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Towards a reversed Faber–Krahn inequality for the truncated Laplacian

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Abstract. We consider the nonlinear eigenvalue problem, with Dirichlet boundary condition, for the very degenerate elliptic operator $\mathcal{P}_1^+$ mapping a function u to the maximum eigenvalue of its Hessian matrix. The aim is to show that, at least for square type domains having fixed volume, the symmetry of the domain maximizes the principal eigenvalue, contrary to what happens for the Laplacian.

1. Introduction

Let us recall that if Ω is a uniformly convex domain and $\lambda_N(X)$ indicates the largest eigenvalue of the symmetric matrix X, then there exist $\mu_1^+ > 0$ and $\varphi(\cdot) > 0$ in Ω such that

$$\begin{cases} \lambda_N(D^2\varphi) + \mu_1^+\varphi = 0 & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega. \end{cases}$$

This was proved in [4]. With a little abuse, but for obvious reasons, we called μ_1^+ and φ respectively the principal eigenvalue and eigenfunction for the operator $\mathcal{P}_1^+(D^2u) = \lambda_N(D^2u)$ in Ω. The value μ_1^+ shares many features with $\mu(\Delta)$, the principal eigenvalue of the Laplacian with homogenous Dirichlet conditions, e.g. the fact that μ_1^+ is a barrier for the validity of the maximum principle. But, strikingly, also many differences. We naturally wondered if other qualitative properties could be extended from $\mu(\Delta)$ to μ_1^+.

Let us start by stating our most surprising result.

> "*Among all rectangles with given measure, the square has the* ***largest*** *eigenvalue* μ_1^+*, and the eigenvalue of the ball of same measure will be even larger than that of the square.*"

This is surprising since, as it is well known, on the contrary, for $\mu(\Delta)$ the principal eigenvalue of the Laplacian, the Faber–Krahn inequality states that

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"*Among all domains with given measure, the ball has the* ***smallest*** *eigenvalue* $\mu(\Delta)$",

which, in its much weaker form, reduces to the obvious fact

"*Among all rectangles with given measure, the square has the* ***smallest*** *eigenvalue* $\mu(\Delta)$."

In [4] we consider a more general class of operators, sometimes called *truncated Laplacians*, which we now describe. For any $N \times N$ symmetric matrix X, let

$$\lambda_1(X) \le \lambda_2(X) \le \cdots \le \lambda_N(X) \tag{1.1}$$

be the ordered eigenvalues of X. For $k \in [1, N]$, k integer, let

$$\mathcal{P}_k^-(D^2u) = \sum_{i=1}^{k} \lambda_i(D^2u) \quad \text{and} \quad \mathcal{P}_k^+(D^2u) = \sum_{i=1}^{k} \lambda_{N+1-i}(D^2u). \tag{1.2}$$

For $k = N$, these operators coincide with the Laplacian, hence we will always consider $k < N$. We want to emphasize that they are fully nonlinear elliptic operators that are degenerate at every point and in every direction.

The truncated Laplacian initially appears in Sha [19], [20] and Wu [21] in order to investigate compact manifolds having *k-convex* boundary, i.e., such that the sum of any k principal curvature functions is positive. Later the operators $\mathcal{P}_k^\pm$ can be found in [1], where Ambrosio and Soner developed a level set theory to the the mean curvature evolution of surfaces with arbitrary codimension. More recently we wish to recall the theory of subequations of Harvey and Lawson, see e.g. [15], [16], which gives a new geometric interpretation of solutions, and the works of Caffarelli, Li and Nirenberg [9], [10] concerning removable singularities along smooth manifolds for Dirichlet problems associated to $\mathcal{P}_k^-$. The extended version of the maximum principle and the study of positive solutions has been done in [2], [14], [13], see also [11] in the case of entire solutions. The case $k = 1$ is treated in the nice paper of Oberman and Silvestre [18] about convex envelope. Blanc and Rossi in [8] consider a similar class of operators, when one takes just one eigenvalue of the Hessian matrix, but not necessarily the first or last one.

Following Berestycki, Nirenberg and Varadhan [3], we have defined in [4] a "candidate" for the principal eigenvalue:

$$\mu_k^- = \sup\{\mu \in \mathbb{R},\ \exists\ \phi > 0 \text{ in } \Omega,\ \mathcal{P}_k^-(D^2\phi) + \mu\phi \le 0\},$$

or

$$\mu_k^+ = \sup\{\mu \in \mathbb{R},\ \exists\ \phi > 0 \text{ in } \Omega,\ \mathcal{P}_k^+(D^2\phi) + \mu\phi \le 0\}.$$

Interestingly, $\mu_k^- = +\infty$ for any bounded domain Ω, while $\mu_k^+ < +\infty$. (For this, see [4], Propositions 4.3 and 4.5 and Remark 4.8). Notice the change of notation, what here is called μ_k^+ and μ_k^- corresponds respectively to μ_k^- and μ_k^+ in [4], since here we consider $\mathcal{P}_1^+$ and there $\mathcal{P}_1^-$). Hence we will concentrate on μ_k^+. As recalled above, in [4] the existence of an eigenfunction was proved only for $k = 1$ and when Ω is uniformly convex. Furthermore, it is the only eigenvalue corresponding to a positive eigenfunction.

