption (2). The latter proves that $N_\lambda = 0$ for every $\lambda \in \sigma(A)$ and so $B(p) = \rho \mathrm{Id}$. In consequence,

$$B(x) = \rho\, H^{s,A}_{p,x}\; H^{s,A}_{x,p} = \rho\; H^{s,A}_{p,x}\; (H^{s,A}_{p,x})^{-1} = \rho \mathrm{Id}$$

for every $x \in M$ (recall the proof of Proposition 3.9). Since $\det B(p) = 1$ then $\rho^d = 1$. This proves that A has trivial centralizer, as desired. □

The proof of the theorem will consist of proving that typical cocycles satisfy the assumptions of Lemma 4.2. We first address item (1) of Lemma 4.2. Assume first that $\mathcal{G}$ is locally non-compact.

Lemma 4.3. *Let $\mathcal{G}$ be a locally non-compact semi-simple Lie group. For every periodic point p for f and every $x \in W^u(p) \cap W^s(p)$ there exists a C^ν-open and dense set of cocycles $\mathcal{O} \subset C^\nu_b(M, \mathcal{G})$ so that $A(p)$ and $H^{s,A}_{x,p} \circ H^{u,A}_{p,x}$ have non-trivial spectrum for every $A \in \mathcal{O}$.*

Proof. Since $\mathcal{G}$ is locally non-compact, the set $\mathcal{G}_* := \{A \in \mathcal{G} \colon \|A\| > 1\}$ is open and dense in $\mathcal{G}$, and it is formed by matrices such that $\|A^{-1}\|^{-1} < \|A\|$. Hence, since (4.6) is a submersion there exists an open and C^ν-open and dense set of cocycles $\mathcal{O} \subset C^\nu_b(M, \mathcal{G})$ so that $(A(p), H^{\mathrm{s},\mathrm{A}}_{f^q(x),p}\, A^{(q)}(x)\, H^{\mathrm{u},\mathrm{A}}_{p,x}) \in \mathcal{G}_* \times \mathcal{G}_*$. This proves the lemma. □

Note that the previous lemma is not true in general. For instance, if $\mathcal{G} = \mathrm{SL}(2, \mathbb{R})$ and A_0 is a rotation cocycle (2.3) with $\alpha(x) \equiv \alpha \notin 2\pi\mathbb{Z}$ and p is a fixed point for f, then $A(p)$ is conformal for every cocycle $A \in C^\nu_b(M, \mathcal{G})$ that is C^0-close to A_0. In this setting one must select periodic points with some pinching behavior, hence leading to a slightly weaker statement. More precisely:

Lemma 4.4. *There exists a C^ν-open and dense set of cocycles $\mathcal{O} \subset C^\nu_b(M, \mathcal{G})$ so that for every $A \in \mathcal{O}$ there exists a periodic point p for f and $x \in W^u(p) \cap W^s(p)$ so that the matrices $A^{\pi(p)}(p)$ and $H^{s,A^{\pi(p)}}_{x,p} \circ H^{u,A^{\pi(p)}}_{p,x}$ have non-trivial spectrum.*

Proof. Recall that $\mathcal{G} \subset \mathrm{SL}(d, \mathbb{K})$ for some $d \geq 2$. In the case that $d \geq 3$, the set $\mathcal{G}^*$ of non-conformal matrices in $\mathcal{G} \subset \mathrm{SL}(d, \mathbb{K})$ is C^0-open and dense and the result follows from Lemma 4.3. It remains to consider the case when $d = 2$. It is well known that hyperbolic homeomorphisms admit finite Markov partitions, hence these are semi-conjugate with subshifts of finite type and admit a maximal entropy measure μ_0 for f with local product structure (cf. [4]). Theorem A in [2] implies that there exists a C^ν-open and dense subset of cocycles $\mathcal{O} \subset C^\nu_b(M, \mathcal{G})$ so that all cocycles in $\mathcal{O}$ have non-zero largest Lyapunov exponent with respect to μ_0. Since f is a homeomorphism satisfying the closing property, Theorem 1.4 in [9] ensures that the Lyapunov exponents of μ_0 are approximated by the Lyapunov exponents of periodic points. Thus, for every $A \in \mathcal{O}$ there exists a sequence of periodic points with nontrivial Lyapunov spectrum, thus there exists a periodic point p of period $\pi(p) \geq 1$ such that $A^{\pi(p)}(p)$ has norm larger than one (cf. Proposition 3.2 in [9]). Consider the cocycle $A^{\pi(p)}$ over $f^{\pi(p)}$ and let x be any homoclinic point associated to p. Consider the open set $\mathcal{G}_* := \{A \in \mathcal{G}\colon \|A\| > 1\}$. If $H^{s,A^{\pi(p)}}_{x,p} \circ H^{u,A^{\pi(p)}}_{p,x} \in \mathcal{G}_*$ we are done. Otherwise $H^{s,A^{\pi(p)}}_{x,p} \circ H^{u,A^{\pi(p)}}_{p,x} \in \mathcal{G}\backslash\mathcal{G}_*$ and there are two cases to consider:

(i) Under the previous condition, if both matrices $H^{s,A^{\pi(p)}}_{x,p}$ and $H^{u,A^{\pi(p)}}_{p,x}$ belong to $\mathcal{G}_*$, then

$$\|H^{s,A^{\pi(p)}}_{x,p}\| = \|H^{u,A^{\pi(p)}}_{p,x}\| > 1 \quad \text{and} \quad H^{s,A^{\pi(p)}}_{x,p}\big(E^+_{H^{u,A^{\pi(p)}}_{p,x}}\big) = E^-_{H^{u,A^{\pi(p)}}_{p,x}}, \tag{4.3}$$

where $E^+_{H^{u,A^{\pi(p)}}_{p,x}}$, (resp. $E^-_{H^{u,A^{\pi(p)}}_{p,x}}$) denote the eigenspace of $H^{u,A^{\pi(p)}}_{p,x}$ associated to the eigenvalue of absolute value larger (resp. smaller) than one. Since stable and unstable holonomies are submersions even when restricted to the subspace of tangent vectors supported on neighborhoods of disjoint open neighborhoods of x and p (cf. relation (3.5) in the proof of Proposition 3.4 in [2]) and $\mathrm{SO}(2,\mathbb{R}) \subset \mathcal{G}$, there exists a C^ν-arbitrarily small perturbation B of A supported in a neighborhood of x such that

$$H^{u,B^{\pi(p)}}_{p,x} = R_\alpha \cdot H^{u,A^{\pi(p)}}_{p,x} \in \mathcal{G}_* \quad \text{and} \quad H^{s,B^{\pi(p)}}_{x,p} = H^{s,A^{\pi(p)}}_{x,p} \in \mathcal{G}_*\,,$$

where R_α is a rotation of a small angle α, and (4.3) does not hold for B. Then $H^{s,B^{\pi(p)}}_{x,p} \circ H^{u,B^{\pi(p)}}_{p,x} \in \mathcal{G}_*$.

(ii) Assume both $H^{s,A^{\pi(p)}}_{x,p}$ and $H^{u,A^{\pi(p)}}_{p,x}$ do not belong to $\mathcal{G}_*$. By Proposition 3.8 item (2),

$$H^{s,A^{\pi(p)}}_{x,p} := \lim_{n\to\infty} [A^{n\,\pi(p)}(p)]^{-1} A^{n\,\pi(p)}(x) = [A^{k\pi(p)}(p)]^{-1} \circ H^{s,A}_{f^{k\pi(p)}(x),p} \circ A^{k\pi(p)}(x) \tag{4.4}$$

for every $k \geq 1$. Note that $\lim_{k\to+\infty} H^{s,A}_{f^{k\pi(p)}(x),p} = \mathrm{Id}$ (recall Proposition 3.8 (3)). Since $A^{\pi(p)}(p) \in \mathcal{G}_*$ then it follows that $\lim_{k\to\infty} \|A^{k\pi(p)}(p)\| = \lim_{k\to\infty} \|A^{k\pi(p)}(x)\| = +\infty$. Moreover, since $\|H^{s,A^{\pi(p)}}_{x,p}\| = 1$, the product of the matrices in the right-hand side of (4.4) has norm one and

$$H^{s,A}_{f^{k\pi(p)}(x),p} \circ A^{k\pi(p)}\big(E^+_{H^{s,A}_{f^{k\pi(p)}(x),p} \circ A^{k\pi(p)}}\big) = E^-_{[A^{k\pi(p)}(p)]^{-1}} = E^+_{A^{k\pi(p)}(p)}. \tag{4.5}$$

