Rev. Mat. Iberoam. (online first)
DOI 10.4171/RMI/1157

On the failure of the Hörmander multiplier theorem in a limiting case

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Abstract. We discuss the Hörmander multiplier theorem for L^p boundedness of Fourier multipliers in which the multiplier belongs to a fractional Sobolev space with smoothness s. We show that this theorem does not hold in the limiting case $|1/p - 1/2| = s/n$.

1. Introduction

Let m be a bounded function on $\mathbb{R}^n$. We define the associated linear operator

$$T_m(f)(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi)\, m(\xi)\, e^{2\pi i x\cdot\xi}\, d\xi, \quad x \in \mathbb{R}^n,$$

where f is a Schwartz function on $\mathbb{R}^n$ and $\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot\xi}\, dx$ is the Fourier transform of f. The problem of characterizing the class of functions m for which the operator T_m admits a bounded extension from $L^p(\mathbb{R}^n)$ to itself for a given $p \in (1,\infty)$ is one of the principal questions in harmonic analysis. We say that m is an L^p Fourier multiplier if the above mentioned property is satisfied. While it is a straightforward consequence of Plancherel's identity that all bounded functions are L^2 Fourier multipliers, the structure of the set of L^p Fourier multipliers for $p \neq 2$ turns out to be significantly more complicated.

A classical theorem of Mikhlin [10] asserts that if the condition

$$|\partial^\alpha m(\xi)| \le C_\alpha\, |\xi|^{-|\alpha|}, \quad \xi \neq 0, \tag{1.1}$$

is satisfied for all multiindices α with size $|\alpha| \le [n/2]+1$, then T_m admits a bounded extension from $L^p(\mathbb{R}^n)$ to itself for all $1 < p < \infty$. A subsequent result by Hörmander [9] showed that the pointwise estimate (1.1) can be replaced by a weaker Sobolev-type condition:

$$\sup_{R>0} R^{-n+2|\alpha|} \int_{\{\xi\in\mathbb{R}^n:\ R<|\xi|<2R\}} |\partial^\alpha m(\xi)|^2\, d\xi < \infty. \tag{1.2}$$

Mathematics Subject Classification (2010): Primary 42B15; Secondary 46E35.
Keywords: Hörmander multiplier theorem, Sobolev space.

Although theorems of Mikhlin and Hörmander admit a variety of applications, their substantial limitation stems from the fact that they can only be applied to functions which are L^p Fourier multipliers for all values of $p \in (1,\infty)$. One can overcome this difficulty using an interpolation argument as in Calderón and Torchinsky [1] or Connett and Schwartz [2], [3]; the conclusion is, roughly speaking, that *the closer p is to 2, the fewer derivatives are needed in conditions* (1.1) *or* (1.2).

To be able to formulate things precisely, let us now recall the notion of fractional Sobolev spaces. For $s>0$ we denote by $(I-\Delta)^{s/2}$ the operator given on the Fourier transform side by multiplication by $(1+4\pi^2|\xi|^2)^{s/2}$. If $1<r<\infty$, then the norm in the fractional Sobolev space L^r_s is defined by

$$\|f\|_{L^r_s} = \|(I-\Delta)^{s/2} f\|_{L^r}.$$

The version of the Mikhlin–Hörmander multiplier theorem due to Calderón and Torchinsky ([1], Theorem 4.7) says that inequality

$$\|T_m f\|_{L^p} \le C \sup_{D\in\mathbb{Z}} \|\phi(\xi)\, m(2^D\xi)\|_{L^r_s} \|f\|_{L^p} \tag{1.3}$$

holds provided that

$$\Big|\frac{1}{p}-\frac{1}{2}\Big| = \frac{1}{r} < \frac{s}{n}. \tag{1.4}$$

Here, ϕ stands for a smooth function on $\mathbb{R}^n$ supported in the set $\{\xi\in\mathbb{R}^n : 1/2 < |\xi| < 2\}$ and satisfying $\sum_{D\in\mathbb{Z}} \phi(2^D\cdot) = 1$. Additionally, it was pointed out in [4] that the equality $|1/p-1/2| = 1/r$ is not essential for (1.3) to be true, and (1.4) can thus be replaced by the couple of inequalities

$$\Big|\frac{1}{p}-\frac{1}{2}\Big| < \frac{s}{n}, \qquad \frac{1}{r} < \frac{s}{n}. \tag{1.5}$$

Let us notice that the latter inequality in (1.5) is dictated by the embedding of L^r_s into the space of essentially bounded functions. Related to this we also mention that the Sobolev-type condition in (1.3) can be further weakened by replacing the Sobolev space L^r_s with $r>n/s$ by the Sobolev space with smoothness s built upon the Lorentz space $L^{n/s,1}$, see [7].

