Rev. Mat. Iberoam. (online first)
DOI 10.4171/RMI/1152

Zariski K3 surfaces

Toshiyuki Katsura and Matthias Schütt

Abstract. We construct Zariski K3 surfaces of Artin invariant 1, 2 and 3 in many characteristics. In particular, we prove that any supersingular Kummer surface is Zariski if $p \not\equiv 1 \bmod 12$. Our methods combine different approaches such as quotients by the group scheme α_p, Kummer surfaces, and automorphisms of hyperelliptic curves.

1. Introduction

Let k be an algebraically closed field of characteristic p, and X be an algebraic variety of dimension n over k. X is said to be unirational if there exists a dominant rational map from $\mathbb{P}^n$ to X. Unirational algebraic curves are automatically rational, and it is a classical fact that the same holds for complex algebraic surfaces as a consequence of Castelnuovo's criterion for rationality. In positive characteristic, however, this is no longer true as was first shown by Zariski ([36], p. 314). In essence, this is due to the impact of inseparable maps which leads to the following definition:

Definition 1.1. A (non-rational) algebraic surface S is called a Zariski surface if there exists a purely inseparable dominant rational map $\mathbb{P}^2 \to S$ of degree p.

In this sense, Zariski surfaces can be considered the first non-rational unirational surfaces. Note that automatically $\mathrm{H}^2_{et}(S, \mathbb{Q}_\ell)$ is spanned by algebraic cycles, i.e., Zariski surfaces are supersingular (cf. Shioda [29], Corollary 2). This leads to the question to what extent the converse may hold true (as initiated by Shioda [30]). Here we concentrate on supersingular K3 surfaces; partly this is due to their striking history in this problem, dating back to the first discoveries of Zariski and unirational K3 surfaces by Artin [1] and Shioda [29], but mostly because the unirationality of supersingular K3 surfaces, known in characteristic 2 by Rudakov–Shafarevich [23], seems within reach in all characteristics by now (though

Mathematics Subject Classification (2010): Primary 14J28; Secondary 14J25, 14J50, 14K02, 14M20.
Keywords: K3 surface, Zariski surface, abelian surface, infinitesimal group scheme, automorphism.

the arguments in Liedtke [15] and Bragg–Lieblich [5] have recently been challenged by Bragg–Lieblich [6]).

Question 1.2. (i) *Is any supersingular* K3 *surface a Zariski surface?*

(ii) *In particular, is any supersingular Kummer surface a Zariski surface?*

While the first question has an affirmative answer if the characteristic $p = 2$ (cf. Rudakov–Shafarevich [23], p. 151, where the proof of the corollary indeed implies the surfaces to be Zariski; see also Shimada [27]), a general answer might be too much to ask for at this time. Meanwhile the second question was prompted by the initial result of Shioda [31], Theorem 1.1, that all supersingular Kummer surfaces are unirational. We will develop an affirmative answer to (ii) for 75% of all characteristics:

Theorem 1.3. *Let $p > 2$ such that $p \not\equiv 1 \bmod 12$. Then any supersingular Kummer surface in characteristic p is a Zariski surface.*

In addition, we will provide a plentitude of new Zariski K3 surfaces. Our results are summarized in the following theorem where only part (i) seems to have been known before (see, e.g., Katsura [13], Theorem 5.10).

Theorem 1.4. *There are Zariski* K3 *surfaces of Artin invariant σ over an algebraically closed field of characteristic p under the following conditions:*

(i) $\sigma = 1$ *and* $p \not\equiv 1 \bmod 12$.

(ii) $\sigma = 2$ *and* $p \not\equiv 1, 49 \bmod 60$.

(iii) $\sigma = 3$ *and* $p \equiv 3, 5 \bmod 7$.

The proofs of Theorems 1.3 and 1.4 proceed by explicit geometric constructions. We combine different approaches such as quotients by the group scheme α_p, Kummer surfaces, lattice theory, and automorphisms of elliptic and hyperelliptic curves. We supplement the theorems with additional results in two directions: an isolated Zariski K3 surface of Artin invariant $\sigma = 1$ over $\mathbb{F}_{13}$ (Example 7.8 – this surface also is Kummer), and an abundance of Zariski elliptic surfaces (Lemma 7.1, Remark 7.7).

2. Preliminaries on supersingular abelian surfaces

For later use, we start by reviewing parts of the theory of abelian surfaces in positive characteristic. Throughout the paper, we fix an algebraically closed field k of characteristic $p > 0$. An abelian surface A is said to be *supersingular* (resp. *superspecial*) if it is isogenous (resp. isomorphic) to a product of two supersingular elliptic curves. In terms of the Néron–Severi lattice, this phrases as

$$\rho(A) = \operatorname{rank} \operatorname{NS}(A) = 6 \quad \text{with discriminant} \ \ -p^{2\sigma}.$$

Here σ is called *Artin invariant* and equals 1 if A is superspecial, and 2 otherwise. By definition a superspecial abelian surface is supersingular (cf. Oort [21], Theorem 2), and a superspecial abelian surface is unique up to isomorphism (cf. Shioda [32], Theorem 3.5). In this section, we recall some results on the Néron–Severi group of the superspecial abelian surface.

Let E be a supersingular elliptic curve defined over k. We consider the superspecial abelian surface $E_1 \times E_2$ with $E_1 = E_2 = E$. We denote by O_E the zero point of E. We put $X = E_1 \times \{O_{E_2}\} + \{O_{E_1}\} \times E_2$, which is a principal polarization on $E_1 \times E_2$. By abuse of notation, we sometimes denote the fibers $E_1 \times \{O_{E_2}\}$ (resp. $\{O_{E_1}\} \times E_2$) by E_1 (resp. by E_2). We set $\mathcal{O} = \mathrm{End}(E)$ and $B = \mathrm{End}^0(E) = \mathrm{End}(E) \otimes \mathbb{Q}$. Here, B is a quaternion division algebra over the rational number field $\mathbb{Q}$ with discriminant p, and $\mathcal{O}$ is a maximal order of B (cf. Mumford [17], §22). For an element $a \in B$, we denote by $\bar{a}$ the image under the canonical involution. We have a natural identification of $\mathrm{End}(E_1 \times E_2)$ with the ring $\mathrm{M}_2(\mathcal{O})$ of two-by-two matrices with coefficients in $\mathcal{O}$. Here, the action of $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{M}_2(\mathcal{O})$ is given by

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : \begin{array}{ccc} E_1 \times E_2 & \longrightarrow & E_1 \times E_2 \\ (x, y) & \mapsto & (\alpha(x) + \beta(y), \gamma(x) + \delta(y)). \end{array}$$

By a divisor L we usually mean the divisor class represented by L in $\mathrm{NS}(E_1 \times E_2)$ if confusion is unlikely to occur. With this convention, a divisor L yields a homomorphism

$$\begin{array}{cccc} \varphi_L : & E_1 \times E_2 & \longrightarrow & \mathrm{Pic}^0(E_1 \times E_2) \\ & x & \mapsto & T_x^* L - L, \end{array}$$

where T_x is the translation by $x \in E_1 \times E_2$ (cf. Mumford [17]). We set

$$H = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{M}_2(\mathcal{O}) \;\middle|\; \alpha, \delta \in \mathbb{Z},\ \gamma, \beta \in \mathcal{O},\ \gamma = \bar{\beta} \right\}.$$

Note that for an automorphism g of $E_1 \times E_2$, we can regard g as an element $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ with $\alpha, \beta, \gamma, \delta \in \mathcal{O}$ of $\mathrm{M}_2(\mathcal{O})$. By this identification, we have ${}^t\bar{g} = \left(\begin{smallmatrix} \bar{\alpha} & \bar{\gamma} \\ \bar{\beta} & \bar{\delta} \end{smallmatrix}\right)$ as an element of $\mathrm{M}_2(\mathcal{O})$. We will use the following theorem, which is well known to specialists (see, for instance, Katsura [14], Theorem 2.1).

Theorem 2.1. *The homomorphism*

$$\begin{array}{cccc} j : & \mathrm{NS}(E_1 \times E_2) & \longrightarrow & H \\ & L & \mapsto & \varphi_X^{-1} \circ \varphi_L \end{array}$$

is bijective. By this correspondence, we have

$$j(E_1) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad j(E_2) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

For $L_1, L_2 \in \mathrm{NS}(E_1 \times E_2)$ such that

$$j(L_1) = \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix}, \quad j(L_2) = \begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{pmatrix},$$

the intersection number $L_1 \cdot L_2$ is given by

$$L_1 \cdot L_2 = \alpha_2 \delta_1 + \alpha_1 \delta_2 - \gamma_1 \beta_2 - \gamma_2 \beta_1.$$

In particular, for $L \in \mathrm{NS}(E_1 \times E_2)$ *such that* $j(L) = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ *we have*

$$L^2 = 2 \det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad L \cdot E_1 = \alpha, \quad L \cdot E_2 = \delta.$$

We have also $j(nD) = nj(D)$ *for an integer* n.

For $L_1, L_2 \in \mathrm{NS}(E_1 \times E_2)$ *with* $j(L_1) = g_1$ *and* $j(L_2) = g_2$ *and for an automorphism* g *of* $E_1 \times E_2$*, we have* $g^* L_1 \equiv L_2$ *if and only if* ${}^t\bar{g} g_1 g = g_2$.