Using once more that the stable holonomy is a submersion as a function of the cocycle, there exists a C^ν-arbitrarily small perturbation B of A supported in a neighborhood of $f^{k\pi(p)}x$ such that relation (4.5) fails for B. Thus

$$H^{u,B^{\pi(p)}}_{p,x} = H^{u,A^{\pi(p)}}_{p,x} \in \mathcal{G} \setminus \mathcal{G}_* \quad \text{and} \quad H^{s,B^{\pi(p)}}_{x,p} \in \mathcal{G}_*$$

and, consequently, $H^{s,B^{\pi(p)}}_{x,p} \circ H^{u,B^{\pi(p)}}_{p,x} \in \mathcal{G}_*$. This proves that there exists a C^ν-dense subset of $C^\nu_b(M,\mathcal{G})$ satisfying the requirements of the lemma. Since these are defined by C^0 open conditions this proves the lemma. □

Remark 4.5. Observe that assumption (i) on the group $\mathcal{G}$ was only used in the proof of the previous lemma to deal with the case when $\mathcal{G} \subset \mathrm{SL}(2,\mathbb{K})$. Moreover, it follows from the previous proof that if $A \in \mathcal{O}$, the periodic point p in the statement of Lemma 4.4 can be chosen uniform for all cocycles C^ν-close to A.

From now on we will study the set of cocycles satisfying item (2) of Lemma 4.2. We will use the following characterization of the set of pairs of matrices whose projective maps have a common invariant measure.

Proposition 4.6 (Corollary 3.13 in [2])**.** *There is a semi-algebraic set $V \subset \mathcal{G} \times \mathcal{G}$ of positive codimension such that for every pair of elements $(A_1, A_2) \in (\mathcal{G} \times \mathcal{G}) \setminus V$, the projective maps P_{A_1} and P_{A_2} admit no common invariant measure on the projective space $\mathbb{P}^{d-1}\mathbb{K}$.*

This is used to prove that there exists an open and dense subset of $C^\nu_b(M, \mathcal{G})$ formed by cocycles that satisfy the requirements of Lemma 4.2 for some suitable periodic and homoclinic points. The precise statement is as follows.

Lemma 4.7. *There exists a C^ν-open and dense subset $\mathscr{G} \subset C^\nu_b(M, \mathcal{G})$ so that the following holds: for every $A \in \mathscr{G}$, there exist an open neighborhood $\mathcal{V}_A$ of A, a periodic point p of period $\pi(p) \geq 1$ for f, and $x \in W^u(p) \cap W^s(p)$ so that the matrices $B^{\pi(p)}(p)$ and $H^{s,B^{\pi(p)}}_{x,p} \circ H^{u,B^{\pi(p)}}_{p,x}$ satisfy items (1) and (2) of Lemma 4.2.*

Proof. Let $\mathcal{O} \subset C^\nu_b(M, \mathcal{G})$ be given by Lemma 4.4. By Remark 4.5, given $A \in \mathcal{O}$ there exists a periodic point p of period $\pi(p) \geq 1$ for f and $x \in W^u(p) \cap W^s(p)$ so that the matrices $B^{\pi(p)}(p)$ and $H^{s,B^{\pi(p)}}_{x,p} \circ H^{u,B^{\pi(p)}}_{p,x}$ have non-trivial spectrum for every cocycle B in a C^ν-neighborhood $\mathcal{N}_A \subset C^\nu_b(M, \mathcal{G})$ of A. Note that p is a fixed point $f^{\pi(p)}$ and that item (1) in Lemma 4.2 holds for the cocycles $B^{\pi(p)}$, for every $B \in \mathcal{N}_A$.

The proof of Lemma 3.2 ensures that the set of pairs of commuting matrices $(A_1, A_2) \in \mathcal{G} \times \mathcal{G}$ with some common proper invariant subspace is contained in those such that their projective maps have a common invariant probability. By Proposition 4.6, these pairs are contained in a closed and positive codimension subset V (hence with empty interior) of $\mathcal{G} \times \mathcal{G}$. Using that the map

$$
\begin{array}{rccc}
(4.6) & C^\nu_b(M, \mathcal{G}) & \longrightarrow & \mathcal{G} \times \mathcal{G} \\
& B & \mapsto & (B^{\pi(p)}(p), H^{\mathrm{s},\mathrm{B}^{\pi(\mathrm{p})}}_{x,p} \circ H^{\mathrm{u},\mathrm{B}^{\pi(\mathrm{p})}}_{p,x})
\end{array}
$$

