



On scattering for the cubic defocusing nonlinear Schrödinger equation on the waveguide $\mathbb{R}^2 \times \mathbb{T}$

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Abstract. In this article, we will show the scattering of the cubic defocusing nonlinear Schrödinger equation on the waveguide $\mathbb{R}^2 \times \mathbb{T}$ in H^1 . We first establish the linear profile decomposition in $H^1(\mathbb{R}^2 \times \mathbb{T})$ motivated by the linear profile decomposition of the mass-critical Schrödinger equation in $L^2(\mathbb{R}^2)$. Then by using the solution of the cubic resonant nonlinear Schrödinger system to approximate the nonlinear profile, we can prove scattering in H^1 by using the concentration-compactness/rigidity method.

1. Introduction

In this article, we consider the cubic nonlinear Schrödinger equation on $\mathbb{R}^2 \times \mathbb{T}$:

$$(1.1) \quad \begin{cases} i \partial_t u + \Delta_{\mathbb{R}^2 \times \mathbb{T}} u = |u|^2 u, \\ u(0) = u_0 \in H^1(\mathbb{R}^2 \times \mathbb{T}), \end{cases}$$

where $\Delta_{\mathbb{R}^2 \times \mathbb{T}}$ is the Laplace–Beltrami operator on $\mathbb{R}^2 \times \mathbb{T}$ and $u: \mathbb{R} \times \mathbb{R}^2 \times \mathbb{T} \rightarrow \mathbb{C}$ is a complex-valued function.

The equation (1.1) has the following conserved quantities:

$$\begin{aligned} \text{mass:} \quad \mathcal{M}(u(t)) &= \int_{\mathbb{R}_x^2 \times \mathbb{T}_y} |u(t, x, y)|^2 dx dy, \\ \text{energy:} \quad \mathcal{E}(u(t)) &= \int_{\mathbb{R}_x^2 \times \mathbb{T}_y} \frac{1}{2} |\nabla u(t, x, y)|^2 + \frac{1}{4} |u(t, x, y)|^4 dx dy. \end{aligned}$$

The equation (1.1) is a special case of the general nonlinear Schrödinger equations on the waveguide $\mathbb{R}^n \times \mathbb{T}^m$:

$$(1.2) \quad \begin{cases} i \partial_t u + \Delta_{\mathbb{R}^n \times \mathbb{T}^m} u = |u|^{p-1} u, \\ u(0) = u_0 \in H^1(\mathbb{R}^n \times \mathbb{T}^m), \end{cases}$$

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where $1 < p < \infty$, $m, n \in \mathbb{Z}$, and $m, n \geq 1$. This kind of equations arise as models in the study of nonlinear optics (propagation of laser beams through the atmosphere or in a plasma), especially in nonlinear optics of telecommunications [21], [22].

We are interested in the range of p for wellposedness and scattering of (1.2) on $\mathbb{R}^n \times \mathbb{T}^m$. On one hand, the wellposedness is intuitively determined by the local geometry of the manifold $\mathbb{R}^n \times \mathbb{T}^m$. Because the manifold is locally just $\mathbb{R}^n \times \mathbb{R}^m$, the wellposedness is believed to be the same as the Euclidean case. That is, when $1 < p \leq 1 + 4/(m + n - 2)$ the wellposedness is expected. Just as the Euclidean case, we call the equation energy-subcritical when $1 < p < 1 + 4/(m + n - 2)$, $m, n \geq 1$ and energy-critical when $p = 1 + 4/(m + n - 2)$, $m + n \geq 3$, $m, n \geq 1$. On the other hand, when the solution exists globally, scattering is expected to be determined by the asymptotic volume growth of ball with radius r in the manifold $\mathbb{R}^n \times \mathbb{T}^m$ when $r \rightarrow \infty$. From the heuristic that linear solutions with frequency $\sim N$ initially localized around the origin will disperse at a time t in the ball of radius $\sim Nt$, and the volume of the ball is asymptotically $(Nt)^n$, as $t \rightarrow \infty$. This is the same as in the \mathbb{R}^n case, so the linear solution is expected to decay at a rate $\sim t^{-n/2}$ when the time t is large enough and based on the scattering theory on \mathbb{R}^n , the solution of (1.2) is expected to scatter for $p \geq 1 + 4/n$. Furthermore, modified scattering in the small data case is expected for $1 + 2/n < p < 1 + 4/n$ when $n \geq 2$ or $2 < p < 5$ when $n = 1$. Therefore, regarding heuristic on the wellposedness and scattering, we expect the solution of (1.2) globally exists and scatters in the range $1 + 4/n \leq p \leq 1 + 4/(m + n - 2)$, $m = 1, 2$. For $1 + 2/n < p < 1 + 4/n$ when $n \geq 2$ or $2 < p < 5$ when $n = 1$, modified scattering is expected as in the Euclidean space case for small data.

The nonlinear Schrödinger equations on the waveguide have been intensively studied in the last decades. There are a lot of results on the global wellposedness for large data in the subcritical case or small data in the critical case. When $n = m = 1$, H. Takaoka and N. Tzvetkov [23] proved global wellposedness for any data in L^2 when $1 < p < 3$ and global wellposedness for small data in L^2 when $p = 3$. S. Herr, D. Tataru and N. Tzvetkov [12] proved global wellposedness for small data in H^s , $s \geq 1$, in the cases $\mathbb{R}^2 \times \mathbb{T}^2$, $\mathbb{R}^3 \times \mathbb{T}$ when $p = 3$. Recently, Z. Hani, B. Pausader, N. Tzvetkov and N. Visciglia [11] considered the cubic nonlinear Schrödinger equation posed on the spatial domain $\mathbb{R} \times \mathbb{T}^m$, where $1 \leq m \leq 4$. They proved modified scattering and constructed modified wave operators for small initial and final data. Scattering for the energy subcritical nonlinear Schrödinger equation on $\mathbb{R}^n \times \mathbb{T}$ in H^1 was shown by N. Tzvetkov and N. Visciglia [28] when $n \geq 1$ and $1 + 4/n < p < 1 + 4/(n - 1)$ by Morawetz estimates. For large data in the critical case, A. D. Ionescu and B. Pausader [15] proved global wellposedness in H^1 for the cubic defocusing nonlinear Schrödinger equation on $\mathbb{R} \times \mathbb{T}^3$ based on profile decompositions on product manifolds. Similar decompositions on hyperbolic spaces and torus have been established in [14], [13]. When $n = 1$, $m = 2$ and $p = 5$, the equation (1.2) is both energy-critical in the sense of scaling and mass-critical in the sense of volume growth. The global wellposedness and scattering was established by Z. Hani and B. Pausader [10] under the assumption of the scattering of the

quintic resonant system

$$i \partial_t u_j + \Delta_{\mathbb{R}} u_j = \sum_{(j_1, j_2, j_3, j_4, j_5) \in \mathcal{Q}(j)} u_{j_1} \bar{u}_{j_2} u_{j_3} \bar{u}_{j_4} u_{j_5}, \quad j \in \mathbb{Z}^2,$$

where

$$\mathcal{Q}(j) = \{(j_1, j_2, j_3, j_4, j_5) \in (\mathbb{Z}^2)^5 : j_1 - j_2 + j_3 - j_4 + j_5 = j, \\ |j_1|^2 - |j_2|^2 + |j_3|^2 - |j_4|^2 + |j_5|^2 = |j|^2\}.$$

The global well-posedness and scattering of this resonant system has been proved by X. Cheng, Z. Guo, and Z. Zhao [5] recently, therefore the scattering result of Z. Hani and B. Pausader is unconditional. We also refer to [26], [27] on the well-posedness and scattering of the nonlinear Schrödinger equation on general waveguide $\mathbb{R}^n \times \mathbb{M}^m$, where \mathbb{M}^m is a compact m -dimensional Riemann manifold.

Our main result addresses the scattering for (1.1) in $H^1(\mathbb{R}^2 \times \mathbb{T})$.

Theorem 1.1 (Scattering in $H^1(\mathbb{R}^2 \times \mathbb{T})$). *For any initial data $u_0 \in H^1(\mathbb{R}^2 \times \mathbb{T})$, there exists a solution $u \in C_t^0 H_{x,y}^1(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{T})$ that is global and scatters in the sense that there exist $u^\pm \in H^1(\mathbb{R}^2 \times \mathbb{T})$ such that*

$$\|u(t) - e^{it\Delta_{\mathbb{R}^2 \times \mathbb{T}}} u^\pm\|_{H^1(\mathbb{R}^2 \times \mathbb{T})} \rightarrow 0, \quad \text{as } t \rightarrow \pm\infty.$$

The proof of Theorem 1.1 is based on the concentration compactness/rigidity method developed by C. E. Kenig and F. Merle [16] and the techniques of approximate profiles by Z. Hani and B. Pausader [10]. However, there are some major differences due to the sub-criticality. First, because the equation (1.1) is energy-subcritical, it suffices for us to work in simpler functional spaces to avoid the introduction of complicated ‘Z-norm’ in [10] which is necessary in the energy-critical case. Since there are more Euclidean directions in our case, stronger dispersion for the linear propagator is available. In fact, a global Strichartz estimate was given in [27], [28]:

$$\|e^{it\Delta_{\mathbb{R}^2 \times \mathbb{T}}} f\|_{L_{t,x}^4 H_y^1 \cap L_t^4 W_x^{1,4} L_y^2(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{T})} \lesssim \|f\|_{H_{x,y}^1}.$$

Therefore, the natural candidate functional space for the well-posedness is $H_{x,y}^1(\mathbb{R}^2 \times \mathbb{T})$. Meanwhile, it is easily seen that the $L_{t,x}^4 H_y^1 \cap L_t^4 W_x^{1,4} L_y^2(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{T})$ norm is a natural spacetime norm for scattering because its finiteness implies scattering even though the ‘Z-norm’ also suffices. Second, the sub-criticality allows us an ϵ loss of regularity in the linear profile decomposition. In fact, to proceed with a concentration-compactness/rigidity argument, a linear profile decomposition is established to describe the defect of the compactness of

$$e^{it\Delta_{\mathbb{R}^2 \times \mathbb{T}}} : H_{x,y}^1(\mathbb{R}^2 \times \mathbb{T}) \hookrightarrow L_{t,x}^4 H_y^1 \cap L_t^4 W_x^{1,4} L_y^2(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{T}).$$

For simplicity, we consider the defect of the compactness of the embedding

$$e^{it\Delta_{\mathbb{R}^2 \times \mathbb{T}}} : H_{x,y}^1(\mathbb{R}^2 \times \mathbb{T}) \hookrightarrow L_{t,x}^4 H_y^1(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{T}).$$

However, by the fact that $H_y^1(\mathbb{T}) \hookrightarrow H_y^1(\mathbb{T})$ cannot be compact even after module any symmetry transformations such as translations or dilations, we cannot expect a linear profile decomposition in $H_{x,y}^1$ with linear propagation of the remainder small in $L_{t,x}^4 H_y^1$. Luckily, we observe that a weaker space-time norm, namely $L_{t,x}^4 H_y^{1-\epsilon_0}$ ($0 < \epsilon_0 < 1/2$), is enough to show the scattering due to the sub-criticality. Therefore, the embedding in consideration is

$$e^{it\Delta_{\mathbb{R}^2 \times \mathbb{T}}} : H_{x,y}^1(\mathbb{R}^2 \times \mathbb{T}) \hookrightarrow L_{t,x}^4 H_y^{1-\epsilon_0}(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{T}).$$

We can establish a linear profile decomposition whose argument is mainly based on the linear profile decomposition of the Schrödinger equation in $L^2(\mathbb{R}^2)$. Note that there is no scaling in the direction of torus in our linear profile decomposition. Moreover, the scaling to zero and Galilean transformation in x -direction are prevented by the boundedness of $H_{x,y}^1(\mathbb{R}^2 \times \mathbb{T})$ norm. Third, in the reduction to the almost-periodic solution, we need to consider the cubic resonant nonlinear Schrödinger system

$$\begin{cases} i\partial_t u_j + \Delta_{\mathbb{R}^2} u_j = \sum_{j_1, j_2, j_3 \in \mathcal{R}(j)} u_{j_1} \bar{u}_{j_2} u_{j_3}, \\ u_j(0) = u_{0,j}, \quad j \in \mathbb{Z}, \end{cases}$$

where $\mathcal{R}(j) = \{j_1, j_2, j_3 \in \mathbb{Z} : j_1 - j_2 + j_3 = j, |j_1|^2 - |j_2|^2 + |j_3|^2 = |j|^2\}$. The global well-posedness and scattering for the system has been established by Yang and Zhao [29] by the method of [8]. Therefore, our whole argument is unconditional.

Remark 1.2. Most of the argument in the proof of Theorem 1.1 does not rely on the structure of \mathbb{T} in the manifold $\mathbb{R}^2 \times \mathbb{T}$. Thus, Theorem 1.1 may be generalized to the cubic nonlinear Schrödinger equation on $\mathbb{R}^2 \times \mathbb{M}$ under the assumption of the scattering for the corresponding resonant system, where \mathbb{M} is a closed curve as in [27].

Remark 1.3 (Recent progress on the nonlinear Schrödinger equation on waveguide). After our article was completed, we learned that global wellposedness and scattering for the cubic nonlinear Schrödinger equation on $\mathbb{R}^2 \times \mathbb{T}^2$ is proved in the recent work of Z. Zhao [30]. Z. Zhao [31] also considered the quintic nonlinear Schrödinger equation posed on $\mathbb{R}^3 \times \mathbb{T}$ and $\mathbb{R}^2 \times \mathbb{T}$. Recently, X. Cheng, Z. Guo, and Z. Zhao [5] also proved the scattering of the quintic nonlinear Schrödinger equation posed on $\mathbb{R} \times \mathbb{T}$, with an argument similar to that in this article. They also prove the scattering on the corresponding quintic resonant nonlinear Schrödinger system by using the argument in [9], and also prove the scattering of the corresponding quintic resonant nonlinear Schrödinger system arising in [10].