Let us now discuss the sharpness of the first condition in (1.5). It is well known that if inequality (1.3) holds, then we necessarily have $|1/p-1/2|\le s/n$, see [8], [17], [11], [12] and [4]. On the critical line $|1/p-1/2| = s/n$, there are positive endpoint results by Seeger [13], [14], [15]. In particular, it is shown in [15] that inequality (1.3) holds when $|1/p-1/2| = s/n$ and $r>n/s$ if the Sobolev space L^r_s is replaced by the Besov space $B^s_{1,r}$, defined by

$$\|f\|_{B^s_{1,r}} = \sum_{k=0}^{\infty} 2^{ks}\, \|(\varphi_k \widehat{f})^\vee\|_{L^r}.$$

Here, φ_0 stands for a Schwartz function on $\mathbb{R}^n$ such that $\varphi_0(x)=1$ if $|x|\le 1$ and $\varphi_0(x)=0$ if $|x|\ge 3/2$, and $\varphi_k(x) = \varphi_0(2^{-k}x) - \varphi_0(2^{1-k}x)$ for $k\in\mathbb{N}$.

We recall that $B^s_{1,r}$ is embedded into L^r_s, thanks to the equivalence

$$\|f\|_{L^r_s} \approx \Big\| \Big(\sum_{k=0}^{\infty} 2^{2ks} \, |(\varphi_k \widehat{f})^{\vee}|^2 \Big)^{1/2} \Big\|_{L^r}$$

and to embeddings between sequence spaces.

In this note we show that Hörmander's condition involving the Sobolev space L^r_s fails to guarantee L^p boundedness of T_m in the limiting case $|1/p - 1/2| = s/n$. In fact, we can even include the more general Lorentz–Sobolev spaces L^{r_1,r_2}_s in our discussion, providing thus a negative answer to the open problem A.2 raised in Appendix A of the recent paper [16]. We recall that the Lorentz–Sobolev space L^{r_1,r_2}_s is defined as

$$\|f\|_{L^{r_1,r_2}_s} = \|(I-\Delta)^{s/2} f\|_{L^{r_1,r_2}},$$

where

$$\|g\|_{L^{r_1,r_2}} = \begin{cases} \left(\int_0^\infty t^{r_2/r_1 - 1} (g^*(t))^{r_2} \, dt\right)^{1/r_2} & \text{if } 1 < r_1 < \infty \text{ and } 1 \le r_2 < \infty, \\ \sup_{t>0} t^{1/r_1} \, g^*(t) & \text{if } 1 < r_1 < \infty \text{ and } r_2 = \infty. \end{cases}$$

Here,

$$g^*(t) = \inf\{\lambda > 0 : \ |\{x \in \mathbb{R}^n : |g(x)| > \lambda\}| \le t\}, \quad t > 0,$$

stands for the nonincreasing rearrangement of g.

Our result has the following form.

Theorem 1. *Let $1 < p < \infty$, $p \neq 2$, and let $s > 0$ be such that*

$$\left| \frac{1}{p} - \frac{1}{2} \right| = \frac{s}{n}. \tag{1.6}$$

Assume that $1 < r_1 < \infty$, $1 \le r_2 \le \infty$ and ϕ is a smooth function on $\mathbb{R}^n$ supported in the set $\{\xi \in \mathbb{R}^n : 1/2 < |\xi| < 2\}$. Then there is no finite constant C such that the inequality

$$\|T_m f\|_{L^p} \le C \sup_{D \in \mathbb{Z}} \|\phi(\xi) \, m(2^D \xi)\|_{L^{r_1,r_2}_s} \|f\|_{L^p} \tag{1.7}$$

holds for all m and f.