is a submersion (cf. Proposition 3.4 in [2] and Proposition 3.8 (2)), there exists a C^ν-open and dense set of cocycles $\mathcal{V}_A \subset \mathcal{N}_A$ so that for every $B \in \mathcal{V}_A$ the matrices $B^{\pi(p)}(p)$ and $H^{s,B^{\pi(p)}}_{x,p} \circ H^{u,B^{\pi(p)}}_{p,x}$ have no common proper invariant subspace, thus satisfy item (2) of Lemma 4.2. In consequence, the set $\mathscr{G} = \bigcup_{A \in \mathcal{O}} \mathcal{V}_A$ is C^ν-open and dense in $C^\nu_b(M, \mathcal{G})$ and satisfy the requirements of the lemma. □

We are now in a position to conclude the proof of Theorem 2.2. The previous results ensure that there exists a C^ν-open and dense set of cocycles $\mathscr{G} \subset C^\nu_b(M, \mathcal{G})$ such that for every $A \in \mathscr{G}$ there exists a C^ν-open neighborhood $\mathcal{V}_A$ of A and a periodic point p of period $\pi(p) \geq 1$ for f such that the cocycle $B^{(\pi(p))}$ satisfies the requirements of Lemma 4.2, hence has trivial centralizer, for every $B \in \mathcal{V}_A$. The result follows as an immediate consequence of the fact that $\mathcal{Z}(A) \subset \mathcal{Z}(A^{(k)})$ for every $k \geq 1$. This proves that there exists a C^ν-open and dense subset of cocycles in $C^\nu_b(M, \mathcal{G})$ with trivial centralizer, and completes the proof of the theorem. □

Remark 4.8. Given a fixed point p for f, the non-existence of common invariant subspaces for matrices $A(p)$ and $H^{s,A}_{x,p} \circ H^{u,A}_{p,x}$ is a C^0-open condition on $C^\nu_b(M,\mathcal{G})$. In particular, there exists a C^0-open and C^ν-dense subset in $C^\nu_b(M,\mathcal{G})$ formed by cocycles with trivial centralizer.

4.3. Proof of Theorem 2.6

Since Anosov diffeomorphisms are structurally stable, there exists a C^1-open neighborhood $\mathcal{W}$ of f formed by Anosov diffeomorphisms and there are $\gamma \in (0,1)$ and a γ-Hölder continuous homeomorphism h_g close to the identity and such that $g \circ h_g = h_g \circ f$ for every $g \in \mathcal{W}$. Let $\eta\colon \mathcal{W} \to (0,1]$ be a continuous function such that any $f \in \mathcal{W}$ is $\eta(f)$-Hölder conjugate to all sufficiently close maps (see e.g. [11] for details). Set $\eta = \eta(f) > 0$. Since the fiber-bunching condition is an open condition (both on the cocycles and the dynamics) then there exists a C^0-open neighborhood $\mathcal{V} \subset \mathrm{Lip}_b(M,\mathcal{G})$ of A such that every $B \in \mathcal{V}$ is fiber-bunched with respect to every $g \in \mathcal{W}$. Given $g \in \mathcal{W}$ and $A \in \mathrm{Lip}(M,\mathcal{G})$,

$$\begin{aligned} A^{(n)}_g(x) &= A(g^{n-1}(x)) \, \ldots \, A(g(x))\, A(x) \\ &= (A \circ h_g)(f^{n-1}(h_g^{-1}(x))) \, \ldots \, (A \circ h_g)(f(h_g^{-1}(x)))\,(A \circ h_g)(h_g^{-1}(x)) \\ &= (A \circ h_g)^{(n)}_f(h_g^{-1}(x)) \end{aligned}$$

for every $x \in M$ and $n \geq 1$. In particular, $C \in \mathcal{Z}_g(A)$ if and only if $C \circ h_g \in \mathcal{Z}_f(A \circ h_g)$. Moreover, the stable holonomies for f and g satisfy the equality

$$\begin{aligned} H^{s,A,g}_{x,y} &:= \lim_{n\to\infty} [A^{(n)}_g(y)]^{-1}\, A^{(n)}_g(x) \\ &= \lim_{n\to\infty} [(A \circ h_g)^{(n)}_f(h_g^{-1}(y))]^{-1}\,(A \circ h_g)^{(n)}_f(h_g^{-1}(x)) = H^{s,\,A\circ h_g,\,f}_{h_g^{-1}(x),h_g^{-1}(y)} \end{aligned} \tag{4.7}$$