The rest of the paper is organized as follows. After introducing some notations and preliminaries, we give the local wellposedness and small data scattering in Section 2. We also give the stability theory in this section. In Section 3, we derive the linear profile decomposition for data in $H^1(\mathbb{R}^2 \times \mathbb{T})$ and analyze the nonlinear profiles. In Section 4, we reduce the non-scattering in H^1 to the existence of an almost-periodic solution, and show the extinction of such an almost-periodic solution in Section 5.

1.1. Notation and preliminaries

We will use the notation $X \lesssim Y$ whenever there exists some constant $C > 0$ so that $X \leq CY$. Similarly, we will use $X \sim Y$ if $X \lesssim Y \lesssim X$.

We define the torus to be $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$. For any $I \subset \mathbb{R}$, and $u(t, x, y) : I \times \mathbb{R}^2 \times \mathbb{T} \rightarrow \mathbb{C}$, we define the space-time norms

$$\begin{aligned} \|u\|_{L_t^p L_x^q L_y^2(I \times \mathbb{R}^2 \times \mathbb{T})} &= \left\| \left(\int_{\mathbb{T}} |u(t, x, y)|^2 dy \right)^{1/2} \right\|_{L_x^q(\mathbb{R}^2)} \Big\|_{L_t^p(I)}, \\ \|u\|_{H_{x,y}^1} &= \|\langle \nabla_x \rangle u\|_{L_{x,y}^2} + \|\langle \nabla_y \rangle u\|_{L_{x,y}^2}. \end{aligned}$$

We will frequently use the partial Fourier transform and partial space-time Fourier transform: for $f(x, y) : \mathbb{R}^2 \times \mathbb{T} \rightarrow \mathbb{C}$,

$$\mathcal{F}_x f(\xi, y) = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{-ix\xi} f(x, y) dx.$$

Given $H : \mathbb{R} \times \mathbb{R}^2 \times \mathbb{T} \rightarrow \mathbb{C}$, we denote the partial space-time Fourier transform as

$$\mathcal{F}_{t,x} H(\omega, \xi, y) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}} \int_{\mathbb{R}^2} e^{i\omega t - i\xi x} H(t, x, y) dx dt.$$

We also define the partial Littlewood–Paley projectors $P_{\leq N}^x$ and $P_{\geq N}^x$ as follows: fix a real-valued radially symmetric bump function $\varphi(\xi)$ satisfying

$$\varphi(\xi) = \begin{cases} 1, & |\xi| \leq 1, \\ 0, & |\xi| \geq 2, \end{cases}$$

and for any dyadic number $N \in 2^{\mathbb{Z}}$, let

$$\begin{aligned} \mathcal{F}_x(P_{\leq N}^x f)(\xi, y) &= \varphi\left(\frac{\xi}{N}\right)(\mathcal{F}_x f)(\xi, y), \\ \mathcal{F}_x(P_{\geq N}^x f)(\xi, y) &= \left(1 - \varphi\left(\frac{\xi}{N}\right)\right)(\mathcal{F}_x f)(\xi, y). \end{aligned}$$

We now define the discrete nonisotropic Sobolev space. For a sequence $\vec{\phi} = \{\phi_k\}_{k \in \mathbb{Z}}$ of real-variable functions, we define

$$\begin{aligned} \|\vec{\phi}\|_{h^{s_2}} &:= \left(\sum_{k \in \mathbb{Z}} \langle k \rangle^{2s_2} |\phi_k|^2 \right)^{1/2}, \\ H_x^{s_1} h^{s_2} &= \left\{ \vec{\phi} = \{\phi_k\} : \|\vec{\phi}\|_{H^{s_1} h^{s_2}} = \|\|\vec{\phi}\|_{h^{s_2}}\|_{H_x^{s_1}} < \infty \right\}, \end{aligned}$$

where $s_1, s_2 \geq 0$. In particular, when $s_1 = 0$, we denote the space $H_x^{s_1} h^{s_2}$ to be $L_x^2 h^{s_2}$. For $\psi \in L_x^2 H_y^1(\mathbb{R}^2 \times \mathbb{T})$, we have the vector $\vec{\psi} = \{\psi_k\} \in L_x^2 h^1$, where ψ_k is the sequence of periodic Fourier coefficients of ψ defined by

$$\psi_k(x) = \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{T}} \psi(x, y) e^{-iky} dy.$$

Finally, for a space-time vector value function $\vec{f}(t, x) = \{f_k(t, x)\}$, we define $\|\vec{f}\|_{h^s L_t^p L_x^q} := \left(\sum \langle k \rangle^{2s} \|f_k\|_{L_t^p L_x^q}^2 \right)^{1/2}$. Throughout the article, $0 < \epsilon_0 < 1/2$ is some fixed number.

2. Local wellposedness and small data scattering

In this section, we will review the local wellposedness and small data scattering, that is Theorem 2.3 and Theorem 2.5. These results have been established in [27], [28]. We also give the stability theory which will be used in showing the existence of a critical element in Section 4.

We first recall the following Strichartz estimate, which is established in [27].

Proposition 2.1 (Strichartz estimate).

$$\begin{aligned} & \|e^{it\Delta_{\mathbb{R}^2 \times \mathbb{T}}} f\|_{L_t^p L_x^q L_y^2} \lesssim \|f\|_{L_{x,y}^2(\mathbb{R}^2 \times \mathbb{T}_y)}, \\ & \left\| \int_0^t e^{i(t-s)\Delta_{\mathbb{R}^2 \times \mathbb{T}}} F(s, x, y) \, ds \right\|_{L_t^p L_x^q L_y^2} \lesssim \|F\|_{L_t^{\tilde{p}'} L_x^{\tilde{q}'} L_y^2}, \end{aligned}$$

where (p, q) , (\tilde{p}, \tilde{q}) satisfy $2/p + 2/q = 1$, $2/\tilde{p} + 2/\tilde{q} = 1$, and $2 < p, \tilde{p} \leq \infty$.

By the fact that $H_y^{1-\epsilon_0}(\mathbb{T})$ is an algebra and Hölder's inequality, we have the following nonlinear estimate, which is useful in showing the local wellposedness.

Proposition 2.2 (Nonlinear estimate).

$$\|u_1 u_2 u_3\|_{L_t^{4/3} L_x^{4/3} H_y^{1-\epsilon_0}} \lesssim \|u_1\|_{L_t^4 L_x^4 H_y^{1-\epsilon_0}} \|u_2\|_{L_t^4 L_x^4 H_y^{1-\epsilon_0}} \|u_3\|_{L_t^4 L_x^4 H_y^{1-\epsilon_0}}.$$

By the Strichartz estimate and the nonlinear estimate, we can give the local wellposedness and small data scattering in $L_x^2 H_y^1(\mathbb{R}^2 \times \mathbb{T})$ and $H_{x,y}^1(\mathbb{R}^2 \times \mathbb{T})$ easily.

Theorem 2.3 (Local wellposedness). *For any $E > 0$, suppose that $\|u_0\|_{L_x^2 H_y^1(\mathbb{R}^2 \times \mathbb{T})} \leq E$. There exists $\delta_0 = \delta_0(E) > 0$ such that if*

$$\|e^{it\Delta_{\mathbb{R}^2 \times \mathbb{T}}} u_0\|_{L_t^4 L_x^4 H_y^{1-\epsilon_0}(I \times \mathbb{R}^2 \times \mathbb{T})} \leq \delta_0,$$

with I a time interval, then there exists a unique solution $u \in C_t^0 L_x^2 H_y^1(I \times \mathbb{R}^2 \times \mathbb{T})$ of (1.1) satisfying

$$\|u\|_{L_t^4 L_x^4 H_y^{1-\epsilon_0}} \leq 2 \|e^{it\Delta_{\mathbb{R}^2 \times \mathbb{T}}} u_0\|_{L_t^4 L_x^4 H_y^{1-\epsilon_0}}, \quad \|u\|_{L_t^\infty L_x^2 H_y^1} \leq C \|u_0\|_{L_x^2 H_y^1}.$$

Moreover, if $u_0 \in H_{x,y}^1(\mathbb{R}^2 \times \mathbb{T})$, then $u \in C_t^0 H_{x,y}^1(I \times \mathbb{R}^2 \times \mathbb{T})$, with

$$\|u\|_{L_t^\infty H_{x,y}^1} \leq C(E) \|u_0\|_{H_{x,y}^1}, \quad \|u\|_{L_t^4 L_x^4 H_y^1 \cap L_t^4 W_x^{1,4} L_y^2} \leq C(\|u_0\|_{H^1}).$$

Arguing as in [4], [25], we can easily obtain the global wellposedness by the Strichartz estimate together with the conservation of mass and energy:

Theorem 2.4 (Global wellposedness in H^1). *For any $E > 0$, if $\|u_0\|_{H_{x,y}^1(\mathbb{R}^2 \times \mathbb{T})} \leq E$, there exists a unique global solution $u \in C_t^0 H_{x,y}^1(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{T})$ of (1.1) satisfying*

$$\|u\|_{L_t^\infty H_{x,y}^1} \leq C(E) \|u_0\|_{H_{x,y}^1}.$$

Theorem 2.5 (Small data scattering in $L_x^2 H_y^1$). *There exists $\delta > 0$ such that if $u_0 \in H_{x,y}^1$ and $\|u_0\|_{L_x^2 H_y^1(\mathbb{R}_x^2 \times \mathbb{T}_y)} \leq \delta$, (1.1) has a unique global solution $u(t, x, y) \in C_t^0 L_x^2 H_y^1 \cap L_t^4 L_x^4 H_y^1$ and there exist $u^\pm \in L_x^2 H_y^1(\mathbb{R}^2 \times \mathbb{T})$ such that*

$$\|u(t, x, y) - e^{it\Delta_{\mathbb{R}^2 \times \mathbb{T}}} u^\pm(x, y)\|_{L_x^2 H_y^1} \rightarrow 0, \quad \text{as } t \rightarrow \pm\infty.$$

We now give the stability theory in $L_x^2 H_y^{1-\epsilon_0}(\mathbb{R}^2 \times \mathbb{T})$ as in [18].

Theorem 2.6 (Stability theory). *Let I be a compact interval and let \tilde{u} be an approximate solution to $i\partial_t u + \Delta_{\mathbb{R}^2 \times \mathbb{T}} u = |u|^2 u$ in the sense that $i\partial_t \tilde{u} + \Delta_{\mathbb{R}^2 \times \mathbb{T}} \tilde{u} = |\tilde{u}|^2 \tilde{u} + e$ for some function e . Assume that*

$$\|\tilde{u}\|_{L_t^\infty L_x^2 H_y^{1-\epsilon_0}} \leq M, \quad \|\tilde{u}\|_{L_t^4 L_x^4 H_y^{1-\epsilon_0}} \leq L$$

for some positive constants M and L . Let $t_0 \in I$ and let $u(t_0)$ obey

$$\|u(t_0) - \tilde{u}(t_0)\|_{L_x^2 H_y^{1-\epsilon_0}} \leq M'$$

for some $M' > 0$. Moreover, assume the smallness conditions

$$\|e^{i(t-t_0)\Delta_{\mathbb{R}^2 \times \mathbb{T}}}(u(t_0) - \tilde{u}(t_0))\|_{L_t^4 L_x^4 H_y^{1-\epsilon_0}} + \|e\|_{L_t^{4/3} L_x^{4/3} H_y^{1-\epsilon_0}} \leq \epsilon,$$

for some $0 < \epsilon \leq \epsilon_1$, where $\epsilon_1 = \epsilon_1(M, M', L) > 0$ is a small constant. Then, there exists a solution u to $i\partial_t u + \Delta_{\mathbb{R}^2 \times \mathbb{T}} u = |u|^2 u$ on $I \times \mathbb{R}^2 \times \mathbb{T}$ with initial data $u(t_0)$ at time $t = t_0$ satisfying

$$\|u - \tilde{u}\|_{L_t^4 L_x^4 H_y^{1-\epsilon_0}} \leq C(M, M', L) \epsilon,$$

$$\|u - \tilde{u}\|_{L_t^\infty L_x^2 H_y^{1-\epsilon_0}} \leq C(M, M', L) M',$$

$$\|u\|_{L_t^\infty L_x^2 H_y^{1-\epsilon_0} \cap L_t^4 L_x^4 H_y^{1-\epsilon_0}} \leq C(M, M', L).$$

Remark 2.7 (Persistence of regularity). The results in the above theorems can be extended to $H^1(\mathbb{R}^2 \times \mathbb{T})$.

The following theorem implies that it suffices to show the finiteness of the solution in $L_{t,x}^4 H_y^{1-\epsilon_0}$ for the scattering of (1.1) in H^1 .

Theorem 2.8 (Weak space-time norm \Rightarrow Scattering). *Suppose that $u \in C_t^0 H_{x,y}^1(\mathbb{R} \times \mathbb{R}_x^2 \times \mathbb{T}_y)$ is a global solution of (1.1) satisfying $\|u\|_{L_t^4 L_x^4 H_y^{1-\epsilon_0}(\mathbb{R} \times \mathbb{R}_x^2 \times \mathbb{T}_y)} \leq L$ and $\|u(0)\|_{H_{x,y}^1} \leq M$ for some positive constants M and L . Then u scatters in $H_{x,y}^1(\mathbb{R}^2 \times \mathbb{T})$. That is, there exist $u_\pm \in H_{x,y}^1(\mathbb{R}^2 \times \mathbb{T})$ such that*

$$\|u(t, x, y) - e^{it\Delta_{\mathbb{R}^2 \times \mathbb{T}}} u_\pm(x, y)\|_{H_{x,y}^1} \rightarrow 0, \quad \text{as } t \rightarrow \pm\infty.$$

Proof. By the classical scattering theory as in [4], [25], we only need to show

$$(2.1) \quad \|u\|_{L_{t,x}^4 H_y^1 \cap L_t^4 W_x^{1,4} L_y^2(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{T})} \leq C(M, L).$$

By Theorem 2.3, it suffices to prove (2.1) as an a priori bound.

Divide the time interval \mathbb{R} into $N \sim (1 + L/\delta)^{10}$ subintervals $I_j = [t_j, t_{j+1}]$ such that

$$(2.2) \quad \|u\|_{L_{t,x}^4 H_y^{1-\epsilon_0}}(I_j \times \mathbb{R}^2 \times \mathbb{T}) \leq \delta,$$

where $\delta > 0$ will be chosen later.