Let E be a supersingular elliptic curve defined over $\mathbb{F}_p$. Such an elliptic curve exists for any $p > 0$ (cf. Waterhouse [35]). We denote by F the relative Frobenius morphism of E. For the local-local group scheme α_p of rank p, we have $\mathrm{End}(\alpha_p) \cong k$. Therefore, for $(i, j) \in k^2$ we have an inclusion

$$\epsilon : \alpha_p \stackrel{(i,j)}{\longrightarrow} \alpha_p \times \alpha_p \subset E \times E.$$

We assume $i/j \notin \mathbb{F}_{p^2}, j \neq 0$. Then, by Oort [21], the quotient surface

$$A = (E \times E)/\epsilon(\alpha_p)$$

is not superspecial. Let

$$\pi : E \times E \longrightarrow A$$

be the quotient map. Considering the dual abelian surface A^t of A and the dual homomorphism π^t, we have a commutative diagram for $D \in \mathrm{NS}(A)$:

$$\begin{array}{ccccc} E \times E & \stackrel{\varphi_{\pi^* D}}{\longrightarrow} & E \times E & \stackrel{\varphi_X^{-1}}{\longrightarrow} & E \times E \\ \pi \downarrow & & \uparrow \pi^t & & \\ A & \stackrel{\varphi_D}{\longrightarrow} & A^t. & & \end{array}$$

Here, we identify the dual abelian surface $(E \times E)^t$ with $E \times E$. We set

$$H' = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{M}_2(\mathcal{O}) \middle| \begin{array}{c} \alpha, \delta \in \mathbb{Z}, \quad \gamma, \beta \in \mathcal{O}, \ \gamma = \bar{\beta}, \\ p \mid \alpha, \ p \mid \delta, \ \beta \in F\mathcal{O} = F\mathrm{End(E)} \end{array} \right\} \subset H \subset \mathrm{M}_2(\mathcal{O}).$$

Proposition 2.2. *The homomorphism*

$$j \circ \pi^* : \mathrm{NS}(A) \longrightarrow H$$

induces an isomorphism $j \circ \pi^* : \mathrm{NS}(A) \longrightarrow H'$ *of additive groups.*

Proof. The proof is essentially the same as the one in Proposition 2.4.1 of Ibukiyama–Katsura–Oort [10]. Since we have $\dim_k \mathrm{Hom}(\alpha_p, A) = \dim_k \mathrm{Hom}(\alpha_p, A^t) = 1$, the subgroup schemes which are isomorphic to α_p are unique in A and A^t, respectively. Therefore, we have $\varphi_D^{-1}(\alpha_p) \supset \alpha_p$ for $D \in \mathrm{NS}(A)$. This implies that $\mathrm{Ker}\varphi_{\pi^* D} \supset \alpha_p \times \alpha_p$. Setting

$$j(\pi^* D) = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

and considering that α and δ are integers, we have $p \mid \alpha$, $p \mid \delta$ and $\beta, \gamma \in F\mathcal{O}$, that is, $j(\pi^* D) \in H'$.

Conversely, for $j(G) \in H'$ with $G \in \mathrm{NS}(E \times E)$, we have $\mathrm{Ker}\varphi_G \supset \alpha_p \times \alpha_p$. Using the notation in Mumford [17], for any subgroup scheme $\epsilon(\alpha_p)$ of $\alpha_p \times \alpha_p$, we have $e^G(\epsilon(\alpha_p), \epsilon(\alpha_p)) = 0$. Therefore, using the descent theory in Mumford [17], there exists a divisor $G' \in \mathrm{NS}(A)$ such that $\pi^*(G') = G$. Hence $j \circ \pi^* : \mathrm{NS}(A) \to H'$ is surjective. □

Let β_i $(i = 1, 2, 3, 4)$ be a basis of $\mathcal{O}$ over $\mathbb{Z}$. Then,

$$\left\langle \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \beta_i \\ \bar{\beta}_i & 0 \end{pmatrix} (i = 1, 2, 3, 4) \right\rangle$$

is a basis of H. There exist D_i $(i = 1, 2, 3, 4)$ such that $j(D_i) = \left(\begin{smallmatrix} 0 & \beta_i \\ \bar{\beta}_i & 0 \end{smallmatrix}\right)$. By Theorem 2.1, $\langle E_1, E_2, D_1, D_2, D_3, D_4 \rangle$ is a basis of $\mathrm{NS}(E \times E)$. Therefore, the determinant of the Gram matrix M of the given basis is equal to $-p^2$. The following proposition is well known (cf. Ogus [20], Prop. 6.9). We give an elementary proof.

Proposition 2.3. *The Artin invariant of A is equal to* 2.

Proof. A basis of H' is given by

$$\left\langle \begin{pmatrix} 0 & 0 \\ 0 & p \end{pmatrix}, \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & F\beta_i \\ \overline{F\beta_i} & 0 \end{pmatrix} (i = 1, 2, 3, 4) \right\rangle$$

By Proposition 2.2, there exist divisors D'_i $(i = 1, 2, \ldots, 6)$ in $\mathrm{NS}(A)$ such that

$$j(\pi^*(D'_1)) = \begin{pmatrix} 0 & 0 \\ 0 & p \end{pmatrix}, \quad j(\pi^*(D'_2)) = \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix},$$
$$j(\pi^*(D'_{i+2})) = \begin{pmatrix} 0 & F\beta_i \\ \overline{F\beta_i} & 0 \end{pmatrix} (i = 1, 2, 3, 4).$$

Here, $\pi^*(D'_i) = pE_i$ $(i = 1, 2)$. Since the Gram matrix $((\pi^* D'_i, \pi^* D'_j)) = (p(D'_i, D'_j))$, we have

$$\begin{aligned} p^6 \det((D'_i, D'_j)) &= \det((\pi^* D'_i, \pi^* D'_j)) \\ &= \det \begin{pmatrix} \begin{matrix} (pE_1)^2 & (pE_1, pE_2) \\ (pE_2, pE_1) & (pE_1)^2 \end{matrix} & 0 \\ 0 & ((\pi^*(D'_{i+2}), \pi^*(D'_{j+2}))_{1 \le i,j \le 4} \end{pmatrix} \\ &= \det \begin{pmatrix} \begin{matrix} 0 & p^2 \\ p^2 & 0 \end{matrix} & 0 \\ 0 & (p(D_i, D_j))_{1 \le i,j \le 4} \end{pmatrix} = p^8 \det M = -p^{10}. \end{aligned}$$

Therefore, $\det((D'_i, D'_j)) = -p^4$, that is, the Artin invariant of A is equal to 2. □

3. Generalized K3 surfaces

It is standard (outside characteristic 2) to associate to an abelian surface a K3 surface by means of the Kummer quotient. In this section, we shall discuss different constructions (inspired by Katsura [13]) which have the benefit of two compatibilities, both with the constructions in Section 2 and with purely inseparable base change as required for Zariski surfaces. We retain the notation from Section 2.

First, we assume $p \equiv 2 \pmod 3$ and consider the supersingular elliptic curve E with j-invariant zero defined by

$$E: \quad y^2 + y = x^3. \tag{3.1}$$

The elliptic curve E is endowed with an automorphism τ of order 3 defined by

$$\tau: \quad x \mapsto \omega x, \quad y \mapsto y$$

where ω denotes a primitive cube root of unity.

Remark 3.1. Some readers may be more familiar with the Weierstrass form

$$y^2 = x^3 - 1$$

for E. Outside characteristic 2, both models are isomorphic, but (3.1) comes with the advantage of being valid in characteristic 2 as well while often also yielding simpler equations.

Fact 3.2. The endomorphism ring of E can be represented as

$$\mathrm{End}(E) = \mathcal{O} = \mathbb{Z} \oplus \mathbb{Z}F \oplus \mathbb{Z}\tau \oplus \mathbb{Z}(1+F)(2+\tau)/3. \tag{3.2}$$

Proof. In characteristic $p > 2, p \equiv 2 \bmod 3$, this is Lemma 5.4 in Katsura [13] (building on Ibukiyama [9]). Meanwhile, for $p = 2$ one may verify directly that the ring $\mathcal{O}$ from (3.2) is isomorphic to the maximal order in the quaternion algebra $\left(\frac{-1,-1}{\mathbb{Q}}\right)$ of discriminant 2, for instance using that both elements $\omega_4, \omega_3 - \omega_4$ (in the notation of loc. cit.) have square -1. This clearly implies the claim. □

In $\mathrm{End}(E)$, we have the relations $F\tau = \tau^2 F$, $\bar{\tau} = \tau^2$, $\bar{F} = -F$ and

$$\overline{(1+F)(2+\tau)}/3 = 1 - \{(1+F)(2+\tau)/3\}.$$

For the sake of simplicity, we set $\eta = (1+F)(2+\tau)/3$. The multiplication is given by the following table:

	1	F	τ	η
1	1	F	τ	η
F	F	$-p$	$3\eta - 2F - \tau - 2$	$\eta - (p+1)(2+\tau)/3$
τ	τ	$-3\eta + F + \tau + 2$	$-\tau - 1$	$-2\eta + F + \tau + 1$
η	η	$(2 - p + (p+1)\tau)/3 + F - \eta$	$\eta - F - 1$	$\eta - (p+1)/3$

The automorphism $\tau \times \tau$ acts on $E \times E$. Since $\tau \times \tau$ preserves the subgroup scheme $\epsilon(\alpha_p)$, $\tau \times \tau$ induces an automorphism θ on the quotient

$$A = (E \times E)/\alpha_p.$$

Since, $\tau \times \tau$ has 9 isolated fixed point on $E \times E$ and π is a finite purely inseparable morphism, we see that θ has also 9 isolated fixed points on A. By a local calculation, the action of θ at the fixed points is given by $(s,t) \mapsto (\omega s, \omega^2 t)$. Therefore, the singularities of the quotient space $A/\langle\theta\rangle$ are all rational double points of type A_2.

Considering the action of θ on the vector space $\mathrm{H}^0(A, \Omega_A^1)$, we see that θ acts symplectically on a non-zero regular 2-form. We denote by $\mathrm{GKm}(A)$ the nonsingular complete minimal model of $A/\langle\theta\rangle$. Since A is a supersingular abelian surface, we conclude that $\mathrm{GKm}(A)$ is a supersingular K3 surface.

Theorem 3.3. *Assume $p \equiv 2 \pmod 3$. Under the notation as above, the Artin invariant of $\mathrm{GKm}(A)$ is equal to 2.*

Remark 3.4. A similar result in characteristic 2 is given in Schröer [24], Theorem 6.4.

Proof. We consider the quotient morphism $\pi\colon E \times E \longrightarrow A$. Since we have a commutative diagram

$$\begin{array}{cccc} \tau\times\tau: & E\times E & \longrightarrow & E\times E \\ & \downarrow & & \downarrow \\ \theta: & A & \longrightarrow & A, \end{array}$$

it should be clear that for $D \in \mathrm{NS}(A)$, $\theta^*(D) = D$ if and only if $(\tau\times\tau)^*(\pi^*D) = \pi^*D$. We calculate the invariants $(H')^{\langle\tau\times\tau\rangle}$. By Theorem 2.1, the action of $\tau\times\tau$ on $\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in H'$ is given by

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \mapsto \begin{pmatrix}\bar\tau & 0\\ 0 & \bar\tau\end{pmatrix}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}\tau & 0\\ 0 & \tau\end{pmatrix}.$$

The elements $\left(\begin{smallmatrix}0 & 0\\ 0 & p\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}p & 0\\ 0 & 0\end{smallmatrix}\right)$ are invariant under this action. Now, let $F(a+bF+c\tau+d\eta)$ be an element of $F\mathcal{O}$ with $a,b,c,d \in \mathbb{Z}$ which satisfies $\tau^2(F(a+bF+c\tau+d\eta)\tau = F(a+bF+c\tau+d\eta)$. Since $\tau^2F = F\tau$, we have $\tau(a+bF+c\tau+d\eta)\tau = (a+bF+c\tau+d\eta)$. Then, using the multiplication table, we have

$$a+c+d=0,\ 2a-c+d=0, \text{that is}, c=a/2,\ d=-3a/2.$$

Hence, a basis of the invariant space $(H')^{\langle\tau\times\tau\rangle}$ is given by

$$\begin{pmatrix}0 & 0\\ 0 & p\end{pmatrix},\quad \begin{pmatrix}p & 0\\ 0 & 0\end{pmatrix},\quad \begin{pmatrix}0 & p\\ p & 0\end{pmatrix},$$
$$\begin{pmatrix}0 & F(2+\tau-3\eta)\\ \overline{F(2+\tau-3\eta)} & 0\end{pmatrix} = \begin{pmatrix}0 & 2p+p\tau\\ 2p+p\bar\tau & 0\end{pmatrix}.$$

We take divisors G_i $(i=1,2,3,4)$ on $E\times E$ such that $j(G_i)$ correspond to elements of this basis in this order. Then, there exist divisors G_i' $(i=1,2,3,4)$ on A such that $\pi^*(G_i') = G_i$. The divisors G_i' $(i=1,2,3,4)$ form a basis of the invariant space $\mathrm{NS}(A)^{\langle\theta\rangle}$. By Theorem 2.1, the Gram matrix of the G_i $(i=1,2,3,4)$ is given by

$$\begin{pmatrix}0 & p^2 & 0 & 0\\ p^2 & 0 & 0 & 0\\ 0 & 0 & -2p^2 & -3p^2\\ 0 & 0 & -3p^2 & -6p^2\end{pmatrix}.$$

Its determinant is equal to $-3p^8$. Therefore, the determinant of the Gram matrix of the basis G_i' $(i=1,2,3,4)$ is equal to $-3p^4$.

