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# On compactness of commutators of multiplication and bilinear pseudodifferential operators and a new subspace of BMO

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**Abstract.** It is known that the compactness of the commutators of pointwise multiplication with bilinear homogeneous Calderón–Zygmund operators acting on product of Lebesgue spaces is characterized by the multiplying function being in the space CMO. This space is the closure in BMO of its subspace of smooth functions with compact support. It is shown in this work that for bilinear Calderón–Zygmund operators arising from smooth (inhomogeneous) bilinear Fourier multipliers or bilinear pseudodifferential operators, one can actually consider multiplying functions in a new subspace of BMO larger than CMO.

## 1. Introduction

The original purpose of this work was to investigate possible versions in the bilinear setting of a somewhat overlooked compactness result of Cordes [9] in the linear situation. In the process we also uncovered a subspace of BMO, which we will denote by MMO. This space does not seem to have been studied in the literature before and it could be of interests in its own. We will show here that the commutators of bilinear smooth multipliers and pseudodifferential operators with pointwise multiplications with functions in MMO (and actually in an even larger space) are compact bilinear operators when acting on products of Lebesgue spaces. We need to briefly recount part of the history of the subject of commutators of pointwise multiplication and singular integral operators to put our result in perspective.

Let  $b(M)$  denote the operator of multiplication by the function  $b$  in  $\mathbb{R}^n$ ,  $b(M)f = bf$  (wherever it is properly defined) and let  $\sigma(D)$  be the Fourier multiplier with symbol  $\sigma$  also in  $\mathbb{R}^n$ ,  $\widehat{\sigma(D)f} = \sigma\widehat{f}$ , where  $\widehat{f}$  is the Fourier transform of the function  $f$  (again defined for appropriate functions). Cordes [9] showed, in particular,

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that if  $b$  is bounded and continuous and satisfies a certain modulus of continuity at infinity (we will not need the precise definition), and  $\sigma$  satisfies for all multi-indices  $\alpha$  the classical differential estimates

$$(1.1) \quad |D^\alpha \sigma(\xi)| \lesssim_\alpha (1 + |\xi|)^{-|\alpha|},$$

then the commutator  $[\sigma(D), b(M)] = \sigma(D)b(M) - b(M)\sigma(D)$  is compact on  $L^p(\mathbb{R}^n)$  for all  $1 < p < \infty$ . It is of interest that this result preceded the better known boundedness and compactness results of the commutators of pointwise multiplications with classical singular integrals obtained, respectively, by Coifman, Rochberg, and Weiss [7] and Uchiyama [22]. See also Janson [13]. The boundedness in  $L^p(\mathbb{R}^n)$  of the commutators considered in [9] is immediate because multipliers satisfying (1.1) produce Calderón–Zygmund operators and the function  $b$  is bounded. Actually it is enough to assume (1.1) for all  $|\alpha| \leq m = [n/2] + 1$  to obtain boundedness and Cordes showed that having the estimates for all  $|\alpha| \leq 2m$  suffices for compactness. We will not track the exact number of derivatives needed in the results presented here. On the other hand, in [7]  $b$  is only assumed to be in BMO, while in [22] it is shown that if instead of  $\sigma(D)$  one considers a classical Calderón–Zygmund operator with non-degenerate homogeneous kernel, then compactness of the commutator holds if and only if  $b$  is in CMO. This last space is the closure in BMO of smooth functions with compact support and it is sometimes referred to as the space of functions with “continuous mean oscillation”. Note that the kernels of the Calderón–Zygmund operators arising from the multipliers in [9] have a fast decay at infinity but fail to satisfy the homogeneity considerations in [22].

Both Cordes and Uchiyama arrived at the spaces of functions  $b$  in their results through density arguments. Since the result of Cordes preceded the one in [7], he considered the closure in  $L^\infty$  of  $A_\infty$ , which is an algebra of functions whose derivatives vanish at infinity (see the next section for precise definitions of the function spaces mentioned in this introduction). Uchiyama, on the other hand, used the closure of  $C_c^\infty$  in BMO. The methods they used are very different too. While in [9] a calculus was developed to use properties of trace class operators, the Fréchet–Kolmogorov’s characterization of compact sets in Lebesgue spaces was used in [22]. It is not clear if the method in [9] admits any extension at all to the bilinear setting, so we will also use the Fréchet–Kolmogorov’s characterization in our approach. We will study the initial algebra of functions employed by Cordes, but viewed as a subspace of BMO not  $L^\infty$ . In fact, the space MMO is precisely the closure of  $A_\infty$  in the BMO topology.

In a preliminary version of the work presented here we proved our compactness result with  $b \in \text{MMO}$ . However, after further investigation we realized that a modification of our arguments allows us to work in and even larger subspace of BMO that we will denote by XMO. We will show that  $\text{CMO} \subsetneq \text{MMO} \subsetneq \text{XMO} \subset \text{VMO}$ , where  $\text{VMO} \subsetneq \text{BMO}$  is the space of functions with “vanishing mean oscillation”.<sup>1</sup>

The situation in the bilinear case is as follows. The boundedness of the commutators of pointwise multiplication with functions in BMO and general bilinear

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<sup>1</sup> For lack of a better name or choice of letter, we selected the notation MMO because ‘M’ is the middle letter in the Latin alphabet between the letters ‘C’ and ‘V’. On the other hand, we use the notation XMO for the larger space because we do not know if  $\text{XMO} = \text{VMO}$ .

Calderón-Zygmund operators was established by Pérez and Torres [16] when the target space is  $L^p$ , with  $1 < p < \infty$ ; and by Tang [18] and Lerner et al [15] for  $1/2 < p \leq 1$ . The compactness when the multiplying functions are in CMO was proved by Bényi and Torres [3] for  $1 < p < \infty$ , while an extension to the case  $1/2 < p \leq 1$  was recently observed by Torres, Xue, and Yan [20]. Moreover, for certain homogeneous bilinear Calderón-Zygmund operators, membership of the multiplying functions in CMO is necessary and sufficient for the compactness of the commutators, as proved by Chaffee et al [5]. We also guide the reader to this last work for additional references to a large and growing literature in the subject of commutators of multilinear singular integral operators.

In this article we will consider a particular type of bilinear Calderón-Zygmund operators  $T$ . We will look at the case of  $T: L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n) \rightarrow L^r(\mathbb{R}^n)$  for all  $1 < p, q < \infty$ , and  $1/p + 1/q = 1/r$ , for which we have the usual representation

$$T(f, g)(x) = \int_{\mathbb{R}^{2n}} K(x, y, z) f(y) g(z) dy dz,$$

for all  $f, g \in C_c^\infty$  and  $x \notin \text{supp } f \cap \text{supp } g$ ; with a kernel  $K$  satisfying the size and regularity conditions

$$(1.2) \quad |\partial^\alpha K(x, y, z)| \lesssim_\alpha (|x - y| + |x - z|)^{-2n - |\alpha|}$$

whenever  $x \neq y$  or  $x \neq z$ , for all multi-indices  $\alpha$  with  $|\alpha| \leq 1$ .