On each I_j , by the Strichartz estimate, the Sobolev embedding and (2.2), we have

$$\begin{aligned} \|u\|_{L_t^4 W_x^{1,4} L_y^2 \cap L_{t,x}^4 H_y^1}(I_j \times \mathbb{R}^2 \times \mathbb{T}) &\leq C(\|u(t_j)\|_{H^1} + \| |u|^2 u \|_{L_{t,x}^{4/3} H_y^1} + \| |u|^2 u \|_{L_t^{4/3} W_x^{1,4/3} L_y^2}) \\ &\leq C(\|u(t_j)\|_{H^1} + \|u\|_{L_{t,x}^4 H_y^{1-\epsilon_0}}^2 \|u\|_{L_{t,x}^4 H_y^1 \cap L_t^4 W_x^{1,4} L_y^2}) \\ &\leq C(\|u(t_j)\|_{H^1} + \delta^2 \|u\|_{L_{t,x}^4 H_y^1 \cap L_t^4 W_x^{1,4} L_y^2}). \end{aligned}$$

Choosing $\delta \leq (\frac{1}{2C})^{1/2}$, we have

$$\|u\|_{L_t^4 W_x^{1,4} L_y^2 \cap L_{t,x}^4 H_y^1}(I_j \times \mathbb{R}^2 \times \mathbb{T}) \leq 2C \|u(t_j)\|_{H_{x,y}^1}.$$

The bound now follows by adding up the bounds on each subintervals I_j , which gives (2.1). \square

3. Profile decomposition

In this section, we will establish the linear profile decomposition in $H^1(\mathbb{R}_x^2 \times \mathbb{T}_y)$ in Subsection 3.1, which will heavily rely on the linear profile decomposition in $L^2(\mathbb{R}^2)$. The linear profile decomposition in $L^2(\mathbb{R}^2)$ for the mass-critical nonlinear Schrödinger equation was established by F. Merle and L. Vega [19]. After the work of R. Carles and S. Keraani [3] on the one-dimensional case, P. Bégout and A. Vargas [1] established the linear profile decomposition of the mass-critical nonlinear Schrödinger equation for arbitrary dimensions by the refined Strichartz inequality [2] and the bilinear restriction estimate [24]. We also refer to [18] for a version of the proof of the linear profile decomposition, and to [17] for a version of the linear profile decomposition of the mass-critical Klein–Gordon equations in the energy space in dimension two. We then analyze the nonlinear profiles in Subsection 3.2, which shows that the solutions of the cubic resonant nonlinear Schrödinger system can approximate the nonlinear profiles after some transformation. This is important to show the existence of the almost-periodic solution in Section 4.

3.1. Linear profile decomposition

In this subsection, we will establish the linear profile decomposition Proposition 3.1. We will prove an inverse Strichartz estimate in Proposition 3.6 by the argument of [18]. By applying this proposition inductively, we can obtain the linear profile decomposition.

Proposition 3.1 (Linear profile decomposition in $H^1(\mathbb{R}^2 \times \mathbb{T})$). *Let $\{u_n\}_{n \geq 1}$ be a bounded sequence in $H_{x,y}^1(\mathbb{R}^2 \times \mathbb{T})$. Then after passing to a subsequence if necessary, there exist $J^* \in \{0, 1, \dots\} \cup \{\infty\}$, functions ϕ^j in $L_x^2 H_y^1(\mathbb{R}^2 \times \mathbb{T})$ and mutually orthogonal frames $(\lambda_n^j, t_n^j, x_n^j, \xi_n^j)_{n \geq 1} \subseteq (0, \infty) \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2$, which means*

$$\left| \log \frac{\lambda_n^j}{\lambda_n^k} \right| + \lambda_n^j \lambda_n^k |\xi_n^j - \xi_n^k|^2 + \frac{|x_n^j - x_n^k|^2}{\lambda_n^j \lambda_n^k} + \frac{|(\lambda_n^j)^2 t_n^j - (\lambda_n^k)^2 t_n^k|}{\lambda_n^j \lambda_n^k} \rightarrow \infty,$$

as $n \rightarrow \infty$, for $j \neq k$, with $\lambda_n^j \equiv 1$ or $\lambda_n^j \rightarrow \infty$, as $n \rightarrow \infty$, $|\xi_n^j| \leq C_j$, for every $1 \leq j \leq J$, and for every $J \leq J^*$ a sequence $r_n^J \in L_x^2 H_y^1(\mathbb{R}^2 \times \mathbb{T})$ such that

$$u_n(x, y) = \sum_{j=1}^J \frac{1}{\lambda_n^j} e^{ix\xi_n^j} (e^{it_n^j \Delta_{\mathbb{R}^2}} P_n^j \phi^j) \left(\frac{x - x_n^j}{\lambda_n^j}, y \right) + r_n^J(x, y),$$

where the projector P_n^j is defined by

$$P_n^j \phi^j(x, y) = \begin{cases} \phi^j(x, y), & \text{if } \lambda_n^j \equiv 1, \\ P_{\leq (\lambda_n^j)^\theta} \phi^j(x, y), & 0 < \theta < 1, \quad \text{if } \lambda_n^j \rightarrow \infty. \end{cases}$$

Moreover,

$$\lim_{n \rightarrow \infty} \left(\|u_n\|_{H_{x,y}^1}^2 - \sum_{j=1}^J \left\| \frac{1}{\lambda_n^j} e^{ix\xi_n^j} (e^{it_n^j \Delta_{\mathbb{R}^2}} P_n^j \phi^j) \left(\frac{x - x_n^j}{\lambda_n^j}, y \right) \right\|_{H_{x,y}^1}^2 - \|r_n^J\|_{H_{x,y}^1}^2 \right) = 0,$$

$$\lambda_n^j e^{-it_n^j \Delta_{\mathbb{R}^2}} (e^{-i(\lambda_n^j x + x_n^j)\xi_n^j} r_n^J (\lambda_n^j x + x_n^j, y)) \rightarrow 0$$

in $L_x^2 H_y^1$, as $n \rightarrow \infty$, and for each $j \leq J$,

$$(3.1) \quad \limsup_{n \rightarrow \infty} \|e^{it \Delta_{\mathbb{R}^2 \times \mathbb{T}}} r_n^J\|_{L_t^4 L_x^4 H_y^{1-\epsilon_0}(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{T})} \rightarrow 0, \quad \text{as } J \rightarrow J^*.$$

Before giving the proof of Proposition 3.1, we will first establish the refined Strichartz estimate in Proposition 3.3. We will collect some basic facts appeared in [18].

Definition 3.2. Given $j \in \mathbb{Z}$, we write \mathcal{D}_j for the set of all dyadic cubes of side-length 2^j in \mathbb{R}^2 ,

$$\mathcal{D}_j = \{[2^j k_1, 2^j(k_1 + 1)) \times [2^j k_2, 2^j(k_2 + 1)) : (k_1, k_2) \in \mathbb{Z}^2\}.$$

We also write $\mathcal{D} = \bigcup_j \mathcal{D}_j$. Given $Q \in \mathcal{D}$, we define $f_Q(x, y)$ by $\mathcal{F}_x(f_Q) = \chi_Q \mathcal{F}_x f$.

By the bilinear Strichartz estimate on \mathbb{R}^2 [24] and Plancherel's theorem, we can prove the following refined Strichartz estimate by using the argument in [18].

Proposition 3.3 (Refined Strichartz estimate).

$$\|e^{it \Delta_{\mathbb{R}^2}} f\|_{L_{t,x,y}^4(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{T})} \lesssim \|f\|_{L_x^2 H_y^{1-\epsilon_0/2}}^{3/4} \left(\sup_{Q \in \mathcal{D}} |Q|^{-3/22} \|e^{it \Delta_{\mathbb{R}^2}} f_Q\|_{L_{t,x,y}^{11/2}} \right)^{1/4}.$$

To prove the inverse Strichartz estimate, we also need the following facts.

Lemma 3.4 (Refined Fatou's lemma). *Suppose that $\{f_n\} \subseteq L^4(\mathbb{R}^3 \times \mathbb{T})$, with $\limsup_{n \rightarrow \infty} \|f_n\|_{L^4} < \infty$. If $f_n \rightarrow f$ almost everywhere, then*

$$\|f_n\|_{L^4}^4 - \|f_n - f\|_{L^4}^4 - \|f\|_{L^4}^4 \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Proposition 3.5 (Local smoothing estimate). *Fix $\epsilon > 0$. We have*

$$\int_{\mathbb{R}} \int_{\mathbb{R}^2 \times \mathbb{T}} \langle x \rangle^{-1-\epsilon} |(|\nabla_x|^{1/2} e^{it\Delta_{\mathbb{R}^2}} f)(x, y)|^2 dx dy dt \lesssim_{\epsilon} \|f\|_{L^2_{x,y}(\mathbb{R}^2 \times \mathbb{T})}^2.$$

Furthermore, if $\epsilon \geq 1$, we have

$$\int_{\mathbb{R}} \int_{\mathbb{R}^2 \times \mathbb{T}} \langle x \rangle^{-1-\epsilon} |(\langle \nabla_x \rangle^{1/2} e^{it\Delta_{\mathbb{R}^2}} f)(x, y)|^2 dx dy dt \lesssim_{\epsilon} \|f\|_{L^2_{x,y}(\mathbb{R}^2 \times \mathbb{T})}^2.$$

We can now prove the inverse Strichartz estimate.

Proposition 3.6 (Inverse Strichartz estimate). *For $\{f_n\} \subseteq H^1_{x,y}$ satisfying*

$$(3.2) \quad \lim_{n \rightarrow \infty} \|f_n\|_{H^1_{x,y}} = A \quad \text{and} \quad \lim_{n \rightarrow \infty} \|e^{it\Delta_{\mathbb{R}^2}} f_n\|_{L^4_{t,x,y}} = \epsilon,$$

there exist a subsequence in n , $\phi \in L^2_x H^1_y(\mathbb{R}^2 \times \mathbb{T})$, $\{\lambda_n\} \subseteq [1, \infty)$, $\xi_n \in \mathbb{R}^2$, $|\xi_n| \lesssim 1$, and $(t_n, x_n, y_n) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{T}$ so that along the subsequence, we have the following:

$$(3.3) \quad \lambda_n e^{-i\xi_n(\lambda_n x + x_n)} (e^{it_n \Delta_{\mathbb{R}^2}} f_n)(\lambda_n x + x_n, y + y_n) \rightarrow \phi(x, y) \text{ in } L^2_x H^1_y, \text{ as } n \rightarrow \infty,$$

$$(3.4) \quad \lim_{n \rightarrow \infty} (\|f_n\|_{H^1_{x,y}}^2 - \|f_n - \phi_n\|_{H^1_{x,y}}^2 - \|\phi_n\|_{H^1_{x,y}}^2) = 0,$$

and $\lim_{n \rightarrow \infty} \|\phi_n\|_{H^1_{x,y}}^2 \gtrsim A \left(\frac{\epsilon}{A}\right)^{12},$

$$(3.5) \quad \limsup_{n \rightarrow \infty} \|e^{it\Delta_{\mathbb{R}^2}} (f_n - \phi_n)\|_{L^4_{t,x,y}}^4 \leq \epsilon^4 \left(1 - c \left(\frac{\epsilon}{A}\right)^{\beta}\right),$$

where c and β are constants,

$$\phi_n(x, y) = \frac{1}{\lambda_n} e^{ix\xi_n} \left(e^{-i\frac{t_n}{\lambda_n} \Delta_{\mathbb{R}^2}} P_n \phi \right) \left(\frac{x - x_n}{\lambda_n}, y - y_n \right),$$

and P_n is the projector defined by

$$P_n \phi(x, y) = \begin{cases} \phi(x, y), & \text{if } \limsup_{n \rightarrow \infty} \lambda_n < \infty, \\ P_{\leq \lambda_n^\theta}^x \phi(x, y), & \text{if } \lambda_n \rightarrow \infty. \end{cases}$$

Proof. By

$$\limsup_{n \rightarrow \infty} \|P_{\geq R}^x f_n\|_{L^2_x H^1_y}^{-\epsilon_0/2} \lesssim \limsup_{n \rightarrow \infty} \langle R \rangle^{-\epsilon_0/2} \|f_n\|_{H^1_{x,y}} \rightarrow 0, \quad \text{as } R \rightarrow \infty,$$

we can replace f_n by $P_{\leq R}^x f_n$ in the assumption of the proposition, for $R = R(A, \epsilon) > 0$ large enough, then by Proposition 3.3, there exists $\{Q_n\} \subseteq \mathcal{D}$, with $|Q_n| \lesssim R^2$, so that

$$(3.6) \quad \epsilon^4 A^{-3} \lesssim \liminf_{n \rightarrow \infty} |Q_n|^{-3/22} \|e^{it\Delta_{\mathbb{R}^2}}(f_n)_{Q_n}\|_{L_{t,x,y}^{11/2}}.$$

Let λ_n to be the inverse of the side-length of Q_n , which implies $|Q_n| = \lambda_n^{-2}$, therefore $\lambda_n \gtrsim R^{-1}$. We also set $\xi_n = c(Q_n)$, which is the center of the cube, and $|\xi_n| = |c(Q_n)| \lesssim R$. By Hölder's inequality, we have

$$\begin{aligned} \liminf_{n \rightarrow \infty} |Q_n|^{-3/22} \|e^{it\Delta_{\mathbb{R}^2}}(f_n)_{Q_n}\|_{L_{t,x,y}^{11/2}} &\lesssim \liminf_{n \rightarrow \infty} |Q_n|^{-3/22} \|e^{it\Delta_{\mathbb{R}^2}}(f_n)_{Q_n}\|_{L_{t,x,y}^4}^{8/11} \|e^{it\Delta_{\mathbb{R}^2}}(f_n)_{Q_n}\|_{L_{t,x,y}^\infty}^{3/11} \\ &\lesssim \liminf_{n \rightarrow \infty} \lambda_n^{3/11} \epsilon^{8/11} \|e^{it\Delta_{\mathbb{R}^2}}(f_n)_{Q_n}\|_{L_{t,x,y}^\infty}^{3/11}. \end{aligned}$$