Observe that studying μ_1^+ in rectangles had a triple interest; on one hand we wished to see, in the simplest case, if the uniform convexity was a necessary condition for the existence of the eigenfunction. On the other hand, we hoped to construct eigenfunctions for $k > 1$. Finally, it was a way to see if one could expect some relationship between the symmetry of the domain and the size of the principal eigenvalue, as in the Faber–Krahn inequalities. We shall now discuss what we have obtained in these three directions.

On this third point we have seen at the beginning that one should, if anything, expect a reversed Faber–Krahn inequality. We wish to point out another feature that cannot be extend from $\mu(\Delta)$ to μ_1^+: it is a famous result of Lieb. He showed, in [17], that if $A, B \subset \mathbb{R}^N$ are two bounded domains, then

$$\inf_{x\in\mathbb{R}^N} \mu\left(\Delta, A \cap B_x\right) < \mu(\Delta, A) + \mu(\Delta, B), \tag{1.3}$$

$\mu(\Delta, \Omega)$ being the principal eigenvalue of the Laplacian with Dirichlet boundary conditions in Ω, and $B_x = x + B$ denoting B translated by $x \in \mathbb{R}^N$.

The inequality (1.3) is not true in general for μ_1^+ , actually it is reversed if A and B are some specific rectangles.

Concerning the first point, we remark that, even though rectangles are not uniformly convex, in Theorem 4 we construct explicitly an eigenfunction and its corresponding eigenvalue; the eigenfunction is a product of functions of one variable. The proof is not at all obvious but it uses only elementary tools from linear algebra and ode.

The question of whether the condition on the uniform convexity is necessary for the existence of the eigenfunction φ was raised in [4]. It was related in particular with the fact that we could prove global Lipschitz regularity for the Dirichlet problem under that hypothesis.

Let us observe that the eigenfunctions that we construct are indeed only Hölder continuous up to the boundary, which confirms that, in general, in order to get Lipschitz regularity up to the boundary, the hypothesis of the strict convexity cannot be removed. In this paper, thanks to the eigenfunctions in squares that we have constructed, we extend the regularity results to domains that are convex but not necessarily strictly convex. Indeed, in that case we shall prove that, under the condition that near the boundary the forcing term is not "too" negative, the solution of the Dirichlet problems exists and it is Hölder continuous up to the boundary. This is done in Theorem 6.

On the other hand, it is not clear if the condition which we require on the forcing term is necessary. For example, suppose that $f \leq -1$ in some domain Ω which is not strictly convex; can we expect that there are solutions of

$$\mathcal{P}_1^+(D^2u) = f \;\text{ in } \Omega,\; u = 0 \text{ on } \partial\Omega\,?$$

A negative answer to this question is given in [5].

Concerning $k \geq 1$, notice that if $\Omega = B_\rho \subset \mathbb{R}^N$ we can construct the eigenvalue μ_k^+ of $\mathcal{P}_k^+$ and a corresponding eigenfunction in terms of those of the Laplacian in space dimension k.

Let $\phi(x) := v(|x|)$ and $\mu(\Delta)$ be respectively the eigenvalue and the eigenfunction of the Laplacian in the ball of radius ρ in $\mathbb{R}^k$. Hence v satisfies

$$\begin{cases} v''(r) + \frac{k-1}{r}v'(r) + \mu(\Delta)v(r) = 0 & \text{for } r \in (0,\rho), \\ v'(0) = 0,\ v(\rho) = 0. \end{cases} \tag{1.4}$$

Since $v' \leq 0$, arguing as in [6],

$$\left(v''(r) - \frac{v'(r)}{r}\right)' \geq -\frac{k}{r}\left(v''(r) - \frac{v'(r)}{r}\right)$$

and $v''(r) \geq v'(r)/r$ for any $r \in (0,\rho)$. Set $u(x) = v(|x|)$ for $x \in B_\rho$. Then

$$\mathcal{P}_k^+(D^2u(x)) + \mu(\Delta)u(x) = v''(|x|) + \frac{k-1}{|x|}v'(|x|) + \mu(\Delta)v(|x|) = 0, \quad x \in B_\rho.$$

This implies that

$$\mu_k^+ = \mu(\Delta) \tag{1.5}$$

and answers the question that there are at least some domains for which the principal eigenfunctions exists even for $k > 1$. On the other hand, for rectangles we do not know if there is a corresponding eigenfunction. Indeed, contrarily to the case $k = 1$ and $k = N$, we prove that for $k = 2, \ldots, N-1$, if it exists, the eigenfunction cannot be a function which is the product of functions of one variable. It is worth pointing out that other fully nonlinear operators for which this is true are the Pucci extremal operators. This was proved in [7].