The proof of Theorem 1 uses the randomization technique in the spirit of [18], Chapter 4, which has been further developed in [4] and [5].

2. Proof of Theorem 1

Let $s > 0$ and let Ψ be a non-identically vanishing Schwartz function on $\mathbb{R}^n$ supported in the set $\{\xi \in \mathbb{R}^n : |\xi| < 1/2\}$. Then for any fixed integer K and for any $t \in [0,1]$, we define

$$m_t(\xi) = \sum_{N=1}^{K} c_N \sum_{k \in \mathbb{N}^n : \ N 2^N < |k| < (N+1/2) 2^N} a_{N,k}(t) \, \Psi(2^N \xi - k),$$

where $a_{N,k}(t)$ denotes the sequence of Rademacher functions indexed by the elements of the countable set $\mathbb{N} \times \mathbb{N}^n$, and $c_N = 2^{-Ns} N^{-s}$.

Lemma 2. *Let $1 < r_1 < \infty$, $1 \leq r_2 \leq \infty$, and let ϕ be as in Theorem 1. Then*

$$\sup_{D \in \mathbb{Z}} \|\phi(\xi)\, m_t(2^D \xi)\|_{L_s^{r_1,r_2}} \leq C,$$

with C independent of t and K.

Proof. We first observe that it is enough to consider the case when $r_1 = r_2$; the general case then follows by real interpolation. For simplicity of notation, we write $r = r_1$ in what follows.

One can verify that $\phi(\xi) m_t(2^D \xi) = 0$ for all ξ if $D < -1$. We can thus assume that $D \geq -1$. For any such D, we denote

$$A_D = \Big\{ \xi \in \mathbb{R}^n : \frac{1}{2} - \frac{1}{4 \cdot 2^D} < |\xi| < 2 + \frac{3}{4 \cdot 2^D} \Big\}.$$

Using the version of the Kato–Ponce inequality from [6], we get

$$\begin{aligned} \|\phi(\xi)\, m_t(2^D \xi)\|_{L_s^r} &= \|(I-\Delta)^{s/2} [\phi(\xi)\, m_t(2^D\xi)]\|_{L^r} \\ &= \|(I-\Delta)^{s/2} [\phi(\xi)\, \chi_{A_D}(\xi)\, m_t(2^D\xi)]\|_{L^r} \\ &\lesssim \|(I-\Delta)^{s/2} [\phi(\xi)]\|_{L^\infty} \|\chi_{A_D}(\xi)\, m_t(2^D\xi)\|_{L^r} \\ &\quad + \|\phi(\xi)\|_{L^\infty} \|(I-\Delta)^{s/2} [\chi_{A_D}(\xi)\, m_t(2^D\xi)]\|_{L^r} \\ &\lesssim \|\chi_{A_D}(\xi)\, m_t(2^D\xi)\|_{L^r} + \|(-\Delta)^{s/2} [\chi_{A_D}(\xi)\, m_t(2^D\xi)]\|_{L^r}, \end{aligned} \tag{2.1}$$

since ϕ is a Schwartz function.

An elementary calculation yields that the last two terms in (2.1) are bounded by a constant independent of D, t and K. Indeed, using support properties of Ψ and the definition of the set A_D, we deduce that

$$\begin{aligned} &\chi_{A_D}(\xi)\, m_t(2^D\xi) \\ &\qquad = \sum_{N=\max(1,2^{D-1})}^{\min(K,2^{D+1})} \sum_{k \in \mathbb{N}^n:\ N2^N < |k| < (N+1/2)2^N} c_N a_{N,k}(t)\, \Psi(2^{N+D}\xi - k). \end{aligned} \tag{2.2}$$

Since $|c_N| \leq 1$ for all N and the functions $\Psi(2^{N+D}\xi - k)$ have pairwise disjoint supports in N and k (for the fixed D), we obtain

$$\|\chi_{A_D}(\xi)\, m_t(2^D\xi)\|_{L^r} \leq C(n, r, \Psi). \tag{2.3}$$

Further, we denote $\Phi = (-\Delta)^{s/2}\Psi$ and observe that

$$\begin{aligned} &(-\Delta)^{s/2} [\chi_{A_D}(\cdot)\, m_t(2^D \cdot)](\xi) \\ &\quad = \sum_{N=\max(1,2^{D-1})}^{\min(K,2^{D+1})} 2^{Ds} N^{-s} \sum_{k \in \mathbb{N}^n:\ N2^N < |k| < (N+1/2)2^N} a_{N,k}(t)\, \Phi(2^{N+D}\xi - k). \end{aligned}$$