Thus by (3.6), there exists $(t_n, x_n, y_n) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{T}$ so that

$$(3.7) \quad \liminf_{n \rightarrow \infty} \lambda_n |(e^{it_n \Delta_{\mathbb{R}^2}}(f_n)_{Q_n})(x_n, y_n)| \gtrsim \epsilon^{12} A^{-11}.$$

By the weak compactness of $L_x^2 H_y^1$, we have

$$\lambda_n e^{-i\xi_n(\lambda_n x + x_n)} (e^{it_n \Delta_{\mathbb{R}^2}} f_n)(\lambda_n x + x_n, y + y_n) \rightharpoonup \phi(x, y) \quad \text{in } L_x^2 H_y^1, \text{ as } n \rightarrow \infty.$$

Moreover, if $\limsup_{n \rightarrow \infty} \lambda_n < \infty$, we have

$$\lambda_n e^{-i\xi_n(\lambda_n x + x_n)} (e^{it_n \Delta_{\mathbb{R}^2}} f_n)(\lambda_n x + x_n, y + y_n) \rightharpoonup \phi(x, y) \quad \text{in } H_{x,y}^1, \text{ as } n \rightarrow \infty,$$

which is exactly (3.3). Define h so that $\mathcal{F}_x h$ is the characteristic function of the cube $[-1/2, 1/2]^2$. Then $h(x)\delta_0(y) \in L_x^2 H_y^{-1}(\mathbb{R}^2 \times \mathbb{T})$. From (3.7), we obtain

$$\begin{aligned} &| \langle h(x)\delta_0(y), \phi(x, y) \rangle_{x,y} | \\ &= \lim_{n \rightarrow \infty} \left| \int_{\mathbb{R}^2} \bar{h}(x) \lambda_n e^{-i\xi_n(\lambda_n x + x_n)} (e^{it_n \Delta_{\mathbb{R}^2}} f_n)(\lambda_n x + x_n, y_n) dx \right| \\ &= \lim_{n \rightarrow \infty} \lambda_n |(e^{it_n \Delta_{\mathbb{R}^2}}(f_n)_{Q_n})(x_n, y_n)| \gtrsim \epsilon^{12} A^{-11}, \end{aligned}$$

which implies ϕ carries non-trivial norm in $H_{x,y}^1$. To show (3.4), we only need to treat the case $\lambda_n \rightarrow \infty$, since the other case is easier. We can see

$$(3.8) \quad +\infty > \lim_{n \rightarrow \infty} \|\phi_n\|_{H_{x,y}^1} \geq \lim_{n \rightarrow \infty} \|P_{\leq \lambda_n^\theta}^x \phi\|_{L_x^2 H_y^1} \gtrsim \epsilon^{12} A^{-11}.$$

For the decoupling of the norm, it comes from the fact $P_{\lambda_n^\theta}^x \rightarrow \mathbb{I}$ in $L_x^2 H_y^1$ and (3.3). We are left to verify (3.5). By Proposition 3.5 and the Rellich–Kondrashov theorem, we have

$$e^{it\Delta_{\mathbb{R}^2}}(\lambda_n e^{-i\xi_n(\lambda_n x + x_n)} (e^{it_n \Delta_{\mathbb{R}^2}} f_n)(\lambda_n x + x_n, y + y_n)) \rightarrow e^{it\Delta_{\mathbb{R}^2}} \phi(x, y),$$

a.e. $(t, x, y) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{T}$, as $n \rightarrow \infty$. Then by applying Lemma 3.4, we obtain

$$\|e^{it\Delta_{\mathbb{R}^2}} f_n\|_{L_{t,x,y}^4}^4 - \|e^{it\Delta_{\mathbb{R}^2}}(f_n - \phi_n)\|_{L_{t,x,y}^4}^4 - \|e^{it\Delta_{\mathbb{R}^2}} \phi_n\|_{L_{t,x,y}^4}^4 \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Together with (3.8), we obtain (3.5). \square

By using Proposition 3.6 repeatedly until the remainder has asymptotically trivial linear evolution in $L^4_{t,x,y}$, we can obtain the following result.

Proposition 3.7 (Linear profile decomposition in $H^1_{x,y}(\mathbb{R}^2 \times \mathbb{T})$). *Let $\{u_n\}$ be a bounded sequence in $H^1_{x,y}(\mathbb{R}^2 \times \mathbb{T})$. Then, after passing to a subsequence if necessary, there exists $J^* \in \{0, 1, \dots\} \cup \{\infty\}$, functions $\phi^j \subseteq L^2_x H^1_y$, $(\lambda_n^j, t_n^j, x_n^j, \xi_n^j)_{n \geq 1} \subseteq [1, \infty) \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2$, and $|\xi_n^j| \leq C_j$, so that for any $J \leq J^*$, defining w_n^J by*

$$\begin{aligned} u_n(x, y) &= \sum_{j=1}^J \frac{1}{\lambda_n^j} e^{ix\xi_n^j} (e^{it_n^j \Delta_{\mathbb{R}^2}} P_n^j \phi^j) \left(\frac{x - x_n^j}{\lambda_n^j}, y - y_n^j \right) + w_n^J(x, y) \\ &=: \sum_{j=1}^J \phi_n^j(x, y) + w_n^J(x, y), \end{aligned}$$

we have the following properties:

(i) As $J \rightarrow \infty$,

$$\limsup_{n \rightarrow \infty} \|e^{it\Delta_{\mathbb{R}^2}} w_n^J\|_{L^4_{t,x,y}(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{T})} \rightarrow 0.$$

(ii) For each $j \leq J$,

$$\lambda_n^j e^{-it_n^j \Delta_{\mathbb{R}^2}} \left(e^{-i(\lambda_n^j x + x_n^j)\xi_n^j} w_n^J(\lambda_n^j x + x_n^j, y + y_n^j) \right) \rightarrow 0$$

in $L^2_x H^1_y(\mathbb{R}^2 \times \mathbb{T})$, as $n \rightarrow \infty$.

(iii)

$$\sup_J \lim_{n \rightarrow \infty} \left(\|u_n\|_{H^1_{x,y}}^2 - \sum_{j=1}^J \|\phi_n^j\|_{H^1_{x,y}}^2 - \|w_n^J\|_{H^1_{x,y}}^2 \right) = 0$$

(iv) For $j \neq j'$, and $n \rightarrow \infty$,

$$\left| \log \frac{\lambda_n^j}{\lambda_n^{j'}} \right| + \lambda_n^j \lambda_n^{j'} |\xi_n^j - \xi_n^{j'}|^2 + \frac{|x_n^j - x_n^{j'}|^2}{\lambda_n^j \lambda_n^{j'}} + |y_n^j - y_n^{j'}| + \frac{|(\lambda_n^j)^2 t_n^j - (\lambda_n^{j'})^2 t_n^{j'}|}{\lambda_n^j \lambda_n^{j'}} \rightarrow \infty.$$

Remark 3.8. By using Plancherel's theorem, interpolation, the Hölder inequality and Proposition 3.7, we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} \|e^{it\Delta_{\mathbb{R}^2 \times \mathbb{T}}} w_n^J\|_{L^4_{t,x} H_y^{1-\epsilon_0}} &= \limsup_{n \rightarrow \infty} \|e^{it\Delta_{\mathbb{R}^2}} w_n^J\|_{L^4_{t,x} H_y^{1-\epsilon_0}} \\ &\lesssim \limsup_{n \rightarrow \infty} \|w_n^J\|_{L^2_x H_y^{1-\epsilon_0/2}}^{\frac{2-2\epsilon_0}{2-\epsilon_0}} \|e^{it\Delta_{\mathbb{R}^2}} w_n^J\|_{L^4_{t,x,y}}^{\frac{\epsilon_0}{2-\epsilon_0}} \rightarrow 0, \quad \text{as } J \rightarrow \infty. \end{aligned}$$

Proof of Proposition 3.1. By Proposition 3.7 and Remark 3.8, we have the decomposition

$$u_n(x, y) = \sum_{j=1}^J \frac{1}{\lambda_n^j} e^{ix\xi_n^j} (e^{it_n^j \Delta_{\mathbb{R}^2}} P_n^j \phi^j) \left(\frac{x - x_n^j}{\lambda_n^j}, y - y_n^j \right) + w_n^J(x, y),$$

with $\lambda_n^j \geq 1$, and $|\xi_n^j| \leq C_j$, for every $1 \leq j \leq J$, and for $j \neq j'$,

$$\left| \log \frac{\lambda_n^j}{\lambda_n^{j'}} \right| + \lambda_n^j \lambda_n^{j'} |\xi_n^j - \xi_n^{j'}|^2 + \frac{|x_n^j - x_n^{j'}|^2}{\lambda_n^j \lambda_n^{j'}} + |y_n^j - y_n^{j'}| + \frac{|(\lambda_n^j)^2 t_n^j - (\lambda_n^{j'})^2 t_n^{j'}|}{\lambda_n^j \lambda_n^{j'}} \rightarrow \infty,$$

as $n \rightarrow \infty$. Since \mathbb{T} is compact, we may assume $y_n^j \rightarrow y_n^{j*}$, as $n \rightarrow \infty$. Then we may replace $\phi^j(\cdot, y)$ by $\phi^j(\cdot, y - y_n^{j*})$, and set $y_n^j \equiv 0$. If λ_n^j does not converge to ∞ , suppose $\lambda_n^j \rightarrow \lambda_\infty^j \in [1, \infty)$. Thus, we may replace $\phi^j(x, \cdot)$ by $\frac{1}{\lambda_\infty^j} \phi^j(x/\lambda_\infty^j, \cdot)$ and set $\lambda_n^j \equiv 1$, whilst retaining the conclusion of Proposition 3.1. \square

3.2. Approximation of profiles

In this subsection, by using the solution of the cubic resonant nonlinear Schrödinger system to approximate the nonlinear Schrödinger equation with initial data being the bubble in the linear profile decomposition, we can show that the nonlinear profile has a bounded space-time norm, where to deal with the error term, we use a normal form argument.

Theorem 3.9 (Large-scale profiles). *For any $\phi \in L_x^2 H_y^1(\mathbb{R}^2 \times \mathbb{T})$, $0 < \theta < 1$, $(\lambda_n, t_n, x_n, \xi_n)_{n \geq 1} \subseteq (0, \infty) \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2$, $\lambda_n \rightarrow \infty$, as $n \rightarrow \infty$, $|\xi_n| \lesssim 1$, there is a global solution $u_n \in C_t^0 L_x^2 H_y^1$ of*

$$(3.9) \quad \begin{cases} i \partial_t u_n + \Delta_{\mathbb{R}^2 \times \mathbb{T}} u_n = |u_n|^2 u_n, \\ u_n(0, x, y) = \frac{1}{\lambda_n} e^{ix\xi_n} (e^{it_n \Delta_{\mathbb{R}^2}} P_{\leq \lambda_n^\theta} \phi) \left(\frac{x - x_n}{\lambda_n}, y \right), \end{cases}$$

for n large enough, satisfying $\|u_n\|_{L_t^\infty L_x^2 H_y^1 \cap L_{t,x}^4 H_y^1(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{T})} \lesssim \|\phi\|_{L_x^2 H_y^1} 1$. Furthermore, assume ϵ_1 is a sufficiently small positive constant, that depends only on $\|\phi\|_{L_x^2 H_y^1}$, $\vec{v}_0 \in H_x^2 h^1$, and such that

$$(3.10) \quad \|\vec{\phi} - \vec{v}_0\|_{L_x^2 h^1} \leq \epsilon_1.$$

Then there exists a solution $\vec{v} \in C_t^0 H_x^2 h^1(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{Z})$ of the cubic resonant Schrödinger system

$$(3.11) \quad i \partial_t v_j + \Delta_{\mathbb{R}^2} v_j = \sum_{(j_1, j_2, j_3) \in \mathcal{R}(j)} v_{j_1} \bar{v}_{j_2} v_{j_3}, \quad \text{where } j \in \mathbb{Z},$$

with

$$\begin{aligned} v_j(0, x) &= v_{0,j}(x), \quad \text{if } t_n = 0, \\ \|v_j(t) - e^{it \Delta_{\mathbb{R}^2}} v_{0,j}\|_{L_x^2 h^1} &\rightarrow 0, \quad \text{as } t \rightarrow \pm\infty, \quad \text{if } t_n \rightarrow \pm\infty. \end{aligned}$$

such that for any $\epsilon > 0$, and for n large enough, it holds that

$$\begin{aligned} \|u_n(t) - W_n(t)\|_{L_t^\infty L_x^2 H_y^1 \cap L_{t,x}^4 H_y^1(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{T})} &\lesssim \|\phi\|_{L_x^2 H_y^1} \epsilon_1, \\ \|u_n\|_{L_t^\infty L_x^2 H_y^1 \cap L_{t,x}^4 H_y^1(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{T})} &\lesssim 1, \end{aligned}$$

where

$$W_n(t, x, y) = e^{-i(t-t_n)|\xi_n|^2} e^{ix\xi_n} V_{\lambda_n}(t, x, y),$$

and V_{λ_n} is defined as

$$V_{\lambda_n}(t, x, y) = \sum_{j \in \mathbb{Z}} \frac{1}{\lambda_n} e^{-it|j|^2} e^{iyj} v_j \left(\frac{t}{\lambda_n^2} + t_n, \frac{x - x_n - 2\xi_n(t - t_n)}{\lambda_n} \right).$$

Before giving the proof, we present a simple lemma which is useful in the proof of the theorem.