for every x,y in the same stable manifold with respect to g (and analogous statement holds for the unstable holonomies). In particular, $\mathcal{Z}_g(A)$ is trivial if and only if $\mathcal{Z}_f(A \circ h_g)$. Using that the map $\psi\colon \mathcal{W}\times\mathcal{V} \to C^\eta_b(M,\mathcal{G})$ given by $\psi(g,A) = A\circ h_g$ is continuous, and that for every fixed $g \in \mathcal{W}$ the map $\mathcal{V} \ni A \mapsto A \circ h_g$ is a submersion, it follows from Theorem 2.2 that the set

$$\begin{aligned} \mathcal{T} &:= \{(g,A) \in \mathcal{W} \times \mathcal{V} \colon \mathcal{Z}_g(A) \text{ is trivial}\} \\ &= \psi^{-1}(\{B \in C^\eta_b(M,\mathcal{G}) : \mathcal{Z}_f(B) \text{ is trivial}\}) \end{aligned}$$

contains an open and dense subset of $\mathcal{W} \times \mathcal{V}$. This completes the proof of the theorem. □

4.4. Proof of Theorem 2.7

Recall that there exists a C^r-generic subset $\mathcal{R}_1 \subset \mathcal{A}^r(M)$ of transitive Anosov diffeomorphisms with trivial C^1-centralizer (cf. [20]). Throughout this section assume that $f \in \mathcal{R}_1$. Given $A \in C^\nu(M,\mathcal{G})$, any skew-product $G_B \in \mathcal{S}$ that commutes with $F_A(x,v) = (f(x),A(x)v)$ is of the form $G_B(x,v) = (g(x),B(x)v)$, for some $g \in \mathrm{Diff}^r(M)$ and $B \in C^\nu(M, \mathcal{G})$. Thus, the relation $F_A \circ G = G \circ F_A$ becomes

$$\begin{cases} f \circ g(x) = g \circ f(x), \\ A(g(x))B(x)v = B(f(x))A(x)v, \end{cases} \tag{4.8}$$

for every $x \in M$ and $v \in \mathbb{K}^d$. The first equality above means that $g \in \mathcal{Z}(f)$ and, as $f \in \mathcal{R}_1$, there exists $\ell \in \mathbb{Z}$ so that $g = f^\ell$. Then the second equality in (4.8) becomes

$$A(f^\ell(x)) = B(f(x))A(x)B(x)^{-1} \tag{4.9}$$

for every $x \in M$, which means that the cocycles $A \circ f^\ell$ and A are conjugated (in other words $A \circ f^\ell \in \mathcal{C}(A)$) and that B is a conjugacy between these cocycles. As the cocycles $A \circ f^\ell$ and A are clearly conjugated (take e.g. the conjugacy $C(x) = A^{(\ell)}(x)$), it remains to determine all possible conjugacies between these cocycles in order to determine all possible B.

Let p be a periodic point for f, say $f^k(p) = p$, and consider the Anosov diffeomorphism f^k and the fiber-bunched cocycles $A^{(k)}$ and $A^{(k)} \circ f^\ell$. If B satisfies (4.9) then

$$A^{(k)}(f^\ell(x)) = B(f^k(x))A^{(k)}(x)B(x)^{-1} \quad \text{for every } x \in M. \tag{4.10}$$

Since $A \in C^\nu_b(M, \mathcal{G})$, the assumptions of Proposition 3.9 are satisfied for the cocycles $A^{(k)}$ and $A^{(k)} \circ f^\ell$. Therefore, every conjugacy $B \in C^\nu(M, \mathcal{G})$ between $A^{(k)}$ and $A^{(k)} \circ f^\ell$ over f^k (as (4.10)) is the unique continuous extension of the cocycle

$$B(x) = H^{s, A^{(k)}\circ f^\ell}_{p,x,f^k} \; C_0 \; H^{s,A^{(k)}}_{x,p,f^k} \quad \text{for every } x \in W^s(p),$$

from $W^s(p)$ to M, where $H^{*,A^{(k)}}_{p,x,f^k}$ denote the $*$-holonomies ($* = s, u$) for the cocycle $A^{(k)}$ over the dynamics f^k and the matrix $C_0 = B(p)$ is determined by the relations