We will require, however, a more restrictive than usual property on  $K$ . Namely, that it satisfies the additional estimates

$$(1.3) \quad |\partial^\alpha K(x, y, z)| \lesssim_{\alpha, N} (|x - y| + |x - z|)^{-2n - N}$$

for all  $|\alpha| \leq 1$ ,  $N = 1, 2, 3$ , and  $|x - y| + |x - z| > 1$ .

Examples of operators with such type of kernels are provided by the (inhomogeneous) Coifman-Meyer bilinear Fourier multipliers defined by

$$T(f, g)(x) = \int_{\mathbb{R}^{2n}} \sigma(\xi, \eta) \widehat{f}(\xi) \widehat{g}(\eta) e^{ix(\xi + \eta)} d\xi d\eta,$$

where the multiplier satisfies

$$(1.4) \quad |D^\alpha \sigma(\xi, \eta)| \lesssim_\alpha (1 + |\xi| + |\eta|)^{-|\alpha|}$$

for all multi-indices  $\alpha$ . More general examples are also given by the bilinear pseudodifferential operators

$$T(f, g)(x) = \int_{\mathbb{R}^{2n}} \sigma(x, \xi, \eta) \widehat{f}(\xi) \widehat{g}(\eta) e^{ix(\xi + \eta)} d\xi d\eta$$

in the sometimes called  $\text{BS}_{1,0}^0$  class. That is, the symbol satisfies now the estimates

$$(1.5) \quad |D_x^\beta D_{\xi, \eta}^\alpha \sigma(x, \xi, \eta)| \lesssim_{\alpha, \beta} (1 + |\xi| + |\eta|)^{-|\alpha|}$$

for all multi-indices  $\alpha$  and  $\beta$ . See Coifman–Meyer [6], Kenig–Stein [14], Grafakos–Torres [11], and Bényi–Torres [2] for boundedness properties of these multipliers and pseudodifferential operators as particular examples of bilinear Calderón–Zygmund operators. The estimates on the corresponding kernels associated to them follow from the similar ones in the linear case. See e.g. [19] or in particular Theorem 5.1 in Bényi et al [1] for an explicit verification of (1.2) and (1.3). Actually for symbols satisfying (1.4) or (1.5), the estimates (1.2) and (1.3) on the corresponding kernels hold for all multi-indices  $\alpha$  and all  $N > 0$ . Under Sobolev type versions of (1.4), Hu [12] proved compactness results for commutators of Fourier multipliers with the multiplying function  $b$  also in CMO.

We can now state our main result.

**Theorem 1.1.** *Let  $T$  be a bilinear Calderón–Zygmund operator whose kernel satisfies (1.2) and (1.3). Then, for all  $1 < p, q < \infty$  and  $1/2 < r < \infty$  verifying  $1/p + 1/q = 1/r$ , and all  $b \in \text{XMO}$ , the commutators*

$$\begin{aligned} [T, b]_1(f, g)(x) &= (T(bf, g) - bT(f, g))(x), \\ [T, b]_2(f, g)(x) &= (T(f, bg) - bT(f, g))(x) \end{aligned}$$

*are compact bilinear operators from  $L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n)$  to  $L^r(\mathbb{R}^n)$ .*

In the next section we revisit certain characterizations of commonly used subspaces of BMO which we compare with the new MMO subspace. For completeness, we include in Section 3 several known facts about compact operators and commutators of Calderón–Zygmund operators in the bilinear setting. The reader familiar with those aspects of the subject may skip this section. In Section 4 we collect several technical results about translation invariant bilinear operators with positive kernels, while finally the proof of Theorem 1.1 is presented in Section 5.

## 2. Several closed subspaces of BMO

There has been some confusion in the literature about alternative characterizations of the original VMO space introduced by Sarason [17] in the torus and the one considered by Coifman and Weiss [8] in Euclidean space. We will denote by CMO the latter. The confusion arose in part because both spaces coincide on the compact case but are different in  $\mathbb{R}^n$ . Fortunately, Bourdaud has clarified in [4] many issues and properties related to these spaces. We will rely heavily on the results of his very nice article and refer the reader to it for further details as well as historical remarks.

Unless specifically indicated otherwise, the function spaces used in this article will be based on  $\mathbb{R}^n$ . We denote by BMO the John–Nirenberg space of function of bounded mean oscillation endowed with its usual norm. The space of  $C^\infty$  functions with compact support is denoted by  $C_c^\infty$ , while  $C_{\text{BMO}}^\infty$  denotes the subspace of  $C^\infty$  of function in BMO whose derivatives of all orders are also in BMO. Following [4], we define

$$\text{CMO} = \overline{C_c^\infty}^{\text{BMO}},$$

the closure of  $C_c^\infty$  in the BMO norm; and

$$\text{VMO} = \overline{C_{\text{BMO}}^\infty}^{\text{BMO}},$$

the closure of  $C_{\text{BMO}}^\infty$  in the BMO norm.

A very useful characterization of CMO due to Uchiyama is the following.

**Theorem 2.1** ([22], [4]). *A function  $b \in \text{BMO}$  is in CMO if and only if*

$$(2.1) \quad \lim_{a \rightarrow 0} \sup_{|Q|=a} \frac{1}{|Q|} \int_Q |b(x) - b_Q| dx = 0,$$

$$(2.2) \quad \lim_{a \rightarrow \infty} \sup_{|Q|=a} \frac{1}{|Q|} \int_Q |b(x) - b_Q| dx = 0,$$

$$(2.3) \quad \lim_{|y| \rightarrow \infty} \frac{1}{|Q|} \int_{Q+y} |b - b_{Q+y}| dx = 0, \quad \text{for each } Q.$$

Denoting by  $C_u$  the space of uniformly continuous functions in  $\mathbb{R}^n$ , the following characterization of VMO also holds.