Lemma 3.10.

$$\sup_{j \in \mathbb{Z}} \langle j \rangle^2 \sum_{\substack{j_1, j_2, j_3 \in \mathbb{Z}, \\ j_1 - j_2 + j_3 = j}} \langle j_1 \rangle^{-2} \langle j_2 \rangle^{-2} \langle j_3 \rangle^{-2} \lesssim 1.$$

Proof. We have

$$\langle j \rangle^2 \sum_{\substack{j_1, j_2, j_3 \in \mathbb{Z}, \\ j_1 - j_2 + j_3 = j}} \langle j_1 \rangle^{-2} \langle j_2 \rangle^{-2} \langle j_3 \rangle^{-2} = \langle j \rangle^2 \sum_{j_2 \in \mathbb{Z}} \langle j_2 \rangle^{-2} \sum_{j_3 \in \mathbb{Z}} \langle j + j_2 - j_3 \rangle^{-2} \langle j_3 \rangle^{-2}.$$

By an easy calculation, we have

$$\begin{aligned} & \sum_{j_3 \in \mathbb{Z}} \langle j + j_2 - j_3 \rangle^{-2} \langle j_3 \rangle^{-2} \\ & \leq \int_{\mathbb{R}} \langle j + j_2 - y \rangle^{-2} \langle y \rangle^{-2} dy \\ & \leq \int_{|j+j_2-y| \geq \frac{|j+j_2|}{2}} \langle j + j_2 - y \rangle^{-2} \langle y \rangle^{-2} dy + \int_{|j+j_2-y| \leq \frac{|j+j_2|}{2}} \langle j + j_2 - y \rangle^{-2} \langle y \rangle^{-2} dy \\ & \lesssim \langle j + j_2 \rangle^{-2} \int_{\mathbb{R}} \langle y \rangle^{-2} dy + \langle j + j_2 \rangle^{-2} \int_{\mathbb{R}} \langle j + j_2 - y \rangle^{-2} dy \lesssim \langle j + j_2 \rangle^{-2}. \end{aligned}$$

Arguing in the same way, we can obtain

$$\begin{aligned} & \langle j \rangle^2 \sum_{j_2 \in \mathbb{Z}} \langle j_2 \rangle^{-2} \sum_{j_3 \in \mathbb{Z}} \langle j + j_2 - j_3 \rangle^{-2} \langle j_3 \rangle^{-2} \\ & \lesssim \langle j \rangle^2 \sum_{j_2 \in \mathbb{Z}} \langle j_2 \rangle^{-2} \langle j + j_2 \rangle^{-2} \lesssim \langle j \rangle^2 \cdot \langle j \rangle^{-2} \lesssim 1. \quad \square \end{aligned}$$

Remark 3.11. By the result in [5], [29], the cubic resonant nonlinear Schrödinger system (3.11) in Theorem 3.9 satisfies

$$\|\vec{v}\|_{L_{t,x}^4 h^1(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{Z})} \leq C,$$

for some constant C depending only on $\|\vec{v}_0\|_{L_x^2 h^1}$. Moreover, there exists $\{v_j^\pm\}_{j \in \mathbb{Z}} \in L_x^2 h^1$ such that

$$\left\| \left(\sum_{j \in \mathbb{Z}} \langle j \rangle^2 |v_j(t) - e^{it\Delta_{\mathbb{R}^2}} v_j^\pm|^2 \right)^{1/2} \right\|_{L_x^2} \rightarrow 0, \quad \text{as } t \rightarrow \pm\infty.$$

Remark 3.12. By the above remark on the scattering theorem, we have that the solution of the cubic resonant nonlinear Schrödinger system (3.11) satisfies the following result as a consequence of the triangle inequality, the Strichartz estimate, Minkowski's inequality, and Lemma 3.10:

$$\begin{aligned}
\|\vec{v}\|_{h^1 L_{t,x}^4} &\lesssim \|\vec{v}(t) - e^{it\Delta_{\mathbb{R}^2}} \vec{v}^+\|_{h^1 L_{t,x}^4} + \|e^{it\Delta_{\mathbb{R}^2}} \vec{v}^+\|_{h^1 L_{t,x}^4} \\
&\lesssim \left\| \sum_{(j_1, j_2, j_3) \in \mathcal{R}(j)} v_{j_1} \bar{v}_{j_2} v_{j_3} \right\|_{h_j^1 L_{t,x}^{4/3}} + \|v_j^+\|_{h_j^1 L_x^2} \\
&\lesssim \left\| \sum_{(j_1, j_2, j_3) \in \mathcal{R}(j)} v_{j_1} \bar{v}_{j_2} v_{j_3} \right\|_{L_{t,x}^{4/3} h_j^1} + \|v_j^+\|_{L_x^2 h_j^1} \\
&\lesssim \left\| \left(\sum_{j \in \mathbb{Z}} \langle j \rangle^2 |v_j(t, x)|^2 \right)^{3/2} \right\|_{L_{t,x}^{4/3}} + \|v_j^+\|_{L_x^2 h^1} \sim \|\vec{v}\|_{L_{t,x}^4 h^1}^3 + \|\vec{v}^+\|_{L_x^2 h^1}.
\end{aligned}$$

Proof of Theorem 3.9. Without loss of generality, we may assume that $x_n = 0$. Using a Galilean transform and the fact that ξ_n is bounded, we may assume that $\xi_n = 0$ for all n . We see

$$W_n(t, x, y) = \sum_{j \in \mathbb{Z}} \frac{1}{\lambda_n} e^{-it|j|^2} e^{iyj} v_j \left(\frac{t}{\lambda_n^2} + t_n, \frac{x}{\lambda_n} \right).$$

When $t_n = 0$, we will show V_{λ_n} is an approximate solution to the cubic nonlinear Schrödinger equation on $\mathbb{R}^2 \times \mathbb{T}$ in the sense of Theorem 2.6. By noting v_j satisfies (3.11) and an easy computation, we have

$$\begin{aligned}
(3.12) \quad e_{\lambda_n} &= (i\partial_t + \Delta_{\mathbb{R}^2 \times \mathbb{T}}) V_{\lambda_n} - |V_{\lambda_n}|^2 V_{\lambda_n} \\
&= - \sum_{j \in \mathbb{Z}} e^{-it|j|^2} e^{iyj} \\
&\quad \times \sum_{(j_1, j_2, j_3) \in \mathcal{N}\mathcal{R}(j)} e^{-it(|j_1|^2 - |j_2|^2 + |j_3|^2 - |j|^2)} (v_{j_1, \lambda_n} \bar{v}_{j_2, \lambda_n} v_{j_3, \lambda_n})(t, x),
\end{aligned}$$

where

$$\mathcal{N}\mathcal{R}(j) = \{(j_1, j_2, j_3) \in \mathbb{Z}^3 : j_1 - j_2 + j_3 - j = 0, |j_1|^2 - |j_2|^2 + |j_3|^2 - |j|^2 \neq 0\}.$$

We first decompose

$$e_{\lambda_n} = P_{\geq 2^{-10}}^x e_{\lambda_n} + P_{\leq 2^{-10}}^x e_{\lambda_n}.$$

By Bernstein's inequality, the Plancherel theorem, (3.12), Leibniz's rule, and Höl-

der's inequality, we have

$$\begin{aligned}
& \|P_{\geq 2^{-10}}^x e\lambda_n\|_{L_t^{4/3} L_x^{4/3} H_y^1(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{T})} \\
& \lesssim \|P_{\geq 2^{-10}}^x \nabla_x e\lambda_n\|_{L_{t,x}^{4/3} H_y^1(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{T})} \\
& \lesssim \frac{1}{\lambda_n} \left\| \sum_{(j_1, j_2, j_3) \in \mathcal{NR}(\cdot)} e^{-i\lambda_n^2 t(|j_1|^2 - |j_2|^2 + |j_3|^2 - |j|^2)} (\nabla_x v_{j_1} \cdot \bar{v}_{j_2} \cdot v_{j_3})(t, x) \right\|_{h^1 L_{t,x}^{4/3}} \\
& \quad + \frac{1}{\lambda_n} \left\| \sum_{(j_1, j_2, j_3) \in \mathcal{NR}(\cdot)} e^{-i\lambda_n^2 t(|j_1|^2 - |j_2|^2 + |j_3|^2 - |j|^2)} (v_{j_1} \cdot \overline{\nabla_x v_{j_2}} \cdot v_{j_3})(t, x) \right\|_{h^1 L_{t,x}^{4/3}} \\
& \quad + \frac{1}{\lambda_n} \left\| \sum_{(j_1, j_2, j_3) \in \mathcal{NR}(\cdot)} e^{-i\lambda_n^2 t(|j_1|^2 - |j_2|^2 + |j_3|^2 - |j|^2)} (v_{j_1} \cdot \bar{v}_{j_2} \cdot \nabla_x v_{j_3})(t, x) \right\|_{h^1 L_{t,x}^{4/3}} \\
& \lesssim \frac{1}{\lambda_n} \left\| \left(\sum_j \langle j \rangle^2 |\nabla_x v_j(t, x)|^2 \right)^{1/2} \right\|_{L_{t,x}^4} \left\| \left(\sum_j \langle j \rangle^2 |v_j(t, x)|^2 \right)^{1/2} \right\|_{L_{t,x}^4}^2 \\
& \lesssim \|\phi\|_{L_x^2 H_y^1} \frac{1}{\lambda_n} \|\bar{v}_0\|_{H_x^1 h^1}^3.
\end{aligned}$$

Thus $P_{\geq 2^{-10}}^x e\lambda_n$ is acceptable when λ_n is large enough depending on $\|\phi\|_{L_x^2 H_y^1}$ and ϵ_1 . We turn to the estimate of $P_{\leq 2^{-10}}^x e\lambda_n$. By integrating by parts, we have

$$\begin{aligned}
& \int_0^t e^{i(t-\tau)\Delta_{\mathbb{R}^2 \times \mathbb{T}}} P_{\leq 2^{-10}}^x e\lambda_n(\tau) d\tau \\
& = - \sum_{\substack{j \in \mathbb{Z}, \\ (j_1, j_2, j_3) \in \mathcal{NR}(j)}} \int_0^t e^{i(t-\tau)\Phi(j_1, j_2, j_3, j)} e^{-it(|j_1|^2 - |j_2|^2 + |j_3|^2)} e^{iyj} \\
& \quad \times P_{\leq 2^{-10}}^x (v_{j_1, \lambda_n} \bar{v}_{j_2, \lambda_n} v_{j_3, \lambda_n})(\tau, x) d\tau \\
& = - \sum_{\substack{j \in \mathbb{Z}, \\ (j_1, j_2, j_3) \in \mathcal{NR}(j)}} i e^{it\Delta_{\mathbb{R}^2 \times \mathbb{T}}} e^{iyj} \frac{P_{\leq 2^{-10}}^x}{\Phi(j_1, j_2, j_3, j)} (v_{j_1, \lambda_n} \bar{v}_{j_2, \lambda_n} v_{j_3, \lambda_n})(0, x) \\
& \quad - \sum_{\substack{j \in \mathbb{Z}, \\ (j_1, j_2, j_3) \in \mathcal{NR}(j)}} i e^{-it(|j_1|^2 - |j_2|^2 + |j_3|^2)} e^{iyj} \frac{P_{\leq 2^{-10}}^x}{\Phi(j_1, j_2, j_3, j)} (v_{j_1, \lambda_n} \bar{v}_{j_2, \lambda_n} v_{j_3, \lambda_n})(t, x) \\
& \quad + \sum_{\substack{j \in \mathbb{Z}, \\ (j_1, j_2, j_3) \in \mathcal{NR}(j)}} i e^{it\Delta_{\mathbb{R}^2}} e^{iyj} \int_0^t \frac{i e^{-i\tau\Phi(j_1, j_2, j_3, j)}}{\Phi(j_1, j_2, j_3, j)} \\
& \quad \times \partial_\tau P_{\leq 2^{-10}}^x (v_{j_1, \lambda_n} \bar{v}_{j_2, \lambda_n} v_{j_3, \lambda_n})(\tau, x) d\tau \\
& =: A_1 + A_2 + A_3,
\end{aligned}$$

where $\Phi(j_1, j_2, j_3, j) = \Delta_{\mathbb{R}^2} + |j_1|^2 - |j_2|^2 + |j_3|^2 - |j|^2$.

For A_1 , by the Strichartz estimate, Plancherel's theorem, Minkowski's inequality, the boundedness of the operator $P_{\leq 2^{-10}}^x / \Phi(j_1, j_2, j_3, j)$ on $L_x^r(\mathbb{R}^2)$, $1 < r < \infty$,

when $(j_1, j_2, j_3) \in \mathcal{NR}(j)$, Hölder's inequality, Lemma 3.10 and the Sobolev embedding, we have

$$\begin{aligned}
& \|A_1\|_{L_t^\infty L_x^2 H_y^1 \cap L_{t,x}^4 H_y^1} \\
& \lesssim \left(\sum_{j \in \mathbb{Z}} \langle j \rangle^2 \left(\sum_{(j_1, j_2, j_3) \in \mathcal{NR}(j)} \left\| \frac{P_{\leq 2^{-10}}^x (v_{j_1, \lambda_n} \bar{v}_{j_2, \lambda_n} v_{j_3, \lambda_n})(0, x)}{\Phi(j_1, j_2, j_3, j)} \right\|_{L_x^2} \right)^2 \right)^{1/2} \\
& \lesssim \frac{1}{\lambda_n^2} \left(\sum_{j \in \mathbb{Z}} \langle j \rangle^2 \left(\sum_{(j_1, j_2, j_3) \in \mathcal{NR}(j)} \|(v_{j_1} \bar{v}_{j_2} v_{j_3})(0, x)\|_{L_x^2} \right)^2 \right)^{1/2} \\
& \lesssim \frac{1}{\lambda_n^2} \left(\sum_{j \in \mathbb{Z}} \sum_{(j_1, j_2, j_3) \in \mathcal{NR}(j)} \langle j_1 \rangle^2 \|v_{j_1}(0, x)\|_{L_x^6}^2 \langle j_2 \rangle^2 \|\bar{v}_{j_2}(0, x)\|_{L_x^6}^2 \langle j_3 \rangle^2 \|v_{j_3}(0, x)\|_{L_x^6}^2 \right)^{1/2} \\
& \quad \cdot \left(\sup_{j \in \mathbb{Z}} \langle j \rangle^2 \sum_{(j_1, j_2, j_3) \in \mathcal{NR}(j)} \langle j_1 \rangle^{-2} \langle j_2 \rangle^{-2} \langle j_3 \rangle^{-2} \right)^{1/2} \\
& \lesssim \frac{1}{\lambda_n^2} \left(\sum_{j \in \mathbb{Z}} \langle j \rangle^2 \|v_{j, \lambda}(0, x)\|_{L_x^6}^2 \right)^{3/2} \lesssim \frac{1}{\lambda_n^2} \|\vec{v}_0\|_{H_x^1 h^1}^3.
\end{aligned}$$