$$\begin{aligned} A^{(k)}(f^\ell(p)) &= C_0 \circ A^{(k)}(p) \circ C_0^{-1}, \\ H^{s,A^{(k)}\circ f^\ell}_{x,p,f^k} \circ H^{u,A^{(k)}\circ f^\ell}_{p,x,f^k} &= C_0 \circ H^{s,A^{(k)}}_{x,p,f^k} \circ H^{u,A^{(k)}}_{p,x,f^k} \circ C_0^{-1} \end{aligned} \tag{4.11}$$

for every $x \in W^s(p) \cap W^u(p)$. Using that $f^k(p) = p$,

$$A^{(\ell)}(p) \circ A^{(k)}(p) = A^{(\ell)}(f^k(p)) \circ A^{(k)}(p) = A^{(k)}(f^\ell(p)) \circ A^{(\ell)}(p),$$

which allows to write the first equality in (4.11) as

$$A^{(k)}(p) = [C_0^{-1}A^{(\ell)}(p)]^{-1} \circ A^{(k)}(p) \circ [C_0^{-1}A^{(\ell)}(p)]. \tag{4.12}$$

The second expression in (4.11) is equivalent to

$$H^{s,A^{(k)}}_{x,p,f^k} \circ H^{u,A^{(k)}}_{p,x,f^k} = [C_0^{-1}A^{(\ell)}(p)]^{-1} \circ H^{s,A^{(k)}}_{x,p,f^k} \circ H^{u,A^{(k)}}_{p,x,f^k} \circ [C_0^{-1}A^{(\ell)}(p)] \tag{4.13}$$

where we used Proposition 3.8 to guarantee

$$\begin{aligned} H^{s,A^{(k)}\circ f^\ell}_{x,p,f^k} &= \lim_{n\to+\infty} [A^{(kn)}(f^\ell(p))]^{-1} A^{(kn)}(f^\ell(x)) \; (= H^{s,A^{(k)}}_{f^\ell(x),f^\ell(p),f^k}) \\ &= A^{(\ell)}(p) H^{s,A^{(k)}}_{x,p,f^k} [A^{(\ell)}(x)]^{-1} \end{aligned}$$

and, using analogous expression for the unstable holonomies,

$$\begin{aligned}H^{s,A^{(k)}\circ f^{\ell}}_{x,p,f^k}\circ H^{u,A^{(k)}\circ f^{\ell}}_{p,x,f^k} &= H^{s,A^{(k)}}_{f^{\ell}(x),f^{\ell}(p),f^k}\circ H^{u,A^{(k)}}_{f^{\ell}(p),f^{\ell}(x),f^k}\\ &= A^{(\ell)}(p)\circ H^{s,A^{(k)}}_{x,p,f^k}\circ[A^{(\ell)}(x)]^{-1}\circ A^{(\ell)}(x)\circ H^{u,A^{(k)}}_{p,x,f^k}\circ[A^{(\ell)}(p)]^{-1}\\ &= A^{(\ell)}(p)\circ[H^{s,A^{(k)}}_{x,p,f^k}\circ H^{u,A^{(k)}}_{p,x,f^k}]\circ[A^{(\ell)}(p)]^{-1}.\end{aligned}$$

Altogether we obtained that $B\in C^{\nu}(M,\mathcal{G})$ is completely determined by a matrix C_0 satisfying (4.12) and (4.13). If $C^{\nu}_f(M,\mathcal{G})\subset C^{\nu}(M,\mathcal{G})$ denotes the open set of cocycles fiber-bunched over f, arguments similar to those in the proof of Theorem 2.2 ensure that there exists a C^{ν}-open and dense subset $\mathscr{G}_f\subset C^{\nu}_f(M,\mathcal{G})$ so that for every $A\in\mathscr{G}_f$ equalities (4.12) and (4.13) hold if and only if $C_0=\lambda A^{(\ell)}(p)$ for some $\lambda\in\mathbb{K}$ such that $\lambda^d=1$.

Finally, observe that the set $\mathcal{R}=\bigcup_{f\in\mathcal{R}_1}\{f\}\times\mathscr{G}_f$ is a residual subset in $\mathcal{S}$ and for every pair $(f,A)\in\mathcal{R}$ there exists $k\geq 1$ so that every element $G_B\in\mathcal{Z}(F^k_A)$ is of the form

$$G_B(x,v)=(f^{\ell}(x),\lambda A^{(\ell)}(x)v)$$

for some $\lambda\in\mathbb{K}$ such that $\lambda^d=1$. This proves the theorem. □

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