**Theorem 2.2** (Theorem 5 in [4]).

$$\text{VMO} = \overline{C_u \cap \text{BMO}}^{\text{BMO}}.$$

The space  $A_\infty$  used in [9] and alluded to in the introduction, is defined to be

$$A_\infty = \{b \in C^\infty \cap L^\infty : D^\alpha b(x) \rightarrow 0, |x| \rightarrow \infty, \alpha \neq 0\}.$$

As we also already mentioned, Cordes considered the closure of  $A_\infty$  in  $L^\infty$ , but we find natural to consider a larger space,

$$\text{MMO} = \overline{A_\infty}^{\text{BMO}}.$$

Moreover, another natural extension is obtained by replacing  $L^\infty$  by BMO also in the definition of  $A_\infty$ . Namely, we define

$$B_\infty = \{b \in C^\infty \cap \text{BMO} : D^\alpha b(x) \rightarrow 0, |x| \rightarrow \infty, \alpha \neq 0\} \subset C_{\text{BMO}}^\infty$$

and

$$\text{XMO} = \overline{B_\infty}^{\text{BMO}}.$$

We obviously have

$$\text{CMO} \subset \text{MMO} \subset \text{XMO} \subset \text{VMO}.$$

Note also that

$$A_\infty \subset C_u \cap L^\infty.$$

It is shown in Proposition 9 in [4] that there exists a function  $b \in B_\infty \setminus L^\infty$  at a positive distance from  $L^\infty$  within BMO. It follows in particular that

$$\overline{C_u \cap L^\infty}^{\text{BMO}} \subsetneq \overline{C_u \cap \text{BMO}}^{\text{BMO}} = \text{VMO},$$

and that  $\text{MMO} \subsetneq \text{XMO}$ . To show that we also have  $\text{CMO} \subsetneq \text{MMO}$ , we present the following example (cf. Proposition 8 in [4]).

**Proposition 2.3.** *There exist a function  $f \in A_\infty \setminus \text{CMO}$ .*

*Proof.* Let  $\phi \in C_c^\infty$  be such that  $\text{supp } \phi \subset B = B(0, 1)$ ,  $\int_B \phi dx = 0$ , and  $\phi \not\equiv 0$ . Define

$$f(x) = \sum_{j=1}^{\infty} \phi\left(\frac{x - L_j}{j}\right),$$

where the vectors  $L_j$  are chosen so that  $\{|L_j|\}_j$  is an increasing sequence satisfying  $\lim_{j \rightarrow \infty} |L_j| = \infty$ , and the balls  $B_j = B(L_j, j)$  are disjoint. Clearly  $f \in C^\infty$  and is bounded. Moreover  $D^\alpha f(x) \rightarrow 0$  as  $|x| \rightarrow \infty$  for all  $|\alpha| > 0$ , so  $f \in A_\infty$ . However,

$$\begin{aligned} \frac{1}{|B_j|} \int_{B_j} \left| f(x) - \frac{1}{|B_j|} \int_{B_j} f dx \right| dx &= \frac{1}{|B_j|} \int_{B_j} \left| \phi\left(\frac{x - L_j}{j}\right) \right| dx \\ &= \frac{1}{|B|} \int_B |\phi(x)| dx = C > 0, \end{aligned}$$

which violates the condition (2.2)

$$\lim_{a \rightarrow \infty} \sup_{|Q|=a} \frac{1}{|Q|} \int_Q |f(x) - f_Q| dx = 0$$

in the alternative characterization of CMO.  $\square$

We conclude this section by pointing out that we do not know which one of the two possibilities  $\text{XMO} \subsetneq \text{VMO}$  or  $\text{XMO} = \text{VMO}$  holds, but we suspect the latter is true. This, however, plays no role in the proof of the compactness result that we will present.

### 3. Compact bilinear operators

We recall several facts about compact bilinear operators and how they are usually applied to study commutators of bilinear Calderón–Zygmund operators.

For a normed space  $X$ , let  $B_{r,X} = \{x \in X : \|x\| \leq r\}$  be the closed ball of radius  $r$  centered at the origin in  $X$ . The following is one of several equivalent definitions of compact bilinear operators.

**Definition 3.1** (Bilinear compact operators, see e.g. [3] and [20]). Let  $X$  and  $Y$  be Banach spaces and let  $Z$  be a Banach or quasi-Banach space. A bilinear operator  $T: X \times Y \rightarrow Z$  is called *compact* if  $T(B_{r_1,X} \times B_{r_2,Y})$  is pre-compact in  $Z$  for all  $r_1, r_2 > 0$ .

Let  $\mathcal{B}(X \times Y, Z)$  denote the collection of all bounded bilinear operators  $T: X \times Y \rightarrow Z$ , endowed with the operator norm, and let  $\mathcal{K}(X \times Y, Z)$  be the collection of all compact bilinear operators  $T: X \times Y \rightarrow Z$ . The following proposition is completely analogous to the linear case situation.

**Proposition 3.1** ([3], [20]).  $\mathcal{K}(X \times Y, Z)$  is a closed linear subspace of  $\mathcal{B}(X \times Y, Z)$ .

As mentioned in the introduction, the boundedness properties of the commutators of bilinear Calderón–Zygmund operators and function in BMO are very well-understood. The following quantitative result is crucial for our purposes.

**Theorem 3.2** (Boundedness of bilinear commutators; see [16], [18], and [15]). *Let  $T$  be a bilinear Calderón–Zygmund operator whose kernel satisfies (1.2) for all multi-indices  $\alpha$  with  $|\alpha| \leq 1$ . Then for  $j = 1, 2$ ,*

$$(3.1) \quad \|[T, b]_j(f, g)\|_{L^r} \lesssim \|b\|_{\text{BMO}} \|f\|_{L^p} \|g\|_{L^q},$$

for all  $1 < p, q < \infty$ , and  $1/p + 1/q = 1/r$ .

The dependance of the operator norm in (3.1) on  $\|b\|_{\text{BMO}}$  and Proposition 3.1 have been used in [3] and several articles that followed it to reduce the study of the compactness of commutators with  $b \in \text{CMO}$  to just  $b \in C_c^\infty$ . Similarly, in the proof of Theorem 1.1 we will reduce the arguments from  $b \in \text{XMO}$  to  $b \in B_\infty$ . In fact, since the operator norm of the commutator depends only on the BMO norm of  $b$  and the limit of compact operators is compact, then once compactness is proved for  $b$  in a subspace of BMO it immediately extends to that subspace closure in the BMO topology.

When dealing with the Lebesgue spaces the following classical Fréchet–Kolmogorov characterization of pre-compact sets, extended by Tsuji in the quasi-Banach case, has proved to be very convenient when studying commutators.

**Theorem 3.3** (Fréchet–Kolmogorov–Tsuji, see [23] and [21]). *A set  $\mathcal{F} \in L^r$  for  $0 < r < \infty$  is pre-compact if and only if*

$$\begin{aligned} \sup_{h \in \mathcal{F}} \|h\|_{L^r} &< \infty, \\ \lim_{A \rightarrow \infty} \|h\|_{L^r(\{|x| > A\})} &= 0 \quad \text{uniformly in } h \in \mathcal{F}, \\ \lim_{t \rightarrow 0} \|h(\cdot + t) - h(\cdot)\|_{L^r} &= 0 \quad \text{uniformly in } h \in \mathcal{F}. \end{aligned}$$

In the proof of Theorem 1.1 we will verify the conditions in the above theorem on the set  $\mathcal{F} = [T, b]_j(B_{r_1, L^p} \times B_{r_1, L^q}) \subset L^r$  with  $1/p + 1/q = 1/r$ . Moreover, as with previous works in the literature, by a limiting argument we may further reduce matters to the study of the image of  $[T, b]_j$  on functions  $(f, g) \in B_{r_1, L^p} \times B_{r_1, L^q}$  with both  $f$  and  $g$  in  $C_c^\infty$ , if so needed.