We can estimate A_2 similarly to A_1 by using Plancherel's theorem, the Minkowski inequality, the Strichartz estimate, and the boundedness of the operator $P_{\leq 2^{-10}}^x / \Phi(j_1, j_2, j_3, j)$ on $L_x^r(\mathbb{R}^2)$, $1 < r < \infty$, when $(j_1, j_2, j_3) \in \mathcal{NR}(j)$:

$$\begin{aligned}
& \|A_2\|_{L_t^\infty L_x^2 H_y^1 \cap L_{t,x}^4 H_y^1} \\
& \lesssim \left\| \sum_{(j_1, j_2, j_3) \in \mathcal{NR}(j)} e^{-it(|j_1|^2 - |j_2|^2 + |j_3|^2)} \right. \\
& \quad \times \left. \frac{P_{\leq 2^{-10}}^x}{\Phi(j_1, j_2, j_3, j)} (v_{j_1, \lambda_n} \bar{v}_{j_2, \lambda_n} v_{j_3, \lambda_n})(t, x) \right\|_{h^1 L_t^\infty L_x^2 \cap h^1 L_{t,x}^4} \\
& \lesssim \left(\sum_{j \in \mathbb{Z}} \left(\sum_{(j_1, j_2, j_3) \in \mathcal{NR}(\cdot)} \left\| \frac{P_{\leq 2^{-10}}^x (v_{j_1, \lambda_n} \bar{v}_{j_2, \lambda_n} v_{j_3, \lambda_n})(t, x)}{\Phi(j_1, j_2, j_3, j)} \right\|_{L_t^\infty L_x^2 \cap L_{t,x}^4} \right)^2 \langle j \rangle^2 \right)^{1/2} \\
& \lesssim \lambda_n^{-2} \left(\sum_{j \in \mathbb{Z}} \left(\sum_{(j_1, j_2, j_3) \in \mathcal{NR}(j)} \|v_{j_1}(0, x) v_{j_2}(0, x) v_{j_3}(0, x)\|_{L_x^2} \right)^2 \langle j \rangle^2 \right)^{1/2} \\
& \quad + \lambda_n^{-2} \left(\sum_{j \in \mathbb{Z}} \left(\sum_{(j_1, j_2, j_3) \in \mathcal{NR}(j)} \|(i \partial_t + \Delta_{\mathbb{R}^2})(v_{j_1} \bar{v}_{j_2} v_{j_3})(t, x)\|_{L_t^1 L_x^2} \right)^2 \langle j \rangle^2 \right)^{1/2} \\
& := \lambda_n^{-2} (A_{21} + A_{22}),
\end{aligned}$$

We see, by Hölder's inequality, Lemma 3.10, and the Sobolev inequality, that

$$\begin{aligned}
A_{21} & \lesssim \left(\sum_{j \in \mathbb{Z}} \left(\sum_{(j_1, j_2, j_3) \in \mathcal{NR}(j)} \|v_{j_1}(0, x)\|_{L_x^2} \|v_{j_2}(0, x)\|_{L_x^\infty} \|v_{j_3}(0, x)\|_{L_x^\infty} \right)^2 \langle j \rangle^2 \right)^{1/2} \\
& \lesssim \left(\sum_{j \in \mathbb{Z}} \langle j \rangle^2 \|v_j(0, x)\|_{H_x^1}^2 \right)^{3/2} \sim \|\vec{v}_0\|_{H_x^1 h^1}^3.
\end{aligned}$$

We also have, by (3.11) and Hölder's inequality,

$$\begin{aligned}
A_{22} &\lesssim \left(\sum_{j \in \mathbb{Z}} \left(\sum_{(j_1, j_2, j_3) \in \mathcal{NR}(j)} \|\Delta_{\mathbb{R}^2} v_{j_1} \bar{v}_{j_2} v_{j_3}\|_{L_t^1 L_x^2} + \|\nabla v_{j_1} \cdot \nabla v_{j_2} \cdot v_{j_3}\|_{L_t^1 L_x^2} \right. \right. \\
&\quad \left. \left. + \left\| v_{j_1} v_{j_2} \sum_{(p_1, p_2, p_3) \in R(j_3)} v_{p_1} \bar{v}_{p_2} v_{p_3} \right\|_{L_t^1 L_x^2} \right)^2 \langle j \rangle^2 \right)^{1/2} \\
&\lesssim \left(\sum_{j \in \mathbb{Z}} \sum_{(j_1, j_2, j_3) \in \mathcal{NR}(j)} \langle j_1 \rangle^2 \|\Delta_{\mathbb{R}^2} v_{j_1}\|_{L_t^3 L_x^6}^2 \langle j_2 \rangle^2 \|v_{j_2}\|_{L_t^3 L_x^6}^2 \langle j_3 \rangle^2 \|v_{j_3}\|_{L_t^3 L_x^6}^2 \right)^{1/2} \\
&\quad + \left(\sum_{j \in \mathbb{Z}} \sum_{(j_1, j_2, j_3) \in \mathcal{NR}(j)} \langle j_1 \rangle^2 \|\nabla v_{j_1}\|_{L_t^3 L_x^6}^2 \langle j_2 \rangle^2 \|\nabla v_{j_2}\|_{L_t^3 L_x^6}^2 \langle j_3 \rangle^2 \|v_{j_3}\|_{L_t^3 L_x^6}^2 \right)^{1/2} \\
&\quad + \left(\sum_{j \in \mathbb{Z}} \left(\sum_{(j_1, j_2, j_3) \in \mathcal{NR}(j)} \left\| v_{j_1} v_{j_2} \sum_{(p_1, p_2, p_3) \in R(j_3)} v_{p_1} \bar{v}_{p_2} v_{p_3} \right\|_{L_t^1 L_x^2} \right)^2 \langle j \rangle^2 \right)^{1/2}
\end{aligned}$$

Hence,

$$\begin{aligned}
A_{22} &\lesssim \left(\sum_j \langle j \rangle^2 \|v_j\|_{L_t^3 W_x^{2,6}}^2 \right)^{3/2} + \left(\sum_j \langle j \rangle^2 \|v_j\|_{L_t^3 W_x^{1,6}}^2 \right)^{3/2} \\
&\quad + \left(\sum_{j \in \mathbb{Z}} \langle j \rangle^2 \left(\sum_{(j_1, j_2, j_3) \in \mathcal{NR}(j)} \|v_{j_1}\|_{L_t^5 L_x^{10}} \|v_{j_2}\|_{L_t^5 L_x^{10}} \right. \right. \\
&\quad \quad \left. \left. \times \sum_{(p_1, p_2, p_3) \in R(j_3)} \|v_{p_1}\|_{L_t^5 L_x^{10}} \|v_{p_2}\|_{L_t^5 L_x^{10}} \|v_{p_3}\|_{L_t^5 L_x^{10}} \right)^2 \right)^{1/2} \\
&\lesssim \left(\sum_j \langle j \rangle^2 \|v_j\|_{L_t^3 W_x^{2,6}}^2 \right)^{3/2} + \left(\sum_j \langle j \rangle^2 \|v_j\|_{L_t^3 W_x^{1,6}}^2 \right)^{3/2} \\
&\quad + \left(\sum_{j \in \mathbb{Z}} \langle j \rangle^2 \|\nabla_x |^{\frac{2}{5}} v_j\|_{L_t^5 L_x^{10/3}}^2 \right)^{5/2}.
\end{aligned}$$

We now consider A_3 . By the Strichartz estimate, Plancherel's theorem, the Minkowski inequality, the boundedness of the operator $P_{\leq 2^{-10}}^x / \Phi(j_1, j_2, j_3, j)$ on $L_x^r(\mathbb{R}^2)$, $1 < r < \infty$, when $(j_1, j_2, j_3) \in \mathcal{NR}(j)$, and (3.11), we have

(3.13)

$$\begin{aligned}
&\|A_3\|_{L_t^\infty L_x^2 H_{t,x}^1 \cap L_{t,x}^4 H_t^1} \\
&\lesssim \left\| \left(\sum_{j \in \mathbb{Z}} \langle j \rangle^2 \left| \sum_{(j_1, j_2, j_3) \in \mathcal{NR}(j)} e^{-it(|j_1|^2 - |j_2|^2 + |j_3|^2)} (i \partial_t + \Phi(j_1, j_2, j_3, j)) \right. \right. \right. \\
&\quad \left. \left. \times \int_0^t \frac{e^{i(t-\tau)\Phi(j_1, j_2, j_3, j)}}{\Phi(j_1, j_2, j_3, j)} \partial_\tau P_{\leq 2^{-10}}^x (v_{j_1, \lambda_n} \bar{v}_{j_2, \lambda_n} v_{j_3, \lambda_n})(\tau, x) d\tau \right|^2 \right)^{1/2} \right\|_{L_t^1 L_x^2} \\
&\sim \left\| \sum_{(j_1, j_2, j_3) \in \mathcal{NR}(\cdot)} e^{-it(|j_1|^2 - |j_2|^2 + |j_3|^2)} \frac{\partial_t P_{\leq 2^{-10}}^x (v_{j_1, \lambda_n} \bar{v}_{j_2, \lambda_n} v_{j_3, \lambda_n})(t, x)}{\Delta_{\mathbb{R}^2} + |j_1|^2 - |j_2|^2 + |j_3|^2 - |j|^2} \right\|_{L_t^1 h^1 L_x^2}
\end{aligned}$$

$$\begin{aligned}
&\lesssim \left\| \left(\sum_{j \in \mathbb{Z}} \langle j \rangle^2 \left(\sum_{(j_1, j_2, j_3) \in \mathcal{NR}(j)} \left\| \frac{\partial_t P_{\leq 2^{-10}}^x(v_{j_1, \lambda_n} \bar{v}_{j_2, \lambda_n} v_{j_3, \lambda_n})(t, x)}{\Delta_{\mathbb{R}^2} + |j_1|^2 - |j_2|^2 + |j_3|^2 - |j|^2} \right\|_{L_x^2} \right)^2 \right)^{1/2} \right\|_{L_t^1} \\
&\lesssim \left\| \left(\sum_{j \in \mathbb{Z}} \langle j \rangle^2 \left(\sum_{(j_1, j_2, j_3) \in \mathcal{NR}(j)} \|\partial_t(v_{j_1, \lambda_n} \bar{v}_{j_2, \lambda_n} v_{j_3, \lambda_n})\|_{L_x^2} \right)^2 \right)^{1/2} \right\|_{L_t^1} \\
&\lesssim \lambda_n^{-2} \left\| \left(\sum_{j \in \mathbb{Z}} \langle j \rangle^2 \left(\sum_{(j_1, j_2, j_3) \in \mathcal{NR}(j)} \left\| \bar{v}_{j_2} v_{j_3} \sum_{(p_1, p_2, p_3) \in R(j_1)} v_{p_1} \bar{v}_{p_2} v_{p_3} \right\|_{L_x^2} \right)^2 \right)^{1/2} \right\|_{L_t^1} \\
&\quad + \lambda_n^{-2} \left\| \left(\sum_{j \in \mathbb{Z}} \langle j \rangle^2 \left(\sum_{(j_1, j_2, j_3) \in \mathcal{NR}(j)} \|\bar{v}_{j_2} v_{j_3} \Delta_{\mathbb{R}^2} v_{j_1}\|_{L_x^2} \right)^2 \right)^{1/2} \right\|_{L_t^1} := \lambda_n^{-2} (A_{31} + A_{32}).
\end{aligned}$$

Using the Hölder inequality and Lemma 3.10,

$$(3.14) \quad A_{32} \lesssim \left\| \left(\sum_{j \in \mathbb{Z}} \langle j \rangle^2 \|\Delta_{\mathbb{R}^2} v_j\|_{L_x^6}^2 \right)^{1/2} \left(\sum_{j \in \mathbb{Z}} \langle j \rangle^2 \|v_j\|_{L_x^6}^2 \right) \right\|_{L_t^1} \lesssim \left(\sum_{j \in \mathbb{Z}} \langle j \rangle^2 \|v_j\|_{L_t^3 W_x^{2,6}}^2 \right)^{3/2}.$$

Similarly, by the Hölder inequality and Lemma 3.10, we also have

$$\begin{aligned}
(3.15) \quad A_{31} &\lesssim \left\| \left(\sum_{j_2} \langle j_2 \rangle^2 \|v_{j_2}\|_{L_x^6}^2 \right)^{1/2} \left(\sum_{j_3} \langle j_3 \rangle^2 \|v_{j_3}\|_{L_x^6}^2 \right)^{1/2} \right. \\
&\quad \times \left. \left(\sum_{j_1} \langle j_1 \rangle^2 \left\| \sum_{(p_1, p_2, p_3) \in R(j_1)} v_{p_1} \bar{v}_{p_2} v_{p_3} \right\|_{L_x^6}^2 \right)^{1/2} \right\|_{L_t^1} \\
&\lesssim \left\| \left(\sum_j \langle j \rangle^2 \|v_j\|_{L_x^6}^2 \right)^{1/2} \right\|_{L_t^3}^2 \left\| \left(\sum_{j_1} \langle j_1 \rangle^2 \left\| \sum_{(p_1, p_2, p_3) \in R(j_1)} v_{p_1} \bar{v}_{p_2} v_{p_3} \right\|_{L_x^6}^2 \right)^{1/2} \right\|_{L_t^3} \\
&\lesssim \left(\sum_j \langle j \rangle^2 \|v_j(t, x)\|_{L_t^3 L_x^6}^2 \right) \cdot \left\| \left(\sum_{p \in \mathbb{Z}} \langle p \rangle^2 \|v_p(t, x)\|_{L_x^{18}}^2 \right)^{1/2} \right\|_{L_t^9}^3 \\
&\lesssim \left(\sum_j \langle j \rangle^2 \|v_j(t, x)\|_{L_t^3 L_x^6}^2 \right) \cdot \left(\sum_{p \in \mathbb{Z}} \langle p \rangle^2 \|v_p(t, x)\|_{L_t^9 W_x^{1,18/7}}^2 \right)^{3/2}.
\end{aligned}$$

Therefore, by (3.13), (3.14) and (3.15), we have

$$\|A_3\|_{L_t^\infty L_x^\infty H_y^1 \cap L_{t,x}^4 H_y^1} \lesssim \|\phi\|_{L_x^2 H_y^1} \lambda_n^{-2} (\|\vec{v}_0\|_{H_x^2 h^1} + \|\vec{v}_0\|_{H_x^2 h^1}^3).$$

So $P_{\leq 2^{-10}}^x e_{\lambda_n}$ is acceptable when λ_n is large enough depending on $\|\phi\|_{L_x^2 H_y^1}$ and ϵ_1 .