Hence, in a similar way to Katsura ([13], Lemma 5.8), we conclude that the discriminant of $\mathrm{NS}(\mathrm{GKm}(A))$ is equal to $-p^4$ as claimed. □

Now, we assume $p \equiv 3 \pmod 4$. We consider a supersingular elliptic curve E with j-invariant 1728 defined by

$$E: \ y^2 = x^3 - x. \tag{3.3}$$

Define an automorphism τ of order 4 on E by

$$\tau: \ x \mapsto -x, \ \ y \mapsto iy.$$

Here i is a primitive fourth root of unity. In this case, we have

$$\mathcal{O} = \mathrm{End}(E) = \mathbb{Z} \oplus \mathbb{Z}\tau \oplus \mathbb{Z}(1+F)/2 \oplus \mathbb{Z}\tau(1+F)/2$$

(cf. Lemma 5.3 in Katsura [13]). One checks that $\tau^2 = -1$, $F\tau = \tau^3 F$, $\bar{\tau} = \tau^3$, $\bar{F} = -F$. For the sake of simplicity, we set $\eta = \tau(1+F)/2$. The multiplication is given by the following table:

	1	τ	$(1+F)/2$	η
1	1	τ	$(1+F)/2$	η
τ	τ	-1	η	$-\eta$
$(1+F)/2$	$(1+F)/2$	$\tau - \eta$	$-(1+p)/4 + (1+F)/2$	$(1+p)\tau/4$
η	η	$-1 + (1+F)/2$	$-(1+p)\tau/4 + \eta$	$-(1+p)/4$

The automorphism $\tau \times \tau$ acts on $E \times E$. Since $\tau \times \tau$ preserves the subgroup scheme $\epsilon(\alpha_p)$, $\tau \times \tau$ induces an automorphism θ on A. By a method similar to above (also see Katsura [13], Lemma 5.8), we conclude that the nonsingular complete minimal model $\mathrm{GKm}(A)$ of $A/\langle\theta\rangle$ is a supersingular K3 surface.

A basis of the invariant space $(H')^{\langle\tau\times\tau\rangle}$ is given by

$$\begin{pmatrix} 0 & 0 \\ 0 & p \end{pmatrix}, \ \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}, \ \begin{pmatrix} 0 & p \\ p & 0 \end{pmatrix}, \ \begin{pmatrix} 0 & p\tau \\ -p\tau & 0 \end{pmatrix}.$$

We take divisors G_i $(i = 1, 2, 3, 4)$ on $E \times E$ such that $j(G_i)$ correspond to elements of this basis in this order. Then, there exist divisors G'_i $(i = 1, 2, 3, 4)$ on A such that $\pi^*(G'_i) = G_i$. The divisors G'_i $(i = 1, 2, 3, 4)$ form a basis of the invariant space $\mathrm{NS}(A)^{\langle\theta\rangle}$. By Theorem 2.1, the Gram matrix of G_i $(i = 1, 2, 3, 4)$ is given by

$$\begin{pmatrix} 0 & p^2 & 0 & 0 \\ p^2 & 0 & 0 & 0 \\ 0 & 0 & -2p^2 & 0 \\ 0 & 0 & 0 & -2p^2 \end{pmatrix}.$$

The determinant of this matrix equals $-4p^8$. It follows that the determinant of the Gram matrix of the basis G'_i $(i = 1, 2, 3, 4)$ is equal to $-4p^4$.

By a similar method to the above and Katsura [13], Lemma 5.8, we obtain the following theorem. We omit the details.

Theorem 3.5. *Assume* $p \equiv 3 \pmod 4$. *Under the notation as above, the Artin invariant of* $\mathrm{GKm}(A)$ *is equal to* 2.

4. Zariski surfaces

We now turn to the problem of Zariski surfaces, in particular for K3 surfaces. We will describe an ad-hoc construction using products of hyperelliptic curves which will be very useful for Theorem 1.4. To this end, we let C be a non-singular complete model of the algebraic curve defined by

$$C: \quad y^2 + y = x^\ell \tag{4.1}$$

with an integer $\ell \geq 3$. Let ζ be a primitive ℓ-th root of unity (so ℓ is not divisible by p). Then, C has an automorphism of order ℓ defined by

$$\tau: \quad x \mapsto \zeta x, \ y \mapsto y. \tag{4.2}$$

Remark 4.1. As in the case of $\ell = 3$ from Remark 3.1, a more standard equation may consist in

$$y^2 = x^\ell - 1.$$

We put $Y = C_1 \times C$ with $C_1 = C$. We assume that the defining equation for C_1 is given by

$$C_1: \quad y_1^2 + y_1 = x_1^\ell.$$

Then, $\tau \times \tau$ acts on Y as in (4.2).

Lemma 4.2. *$Y/\langle\tau \times \tau\rangle$ is a rational surface.*

Proof. The group $G = \langle\tau \times \tau\rangle$ acts on the function field of Y via its natural action on $k(x_1, y_1, x, y)$. We set

$$z = x/x_1.$$

Then, the invariant field $k(Y)^G$ is given by $k(y, y_1, z)$ with the relation

$$z^\ell(y_1^2 + y_1) = y^2 + y.$$

This endows Y/G with the structure of a conic fibration over $\mathbb{P}^1$ with the parameter z and section $(0, 0)$, say. Therefore, this is a rational surface. □

Proposition 4.3. *Assume $p \equiv i \mod \ell$ $(2 \leq i \leq p-1)$. Then, $Y/\langle\tau \times \tau^i\rangle$ is a Zariski surface.*

Proof. We have the following commutative diagram:

$$\begin{array}{ccc} Y & \stackrel{\tau\times\tau}{\longrightarrow} & Y \\ \mathrm{id}\times F \downarrow & & \downarrow \mathrm{id}\times F \\ Y & \stackrel{\tau\times\tau^i}{\longrightarrow} & Y. \end{array}$$

Therefore, we have a purely inseparable rational map $Y/\langle\tau \times \tau\rangle \longrightarrow Y/\langle\tau \times \tau^i\rangle$. Since $Y/\langle\tau \times \tau\rangle$ is a rational surface by Lemma 4.2, we conclude that $Y/\langle\tau \times \tau^i\rangle$ is a Zariski surface. □

Now let X be a K3 surface defined over k. The surface X is said to be *supersingular* if the Picard number satisfies

$$\rho(X) = b_2(X) = 22$$

with b_2 the second Betti number. If X is supersingular, then the discriminant of the Néron–Severi group is of the form $-p^{2\sigma}$ with a positive integer $\sigma \in \{1, \ldots, 10\}$ called the *Artin invariant* (cf. Artin [1], (4.6)). In analogy with abelian surfaces, a supersingular K3 surface with $\sigma = 1$ is called superspecial. If $p \neq 2$, it is known that any superspecial K3 surface is isomorphic to the Kummer surface $\mathrm{Km}(E \times E)$ with E a supersingular elliptic curve, and that any supersingular K3 surface with Artin invariant 2 is isomorphic to a Kummer surface $\mathrm{Km}((E \times E)/\alpha_p)$ with E a supersingular elliptic curve and a suitable embedding $\alpha_p \to E \times E$ (cf. Oort [21], Ogus [20], Theorem 7.10, and see also Shioda [32]). Here, α_p is the local-local group scheme of rank 1 discussed in Section 2.

We give a natural proof for the following known theorem (see Theorem 5.10 in Katsura [13], for instance) which will later be generalized in different directions.

Theorem 4.4. *Assume* $p \not\equiv 1 \pmod{12}$. *Then, the supersingular* K3 *surface with Artin invariant* $\sigma = 1$ *is a Zariski surface.*

Note that Theorem 4.4 covers (i) of Theorem 1.4. For the first missing case of characteristic $p = 13$, see Example 7.8.

Proof. First, we assume $p \equiv 2 \bmod 3$ and take E from (3.1) – or from (4.1) with automorphism τ as in (4.2) with $\ell = 3$. Then, the minimal resolution of $(E \times E)/\langle \tau \times \tau^2 \rangle$ is isomorphic to a Kummer surface $\mathrm{Km}(E \times E)$ (cf. Katsura [13], Theorem 5.9). Hence, by Proposition 4.3, $\mathrm{Km}(E \times E)$ is a Zariski surface with $C = E$ and $i = 2$.

Secondly, we assume $p \equiv 3 \pmod 4$. It follows that E from (3.3) is a supersingular elliptic curve which is isomorphic to the elliptic curve defined by (4.1) with an automorphism σ as in (4.2) with $\ell = 4$. Then, the minimal resolution of $(E \times E)/\langle \sigma \times \sigma^3 \rangle$ is isomorphic to a Kummer surface $\mathrm{Km}(E \times E)$ (cf. Katsura [13], Theorem 5.9), and it has Artin invariant $\sigma = 1$. As before, it is also a Zariski surface by Proposition 4.3 (with $i = 3$). This completes our proof. □

In fact, we have already enough information to prove a big portion (in terms of the congruence $p \not\equiv 1, 49 \bmod 60$) of Theorem 1.4 (ii).