## 4. Translation invariant bilinear operators with positive kernels

We collect in this section several results about bilinear operators with positive kernels which will be needed in the proof of our main result. We will rely on results and ideas from the work of Grafakos and Soria [10].

**Definition 4.1** (Translation invariant positive operators). We say that a bilinear operator  $L$  acting on pairs of Lebesgue spaces is a *translation invariant positive*

operator if it admits a representation of the form

$$L(f, g)(x) = \int_{\mathbb{R}^{2n}} H(x-y, x-z) f(y) g(z) dy dz = \int_{\mathbb{R}^{2n}} H(y, z) f(x-y) g(x-z) dy dz$$

as an absolutely convergent integral with some kernel  $H \geq 0$ .

We will sometimes write  $L = L_H$  to identify the kernel of the operator. It is trivial to observe that if  $L_H$  is a translation invariant positive operator and  $H \in L^1(\mathbb{R}^{2n})$  then, by Minkowski's inequality,

$$L_H : L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n) \rightarrow L^r(\mathbb{R}^n)$$

for all  $1 \leq p, q \leq \infty$  and  $1/p + 1/q = 1/r \leq 1$ . Moreover, in such a case one has

$$\|L_H\| \leq \|H\|_{L^1(\mathbb{R}^{2n})}.$$

However, for  $1/2 < r < 1$  the situation is more complicated. This has been studied in [10]. They showed that the integrability of  $H$  is actually necessary for boundedness of positive operators, but it is not sufficient for the range  $1/2 < r < 1$ . They provide the following sufficient condition in such a case. We state the result in a way convenient for our purposes.

**Theorem 4.1** (Theorem 3.2 and Remark 3.3 in [10]). *Let  $1 < p, q < \infty$  and  $1/p + 1/q = 1/r > 1$ . Suppose that  $L_H$  is a translation invariant positive bilinear operator whose kernel  $H$  verifies the following monotonicity conditions:  $|H(y, z)| \leq |H(y', z)|$  whenever  $|y'| \leq |y|$ , and  $|H(y, z)| \leq |H(y, z')|$  whenever  $|z'| \leq |z|$ . If, in addition,*

$$C_r = \left( \int_{\mathbb{R}^{2n}} \frac{|H(y, z)|^r}{(|y||z|)^{n(1-r)}} dy dz \right)^{1/r} < \infty,$$

then  $L_H : L^p \times L^q \rightarrow L^r$  and  $\|L_H\|_{L^p \times L^q \rightarrow L^r} \lesssim C_r$ .

We will need a small modification of this result for some very specific functions  $H$  which do not necessarily satisfy the monotonicity conditions of Theorem 4.1 in all of  $\mathbb{R}^{2n}$ . It may be possible to obtain the following lemma from some other results in [10] via interpolation, but we find easier to obtain explicitly quantitative norm estimates on the operators we need to consider by essentially repeating the arguments employed in Theorem 3.2 in [10].

**Lemma 4.2.** *Let  $1 < p, q < \infty$  and  $1/p + 1/q = 1/r > 1$ . For  $A > 0$ ,  $0 < \delta \ll 1$ , and  $\epsilon > 0$ , consider the kernel functions*

$$(4.1) \quad H_1(y, z) = \begin{cases} \frac{1}{(|y| + |z|)^{2n+\epsilon}} & \text{if } |y| + |z| \geq A, \\ 0 & \text{otherwise;} \end{cases}$$

$$(4.2) \quad H_2(y, z) = \begin{cases} \frac{1}{(|y| + |z|)^{2n-\epsilon}} & \text{if } |y| + |z| \leq A, \\ 0 & \text{otherwise;} \end{cases}$$



$$(4.3) \quad H_3(y, z) = \begin{cases} \frac{1}{(|y| + |z|)^{2n}} & \text{if } \delta \leq |y| + |z| \leq 1, \\ 0 & \text{otherwise;} \end{cases}$$

and

$$(4.4) \quad H_4(y, z) = \begin{cases} \frac{1}{(|y| + |z|)^{2n-\epsilon}} & \text{if } |y| + |z| \leq A, \\ \frac{1}{(|y| + |z|)^{2n+\epsilon}} & \text{if } |y| + |z| > A. \end{cases}$$

Then, for  $l = 1, 2, 3, 4$ ,

$$L_{H_l}(f, g)(x) = \int_{\mathbb{R}^{2n}} H_l(y, z) f(x-y) g(x-z) dy dz$$

is bounded from  $L^p \times L^q$  to  $L^r$  and the following operator norm estimates hold:

$$(4.5) \quad \|L_{H_1}\|_{L^p \times L^q \rightarrow L^r} \lesssim A^{-\epsilon},$$

$$(4.6) \quad \|L_{H_2}\|_{L^p \times L^q \rightarrow L^r} \lesssim A^\epsilon,$$

$$(4.7) \quad \|L_{H_3}\|_{L^p \times L^q \rightarrow L^r} \lesssim \delta^{-n},$$

$$(4.8) \quad \|L_{H_4}\|_{L^p \times L^q \rightarrow L^r} \lesssim \max(A, A^{-1})^\epsilon,$$

*Proof.* To study  $L_{H_1}$  we cannot use Theorem 4.1 directly because  $H_1$  is not always decreasing in each variable, but as mentioned we borrow the arguments from the proof in [10]. Let  $I_j = \{y : 2^j < |y| \leq 2^{j+1}\}$  and  $J_k = \{z : 2^k < |z| \leq 2^{k+1}\}$ . We can write

$$|L_{H_1}(f, g)(x)| \leq \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \int_{I_j \times J_k} H_1(y, z) |f(x-y) g(x-z)| dy dz \leq \left( \sum_{(1)} + \sum_{(2)} + \sum_{(3)} \right),$$

where  $\sum_{(1)}$  represents the double sum of terms  $\int_{I_j \times J_k} H_1(y, z) |f(x-y) g(x-z)| dy dz$  with  $2^j \gtrsim A$  and  $2^k \gtrsim A$ ;  $\sum_{(2)}$  the one with  $2^j \lesssim A$  and  $2^k \gtrsim A$ ; and  $\sum_{(3)}$  the one with  $2^j \gtrsim A$  and  $2^k \lesssim A$ ; since the other terms are zero. Using the definition of  $H_1$  we can estimate

$$\begin{aligned} \sum_{(1)} &\lesssim \sum_{2^j \gtrsim A} \sum_{2^k \gtrsim A} \int_{I_j \times J_k} H_1(y, z) |f(x-y) g(x-z)| dy dz \\ &\lesssim \sum_{2^j \gtrsim A} \sum_{2^k \gtrsim A} 2^{-j(n+\epsilon/2)} 2^{-k(n+\epsilon/2)} \int_{I_j} |f(x-y)| dy \int_{J_k} |g(x-z)| dz. \end{aligned}$$