Therefore, $\int_0^t e^{i(t-\tau)\Delta_{\mathbb{R}^2 \times \mathbb{T}}} e_{\lambda_n}(\tau) d\tau$ is small enough in $L_{t,x}^4 H_y^1 \cap L_t^\infty L_x^2 H_y^1$, for λ_n large enough. We still need to verify the easier assumptions of Theorem 2.6.

By Plancherel's theorem, (3.11), Lemma 3.10, Hölder's inequality, and the scattering theorem of the cubic resonant nonlinear Schrödinger system (cf. Re-

mark 3.11), we have

$$\begin{aligned} \|V_{\lambda_n}\|_{L_t^\infty L_x^2 H_y^1(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{T})} &\lesssim \left(\sum_{j \in \mathbb{Z}} \langle j \rangle^2 \left(\|v_j(0, x)\|_{L_x^2}^2 + \left\| \sum_{(j_1, j_2, j_3) \in \mathcal{R}(j)} v_{j_1} \bar{v}_{j_2} v_{j_3} \right\|_{L_{t,x}^{4/3}}^2 \right) \right)^{1/2} \\ &\lesssim \|\vec{v}_0\|_{L_x^2 h^1} + \left(\sum_j \langle j \rangle^2 \|v_j\|_{L_{t,x}^4}^2 \right)^{3/2} \lesssim \|\vec{v}_0\|_{L_x^2 h^1} + \|\vec{v}_0\|_{L_x^2 h^1}^3, \end{aligned}$$

and

$$\begin{aligned} \|V_{\lambda_n}\|_{L_t^4 L_x^4 H_y^1(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{T})} &\lesssim \|V_{\lambda_n}(0, x, y)\|_{L_x^2 H_y^1} + \|(i\partial_t + \Delta_{\mathbb{R}^2 \times \mathbb{T}})V_{\lambda_n}\|_{L_{t,x}^{4/3} H_y^1} \\ &\sim \left(\sum_{j \in \mathbb{Z}} \langle j \rangle^2 \|v_j(0, x)\|_{L_x^2}^2 \right)^{1/2} + \left\| \left(\sum_{j \in \mathbb{Z}} \langle j \rangle^2 |(i\partial_t + \Delta_{\mathbb{R}^2})v_j(t, x)|^2 \right)^{1/2} \right\|_{L_{t,x}^{4/3}} \\ &\lesssim \|\vec{v}_0\|_{L_x^2 h^1} + \left\| \left(\sum_j \langle j \rangle^2 |v_j(t, x)|^2 \right)^{1/2} \right\|_{L_{t,x}^4}^3 \lesssim \|\vec{v}_0\|_{L_x^2 h^1} + \|\vec{v}_0\|_{L_x^2 h^1}^3. \end{aligned}$$

Moreover, by Plancherel's theorem and (3.10), we have

$$\|u_n(0) - V_{\lambda_n}(0)\|_{L_x^2 H_y^1} = \|P_{\leq \lambda_n^q} \vec{\phi} - \vec{v}_0\|_{L_x^2 h^1} \lesssim \epsilon_1,$$

when n large enough. Applying Theorem 2.6, we conclude that for λ_n (depending on \vec{v}_0) large enough, the solution u_n of (3.9) exists globally and

$$\|u_n - V_{\lambda_n}\|_{L_t^\infty L_x^2 H_y^1 \cap L_{t,x}^4 H_y^1(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{T})} \lesssim \epsilon_1,$$

which ends the proof in the case $t_n = 0$.

When $t_n \rightarrow \pm\infty$, v_j is the solution of the cubic resonant nonlinear Schrödinger system with

$$\|v_j(t) - e^{it\Delta_{\mathbb{R}^2}} v_{0,j}\|_{L_x^2 h^1} \rightarrow 0, \text{ as } t \rightarrow \pm\infty.$$

By the argument in the previous case, we can also obtain the result.

4. Existence of an almost-periodic solution

In this section, we will show the existence in $H_{x,y}^1(\mathbb{R}^2 \times \mathbb{T})$ of an almost-periodic solution by the profile decomposition. By Theorem 2.8, to prove the scattering of the solution of (1.1), we only need to show the finiteness of the space-time norm $\|\cdot\|_{L_{t,x}^4 H_y^{1-\epsilon_0}}$ of the solution u of (1.1). Define

$$\Lambda(L) = \sup \|u\|_{L_t^4 L_x^4 H_y^{1-\epsilon_0}(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{T})},$$

where the supremum is taken over all global solutions $u \in C_t^0 H_{x,y}^1$ of (1.1) obeying $\mathcal{E}(u(t)) + \frac{1}{2}\mathcal{M}(u(t)) \leq L$.

By the local wellposedness theory, $\Lambda(L) < \infty$ for L sufficiently small. In addition, define $L_{\max} = \sup\{L : \Lambda(L) < \infty\}$. Our goal is to prove $L_{\max} = \infty$. Suppose to the contradiction $L_{\max} < \infty$, we will show a Palais–Smale type theorem.

Proposition 4.1 (Palais–Smale condition modulo symmetries in $H_{x,y}^1(\mathbb{R}^2 \times \mathbb{T})$). *Assume that $L_{\max} < \infty$. Let $\{t_n\}_n$ be an arbitrary sequence of real numbers, and let $\{u_n\}$ be a sequence of solutions to (1.1) satisfying, as $n \rightarrow \infty$,*

$$(4.1) \quad \mathcal{E}(u_n(t)) + \frac{1}{2}\mathcal{M}(u_n(t)) \rightarrow L_{\max},$$

$$(4.2) \quad \|u_n\|_{L_{t,x}^4 H_y^{1-\epsilon_0}((-\infty, t_n) \times \mathbb{R}^2 \times \mathbb{T})} \rightarrow \infty, \quad \|u_n\|_{L_{t,x}^4 H_y^{1-\epsilon_0}((t_n, \infty) \times \mathbb{R}^2 \times \mathbb{T})} \rightarrow \infty,$$

and such that $u_n \in C_t^0 H_{x,y}^1((-\infty, \infty) \times \mathbb{R}^2 \times \mathbb{T})$. Then, after passing to a subsequence, there exists a sequence $x_n \in \mathbb{R}^2$ and $w \in H^1(\mathbb{R}^2 \times \mathbb{T})$ such that

$$u_n(x + x_n, y, t_n) \rightarrow w(x, y) \quad \text{in } H_{x,y}^1(\mathbb{R}^2 \times \mathbb{T}), \quad \text{as } n \rightarrow \infty.$$

Proof. By replacing $u_n(t)$ with $u_n(t + t_n)$, we may assume $t_n = 0$. Applying Proposition 3.1 to $\{u_n(0)\}_n$, after passing to a subsequence, we have

$$u_n(0, x, y) = \sum_{j=1}^J \frac{1}{\lambda_n^j} e^{ix\xi_n^j} (e^{it_n^j \Delta_{\mathbb{R}^2}} P_n^j \phi^j) \left(\frac{x - x_n^j}{\lambda_n^j}, y \right) + w_n^J(x, y).$$

The remainder has asymptotically trivial linear evolution:

$$(4.3) \quad \limsup_{n \rightarrow \infty} \|e^{it \Delta_{\mathbb{R}^2 \times \mathbb{T}}} w_n^J\|_{L_t^4 L_x^4 H_y^{1-\epsilon_0}} \rightarrow 0, \quad \text{as } J \rightarrow \infty,$$

and we also have the asymptotic decoupling of energy and mass, for arbitrary J :

$$(4.4) \quad \lim_{n \rightarrow \infty} \left(\mathcal{E}(u_n(0)) - \sum_{j=1}^J \mathcal{E} \left(\frac{1}{\lambda_n^j} e^{ix\xi_n^j} (e^{it_n^j \Delta_{\mathbb{R}^2}} P_n^j \phi^j) \left(\frac{x - x_n^j}{\lambda_n^j}, y \right) \right) - \mathcal{E}(w_n^J) \right) = 0,$$

$$(4.5) \quad \lim_{n \rightarrow \infty} \left(\mathcal{M}(u_n(0)) - \sum_{j=1}^J \mathcal{M} \left(\frac{1}{\lambda_n^j} e^{ix\xi_n^j} (e^{it_n^j \Delta_{\mathbb{R}^2}} P_n^j \phi^j) \left(\frac{x - x_n^j}{\lambda_n^j}, y \right) \right) - \mathcal{M}(w_n^J) \right) = 0.$$

There are two possibilities:

Case 1.

$$\sup_j \limsup_{n \rightarrow \infty} \left(\mathcal{E} + \frac{1}{2}\mathcal{M} \right) \left(\frac{1}{\lambda_n^j} e^{ix\xi_n^j} (e^{it_n^j \Delta_{\mathbb{R}^2}} P_n^j \phi^j) \left(\frac{x - x_n^j}{\lambda_n^j}, y \right) \right) = L_{\max}.$$

Combining (4.4), (4.5) with the fact that ϕ^j are nontrivial in $L_x^2 H_y^1$, we deduce that

$$u_n(0, x, y) = \frac{1}{\lambda_n} e^{ix\xi_n} (e^{it_n \Delta_{\mathbb{R}^2}} \phi) \left(\frac{x - x_n}{\lambda_n}, y \right) + w_n(x, y),$$

with $\lim_{n \rightarrow \infty} \|w_n\|_{H_{x,y}^1} = 0$. Assume $\lambda_n \rightarrow \infty$. Proposition 3.9 implies that for all large n , there exists a unique solution u_n on \mathbb{R} with

$$u_n(0, x, y) = \frac{1}{\lambda_n} e^{ix\xi_n} (e^{it_n \Delta_{\mathbb{R}^2}} \phi) \left(\frac{x - x_n}{\lambda_n}, y \right)$$

and

$$\limsup_{n \rightarrow \infty} \|u_n\|_{L_t^4 L_x^4 H_y^{1-\epsilon_0}(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{T})} \leq C(L_{\max}),$$

which is a contradiction with (4.2).

Therefore, $\lambda_n \equiv 1$, and $u_n(0, x, y) = e^{ix\xi_n} (e^{it_n \Delta_{\mathbb{R}^2}} \phi)(x - x_n, y) + w_n(x, y)$. If $t_n \equiv 0$, by the fact ξ_n is bounded, this is precisely the conclusion. If $t_n \rightarrow -\infty$, by the Galilean transform, we observe

$$\begin{aligned} & \left\| e^{it \Delta_{\mathbb{R}^2} \times \mathbb{T}} (e^{ix\xi_n} (e^{it_n \Delta_{\mathbb{R}^2}} \phi)(x - x_n, y)) \right\|_{L_{t,x}^4 H_y^{1-\epsilon_0}((-\infty, 0) \times \mathbb{R}^2 \times \mathbb{T})} \\ &= \left\| e^{-it|\xi_n|^2} e^{ix\xi_n} (e^{it \Delta_{\mathbb{R}^2}} (e^{it_n \Delta_{\mathbb{R}^2}} \phi)(\cdot - x_n, y))(x - 2t\xi_n) \right\|_{L_{t,x}^4 H_y^{1-\epsilon_0}((-\infty, 0) \times \mathbb{R}^2 \times \mathbb{T})} \\ &= \left\| e^{i(t+t_n) \Delta_{\mathbb{R}^2}} \phi \right\|_{L_{t,x}^4 H_y^{1-\epsilon_0}((-\infty, 0) \times \mathbb{R}^2 \times \mathbb{T})} = \left\| e^{it \Delta_{\mathbb{R}^2}} \phi \right\|_{L_{t,x}^4 H_y^{1-\epsilon_0}((-\infty, t_n) \times \mathbb{R}^2 \times \mathbb{T})} \rightarrow 0, \end{aligned}$$

as $n \rightarrow \infty$. Using Theorem 2.3, we see that, for n large enough,

$$\|u_n\|_{L_{t,x}^4 H_y^{1-\epsilon_0}((-\infty, 0) \times \mathbb{R}^2 \times \mathbb{T})} \leq 2\delta_0 < \infty,$$

which contradicts (4.2). The case $t_n \rightarrow \infty$ is similar.

Case 2. For some $\delta > 0$,

$$\sup_j \limsup_{n \rightarrow \infty} \left(\mathcal{E} + \frac{1}{2} \mathcal{M} \right) \left(\frac{1}{\lambda_n^j} e^{ix\xi_n^j} (e^{it_n^j \Delta_{\mathbb{R}^2}} P_n^j \phi^j) \left(\frac{x - x_n^j}{\lambda_n^j}, y \right) \right) \leq L_{\max} - 2\delta.$$

By the definition of L_{\max} , there exist global solution v_n^j to

$$\begin{cases} i \partial_t v_n^j + \Delta_{\mathbb{R}^2 \times \mathbb{T}} v_n^j = |v_n^j|^2 v_n^j, \\ v_n^j(0, x, y) = \frac{1}{\lambda_n^j} e^{ix\xi_n^j} (e^{it_n^j \Delta_{\mathbb{R}^2}} \phi^j) \left(\frac{x - x_n^j}{\lambda_n^j}, y \right), \end{cases}$$

satisfying

$$(4.6) \quad \|v_n^j\|_{L_{t,x}^4 H_y^{1-\epsilon_0}}^2 \lesssim_{L_{\max}, \delta} \left(\mathcal{E} + \frac{1}{2} \mathcal{M} \right) \left(\frac{1}{\lambda_n^j} e^{ix\xi_n^j} (e^{it_n^j \Delta_{\mathbb{R}^2}} P_n^j \phi^j) \left(\frac{x - x_n^j}{\lambda_n^j}, y \right) \right).$$

Put

$$u_n^J = \sum_{j=1}^J v_n^j + e^{it \Delta_{\mathbb{R}^2} \times \mathbb{T}} w_n^J.$$

Then we have $u_n^J(0) = u_n(0)$. We claim that for sufficiently large J and n , u_n^J is an approximate solution to u_n in the sense of the Theorem 2.6. Then we have the finiteness of $\|u_n\|_{L_{t,x}^4 H_y^{1-\epsilon_0}(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{T})}$, which contradicts with (4.2).