Theorem 4.5. *Assume* $p \not\equiv 1 \pmod{12}$. *Then, there exists a Zariski supersingular* K3 *surface with Artin invariant* 2*.*

Proof. Assume $p \equiv 2 \pmod 3$ and let E denote the supersingular elliptic curve from (3.1), (4.1) with automorphism τ as in (4.2) with $\ell = 3$. Consider the quotient morphism $\pi \colon E \times E \longrightarrow A$ as in Section 3. This morphism induces

$$(E \times E)/\langle \tau \times \tau \rangle \longrightarrow A/\langle \theta \rangle.$$

The minimal model of $A/\langle\theta\rangle$ is $\mathrm{GKm}(A)$. By Proposition 4.2, $(E\times E)/\langle\tau\times\tau\rangle$ is rational, so $\mathrm{GKm}(A)$ is Zariski, while by Theorem 3.3, the Artin invariant of $\mathrm{GKm}(A)$ is equal to 2.

Secondly, assume $p\equiv 3 \pmod 4$. Fix the supersingular elliptic curve E from (4.1) with automorphism τ as in (4.2) with $\ell=4$. Recall that E is isomorphic to the elliptic curve from (3.3) with automorphism τ introduced in Section 3. Arguing as above, but with Theorem 3.5 instead, we complete our proof. □

5. Kummer surfaces with Artin invariant 2

In the previous section, we showed that there exist Zariski supersingular K3 surfaces with Artin invariant 2 if $p\not\equiv 1 \pmod{12}$. In this section, we will prove Theorem 1.3 by showing directly that any supersingular K3 surface with Artin invariant 2 is Zariski if $p\not\equiv 1 \pmod{12}$. Since Rudakov and Shafarevich already showed that all supersingular K3 surfaces are Zariski in characteristic 2 (Rudakov–Shafarevich [23]), we may assume $p\neq 2$. We start with a few preparations.

Lemma 5.1. *Let A be an abelian surface, and let τ be an automorphism of A of order m ($m>2$, $m\neq 4$). Let ζ be a primitive m-th root of unity. Assume that τ acts as the multiplication by ζ on the vector space $\mathrm{H}^0(A,\Omega^1_A)$ of regular 1-forms on A. Then, the quotient surface $A/\langle\tau\rangle$ is rational.*

Proof. Let X be a nonsingular model of $A/\langle\tau\rangle$. Then, there exists a dominant rational map

$$\varphi : A \longrightarrow X.$$

Suppose that the Albanese variety $\mathrm{Alb}(X)$ of X is nontrivial. Then, there exists a non-zero regular 1-form on $\mathrm{Alb}(X)$. Pulling back the regular 1-form to X, we have a non-zero regular 1-form ω on X. Then, $\varphi^*(\omega)$ is a non-zero τ-invariant regular 1-form on A, which contradicts our assumption. Therefore, we see the irregularity $q(X)=0$.

Now, suppose there exists a non-zero regular 2-ple 2-form Ω on X. Then, $\varphi^*\Omega$ gives a non-zero τ-invariant regular 2-ple 2-form on A. However, since

$$\mathrm{H}^0(A,(\Omega^2_A)^{\otimes 2}) \cong (\wedge^2\mathrm{H}^0(A,\Omega^1_A))^{\otimes 2} \cong k,$$

the action of τ on the space of regular 2-ple 2-forms on A is given by multiplication by ζ^4, which is not 1 by assumption, a contradiction. Hence, we conclude that X is rational by Castelnuovo's criterion of rationality as in Zariski [36]. □

Remark 5.2. It is posible to weaken the assumption of Lemma 5.1 as long as none of the induced actions of τ on differential forms is trivial. In order to cover the case $m=4$, however, we will need to throw in some extra work in Section 5.2.

Let X be an algebraic surface with $\dim\mathrm{H}^2(X,\mathcal{O}_X)=1$. We assume that the formal Brauer group of X is prorepresentable by a one-dimensional formal group (cf. Artin–Mazur [2]). We denote the formal Brauer group by Φ_X.

Lemma 5.3. *Let X (respectively, Y) be an algebraic surface with* $\dim \mathrm{H}^2(X, \mathcal{O}_X) = 1$ (*respectively,* $\dim \mathrm{H}^2(Y, \mathcal{O}_Y) = 1$). *Assume that their formal Brauer groups are prorepresentable by one-dimensional formal groups Φ_X, Φ_Y, respectively. Moreover, assume there exists a dominant separable rational map $f\colon Y \longrightarrow X$ such that the degree of the Galois closure of f is prime to p. Then, the height of Φ_X is equal to the height of Φ_Y. In particular, if the degree of f is smaller than p, then the height of Φ_X is equal to the height of Φ_Y.*

Proof. Since the formal Brauer group is stable under blowing-up by Artin–Mazur (see [2], p. 122), we have a homomorphism $f^*\colon \Phi_X \longrightarrow \Phi_Y$. We also have a non-zero homomorphism $f^*\colon \mathrm{H}^2(X, \mathcal{O}_X) \longrightarrow \mathrm{H}^2(Y, \mathcal{O}_Y)$, which is an isomorphism since each space is 1-dimensional and the trace map $\mathrm{H}^2(Y, \mathcal{O}_Y) \longrightarrow \mathrm{H}^2(X, \mathcal{O}_X)$ is nontrivial. Since $\mathrm{H}^2(X, \mathcal{O}_X)$ (respectively, $\mathrm{H}^2(Y, \mathcal{O}_Y)$) is the tangent space of Φ_X (respectively,Φ_Y), the homomorphism $f^*\colon \Phi_X \longrightarrow \Phi_Y$ is nontrivial. Therefore, the height of Φ_X is equal to the height of Φ_Y. If the degree of f is smaller than p, then f is separable and the degree of Galois closure of f is prime to p. Therefore, the result follows. □

Now, we recall the theory of *a-number* (for the details of a-number for algebraic varieties, see van der Geer–Katsura [8], Definition 2.1). For a nonsingular complete algebraic surface X, we denote by $\mathrm{H}^2_{dR}(X)$ the second De Rham cohomology group of X. From here on, we consider only algebraic surfaces such that the Hodge-to-De Rham spectral sequence is degenerate at E_1-term. (For instance, this holds if the characteristic satisfies $p > 2$ and if X can be lifted to the Witt ring $W_2(k)$, and in particular for K3 surfaces.) Then, we have the Hodge filtration

$$\mathrm{H}^2_{dR}(X) = \mathrm{F}_0 \supset \mathrm{F}_1 \supset \mathrm{F}_2 \supset 0$$

such that $F_0/F_1 = \mathrm{H}^2(X, \mathcal{O}_X)$, $F_1/F_2 = \mathrm{H}^1(X, \Omega^1_X)$ and $F_2 = \mathrm{H}^0(X, \Omega^2_X)$. The absolute Frobenius map F acts on $\mathrm{H}^2_{dR}(X)$, and the kernel of F is F_1. Therefore, we have an injective map

$$F : \mathrm{H}^2(X, \mathcal{O}_X) \longrightarrow \mathrm{H}^2_{dR}(X).$$

Then, the a-number $a(X)$ of X is defined by

$$a(X) = \max\{i \mid (\mathrm{Im} F) \cap \mathrm{F}_i \neq 0\}.$$

Here, max means the largest number in the set. Note that if X is an abelian surface, $a(X)$ coincides with the usual a-number defined by Oort [21], Notation 1 (cf. van der Geer–Katsura [8], Proposition 2.2). For instance, a supersingular abelian surface A of Artin invariant 2 has a-number 1, since α_p embeds uniquely into A.

Lemma 5.4. *Let X (respectively, Y) be an algebraic surface with* $\dim \mathrm{H}^2(X, \mathcal{O}_X) = 1$ (*respectively,* $\dim \mathrm{H}^2(Y, \mathcal{O}_Y) = 1$). *Assume there exists a dominant separable rational map $f\colon Y \longrightarrow X$ such that the degree of the Galois closure of f is prime to p. Then, we have $a(X) = a(Y)$. In particular, if the degree of f is smaller than p, then we have $a(X) = a(Y)$.*

Proof. If necessary, we blow up Y, and we may assume that f is a morphism. We have a commutative diagram:

$$\begin{array}{ccc} \mathrm{H}^2(Y, \mathcal{O}_X) & \stackrel{F}{\longrightarrow} & \mathrm{H}^2_{dR}(Y) \\ f^* \uparrow & & f^* \uparrow \\ \mathrm{H}^2(X, \mathcal{O}_X) & \stackrel{F}{\longrightarrow} & \mathrm{H}^2_{dR}(X). \end{array}$$

Here, the second up-arrow preserves the Hodge filtrations. Since the degree of the Galois closure of f is prime to p, the first up-arrow is an isomorphism. Therefore, we have $a(X) = a(Y)$. □

Lemma 5.5. *Let X be a supersingular* K3 *surface with Artin invariant* 2 *in characteristic $p > 2$. Then the a-number of X is equal to* 1*.*

Proof. Since X is a supersingular K3 surface with Artin invariant 2, X is isomorphic to a Kummer surface $\mathrm{Km}(B)$, where B is a supersingular abelian surface with Artin invariant 2 (as we have mentioned before). Clearly, the a-number of B is equal to 1. Since we have a dominant separable rational map $f: B \longrightarrow X$ of degree 2 (which is smaller than p), Lemma 5.4 implies that $a(X) = 1$. □

5.1. Proof of Theorem 1.3 for $p \equiv 2 \bmod 3$

First we assume $p \equiv 2 \pmod 3$ and $p \neq 2$. Let E be the supersingular elliptic curve defined by $y^2 + y = x^3$ and τ the automorphism defined by $x \mapsto \omega x$, $y \mapsto y$ with ω a primitive cube root of unity. We take an immersion

$$\epsilon : \alpha_p \stackrel{(i,j)}{\longrightarrow} \alpha_p \times \alpha_p \longrightarrow E \times E$$

with $i/j \notin \mathbb{F}_{p^2}$. Then, as we already showed, the automorphism $\tau \times \tau$ of $E \times E$ induces an order 3 automorphism θ of $A = (E \times E)/\epsilon(\alpha_p)$ and the nonsingular minimal model $\mathrm{GKm}(A)$ of $A/\langle\theta\rangle$ is a supersingular K3 surface with Artin invariant 2.

In spirit, our approach follows closely Shioda (proof of Theorems 4.2, 4.3 in [32]). It is also similar to what was done in [3] (over **C**) and in [13]. In fact, it can be adapted for the congruence class $p \equiv 3 \bmod 4$ (in 5.2) and goes roughly as follows:

(1) set up a smooth covering corresponding to the quotient map $A \to \mathrm{GKm}(A)$;
(2) translate the information into lattices which thus carry over to any supersingular K3 surface X of Artin invariant 2;
(3) recover a cover of X which leads to a supersingular abelian surface Y of Artin invariant 2;
(4) facilitate the unique embedding $\alpha_p \hookrightarrow Y$ to derive that X is a Zariski surface.