Let $\alpha > n + n/r$ be an integer. Since Φ is a Schwartz function, we have

$$|\Phi(\xi)| \lesssim (1 + |\xi|)^{-\alpha}.$$

This yields

$$\begin{aligned}
&|(-\Delta)^{s/2} [\chi_{A_D}(\cdot)\, m_t(2^D \cdot)](\xi)| \\
&\qquad \lesssim \sum_{N=\max(1,2^{D-1})}^{\min(K,2^{D+1})} \sum_{k\in\mathbb{N}^n:\ N2^N<|k|<(N+1/2)2^N} (1 + |2^{N+D}\xi - k|)^{-\alpha} \\
&\qquad \approx \sum_{N=\max(1,2^{D-1})}^{\min(K,2^{D+1})} \int_{\{z\in\mathbb{R}^n:\ N2^N<|z+2^{N+D}\xi|<(N+1/2)2^N\}} (1 + |z|)^{-\alpha}\, dz.
\end{aligned}$$

Now, if $z \in \mathbb{R}^n$ satisfies $N2^N < |z + 2^{N+D}\xi| < (N + 1/2)2^N$ then it can be verified that $|z| > 2^N$ holds for all but three values of N (the exceptional N's are those close to $2^D|\xi|$). In addition, if $|\xi|$ is large (say, $|\xi| \geq 6$) and $N \leq 2^{D+1}$ then $|z| > 2^{N-2}|\xi|$. This yields

$$|(-\Delta)^{s/2} [\chi_{A_D}(\cdot)\, m_t(2^D \cdot)](\xi)| \leq C(n, \alpha, s, \Psi), \quad \xi \in \mathbb{R}^n, \tag{2.4}$$

and

$$|(-\Delta)^{s/2} [\chi_{A_D}(\cdot)\, m_t(2^D \cdot)](\xi)| \leq C(n, \alpha, s, \Psi)\, |\xi|^{n-\alpha}, \quad |\xi| \geq 6. \tag{2.5}$$

A combination of estimates (2.4) and (2.5) then implies

$$\|(-\Delta)^{s/2} [\chi_{A_D}(\xi)\, m_t(2^D \xi)]\|_{L^r} \leq C(n, r, s, \Psi). \tag{2.6}$$

Finally, combining estimates (2.1), (2.3) and (2.6), we obtain the desired conclusion. □

Proof of Theorem 1. We may assume, without loss of generality, that $p < 2$; the result in the case $p > 2$ will then follow by duality.

Let t, K and m_t be as described at the beginning of this section, and let φ be a Schwartz function such that $\varphi(\xi) = 1$ if $|\xi| \leq 2$. Define a function f via its Fourier transform by $\widehat{f}(\xi) = \varphi(\xi/K)$. Then $\widehat{f}(\xi) = 1$ if $|\xi| \leq 2K$. It is straightforward to verify that $m_t(\xi)$ is supported in the set $|\xi| < 2K$. Therefore, we have

$$m_t(\xi)\, \widehat{f}(\xi) = m_t(\xi),$$

and so

$$T_{m_t} f(x) = \sum_{N=1}^{K} c_N \sum_{k\in\mathbb{N}^n:\ N2^N<|k|<(N+1/2)2^N} a_{N,k}(t)\, 2^{-nN} (\mathcal{F}^{-1}\Psi) \left(\frac{x}{2^N}\right) e^{2\pi i x\cdot \frac{k}{2^N}}.$$