To verify the claim, we need to check that u_n^J satisfies the following properties:

- (i) $\limsup_{n \rightarrow \infty} \|u_n^J\|_{L_t^4 L_x^4 H_y^{1-\epsilon_0}} \lesssim_{L_{\max}, \delta} 1$, uniformly in J ;
- (ii) $\limsup_{n \rightarrow \infty} \|e_n^J\|_{L_{t,x}^{4/3} H_y^{1-\epsilon_0}} \rightarrow 0$, as $J \rightarrow J^*$, where $e_n^J = (i \partial_t + \Delta_{\mathbb{R}^2 \times \mathbb{T}}) u_n^J - |u_n^J|^2 u_n^J$.

The verification of (i) relies on the asymptotic decoupling of the nonlinear profiles v_n^j , which we record in the following lemma. Similarly to the proof in [10] to deal with the quintic nonlinear Schrödinger equation on $\mathbb{R} \times \mathbb{T}^2$, we can obtain the following lemma from Proposition 3.9. We also refer to [5], [6] for similar arguments.

Lemma 4.2 (Decoupling of nonlinear profiles). *Let v_n^j be the nonlinear solutions defined above. Then, for $j \neq k$,*

$$(4.7) \quad \|\langle \nabla_y \rangle^{1-\epsilon_0} v_n^j \cdot \langle \nabla_y \rangle^{1-\epsilon_0} v_n^k\|_{L_{t,x}^2 L_y^1} \rightarrow 0, \quad \text{and} \quad \|v_n^k \cdot \langle \nabla_y \rangle^{1-\epsilon_0} v_n^j\|_{L_{t,x}^2 L_y^2} \rightarrow 0,$$

as $n \rightarrow \infty$.

Let us verify claim (i) above. By (4.6) and Lemma 4.2, we have

$$\begin{aligned} \left\| \sum_{j=1}^J v_n^j \right\|_{L_{t,x}^4 H_y^{1-\epsilon_0}}^4 &\lesssim \left(\sum_{j=1}^J \|v_n^j\|_{L_{t,x}^4 H_y^{1-\epsilon_0}}^2 + \sum_{j \neq k} \|\langle \nabla_y \rangle^{1-\epsilon_0} v_n^j \cdot \langle \nabla_y \rangle^{1-\epsilon_0} v_n^k\|_{L_{t,x}^2 L_y^1} \right)^2 \\ &\lesssim \left(\sum_{j=1}^J \left(\mathcal{E} + \frac{1}{2} \mathcal{M} \right) \left(\frac{1}{\lambda_n^j} e^{ix\xi_n^j} (e^{it_n^j \Delta_{\mathbb{R}^2}} P_n^j \phi^j) \left(\frac{x - x_n^j}{\lambda_n^j}, y \right) \right) \right)^2 + o_J(1). \end{aligned}$$

The energy and mass decoupling implies

$$\sum_{j=1}^J \left(\mathcal{E} + \frac{1}{2} \mathcal{M} \right) \left(\frac{1}{\lambda_n^j} e^{ix\xi_n^j} (e^{it_n^j \Delta_{\mathbb{R}^2}} P_n^j \phi^j) \left(\frac{x - x_n^j}{\lambda_n^j}, y \right) \right) \leq L_{\max}.$$

Together with (4.3), we obtain

$$(4.8) \quad \lim_{J \rightarrow J^*} \limsup_{n \rightarrow \infty} \|u_n^J\|_{L_t^4 L_x^4 H_y^{1-\epsilon_0}} \lesssim_{L_{\max}, \delta} 1.$$

It remains to check property (ii) above. By the definition of u_n^J , we decompose

$$\begin{aligned} e_n^J &= (i\partial_t + \Delta_{\mathbb{R}^2 \times \mathbb{T}}) u_n^J - |u_n^J|^2 u_n^J \\ &= \sum_{j=1}^J |v_n^j|^2 v_n^j - \left| \sum_{j=1}^J v_n^j \right|^2 \sum_{j=1}^J v_n^j + |u_n^J - e^{it\Delta_{\mathbb{R}^2 \times \mathbb{T}}} w_n^J|^2 (u_n^J - e^{it\Delta_{\mathbb{R}^2 \times \mathbb{T}}} w_n^J) - |u_n^J|^2 u_n^J. \end{aligned}$$

First consider

$$\sum_{j=1}^J |v_n^j|^2 v_n^j - \left| \sum_{j=1}^J v_n^j \right|^2 \sum_{j=1}^J v_n^j.$$

Thus by the fractional chain rule, the Minkowski and the Hölder inequalities, the Sobolev embedding, (4.7) and (4.6), we have

$$\begin{aligned} &\left\| \sum_{j=1}^J |v_n^j|^2 v_n^j - \left| \sum_{j=1}^J v_n^j \right|^2 \sum_{j=1}^J v_n^j \right\|_{L_{t,x}^{4/3} H_y^{1-\epsilon_0}} \\ &\lesssim \sum_{j \neq k} \|v_n^k \langle \nabla_y \rangle^{1-\epsilon_0} v_n^j\|_{L_{t,x,y}^2} \|v_n^k\|_{L_{t,x}^4 H_y^{1-\epsilon_0}} + \sum_{j \neq k} \|v_n^k \langle \nabla_y \rangle^{1-\epsilon_0} v_n^j\|_{L_{t,x,y}^2} \|v_n^j\|_{L_{t,x}^4 H_y^{1-\epsilon_0}} \\ &\lesssim o_J(1), \quad \text{as } n \rightarrow \infty. \end{aligned}$$

We now estimate $|u_n^J - e^{it\Delta_{\mathbb{R}^2 \times \mathbb{T}} w_n^J}|^2 (u_n^J - e^{it\Delta_{\mathbb{R}^2 \times \mathbb{T}} w_n^J}) - |u_n^J|^2 u_n^J$. By the fractional chain rule, Hölder's inequality, and the Sobolev embedding, we have

$$\begin{aligned} & \left\| |u_n^J - e^{it\Delta_{\mathbb{R}^2 \times \mathbb{T}} w_n^J}|^2 (u_n^J - e^{it\Delta_{\mathbb{R}^2 \times \mathbb{T}} w_n^J}) - |u_n^J|^2 u_n^J \right\|_{L_t^{4/3} L_x^{4/3} H_y^{1-\epsilon_0}} \\ & \lesssim \left(\|u_n^J\|_{H_y^{1-\epsilon_0}}^2 + \|e^{it\Delta_{\mathbb{R}^2 \times \mathbb{T}} w_n^J}\|_{H_y^{1-\epsilon_0}}^2 \right) \|e^{it\Delta_{\mathbb{R}^2 \times \mathbb{T}} w_n^J}\|_{H_y^{1-\epsilon_0}} \|u_n^J\|_{L_{t,x}^{4/3}} \\ & \lesssim \left(\|u_n^J\|_{L_{t,x}^4 H_y^{1-\epsilon_0}}^2 + \|e^{it\Delta_{\mathbb{R}^2 \times \mathbb{T}} w_n^J}\|_{L_{t,x}^4 H_y^{1-\epsilon_0}}^2 \right) \|e^{it\Delta_{\mathbb{R}^2 \times \mathbb{T}} w_n^J}\|_{L_{t,x}^4 H_y^{1-\epsilon_0}}. \end{aligned}$$

Using (4.8), and the decay property (4.3), we get

$$\limsup_{n \rightarrow \infty} \left\| |u_n^J - e^{it\Delta_{\mathbb{R}^2 \times \mathbb{T}} w_n^J}|^2 (u_n^J - e^{it\Delta_{\mathbb{R}^2 \times \mathbb{T}} w_n^J}) - |u_n^J|^2 u_n^J \right\|_{L_{t,x}^4 H_y^{1-\epsilon_0}} \rightarrow 0$$

as $J \rightarrow J^*$. \square

Arguing as in [6], the proof of Proposition 4.1 implies the following result.

Theorem 4.3 (Existence of the almost-periodic solution). *Assume that $L_{\max} < \infty$. Then there exists $u_c \in C_t^0 H_{x,y}^1(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{T})$ solving (1.1) satisfying*

$$\mathcal{E}(u_c) + \frac{1}{2} \mathcal{M}(u_c) = L_{\max}, \quad \|u_c\|_{L_{t,x}^4 H_y^{1-\epsilon_0}(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{T})} = \infty.$$

Furthermore, u_c is almost periodic in the sense that $\forall \eta > 0$, there is $C(\eta) > 0$ such that

$$(4.9) \quad \int_{|x+x(t)| \geq C(\eta)} \|u_c(t, x, y)\|_{H_y^1(\mathbb{T})}^2 dx + \int_{|x+x(t)| \geq C(\eta)} \|\nabla_x u_c(t, x, y)\|_{L_y^2(\mathbb{T})}^2 dx < \eta$$

for all $t \in \mathbb{R}$, where $x : \mathbb{R} \rightarrow \mathbb{R}^2$ is a Lipschitz function with $\sup_{t \in \mathbb{R}} |x'(t)| \lesssim 1$.

5. Rigidity theorem

In this section, we will exclude the almost-periodic solution in Theorem 4.3 by using the interaction Morawetz action with the weight function taken appropriately.

Proposition 5.1 (Non-existence of the almost-periodic solution). *The almost-periodic solution u_c in Theorem 4.3 does not exist.*

Proof. Define the interaction Morawetz action

$$M(t) = \int_{\mathbb{R}^2 \times \mathbb{T}} \int_{\mathbb{R}^2 \times \mathbb{T}} \Im(\bar{u}_c(t, x, y) \nabla_x u_c(t, x, y)) \nabla_x a(x - \tilde{x}) |u_c(t, \tilde{x}, \tilde{y})|^2 dx dy d\tilde{x} d\tilde{y},$$

where a is a radial function on \mathbb{R}^2 defined, as in [7], [20], by

$$a(r) = \begin{cases} \frac{r^2}{2r_0} (1 + \frac{1}{2} \log \frac{r_0}{r}), & r < r_0, \\ r - r_0/2, & r \geq r_0, \end{cases}$$

for some positive constant r_0 . We have $\Delta a \geq 0$ and $\left\| \frac{3}{4r_0} + \frac{1}{2r_0} \log \frac{r_0}{r} \right\|_{L^2(r \leq r_0)} \lesssim 1$, and for $r > r_0$, $\Delta a = 1/r$.

Letting $r_0 \rightarrow 0$, we have $\forall T_0 > 0$,

$$(5.1) \quad \int_{-T_0}^{T_0} \int_{\mathbb{R}^2 \times \mathbb{T}} \left| |\nabla_x|^{1/2} (|u_c(t, x, y)|^2) \right|^2 dx dy dt \lesssim \int_{-T_0}^{T_0} M'(t) dt \lesssim \sup_{t \in [-T_0, T_0]} |M(t)|.$$

By the Hölder, the Minkowski, and the Sobolev inequalities, and the fact that \mathbb{T} is compact, we have

$$(5.2) \quad \begin{aligned} & \int_{|x+x(t)| \leq C \left(\frac{m_0}{100}\right)} \|u_c(t, x, y)\|_{L_y^2}^2 dx \lesssim C \left(\frac{m_0}{100}\right)^{3/2} \|u_c(t, x, y)\|_{L_x^8 L_y^2}^2 \\ & \lesssim C \left(\frac{m_0}{100}\right)^{3/2} \|u_c(t, x, y)\|_{L_y^2 L_x^8}^2 \lesssim C \left(\frac{m_0}{100}\right)^{3/2} \left\| |\nabla_x|^{1/2} (|u_c(t, x, y)|^2) \right\|_{L_y^4 L_x^2}^2 \\ & \lesssim C \left(\frac{m_0}{100}\right)^{3/2} \left\| |\nabla_x|^{1/2} (|u_c(t, x, y)|^2) \right\|_{L_{x,y}^2}, \end{aligned}$$

where $m_0 = \mathcal{M}(u_c)$.

By (4.9) and the conservation of mass, we have

$$(5.3) \quad \frac{m_0}{2} \leq \int_{|x+x(t)| \leq C(m_0/100)} \|u_c(t, x, y)\|_{L_y^2}^2 dx.$$

By Hölder's inequality, the fact that $|\nabla a|$ is bounded, and $u_c \in C_t^0 H_{x,y}^1(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{T})$,

$$(5.4) \quad |M(t)| \lesssim \|u_c\|_{L_t^\infty L_{x,y}^2}^3 \|\nabla_x u_c\|_{L_t^\infty L_{x,y}^2} \lesssim 1.$$

Therefore, $\forall T_0 > 0$, by (5.3), (5.2), (5.1) and (5.4),

$$\begin{aligned} m_0^2 T_0 &= \int_0^{T_0} m_0^2 dt \lesssim \int_{-T_0}^{T_0} \left(\int_{|x+x(t)| \leq C(m_0/100)} \|u_c(t, x, y)\|_{L_y^2}^2 dx \right)^2 dt \\ &\lesssim C \left(\frac{m_0}{100}\right) \left\| |\nabla_x|^{1/2} (|u_c(t, x, y)|^2) \right\|_{L_{t,x,y}^2}^2 \lesssim C \left(\frac{m_0}{100}\right) \sup_{t \in [-T_0, T_0]} |M(t)| \lesssim C \left(\frac{m_0}{100}\right). \end{aligned}$$

Letting $T_0 \rightarrow \infty$, we obtain a contradiction unless $u_c \equiv 0$, which is impossible due to $\|u_c\|_{L_{t,x}^4 H_y^{1-\epsilon_0}(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{T})} = \infty$. \square

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