The numbering of the subsections to follow reflects the above steps.

5.1.1. First, we recall how to construct $\mathrm{GKm}(A)$, following Katsura [13], Section 5. Since θ has 9 fixed points on A, we blow up at these 9 points:

$$\psi_1 : A_1 \longrightarrow A.$$

We denote by G_i $(i = 1, \ldots, 9)$ the exceptional curves. Then, θ induces an automorphism θ_1 on A_1 which has 2 fixed points on each exceptional curve. We once more blow up these 18 fixed points:

$$\psi_2 : A_2 \longrightarrow A_1.$$

Abusing notation, we denote again by G_i the proper transform of G_i, and by D_i, F_i $(i = 1, 2, \ldots, 9)$ the exceptional curves such that

$$(D_i, G_i) = (F_i, G_i) = 1, \quad (D_i, F_i) = 0 \quad (i = 1, 2, \ldots, 9).$$

We have $D_i^2 = -1$, $F_i^2 = -1$, and $G_i^2 = -3$. The automorphism θ_1 on A_1 induces an automorphism θ_2 on order 3 on A_2 which acts as identity map on D_i and F_i, and induces an automorphism of order 3 on G_i. Therefore, the quotient surface $A_2/\langle\theta_2\rangle$ is nonsingular and we have a diagram:

$$\begin{array}{ccccc} A_2 & \xrightarrow{\psi_2} & A_1 & \xrightarrow{\psi_1} & A \\ \pi \downarrow & & & & \\ A_2/\langle\theta_2\rangle & \xrightarrow{\psi_3} & \mathrm{GKm}(A). & & \end{array}$$

Here, π is the quotient map and ψ_3 will be described momentarily. We set $D_i' = \pi(D_i)$, $F_i' = \pi(F_i)$ and $G_i' = \pi(G_i)$. Then, we have $\pi^*(D_i') = 3D_i$, $\pi^*(F_i') = 3F_i$ and $\pi^*(G_i') = G_i$. Using these relations, we have $(D_i')^2 = -3$, $(F_i')^2 = -3$ and $(G_i')^2 = -1$. Then the morphism ψ_3 is simply the blowing-down of the 9 curves G_i' $(i = 1, \ldots, 9)$.

5.1.2. Since $\pi\colon A_2 \longrightarrow A_2/\langle\theta_2\rangle$ is a cyclic covering of degree 3 with smooth ramification locus comprising the D_i and F_i, it induces an effective branch divisor

$$R = \sum_{i=1}^{9} (D_i' + 2F_i')$$

which is divisible by 3 in $\mathrm{NS}(A_2/\langle\theta_2\rangle)$; that is, there exists a divisor R' on $A_2/\langle\theta_2\rangle$ such that $3R' = R$. (The coefficients of the components of R guarantee that R' has integral intersection number with each G_i'. They can also be derived from the action of θ_2 on the ramification locus in A_2 which, by piecing the local information together, leads to an invariant one-cycle in $\mathrm{H}^1(\mathcal{O}^*)$, i.e., to the invertible sheaf $\mathcal{O}(R)$ on the quotient.)

Now, let X be any supersingular K3 surface (or Kummer surface, for that matter) with Artin invariant 2. Then there is an isometry

$$\varphi : \mathrm{NS}(\mathrm{GKm}(A)) \longrightarrow \mathrm{NS}(X)$$

which maps effective cycles to effective cycles (cf. Piatetskij-Shapiro–Shafarevich [22], Section 6; Shioda [32], p. 585). The nine pairs of (-2)-curves $\{\psi_3(D_i'), \psi_3(F_i')\}$ $(i = 1, \ldots, 9)$ in $\mathrm{NS}(\mathrm{GKm}(A))$ thus correspond to pairs $\{\varphi(\psi_3(D_i')), \varphi(\psi_3(F_i'))\}$ $(i = 1, \ldots, 9)$ in $\mathrm{NS}(X)$. For the sake of simplicity, we set $D_i'' = \varphi(\psi_3(D_i'))$ and $F_i'' = \varphi(\psi_3(F_i'))$. They are nonsingular rational curves which intersect transversally at a single point.

Reversing the above construction, we blow up at each intersection point of D_i'' and F_i''. Let

$$\psi_3' : \tilde{X} \longrightarrow X$$

be the blowing-up. We denote by G_i'' ($i = 1, \dots, 9$) the exceptional curves while we again use the same notation for the proper transforms of D_i'' and F_i''. Then, G_i'', D_i'' and F_i'' form a triple with the same intersection numbers as before (i.e., $(G_i'', D_i'') = 1$, $(G_i'', F_i'') = 1$, $(D_i'', F_i'') = 0$, $(D_i'')^2 = -3$, $(F_i'')^2 = -3$ and $(G_i'')^2 = -1$). We set

$$R'' = \sum_{i=1}^{9} (D_i'' + 2F_i'').$$

We can naturally extend the isometry φ to

$$\tilde{\varphi} : \mathrm{NS}(\mathrm{GKm}(A)) \oplus_{i=1}^{9} \mathbb{Z} G_i' \longrightarrow \mathrm{NS}(X) \oplus_{i=1}^{9} \mathbb{Z} G_i'',$$

and we have $\tilde{\varphi}(R) = R''$. Since R is divisible by 3, so is R''; that is, there exists a divisor R''' on $\tilde{X}$ such that $R'' = 3R'''$.

5.1.3. Using R''', we can construct a cyclic covering

$$\pi' : \tilde{Y} \longrightarrow \tilde{X}$$

of degree 3 with (smooth) branch locus the support of R''. A local computation reveals that $\tilde{Y}$ can be taken to be smooth after a normalization (which also follows from the general theory of triple covers in Miranda [16]; for K3 surfaces, see also Bertin [3]). By construction, $\tilde{Y}$ comes with an induced order 3 automorphism $\tilde{\eta}$ of $\tilde{Y}$ such that

$$\tilde{X} \cong \tilde{Y}/\langle\tilde{\eta}\rangle.$$

Denoting pullbacks by tildes, we have

$$\pi'^*(G_i'') = \tilde{G_i''}, \quad \pi'^*(D_i'') = 3\tilde{D_i''} \quad \text{and} \quad \pi'^*(F_i'') = 3\tilde{F_i''},$$

with $(\tilde{G_i''})^2 = -3$, $(\tilde{D_i''})^2 = -1$ and $(\tilde{F_i''})^2 = -1$. Contracting $\tilde{D_i''}$, $\tilde{F_i''}$, and subsequently $\tilde{G_i''}$, we derive an algebraic surface Y, and we see that $\tilde{\eta}$ induces an automorphism η of order 3 on Y.

We will show that Y is an abelian surface. The canonical divisor of $\tilde{X}$ is given by $K_{\tilde{X}} = \sum_{i=1}^{9} G_i''$. Therefore, the canonical divisor of $\tilde{Y}$ is given by adding the ramification divisor:

$$K_{\tilde{Y}} = \sum_{i=1}^{9} \tilde{G_i''} + \sum_{i=1}^{9} 2\tilde{D_i''} + \sum_{i=1}^{9} 2\tilde{F_i''}.$$

Hence, the canonical divisor of Y is trivial.

The Euler–Poincaré characteristic of Y can be computed using a topological argument for the involved coverings and blow-ups. Essentially this works like over $\mathbf{C}$ by a standard argument, except that we have to use étale cohomology with compact support. For brevity, we omit the details leading to $\chi(Y) = 0$. Hence, considering the fact that K_Y is trivial (and $p \geq 5$), we conclude that Y is an abelian surface. Since we have a dominant separable rational map from Y to X of degree 3, by Lemma 5.3 the height of the formal Brauer group Φ_Y is equal to the height of Φ_X. Since $\Phi_X = \infty$, we have $\Phi_Y = \infty$. Therefore, Y is a supersingular abelian surface. By Lemma 5.5, the a-number of X is equal to 1. Therefore, by Lemma 5.4, the a-number of Y is equal to 1, and Y has Artin invariant 2.

5.1.4. Recall that $Y/\langle\eta\rangle$ is birational to X and we have the following diagram:

$$\begin{array}{ccccc} & \tilde{Y} & \longrightarrow & Y & \\ \swarrow & & & & \searrow \\ \tilde{Y}/\langle\tilde{\eta}\rangle = \tilde{X} & \longrightarrow & X & \longrightarrow & Y/\langle\eta\rangle \end{array}$$

From the diagram of exceptional curves we see that the singularities of $Y/\langle\eta\rangle$ are of type A_2. Since the a-number of Y is equal to 1, the subgroup scheme α_p embeds uniquely into Y. Therefore, η preserves α_p. Let P be a fixed point of η. Let $\mathcal{O}_P$ be the local ring at the point P, and m_P the maximal ideal. We take the local parameter s' in the direction of the subgroup scheme α_p. Then, $\eta^*(s') = \gamma s'$ mod m_P^2. Since η is of order 3 and P is an isolated fixed point, we see that γ is a primitive cube root of unity. Setting

$$s = s' + \gamma^{-1}\eta^*(s') + \gamma^{-2}(\eta^*)^2 s',$$

we see $\eta^* s = \gamma s$ and s is a nonzero element of m_P/m_P^2 by $p \neq 3$. Since the quotient singularity is of type A_2 and the representation of $\mathbb{Z}/3\mathbb{Z}$ on m_P/m_P^2 is completely reducible, we can take an element t of m_P such that s, t form a basis of m_P/m_P^2 and such that $\eta^* s = \gamma s$, $\eta^* t = \gamma^2 t$.