By Fubini's theorem and Khintchine's inequality, we obtain

$$\begin{aligned}
\int_0^1 \|T_{m_t} f(x)\|_{L^p}^p \, dt &= \int_{\mathbb{R}^n} \int_0^1 |T_{m_t} f(x)|^p \, dt \, dx \\
&\approx \int_{\mathbb{R}^n} \Big(\sum_{N=1}^K \sum_{k \in \mathbb{N}^n: \; N2^N < |k| < (N+1/2)2^N} c_N^2 \, 2^{-2nN} \Big| (\mathcal{F}^{-1}\Psi)\Big(\frac{x}{2^N}\Big) \Big|^2 \Big)^{p/2} dx \\
&\approx \int_{\mathbb{R}^n} \Big(\sum_{N=1}^K c_N^2 \, N^{n-1} \, 2^{-nN} \Big| (\mathcal{F}^{-1}\Psi)\Big(\frac{x}{2^N}\Big) \Big|^2 \Big)^{p/2} dx.
\end{aligned}$$

Let $A > 0$ be such that $\mathcal{F}^{-1}\Psi$ does not vanish in $\{y \in \mathbb{R}^n : A \le |y| < 2A\}$. Then

$$\begin{aligned}
&\int_0^1 \|T_{m_t} f(x)\|_{L^p}^p \, dt \\
&\quad \gtrsim \int_{\mathbb{R}^n} \Big(\sum_{N=1}^K c_N^2 \, N^{n-1} \, 2^{-nN} \Big| (\mathcal{F}^{-1}\Psi)\Big(\frac{x}{2^N}\Big) \Big|^2 \chi_{\{x: \; A \le |x|/2^N < 2A\}}(x) \Big)^{p/2} dx \\
&\quad \approx \sum_{N=1}^K c_N^p \, N^{(n-1)p/2} \, 2^{-nNp/2} \int_{\{x: \; A \le |x|/2^N < 2A\}} \Big| (\mathcal{F}^{-1}\Psi)\Big(\frac{x}{2^N}\Big) \Big|^p dx \\
&\quad \approx \sum_{N=1}^K c_N^p \, N^{(n-1)p/2} \, 2^{nN(1-p/2)} \int_{\{y: \; A \le |y| < 2A\}} \big| (\mathcal{F}^{-1}\Psi)(y) \big|^p dy \\
&\quad \approx \sum_{N=1}^K c_N^p \, N^{(n-1)p/2} \, 2^{nN(1-p/2)} \approx \sum_{N=1}^K N^{(n-1)p/2 - sp} \, 2^{N(n - np/2 - sp)} \\
&\quad = \sum_{N=1}^K N^{np-n-p/2},
\end{aligned}$$

where the last equality follows from (1.6). We observe that $np - n - p/2 > -1$ as

$$p > 1 > \frac{n-1}{n-1/2}.$$

Thus,

$$\int_0^1 \|T_{m_t} f(x)\|_{L^p}^p \, dt \gtrsim K^{np-n-p/2+1}. \tag{2.7}$$

Let us now estimate the L^p-norm of f. Since $f(x) = K^n (\mathcal{F}^{-1}\varphi)(Kx)$, we obtain

$$\begin{aligned}
\|f\|_{L^p}^p &= K^{np} \int_{\mathbb{R}^n} |(\mathcal{F}^{-1}\varphi)(Kx)|^p \, dx \\
&= K^{np-n} \int_{\mathbb{R}^n} |(\mathcal{F}^{-1}\varphi)(y)|^p \, dy \approx K^{np-n}.
\end{aligned} \tag{2.8}$$

Assume that inequality (1.7) is satisfied. Then, applying (1.7) with $m = m_t$, integrating with respect to t and using Lemma 2, we get

$$\int_0^1 \|T_{m_t} f(x)\|_{L^p}^p \, dt \leq C \, \|f\|_{L^p}^p,$$

which implies, via (2.7) and (2.8), that

$$K^{np-n-p/2+1} \leq C \, K^{np-n},$$

or, equivalently,

$$K^{1-p/2} \leq C. \tag{2.9}$$

As $p < 2$, we have $\lim_{K\to\infty} K^{1-p/2} = \infty$, which contradicts (2.9). The proof is complete. □

Acknowledgments. I am grateful to Andreas Seeger for useful discussions and to Loukas Grafakos for careful reading of this paper and valuable comments. I also thank the referees for their comments which helped to improve the paper.

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Received June 11, 2018. Published online December 17, 2019.

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