Computing the integral of the  $r$ -power of the above, bringing the  $r$  inside the sums (since  $r < 1$ ), and using Hölder's inequality twice, we get

$$\begin{aligned} \left\| \sum_{(1)} \right\|_{L^r}^r &\lesssim \sum_{2^j \gtrsim A} \sum_{2^k \gtrsim A} 2^{-jr(n+\epsilon/2)} 2^{-kr(n+\epsilon/2)} \\ &\quad \times \left( \int_{\mathbb{R}^n} \left( \int_{I_j} |f(x-y)| dy \right)^p dx \right)^{r/p} \left( \int_{\mathbb{R}^n} \left( \int_{J_k} |g(x-z)| dz \right)^q dx \right)^{r/q} \end{aligned}$$

$$\begin{aligned}
&\lesssim \sum_{2^j \gtrsim A} \sum_{2^k \gtrsim A} 2^{-jr(n+\epsilon/2)} 2^{-kr(n+\epsilon/2)} \\
&\quad \times \left( \int_{\mathbb{R}^n} \int_{I_j} |f(x-y)|^p dy 2^{jn(p-1)} dx \right)^{r/p} \left( \int_{\mathbb{R}^n} \int_{J_k} |g(x-z)|^q dz 2^{kn(q-1)} dx \right)^{r/q} \\
&\lesssim \sum_{2^j \gtrsim A} \sum_{2^k \gtrsim A} 2^{-jr(n+\epsilon/2)} 2^{-kr(n+\epsilon/2)} 2^{jnr} 2^{knr} \|f\|_{L^p}^r \|g\|_{L^q}^r \\
&\lesssim \sum_{2^j \gtrsim A} \sum_{2^k \gtrsim A} 2^{-jr\epsilon/2} 2^{-kr\epsilon/2} \|f\|_{L^p}^r \|g\|_{L^q}^r \lesssim A^{-r\epsilon} \|f\|_{L^p}^r \|g\|_{L^q}^r.
\end{aligned}$$

To control  $\sum_{(2)}$ , we use the same computations but with the estimate

$$H_1(y, z) \lesssim \frac{1}{2^{k(2n+\epsilon)}}$$

for  $(y, z)$  in the  $I_j \times J_k$  boxes appearing in the sum. Hence, we can arrive at

$$\begin{aligned}
\left\| \sum_{(2)} \right\|_{L^r}^r &\lesssim \sum_{2^j \lesssim A} \sum_{2^k \gtrsim A} 2^{-kr(2n+\epsilon)} 2^{jnr} 2^{knr} \|f\|_{L^p}^r \|g\|_{L^q}^r \\
&\lesssim A^{nr} A^{-r(n+\epsilon)} \|f\|_{L^p}^r \|g\|_{L^q}^r = A^{-r\epsilon} \|f\|_{L^p}^r \|g\|_{L^q}^r.
\end{aligned}$$

The estimate for  $\sum_{(3)}$  is completely symmetric to the one for  $\sum_{(2)}$ , so we skip the details. The norm estimate (4.5) now follows.

The computations for  $L_{H_2}$  are also completely analogous, but actually its boundedness and the norm estimate (4.6) can be directly obtained from Theorem 4.1 since  $H_2$  is decreasing in each variable separately.

For  $L_{H_3}$ , we observe that

$$\begin{aligned}
|L_{H_3}(f, g)(x)| &\leq \sum_{(4)} + \sum_{(5)} \\
&:= \left( \left( \sum_{\delta \lesssim 2^j \lesssim 1} \sum_{2^k \lesssim 1} \right) + \left( \sum_{2^j \lesssim 1} \sum_{\delta \lesssim 2^k \lesssim 1} \right) \right) \int_{I_j \times J_k} \frac{1}{(2^j + 2^k)^{2n}} |f(x-y)g(x-z)| dy dz,
\end{aligned}$$

since integration over other  $I_j \times J_k$  is zero. Then, proceeding as in the previous cases, we can arrive to

$$\begin{aligned}
\left\| \sum_{(4)} \right\|_{L^r}^r &\lesssim \sum_{\delta \lesssim 2^j \lesssim 1} \sum_{2^k \lesssim 1} 2^{-j2nr} 2^{jnr} 2^{knr} \|f\|_{L^p}^r \|g\|_{L^q}^r \\
&\lesssim \delta^{-nr} \|f\|_{L^p}^r \|g\|_{L^q}^r.
\end{aligned}$$

The estimate for  $\sum_{(5)}$  follows again by symmetry.

Finally notice that  $L_{H_4} = L_{H_1} + L_{H_2}$ .  $\square$

**Remark 4.2.** Note that since  $H_j \in L^1(\mathbb{R}^{2n})$  for  $j = 1, 2, 3, 4$ , the operators  $L_{H_j}$  are also bounded from  $L^p \times L^q$  to  $L^r$  when  $1 < p, q < \infty$  and  $1/p + 1/q = 1/r \leq 1$  and the operator norms have the same control above. Actually, when  $r \geq 1$  the operator norm of  $L_{H_3}$  is easily seen to be control by just  $|\log \delta|$ , but the estimate in (4.7) suffices for our purposes.

## 5. The proof of Theorem 1.1

*Proof.* It is enough to study the commutator in, say, the first entry which for simplicity we will denote by  $\mathcal{T}_b$ . As explained after Theorem 3.2 we may assume  $b \in B_\infty$ . We will actually show the following two statements:

a) *Given  $\epsilon > 0$ , there exists  $A > 0$  such that*

$$(5.1) \quad \|\mathcal{T}_b(f, g)\|_{L^r(\{|x|>A\})} \lesssim \epsilon \|f\|_{L^p} \|g\|_{L^q}.$$

b) *If  $|t| > 0$  is small enough, then also*

$$(5.2) \quad \|\mathcal{T}_b(f, g)(\cdot + t) - \mathcal{T}_b(f, g)(\cdot)\|_{L^r} \lesssim \epsilon \|f\|_{L^p} \|g\|_{L^q}.$$

The above estimates, combined with (3.1) give, via Theorem 3.3, the compactness result we want to prove. To verify (5.1) and (5.2), we will decompose over several regions the expressions in the  $L^r$  norms in the left side of the equations and control them by sums of operators to which we can apply the norm estimates obtained in Lemma 4.2.