Now, go to the quotient $Y' = Y/\alpha_p$. Then, $Y' \cong E \times E$ and η induces an automorphism η' of Y'. We may assume that s^p, t give a local parameters at a fixed point of η'. Therefore, the action of η' at the fixed point is given by $s^p \mapsto \gamma^2 s^p$, $t \mapsto \gamma^2 t$. Then, taking the Frobenius pull-back of these structures, we have the following commutative diagram:

$$\begin{array}{ccc} Y'^{(1/p)} & \stackrel{\eta'^{(1/p)}}{\longrightarrow} & Y'^{(1/p)} \\ \downarrow & & \downarrow \\ Y & \stackrel{\eta}{\longrightarrow} & Y \\ \downarrow & & \downarrow \\ Y' & \stackrel{\eta'}{\longrightarrow} & Y'. \end{array}$$

Here, all vertical arrows depict quotient morphisms by the subscheme α_p (suitably embedded), and $Y'^{(1/p)} = Y' = E \times E$. We set $\tau = \eta''^{(1/p)}$. Let Q be a fixed point of τ, and let $\mathcal{O}_Q$ be the local ring at the point Q, and m_Q the maximal ideal. Then, by our construction, we have local parameters u, v of m_Q such that the action of τ is given by $u \mapsto \gamma u$, $v \mapsto \gamma v$. Since the cotangent space m_Q/m_Q^2 is isomorphic to $\mathrm{H}^0(Y'^{(1/p)}, \Omega^1_{Y'^{(1/p)}})$, we see that the action of τ on the space $\mathrm{H}^0(Y'^{(1/p)}, \Omega^1_{Y'^{(1/p)}})$ is given by the multiplication by γ, which is a primitive cube root of unity. Hence Lemma 5.1 shows that $Y'^{(1/p)}/\langle\tau\rangle$ is rational. Since the construction provides a purely inseparable morphism

$$Y'^{(1/p)}/\langle\tau\rangle \longrightarrow Y/\langle\eta\rangle$$

of degree p, and $Y/\langle\eta\rangle$ is birational to X, we conclude that X is a Zariski surface as claimed. $\square$

5.2. Proof of Theorem 1.3 for $p \equiv 3 \bmod 4$

If $p \equiv 3 \bmod 4$, then the proof of Theorem 1.3 proceeds very much along the lines of Section 5.1, except that there are a few subtleties to overcome since our automorphism does not have prime order and Lemma 5.1 does not apply. As before, we start with a supersingular abelian surface A with $\sigma = 2$ endowed with the automorphism θ of order 4 from Section 3. Consider $\mathrm{GKm}(A)$, the minimal resolution of the quotient $A/\langle\theta\rangle$ with singularities of types $4A_3 + 6A_1$ (cf. Katsura [13], p. 17). Thus $\mathrm{GKm}(A)$ carries natural configurations of smooth rational curves C_i, D_i, E_i $(i = 1, 2, 3, 4)$, forming A_3 root lattices, and 6 disjoint (-2)-curves F_i $(i = 1, \dots, 6)$. In particular, $\mathrm{GKm}(A)$ contains an effective 4-divisible divisor

$$R = \sum_{i=1}^{4}(C_i + 2D_i + 3E_i) + 2\sum_{i=1}^{6} F_i,$$

but the corresponding cover $A_0 \to \mathrm{GKm}(A)$ is not smooth since $\mathrm{supp}(R)$ is not. We will see momentarily how to overcome this without any additional blow-ups (one of the advantages over the direct approach from Section 5.1).

Now consider a supersingular K3 surface X of Artin invariant 2. Then, as before, $\mathrm{NS}(X)$ contains the same configuration of (-2)-curves, and the same 4-divisible divisor R (using the same notation as for $\mathrm{GKm}(A)$). We search for a smooth birational model of the corresponding cyclic degree 4 cover Y_0. To this end, we first consider the smooth degree 2 cover

$$\tilde{W} \to X$$

corresponding to the 2-divisible branch divisor $\sum_{i=1}^{4}(C_i + E_i)$ with smooth support. Clearly the C_i, E_i pull-back to (-1)-curves C_i', E_i' on $\tilde{W}$ while

$$K_{\tilde{W}} = \sum_{i=1}^{4}(C_i' + E_i').$$

Contracting these disjoint (-1)-curves, we obtain a smooth surface W with

$$K_W = 0, \quad \chi(W) = 24,$$

thus a K3 surface. This comes equipped with (-2)-curves D_i' mapping to D_i, and with effective (-4)-divisors mapping $2:1$ to the F_i. It follows that these decompose into two disjoint (-2)-curves $F_{i,1} + F_{i,2}$ each.

On $\tilde{W}$, R pulls back to twice the divisor $R' + 2\sum_{i=1}^{9} E_i'$ where

$$R' = \sum_{i=1}^{4}(C_i' + D_i' + E_i') + \sum_{i=1}^{6}(F_{i,1} + F_{i,2})$$

is still 2-divisible by construction; the corresponding cover remains birational to Y_0. Push forward of R' to the K3 surface W yields the 2-divisible divisor

$$R'' = \sum_{i=1}^{4} D_i' + \sum_{i=1}^{6}(F_{i,1} + F_{i,2})$$

which again is smooth. As before, the corresponding cover is the blow-up $\tilde{Y}$ of an abelian surface Y, this time in 16 points, and indeed W is the Kummer surface of Y. (To rule out (quasi-)bielliptic surfaces (in characteristic 3), one may note that by construction, W is supersingular and hence $\rho(Y) \geq 6$.) The whole construction endows Y with an automorphism η of order 4 whose quotient is birational to X. We sketch the resulting maps in the following diagram:

$$\begin{array}{ccccc} & & \tilde{Y} & \rightarrow & Y \\ & & \downarrow & & \downarrow \\ \tilde{W} & \rightarrow & W & \rightarrow & Y/\langle \eta^2 \rangle \\ \downarrow & & & & \downarrow \\ X & & \rightarrow & & Y/\langle \eta \rangle \end{array}$$

Now we can proceed exactly as before, with intermediate step from X to Y going through W, to deduce that all these varieties are supersingular with a-number 1. It follows that α_p admits a unique embedding into Y which is thus compatible with the action of η. This induces an order 4 automorphism η' on the quotient $Y' = Y/\alpha_p$. However, we cannot infer from Lemma 5.1 that $Y'/\langle \eta' \rangle$ is rational, so we have to pursue a different line of reasoning. To this end, we once again factorize the quotient map. Namely, we first consider the quotient

$$V' = Y'/\langle \eta'^2 \rangle$$

Note that $\eta^2 = -\mathrm{id}$ on Y, so the same holds for η'^2 on Y'. That is, V' is birational to $\mathrm{Km}(Y')$, a K3 surface. Then η' induces an involution $\imath$ on V'. Since $\imath$ kills the regular 2-form on V' by construction, the quotient

$$V = V'/\langle \imath \rangle \sim Y'/\langle \eta' \rangle$$

cannot be (birationally) K3 again. It follows that V is either rational or Enriques. But then V admits a purely inseparable map of degree $p > 2$ to the K3 surface X by construction, so V cannot have fundamental group $\mathbb{Z}/2\mathbb{Z}$. Hence V is rational, and X is Zariski as claimed. This completes the proof of Theorem 1.3. □

Remark 5.6. It may be feasible to pursue an alternative approach to prove Theorem 1.3 based on the results of Blass–Levine [4], choosing a suitable polarization etc. However, given the explicit geometric arguments which supersingular Kummer surfaces lend themselves to, we decided to pursue the above reasoning.

6. Supersingular K3 surfaces

In order to deal with other supersingular K3 surfaces and compute their Artin invariants (as required for Theorem 1.4 (ii), (iii)), we need a little preparation on the lattice theoretic side. To this end, we assume that S is a supersingular K3 surface, endowed with a certain sublattice L embedding into the Néron–Severi group $\mathrm{NS}(S)$. For instance, S could be the minimal resolution of some singular surface with L generated by the exceptional curves above the singularities.

Theorem 6.1. *In the notation above, let n be the rank of L. We assume that the discriminant of L is prime to the characteristic p. Then the Artin invariant σ of S is smaller than or equal to $(22-n)/2$.*

Proof. We denote by $\mathrm{NS}(S)^*$ the dual lattice of $\mathrm{NS}(S)$, and likewise for L etc. Then, by the result of M. Artin ([1], (4.6)ff.), $\mathrm{NS}(S)^*/\mathrm{NS}(S)$ is a p-elementary group, and $\mid \mathrm{NS}(S)^*/\mathrm{NS}(S) \mid = p^{2\sigma}$. By replacing L with its saturation in $\mathrm{NS}(S)$, we may assume without loss of generality that L embeds primitively into $\mathrm{NS}(S)$. Note that by definition, its orthogonal complement $L^\perp$ embeds primitively into $\mathrm{NS}(S)$ as well.

Consider the embedding

$$L \oplus L^\perp \hookrightarrow \mathrm{NS}(S)$$

of finite index m, say. The lattices L and $L^\perp$ are glued together via an isomorphism of subgroups of the discriminant groups,

$$L^*/L \supseteq H_1 \cong H_2 \subseteq (L^\perp)^*/L^\perp,$$

such that the induced intersection form (modulo $2\mathbf{Z}$) agrees up to sign. We use two related properties: on the one hand, from lattice theory,

$$|H_1| = |H_2| = m;$$

on the other hand, as a subgroup,

$$|H_1| \mid |L^*/L| = |\mathrm{disc} L|.$$

In particular, our assumption implies that m is prime to p. From this we aim to deduce that (the p-part of) $\mathrm{NS}(S)^*/\mathrm{NS}(S)$ is isometric (with respect to discriminant forms) to a subgroup of $(L^\perp)^*/L^\perp$. To this end, consider the sequence of $\mathbb{Z}$-modules:

$$L^\perp \subset L \oplus L^\perp \subset \mathrm{NS(S)} \subset \mathrm{NS(S)}^* \subset (L \oplus L^\perp)^* \cong L^* \oplus (L^\perp)^* \supset (L^\perp)^*,$$

where the finite index inclusions in the middle have index $m, p^{2\sigma}$ and m, respectively. It follows that $m^2p(L \oplus L^\perp)^* \subset L \oplus L^\perp$, and thus, restricting to the orthogonal summand $L^\perp$, also $m^2p(L^\perp)^* \subset L^\perp$. Since rank $L^\perp = 22-n$, we infer from the elementary divisor theorem that

$$p^{23-n} \not| \,\mathrm{disc}\ L^\perp. \tag{6.1}$$

Putting everything together, we use

$$-p^{2\sigma} = \mathrm{disc}\ \mathrm{NS}(S) = \frac{(\mathrm{disc}\ L)(\mathrm{disc}\ L^\perp)}{m^2}$$

to deduce, from our assumption that disc L and thus m is prime to p, and from (6.1), that $2\sigma \leq 22-n$ as claimed. □

Example 6.2. Let k be an algebraically closed field of characteristic $p \neq 2, 3$, and let A be a supersingular abelian surface over k. Then A has a principal polarization Θ. Since $p \neq 2, 3$, we can choose a nonsingular curve of genus 2 as the principally polarization (cf. Proposition 3.1 in Ibukiyama–Katsura–Oort [10], and Corollary 6.15 in Ogus [20]). Consider the linear system $\mid 2\Theta \mid$. Then the associated rational map $\varphi_{|2\Theta|}$ is a morphism (cf. Mumford [17], §6, Application 1), and, as is well known, the image of $\varphi_{|2\Theta|}$ is a quartic surface with 16 rational double points of type A_1 in the projective plane $\mathbb{P}^3$, which is isomorphic to the quotient surface $A/\langle\iota\rangle$. Here, ι is the inversion of A. The minimal resolution of the surface $A/\langle\iota\rangle$ is the Kummer surface $\mathrm{Km}(A)$. We denote by

$$\pi : \mathrm{Km}(A) \longrightarrow A/\langle\iota\rangle$$

the resolution. We take a generic hyperplane section H of $A/\langle\iota\rangle$ and pull-back $D = \pi^*H$. Then, D does not intersect the exceptional divisors and $D^2 = 4$, which is prime to p. Consider the lattice $L \subset \mathrm{NS}(\mathrm{Km}(A))$ generated by the exceptional curves and D. Then, we have rank $L = 17$. By the same argument as in Theorem 6.1, we see that the Artin invariant satisfies

$$\sigma(\mathrm{Km}(A)) \leq \frac{22-17}{2}.$$

Thus we obtain an alternative reasoning for the well-known result $\sigma(\mathrm{Km}(A)) \leq 2$ (cf. Ogus [20]).

Example 6.3. Analogous arguments apply to the generalized Kummer surfaces from Section 3 to prove that they have Artin invariant $\sigma \leq 2$.

We continue with another application to Kummer surfaces which is a kind of converse of Example 6.2. Recall that Nikulin showed that a complex Kähler K3 surface X is a Kummer surface if and only if there exist 16 nonsingular rational curves on X which do not intersect each other (cf. Nikulin [18], Theorem 1). For supersingular K3 surfaces, as an application to Theorem 6.1, we have the following.

Theorem 6.4. *Let X be a supersingular* K3 *surface with a divisor D such that D^2 is prime to the characteristic p. Assume that there exist* 16 *nonsingular rational curves E_i which do not intersect each other, and assume that $(D, E_i) = 0$ $(i = 1, 2, \ldots, 16)$. Then, X is a Kummer surface.*

Proof. We consider the lattice $L \subset \mathrm{NS}(X)$ generated by D and E_i's. The lattice L is of rank 17 and the discriminant is prime to p. Therefore, by Theorem 6.1, the Artin invariant σ is $\leq (22-17)/2$. Therefore, we have $\sigma \leq 2$. Hence, X is a Kummer surface (cf. Ogus [20], Theorem 7.10, Shioda [32], Theorem 4.3). □

We proceed by giving a direct construction covering the remaining part of Theorem 1.4 (ii). We emphasize that this does not require any further machinery; in the next section it will be generalized in the context of elliptic surfaces (Lemma 7.1, etc.).

Lemma 6.5. *Assume $\ell = 5$ in (4.1). Then, $(C \times C)/\langle\tau \times \tau^2\rangle$ is birational to a* K3 *surface S, and $(C \times C)/\langle\tau \times \tau^3\rangle$ is birational to the same* K3 *surface.*

Proof. The group $G = \langle\tau \times \tau^2\rangle$ acts on the function field of $C \times C$ via its natural action on $k(x, y, x_1, y_1)$. We set

$$z = x x_1^2.$$

Then, z is invariant under G and the invariant field $k(C\times C)^G$ is given by $k(y_1,y,z)$ with the equation $z^5=(y^2+y)(y_1^2+y_1)^2$. We set

$$w=y\,(y_1^2+y_1).$$

Then the relation translates as

$$w^2+w(y_1^2+y_1)=z^5, \tag{6.2}$$

which gives a birational equation for the quotient surface S. Outside characteristic 2, S is thus birational to the double cover of $\mathbb{P}^2$ branched along the sextic curve

$$C:\quad z^5u+\frac{u^2(y_1^2+y_1u)^2}{4}=0, \tag{6.3}$$

where y_1,z,u denote homogeneous coordinates of $\mathbb{P}^2$. Obviously C is reducible, but the singularities are only isolated rational double points. It follows that the minimal resolution of the double cover is a K3 surface as claimed.

In characteristic 2, (6.2) still defines a separable double cover of $\mathbb{P}^2$, but due to the presence of wild ramification, the branch locus degenerates to the cubic curve $(y_1^2+y_1u)u$. Yet the singularities and the underlying invariants are preserved, so we obtain a K3 surface as before.

Since τ is of order 5, $(\tau\times\tau^2)^3=(\tau^3\times\tau)$ is a generator of the group $\langle\tau\times\tau^2\rangle$. Therefore, by exchanging the components of S, we have an isomorphism from $(S)/\langle\tau\times\tau^2\rangle$ to $(S)/\langle\tau\times\tau^3\rangle$. This concludes the proof of Lemma 6.5. □

We are now in the position to prove the remaining part of Theorem 1.4 (ii).

Theorem 6.6. *Assume* $p\equiv 2$ or $3 \pmod 5$. *Then the* K3 *surface* S *from Lemma* 6.5 *is Zariski with Artin invariant* 2.

Proof. By Proposition 4.3 and Lemma 6.5, S is a Zariski K3 surface. In order to exhibit a suitable sublattice L of $\mathrm{NS}(S)$, we study the singularities of the double covering of $\mathbb{P}^2$ from the proof of Lemma 6.5. In the affine chart (6.2), there are two singularities at $(w,y_1,z)=(0,0,0),(0,-1,0)$; visibly, each is a rational double point of type A_4. The chart $y_1\neq 0$ with affine equation

$$w^2+w(1+u)u=uz^5$$

has another rational double point at $(w,u,z)=(0,0,0)$, this time of type A_9. The minimal resolution S is thus endowed with the sublattice $L\subset\mathrm{NS}(S)$ generated by the exceptional curves above the singularities. Presently L has rank 17 and discriminant 250. In particular, the discriminant is prime to p if $p>2$, so using Theorem 6.1, we see that the Artin invariant σ of S is smaller than or equal to 2. To establish the same claim in characteristic two, it suffices to enhance the lattice L by the classes of the strict transforms of the two 'lines' $\{w=z=0\}$ and $\{w=u=0\}$. One easily checks that the resulting overlattice L' has rank 18 and discriminant -5, so we conclude $\sigma\leq 2$ as before.

In order to prove the equality $\sigma = 2$, we appeal to work of Nygaard, [19], Theorem 2.1 (recently generalized by Jang in Theorem 1.1 of [11]), studying the image of the natural representation

$$\mathrm{Aut}(S) \to \mathrm{GL}(H^0(S, \Omega^2_S)). \tag{6.4}$$

In detail, Nygaard proves in characteristic $p > 3$ that the image of (6.4) is a cyclic group of size dividing $p^\sigma + 1$. Presently, the automorphism $(w, z, y_1) \mapsto (w, \zeta z, y_1)$ induced by τ acts primitively of order 5 on the regular two-form $dz \wedge dy_1/w$, so the initial congruence assumption for p implies $\sigma = 2$ as claimed.

In characteristics 2 and 3, where Nygaard's results may not be valid, one can complete the proof without difficulty using elliptic fibrations. We postpone their treatment until Remark 7.4. □

Note that the proof of Theorem 1.4 (ii) is now complete since the two characteristics so far missing from the proof of Theorem 6.6 are covered by Theorem 4.5.

7. Zariski elliptic surfaces

In this section, we continue to argue with the curve C from (4.1) with automorphism τ for odd ℓ and consider the quotient

$$S = (C \times C)/\langle \tau \times \tau^i \rangle \quad (i \in \{1, \dots, \ell - 1\}).$$

For starters, we briefly leave the restricted area of K3 surfaces:

Lemma 7.1. *Let $i = (\ell - 1)/2$. If $p \equiv i \mod \ell$ or $p \equiv i^{-1} \mod \ell$, then S is a Zariski surface admitting an elliptic fibration.*

Proof. It is immediate that S is still birationally given by the degree ℓ analogue of the affine equation (6.2):

$$S: \quad w^2 + w(y_1^2 + y_1) = z^\ell. \tag{7.1}$$

Interpreting this as a cubic over $k(z)$, we obtain the claim (with sections at ∞, indeed). □

Remark 7.2. Of course, there are Zariski elliptic surfaces in the literature, but to our knowledge mostly arising by purely inseparable base change from a rational elliptic surface (Shioda [31], Example 4.2, and Katsura [12], Theorem I) −just like in Example 7.8.

Transferring (7.1) to a Weierstrass form is easily achieved: homogenize as a cubic in $\mathbb{P}^2_{k(z)}$ by a variable u, say and switch to the affine chart $w \neq 0$ to derive

$$S: \quad y_1^2 + uy_1 = z^\ell u^3 - u.$$

Then multiplying the equation by $z^{2\ell}$ and dividing variables by z^ℓ, we arrive at the normalized Weierstrass form

$$S: \quad y_1^2 + uy_1 = u^3 - z^\ell u. \tag{7.2}$$

Note the two-torsion section at $(0,0)$, and the automorphism τ induced by the action $z \mapsto \zeta z$ on the base.

Lemma 7.3. *In the above setting, S is birational to a* K3 *surface if and only if $\ell = 5$ or 7.*

Proof. This is a standard argument using the theory of elliptic fibrations, see e.g. Schütt–Shioda [25], §12.7. To give some details, we compute the basic invariants of (the Kodaira–Néron model of) S, starting from the zero irregularity (which follows from the base curve being $\mathbb{P}^1$). The fibration (7.2) has singular fibers of Kodaira types

$$I_{2\ell}/z = 0, \quad I_1/64z^\ell = -1, \quad \begin{cases} III/\infty, & \ell \equiv 3 \bmod 4, \\ III^*/\infty, & \ell \equiv 1 \bmod 4, \end{cases}$$

except that in characteristic two, the I_1 fibers are absorbed by the wild ramification at ∞. It follows that S has Euler–Poincaré characteristic $e(S) = 3\ell+3$ resp. $e(S) = 3\ell + 9$ and geometric genus $\lfloor \ell/4 \rfloor$. Hence S is a K3 surface exactly for $\ell = 5$ and 7, as claimed. □

Remark 7.4. As a first application, we explain how to infer that for $\ell = 5$ and $p = 2$ or 3, the Artin invariant of S cannot be $\sigma = 1$ (as stated in Theorem 6.6). To see this, it suffices to go through the classification of elliptic fibrations on the superspecial K3 surface in each characteristic: by inspection neither Elkies–Schütt [7] nor Sengupta [26] lists a fibration with both given reducible fibers of type III^* and I_{10}.

We now proceed to prove Theorem 1.4 (iii).