To show (5.1) we start by splitting  $\mathbb{R}^{2n}$ , for each  $A \gg 1$  and  $x$  with  $|x| > A$ , into three subsets as follows:

$$\begin{aligned} E_1 &= \{(y, z) \in \mathbb{R}^{2n} : |x - y| + |x - z| \leq 1\}, \\ E_2 &= \{(y, z) \in \mathbb{R}^{2n} : 1 < |x - y| + |x - z| < A/2\}, \\ E_3 &= \{(y, z) \in \mathbb{R}^{2n} : |x - y| + |x - z| > A/2\}, \end{aligned}$$

and we write

$$\begin{aligned} |\mathcal{T}_b(f, g)(x)| &\leq \int_{E_1} |(b(x) - b(y)) K(x, y, z) f(y) g(z)| dy dz \\ &\quad + \int_{E_2} |(b(x) - b(y)) K(x, y, z) f(y) g(z)| dy dz \\ &\quad + \int_{E_3} |(b(x) - b(y)) K(x, y, z) f(y) g(z)| dy dz \\ &:= T_1(x) + T_2(x) + T_3(x). \end{aligned}$$

To estimate  $T_1$ , we note that there exists a point  $\xi$  on the segment  $\overline{xy}$ , such that  $|b(x) - b(y)| = |\nabla b(\xi)| |x - y|$  and, since  $|x| \approx |y| \approx |\xi|$ , we get using the size estimate on the kernel  $K$ ,

$$\begin{aligned} T_1(x) &\lesssim \sup_{|\xi|>A/2} |\nabla b(\xi)| \int_{|x-y|+|x-z|\leq 1} \frac{1}{(|x-y|+|x-z|)^{2n-1}} |f(y)g(z)| dy dz \\ &= \sup_{|\xi|>A/2} |\nabla b(\xi)| \int_{|y|+|z|\leq 1} \frac{1}{(|y|+|z|)^{2n-1}} |f(x-y)g(x-z)| dy dz. \end{aligned}$$

Therefore we can write

$$T_1(x) \lesssim \sup_{|\xi|>A/2} |\nabla b(\xi)| L_{H_2}(f, g)(x),$$

where  $H_2$  is positive kernel like the type considered in (4.2).

Note that  $\sup_{|\xi|>A/2} |\nabla b(\xi)| \rightarrow 0$  as  $A \rightarrow \infty$ . It follows then that given  $\epsilon > 0$  we obtain

$$\|T_1\|_{L^r(\{|x|>A\})} \lesssim \sup_{|\xi|>A/2} |\nabla b(\xi)| \|L_{H_2}(f, g)\|_{L^r} \lesssim \epsilon \|f\|_{L^p} \|g\|_{L^q}$$

if  $A$  is large enough (depending on  $\epsilon$  but not on  $f$  or  $g$ ).

For any point  $(y, z) \in E_2$  there still exists a point  $\xi$  in the segment  $\overline{xy}$ , such that  $|b(x) - b(y)| = |\nabla b(\xi)| |x - y|$  and  $|x| \approx |y| \approx |\xi|$ . Then, taking  $N = 2$  in (1.3),

$$\begin{aligned} T_2(x) &\lesssim \sup_{|\xi|>A/2} |\nabla b(\xi)| \int_{|x-y|+|x-z|>1} \frac{1}{(|x-y|+|x-z|)^{2n+1}} |f(y)g(z)| dy dz \\ &= \sup_{|\xi|>A/2} |\nabla b(\xi)| \int_{|y|+|z|>1} \frac{1}{(|y|+|z|)^{2n+1}} |f(x-y)g(x-z)| dy dz \\ &= \sup_{|\xi|>A/2} |\nabla b(\xi)| L_{H_1}(f, g)(x), \end{aligned}$$

where now  $H_1$  is positive kernel like the ones in (4.1). We obtain, as before,

$$\|T_2\|_{L^r(\{|x|>A\})} \lesssim \epsilon \|f\|_{L^p} \|g\|_{L^q}$$

for sufficiently large  $A$ .

To control the last term  $T_3(x)$ , we simply use (1.3) with  $N = 2$  to estimate

$$\begin{aligned} T_3(x) &\lesssim \|\nabla b\|_{L^\infty} \int_{|x-y|+|x-z|>A/2} \frac{1}{(|x-y|+|x-z|)^{2n+1}} |f(y)g(z)| dy dz \\ &\lesssim \int_{|y|+|z|>A/2} \frac{1}{(|y|+|z|)^{2n+1}} |f(x-y)g(x-z)| dy dz \\ &\lesssim L_{H_{1,A}}(f, g)(x), \end{aligned}$$

again with a kernel  $H_{1,A}$  as in (4.1), but now depending on  $A$ . We can finally compute

$$\|T_3\|_{L^r(\{|x|>A\})} \lesssim \|L_{H_{1,A}}(f, g)\|_{L^r(\{|x|>A\})} \lesssim \|L_{H_{1,A}}(f, g)\|_{L^r} \lesssim A^{-1} \|f\|_{L^p} \|g\|_{L^q},$$

which is again the right estimate when  $A \rightarrow \infty$ .

We are now going to show (5.2). We use Taylor's formula with integral remainder and write

$$b(x) - b(y) = \sum_{|\alpha|=1} \frac{D^\alpha b(y)}{\alpha!} (x-y)^\alpha + \sum_{|\alpha|=2} (x-y)^\alpha r_{2,\alpha}(x, y),$$

where

$$r_{2,\alpha}(x, y) = (-1)^2 \frac{2}{\alpha!} \int_0^1 s D^\alpha b((1-s)x + sy) ds,$$

We have then

$$\begin{aligned} \mathcal{T}_b(f, g)(x) &= \sum_{|\alpha|=1} \int_{\mathbb{R}^{2n}} D^\alpha b(y) \frac{(x-y)^\alpha}{\alpha!} K(x, y, z) f(y) g(z) dy dz \\ &\quad + \sum_{|\alpha|=2} \int_{\mathbb{R}^{2n}} (x-y)^\alpha r_{2,\alpha}(x, y) K(x, y, z) f(y) g(z) dy dz \\ &:= I(x) + J(x). \end{aligned}$$

For  $0 < \delta \ll 1$  and  $|t| < \delta/4$ , denote by  $E_4$  and  $E_5$  the sets

$$E_4 = \{(y, z) : |x-y| + |x-z| < \delta, |x+t-y| + |x+t-z| \leq 2\delta\}$$

and

$$E_5 = \{(y, z) : |x-y| + |x-z| > \delta, \text{ or } |x+t-y| + |x+t-z| > 2\delta\}.$$