Theorem 7.5. *Let $\ell = 7$ and $p \equiv 3, 5 \bmod 7$. Then S is a Zariski* K3 *surface with Artin invariant $\sigma = 3$.*

Proof. The surface S is Zariski by Theorem 4.3 and K3 by Lemma 7.3. We proceed by exhibiting a suitable sublattice $L \subset \mathrm{NS}(S)$. To this end, consider the lattice L' generated by fiber components and zero section,

$$L' = U \oplus A_1 \oplus A_{13},$$

where U denotes the hyperbolic plane generated by zero section O and general fiber. The two-torsion section provides an index 2 overlattice $L' \subset L \subset \mathrm{NS}(S)$ of rank 16 and discriminant -7. Hence Theorem 6.1 applies to show that $\sigma \leq 3$. If $p > 3$, then we conclude following Nygaard as before. For $p = 3$, in contrast, we pursue a direct approach by exhibiting a full set of generators of $\mathrm{NS}(S)$ using the theory of Mordell–Weil lattices after Shioda [33]. In practice, we search for a section P of small height. This soon leads to P being integral (i.e., disjoint from the zero section O) and meeting both reducible fibers in components adjacent to the identity component. This means that $P = (tU, tV)$ for polynomials $U, V \in k[z]$ of degree 2 resp. 4 with t not dividing U. In fact, the special shape of the Weierstrass form (7.2), in particular the presence of the two-torsion section, implies that U has to be a square in $k[z]$.

Given this, one can solve directly for P to find, uniquely up to symmetry,

$$P = (-t(t+1)^2, t^2(t+1)(t-1)^2).$$

Comparing the seven sections $P_j = \tau^j P$ ($j = 0, \dots, 6$), we find that any two of them intersect transversally in a single point. Hence the height pairing evaluates as

$$h(P_j) = 4 + 2\underbrace{(P_j.O)}_{=0} - \frac{13}{14} - \frac{1}{2} = \frac{18}{7} \quad (0 \leq j \leq 6)$$

$$\langle P_j, P_m \rangle = 2 - \underbrace{(P_j.P_m)}_{=1} - \frac{13}{14} - \frac{1}{2} = -\frac{3}{7} \quad (0 \leq j \neq m \leq 6),$$

where the correction terms are read off from the fiber components met. From the resulting Gram matrix, we infer that the P_i generate a sublattice M of the Mordell–Weil lattice of S of rank 6 and discriminant $3^6/7$, in perfect agreement with the fact $\sigma \leq 3$. Proving equality thus amounts to showing that M equals the full Mordell–Weil lattice, i.e., that there cannot be any divisibilities among the given sections. This can be verified in multiple ways, for instance using the fact that the automorphism τ makes M an (irreducible) $\mathbb{Z}[\zeta]$-module of rank one. Hence a single divisibility would cause several independent others – too many in fact for the discriminant of $\mathrm{NS}(S)$ to stay integral. □

Remark 7.6. One could also argue without appealing to Mordell–Weil lattices, just using intersection numbers and the rank formula often attributed to Shioda–Tate ([28], Corollary 1.5). The above reasoning, in contrast, seems more streamlined and conceptual.

Remark 7.7. Similar results can be derived for the Zariski elliptic surfaces with $e > 24$ from Lemma 7.1. In particular, the length of the discriminant group (generalizing twice the Artin invariant) is always bounded by $\ell - 1$ (and even smaller when ℓ is not prime). We emphasize that our approach is not limited to characteristics satisfying the standard condition

$$\exists\, \nu : \quad p^\nu \equiv -1 \mod \ell \tag{7.3}$$

from the Fermat surface case (Shioda [29], Proposition 1, Shioda–Katsura [34], Theorem II). Indeed, for $\ell = 11$, for instance, we obtain (non-rational) Zariski surfaces in characteristics congruent to $5, 9$ modulo 11 which do not satisfy (7.3).

As a supplement to Theorem 1.4 (i), we conclude this paper by providing a Zariski K3 surface in characteristic $p = 13$ of Artin invariant $\sigma = 1$ in the vein of Shioda [31], Example 4.2.

Example 7.8. Assume $p = 13$. Consider the rational elliptic surface given in Weierstrass form

$$y^2 = x^3 + tx - t.$$

It has singular fibers of Kodaira type III^* at ∞, II at $t = 0$ and I_1 at $t = -27/4$. Applying the purely inseparable base change $t = s^{13}$, we obtain an elliptic K3 surface X with the same additive fibers, but I_1 replaced by I_{13}.

By construction, X is Zariski and furnished with a sublattice

$$L = U \oplus A_{12} \oplus E_7 \subset \mathrm{NS}(X)$$

of rank 21 and discriminant 26. While this is not relatively prime to the characteristic, Theorem 6.1 is easily adjusted to prove that X has Artin invariant $\sigma \leq 1$. Hence equality holds. (Alternatively this can be inferred from the section $P = (1, 1)$ of height 1/2 on the rational elliptic surface.)

Acknowledgement. We thank T. Shioda for helpful comments and discussions.

References

[1] Artin, M.: Supersingular K3 surfaces. *Ann. Sci. Ecole Norm. Sup.* **7** (1974), 543–567.

[2] Artin, M. and Mazur, B.: Formal groups arising from algebraic varieties. *Ann. Sci. École Norm. Sup. (4)* **10** (1977), no. 1, 87–132.

[3] Bertin, J.: Réseaux de Kummer et surfaces K3. *Invent. Math.* **93** (1988), 267–284.

[4] Blass, P. and Levine, M.: Families of Zariski surfaces. *Duke Math. J.* **49** (1982), no. 1, 129–136.

[5] Bragg, D. and Lieblich, M.: Twistor spaces for supersingular K3 surfaces. Preprint, arXiv: 1804.07282, 2018.

[6] Bragg, D. and Lieblich, M.: Perfect points on genus one curves and consequences for supersingular K3 surfaces. Preprint, arXiv: 1904.04803, 2019.

[7] Elkies, N. D. and Schütt, M.: Genus 1 fibrations on the supersingular K3 surface in characteristic 2 with Artin invariant 1. *Asian J. Math.* **19** (2015), no. 3, 555–582.

[8] van der Geer, G. and Katsura, T.: Relations between some invariants of algebraic varieties in positive characteristic. *Rend. Circ. Mat. Palermo* **62** (2013), 111–125.

[9] Ibukiyama, T.: A basis for the algebra of quaternions over the field of rational numbers and its maximal orders. *Sūgaku* **24** (1972), no. 4, 316–318.

[10] Ibukiyama, T., Katsura, T. and Oort, F.: Supersingular curves of genus two and class numbers. *Compositio Math.* **57** (1986), no. 2, 127–152.

[11] Jang, J.: The non-symplectic index of supersingular K3 surfaces. To appear in *Taiwanese J. Math.*, 2019, doi:10.11650/tjm/181202.

[12] Katsura, T.: Unirational elliptic surfaces in characteristic p. *Tohoku Math. J.* **33** (1981), no. 4, 521–553.

[13] Katsura, T.: Generalized Kummer surfaces and their unirationality in characteristic p. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **34** (1987), no. 1, 1–41.

[14] Katsura, T.: The Chern class map on abelian surfaces. *J. Algebra* **425** (2015), 85–106.

[15] Liedtke, C.: Supersingular K3 surfaces are unirational. *Invent. Math.* **200** (2015), no. 3, 979–1014.

[16] Miranda, R.: Triple covers in algebraic geometry. *Amer. J. Math.* **107** (1985), no. 5, 1123–1158.

[17] Mumford, D.: *Abelian varieties.* Oxford Univ. Press, London/New York, 1970.

[18] Nikulin, V.: On Kummer surfaces. *Math. USSR Izv.* **9** (1975), 261–275.

[19] Nygaard, N. O.: Higher de Rham–Witt complexes of supersingular K3 surfaces. *Compositio Math.* **42** (1980/81), no. 2, 245–271.

[20] Ogus, A.: Supersingular K3 crystals. In *Journées de géométrie algébrique de Rennes (Rennes, 1978), Vol. II,* 3–86. Astérisque 64, Société Mathématique de France, Paris.

[21] Oort, F.: Which abelian surfaces are products of elliptic curves? *Math. Ann.* **214** (1975), 35–47.

[22] Piatetskij-Shapiro, I. I. and Shafarevich, I. R.: A Torelli theorem for algebraic surfaces of type K3. *Math. USSR, Izv.* **5** (1972), 547–588.

[23] Rudakov, A. N. and Shafarevich, I. R.: Supersingular K3 surfaces over fields of characteristic 2. *Math. USSR, Izv.* **13** (1979), 147–165.

[24] Schröer, S.: Some Calabi–Yau threefolds with obstructed deformations over the Witt vectors. *Compos. Math.* **140** (2004), no. 6, 1579–1592.

[25] Schütt, M. and Shioda, T.: Elliptic surfaces. In *Algebraic geometry in East Asia (Seoul 2008),* 51–160. Advanced Studies in Pure Math. 60, Amer. Math. Soc., 2010.

[26] Sengupta, T.: Elliptic fibrations on supersingular K3 surface with Artin invariant 1 in characteristic 3. In *Analytic and algebraic geometry,* 239–292. Hindustan Book Agency, New Delhi, 2017.

[27] Shimada, I.: Supersingular K3 surfaces in characteristic 2 as double covers of a projective plane. *Asian J. Math.* **8** (2004), no. 3, 531–586.

[28] Shioda, T.: On elliptic modular surfaces. *J. Math. Soc. Japan* **24** (1972), 20–59.

[29] Shioda, T.: An example of unirational surfaces in characteristic p. *Math. Ann.* **211** (1974), 233–236.

[30] Shioda, T.: On unirationality of supersingular surfaces. *Math. Ann.* **225** (1977), no. 2, 155–159.

[31] Shioda, T.: Some results on unirationality of surfaces. *Math. Ann.* **230** (1977), no. 2, 153–168.

[32] Shioda, T.: Supersingular K3 surfaces. In *Algebraic geometry (Copenhagen, 1978),* 564–591. Lecture Notes in Math. 732, Springer-Verlag, Berlin, 1979.

[33] Shioda, T.: On the Mordell–Weil lattices. *Comment. Math. Univ. St. Pauli* **39** (1990), no. 2, 211–240.

[34] Shioda, T. and Katsura, T.: On Fermat varieties. *Tohoku Math. J.* **31** (1979), no. 1, 97–115.

[35] Waterhouse, W. C.: Abelian varieties over finite fields. *Ann. Sci. École Norm. Sup. (4)* **2** (1969), 521–560.

[36] Zariski, O.: On Castelnuovo's criterion of rationality $p_a = P_2 = 0$ of an algebraic surface. *Illinois J. Math.* **2** (1958), 303–315.

Received May 18, 2018. Published online November 11, 2019.

Toshiyuki Katsura: Graduate School of Mathematical Sciences, The University of Tokyo, Meguro-ku, Tokyo 153-8914, Japan.
E-mail: tkatsura@ms.u-tokyo.ac.jp

Matthias Schütt: Institut für Algebraische Geometrie, Leibniz Universität Hannover, Welfengarten 1, 30167 Hannover, Germany; *and* Riemann Center for Geometry and Physics, Appelstrasse 2, 30167 Hannover, Germany.
E-mail: schuett@math.uni-hannover.de