Then,

$$\begin{aligned} &I(x+t) - I(x) \\ &= \sum_{|\alpha|=1} \int_{E_4} \frac{D^\alpha b(y)}{\alpha!} \left[ (x+t-y)^\alpha K(x+t, y, z) - (x-y)^\alpha K(x, y, z) \right] f(y) g(z) dy dz \\ &\quad + \sum_{|\alpha|=1} \int_{E_5} \frac{D^\alpha b(y)}{\alpha!} \left[ (x+t-y)^\alpha K(x+t, y, z) - (x-y)^\alpha K(x+t, y, z) \right. \\ &\quad \left. + (x-y)^\alpha K(x+t, y, z) - (x-y)^\alpha K(x, y, z) \right] f(y) g(z) dy dz. \end{aligned}$$

Using the fact that  $|D^\alpha b|$  is bounded, we can estimate

$$\begin{aligned} &|I(x+t) - I(x)| \\ &\lesssim \sum_{|\alpha|=1} \int_{E_4} |x+t-y|^{|\alpha|} |K(x+t, y, z)| |f(y) g(z)| dy dz \\ &\quad + \sum_{|\alpha|=1} \int_{E_4} |x-y|^{|\alpha|} |K(x, y, z)| |f(y) g(z)| dy dz \\ &\quad + \sum_{|\alpha|=1} \int_{E_5} |(x+t-y)^\alpha - (x-y)^\alpha| |K(x+t, y, z)| |f(y) g(z)| dy dz \\ &\quad + \sum_{|\alpha|=1} \int_{E_5} |x-y|^{|\alpha|} |K(x+t, y, z) - K(x, y, z)| |f(y) g(z)| dy dz \\ &:= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

By (1.2),

$$I_1 \lesssim \sum_{|\alpha|=1} \int_{|x+t-y|+|x+t-z| \leq 2\delta} \frac{1}{(|x+t-y| + |x+t-z|)^{2n-1}} |f(y) g(z)| dy dz,$$

and

$$I_2 \lesssim \sum_{|\alpha|=1} \int_{|x-y|+|x-z|\leq\delta} \frac{1}{(|x-y|+|x-z|)^{2n-1}} |f(y)g(z)| dy dz.$$

Hence, using what are by now familiar arguments involving Lemma 4.2, we can arrive to

$$\|I_j\|_{L^r} \lesssim \delta \|f\|_{L^p} \|g\|_{L^q},$$

for  $j = 1, 2$ . For  $I_3$  we further consider

$$E_6 = \{(y, z) : |x-y| \leq 2t, \delta < |x+t-y| + |x+t-z| \leq 1\},$$

$$E_7 = \{(y, z) : |x-y| \leq 2t, |x+t-y| + |x+t-z| > 1\},$$

$$E_8 = \{(y, z) : |x-y| > 2t, \delta < |x+t-y| + |x+t-z| \leq 1\},$$

$$E_9 = \{(y, z) : |x-y| > 2t, |x+t-y| + |x+t-z| > 1\}.$$

Note that we have  $|x-y| \leq 2$  in  $E_8$ , while  $|x-y| \leq 2|x+t-y|$  in  $E_9$ .

Appropriately using (1.2) and (1.3), we have

$$\begin{aligned} I_3 &\lesssim \sum_{|\alpha|=1} \left[ \int_{E_6} |(x+t-y)^\alpha - (x-y)^\alpha| |K(x+t, y, z)| |f(y)g(z)| dy dz \right. \\ &\quad \left. + \int_{E_7} |(x+t-y)^\alpha - (x-y)^\alpha| |K(x+t, y, z)| |f(y)g(z)| dy dz \right] \\ &\quad + \sum_{|\alpha|=1} \left[ \int_{E_8} |(x+t-y)^\alpha - (x-y)^\alpha| |K(x+t, y, z)| |f(y)g(z)| dy dz \right. \\ &\quad \left. + \int_{E_9} |(x+t-y)^\alpha - (x-y)^\alpha| |K(x+t, y, z)| |f(y)g(z)| dy dz \right] \\ &\lesssim \sum_{|\alpha|=1} \left[ \int_{\delta < |x+t-y|+|x+t-z| < 1} \frac{|t|}{(|x+t-y|+|x+t-z|)^{2n}} |f(y)g(z)| dy dz \right. \\ &\quad \left. + \int_{|x+t-y|+|x+t-z| > 1} \frac{|t|}{(|x+t-y|+|x+t-z|)^{2n+1}} |f(y)g(z)| dy dz \right] \\ &\quad + \sum_{|\alpha|=1} \left[ \int_{\delta < |x+t-y|+|x+t-z| < 1} \frac{|t|}{(|x+t-y|+|x+t-z|)^{2n}} |f(y)g(z)| dy dz \right. \\ &\quad \left. + \int_{|x+t-y|+|x+t-z| > 1} \frac{|t|}{(|x+t-y|+|x+t-z|)^{2n+1}} |f(y)g(z)| dy dz \right], \end{aligned}$$

and employing again Lemma 4.2,

$$\|I_3\|_{L^r} \lesssim |t| \delta^{-n} + |t| + |t| \delta^{-n} + |t|.$$

Consider now  $I_4$ . We note that if  $|t| < \delta/4$  then we have that in  $E_5$  it always hold  $|x-y| + |x-z| > \delta$ . Thus,  $|t| < |x-y|/2$  or  $|t| < |x-z|/2$ . In either case

$$|K(x+t, y, z) - K(x, y, z)| \lesssim \frac{|t|}{(|x-y|+|x-z|)^{2n+1}}.$$

There are two possible cases to study. If  $|x - y| + |x - z| < 1$ , then

$$I_4(x) \lesssim \sum_{|\alpha|=1} \int_{\delta < |x-y|+|x-z| < 1} \frac{|x-y|^{|\alpha|}|t|}{(|x-y|+|x-z|)^{2n+1}} |f(y)g(z)| dy dz,$$

which yields

$$\|I_4\|_{L^r} \lesssim |t| \frac{1}{\delta^n}.$$

On the other hand, if  $|x - y| + |x - z| > 1$ , then

$$I_4(x) \lesssim \sum_{|\alpha|=1} \int_{|x-y|+|x-z| > 1} \frac{|x-y|^{|\alpha|}|t|}{(|x-y|+|x-z|)^{2n+|\alpha|+1}} |f(y)g(z)| dy dz,$$

which leads to

$$\|I_4\|_{L^r} \lesssim |t|.$$

Recall that  $J(x)$ , is given by

$$J(x) = \sum_{|\alpha|=2} \frac{2}{\alpha!} \int_{\mathbb{R}^{2n}} (x-y)^\alpha K(x, y, z) \int_0^1 s D^\alpha b((1-s)x + sy) f(y) g(z) ds dy dz.$$

Let  $F_\alpha(x, y, z) = (x-y)^\alpha K(x, y, z)$ . We have

$$\begin{aligned} & |J(x+t) - J(x)| \\ & \lesssim \sum_{|\alpha|=2} \int_{\mathbb{R}^{2n}} |F_\alpha(x+t, y, z)| \int_0^1 |D^\alpha b((1-s)(x+t) + sy) - D^\alpha b((1-s)x + sy)| \\ & \quad \times |f(y)g(z)| ds dy dz \\ & + \sum_{|\alpha|=2} \int_{\mathbb{R}^{2n}} |F_\alpha(x+t, y, z) - F_\alpha(x, y, z)| \int_0^1 |D^\alpha b((1-s)x + sy)| |f(y)g(z)| ds dy dz \\ & =: \sum_{|\alpha|=2} J_{1,\alpha}(x) + \sum_{|\alpha|=2} J_{2,\alpha}(x). \end{aligned}$$

Because of the smoothness of  $b$  and the boundedness of its derivatives, and the properties (1.2) and (1.3) of  $K$ ,

$$\begin{aligned} J_{1,\alpha}(x) & \lesssim \|\nabla D^\alpha b\|_{L^\infty} |t| \int_{\mathbb{R}^{2n}} |x+t-y|^2 |K(x+t, y, z)| |f(y)+g(z)| dy dz \\ & \lesssim |t| \int_{|x+t-y|+|x+t-z| \leq 1} \frac{|x+t-y|^2}{(|x+t-y|+|x+t-z|)^{2n}} |f(y)+g(z)| dy dz \\ & \quad + |t| \int_{|x+t-y|+|x+t-z| > 1} \frac{|x+t-y|^2}{(|x+t-y|+|x+t-z|)^{2n+N}} |f(y)+g(z)| dy dz. \end{aligned}$$

If we now chose  $N = 3$ , we obtain

$$J_{1,\alpha}(x) \lesssim |t| \int_{\mathbb{R}^{2n}} H(y, z) |f(x+t-y)g(x+t-z)| dy dz,$$

where  $H$  is a kernel of type  $H_4$  in Lemma 4.2. It follows that

$$\|J_{1,\alpha}(f,g)\|_{L^r} \lesssim |t| \|f(t \cdot)\|_{L^p} \|g(t \cdot)\|_{L^q} = |t| \|f\|_{L^p} \|g\|_{L^q}.$$

Before we estimate the terms  $J_{2,\alpha}$ , we observe that if  $|x - y| + |x - z| \leq 4t$ , then

$$|F_\alpha(x+t, y, z) - F_\alpha(x, y, z)| \lesssim \frac{|x-y|^2}{(|x-y|+|x-z|)^{2n}} + \frac{|x+t-y|^2}{(|x+t-y|+|x+t-z|)^{2n}}.$$

On the other hand, if  $|x - y| + |x - z| > 4t$  and  $|x - y| + |x - z| \leq 1$ ,

$$\begin{aligned} & |F_\alpha(x+t, y, z) - F_\alpha(x, y, z)| \\ & \lesssim |(x+t-y)^\alpha - (x-y)^\alpha| |K(x+t, y, z)| + |(x-y)^\alpha| |K(x+t, y, z) - K(x, y, z)| \\ & \lesssim |t| \frac{(|x-y|+|x-z|)}{(|x-y|+|x-z|)^{2n}} + |t| \frac{|x-y|^2}{(|x-y|+|x-z|)^{2n+1}}. \end{aligned}$$

While if  $|x - y| + |x - z| > 1$ , which also means  $|x - y| + |x - z| > 4t$  since we are assuming  $|t| < \delta/4 \ll 1$ , we can estimate using (1.3),

$$\begin{aligned} & |F_\alpha(x+t, y, z) - F_\alpha(x, y, z)| \\ & \lesssim |(x+t-y)^\alpha - (x-y)^\alpha| |K(x+t, y, z)| + |(x-y)^\alpha| |K(x+t, y, z) - K(x, y, z)| \\ & \lesssim |t| \frac{(|x-y|+|x-z|)}{(|x-y|+|x-z|)^{2n+2}} + |t| \frac{|x-y|^2}{(|x-y|+|x-z|)^{2n+3}}. \end{aligned}$$

We finally have

$$\begin{aligned} J_{2,\alpha}(x) & \lesssim \int_{|x-y|+|x-z| \leq 4t} \frac{1}{(|x-y|+|x-z|)^{2n-2}} |f(y)g(z)| dy dz \\ & \quad + \int_{|x+t-y|+|x+t-z| \leq 6t} \frac{1}{(|x+t-y|+|x+t-z|)^{2n-2}} |f(y)g(z)| dy dz \\ & \quad + |t| \int_{|x-y|+|x-z| \leq 1} \frac{1}{(|x-y|+|x-z|)^{2n-1}} |f(y)g(z)| dy dz \\ & \quad + |t| \int_{|x-y|+|x-z| > 1} \frac{1}{(|x-y|+|x-z|)^{2n+1}} |f(y)g(z)| dy dz. \end{aligned}$$

Applying one more time Lemma 4.2 we conclude

$$\|J_{2,\alpha}(f,g)\|_{L^r} \lesssim (|t|^2 + |t|) \|f\|_{L^p} \|g\|_{L^q}.$$

Collecting all the  $L^r$  estimates, we see that given  $0 < \epsilon \ll 1$ , if we let  $\delta = \epsilon$  and  $|t| < \delta^{n+1}$ , then (5.2) follows.  $\square$

**Remark 5.1.** The interested reader could easily extend our result and proof to the case of  $m$ -linear operators, as it would only require notational complications. We also note that in the linear case our result considerably improves the one in Theorem  $C_p$  in [9], which is proved for Hörmander–Mihlin multipliers and  $b \in \overline{A_\infty}^{L^\infty}$ .



**Remark 5.2.** Starting with [3], there have been several works in the literature about compactness of higher order commutators of bilinear singular integrals. For example, one can consider commutators of the form  $[[T, b_1]_1, b_2]_2$  for two functions  $b_1, b_2$  in CMO. As mentioned by one of the reviewers, it may be possible to consider higher order commutators in the context of this article with  $b_j$ s in XMO, but we have not investigated this possibility. We would like to mention, however, that compactness results for higher order commutators often turn out to be easier to obtain than those for the first order commutator. This is because the factor  $(b_1(x) - b_1(y))(b_2(x) - b_2(z))$  present in the kernels of higher order commutators has now additional cancellation when the  $b_j$ s are smooth. We shall leave such investigations to the interested reader.

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