The paper is organized in the following way. The next section is preliminary. In Section 3 we construct the explicit eigenfunctions for $k = 1$ and we treat also the case $k > 1$. Section 4 is devoted to existence and Hölder regularity in convex domains of the Dirichlet problem.

2. Preliminaries

We denote by $\mathbb{S}^N$ the set of $N \times N$ symmetric real matrices equipped with its usual partial order. The eigenvalues of $X \in \mathbb{S}^N$ will be henceforth arranged in the nondecreasing order (1.1). The norm of X is

$$\|X\| = \max_{i=1,\ldots,N} |\lambda_i(X)|\,.$$

The operators $\mathcal{P}_k^\pm$, which are fully nonlinear degenerate elliptic operators, can be equivalently defined either by the partial sums (1.2) or by the representation formulas

$$\begin{aligned} \mathcal{P}_k^-(X) &= \min\left\{\sum_{i=1}^k \langle Xv_i, v_i\rangle \mid v_i \in \mathbb{R}^N \text{ and } \langle v_i, v_j\rangle = \delta_{ij}, \text{ for } i,j=1,\ldots,k\right\}, \\ \mathcal{P}_k^+(X) &= \max\left\{\sum_{i=1}^k \langle Xv_i, v_i\rangle \mid v_i \in \mathbb{R}^N \text{ and } \langle v_i, v_j\rangle = \delta_{ij}, \text{ for } i,j=1,\ldots,k\right\}. \end{aligned} \tag{2.1}$$

From (2.1) one deduces the inequalities

$$\mathcal{P}_k^-(X-Y) \le \mathcal{P}_k^\pm(X) - \mathcal{P}_k^\pm(Y) \le \mathcal{P}_k^+(X-Y)$$

and the Lipschitz continuity of $\mathcal{P}_k^\pm : \mathbb{S}^N \mapsto \mathbb{R}$: for any $X, Y \in \mathbb{S}^N$,

$$\left|\mathcal{P}_k^\pm(X) - \mathcal{P}_k^\pm(Y)\right| \le k \left\|X-Y\right\|. \tag{2.2}$$

The following elementary linear algebra lemma will play a key role.

Lemma 1. *Let $a, b \in \mathbb{R}$ and let us consider the symmetric matrix*

$$M(a,b) = \begin{pmatrix} a & b & b & \cdots & b \\ b & a & b & \cdots & b \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ b & \cdots & b & a & b \\ b & b & \cdots & b & a \end{pmatrix}. \tag{2.3}$$

Then, for $b \neq 0$, the eigenvalues of $M(a,b)$ are

- *$a-b$ with multiplicity $N-1$, and its eigenspace is*
$$V = \Big\{x \in \mathbb{R}^N \,:\, \sum_{i=1}^N x_i = 0\Big\};$$
- *$a+(N-1)b$, which is simple, and its eigenspace $V^\perp$ spanned by $(1,\ldots,1)^\mathsf{T}$.*

3. Construction of eigenfunctions

To begin with, consider the eigenvalue problem

$$\begin{cases} \mathcal{P}_k^+(D^2u) + \mu u = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

We claim here that if Ω is convex and (3.1) has a solution $u \in C(\overline{\Omega})$, then $\mu = \mu_k^+$. Indeed, by definition of μ_k^+, we have $\mu \le \mu_k^+$. Under the convexity assumption for $\mu < \mu_k^+$ the operator $\mathcal{P}_k^+(D^2\cdot) + \mu\cdot$ satisfies the maximum principle (see Theorem 4.1 and Remark 4.8 in [4]). This implies $\mu = \mu_k^+$.

3.1. Cube

Let Q_{2R} be the N-dimensional open cube with center 0 and side length $2R$, i.e.,

$$Q_{2R} = (-R,R)^N.$$

We start by computing the principal eigenvalue μ_1^+ in Q_{2R} for the operator $\mathcal{P}_1^+$ by constructing a positive eigenfunction having the multiplicative form

$$u(x) = \prod_{i=1}^N f(x_i), \quad x \in Q_{2R}, \tag{3.2}$$

with f a positive smooth function to be determined. By homogeneity we assume $u(0)=1$, hence $f(0)=1$. To find out f, we compute

$$\partial_{ii}u(x)=f''(x_i)\prod_{k\neq i}f(x_k) \qquad \text{for } i=1,\dots,N,$$
$$\partial_{ij}u(x)=f'(x_i)\,f'(x_j)\prod_{k\neq i,j}f(x_k) \qquad \text{for } i,j=1,\dots,N \text{ and } i\neq j.$$

In particular, on the diagonal $\mathcal{D}=\{x\in Q_{2R}\,:\,x_1=\cdots=x_N\}$ we deduce that

$$D^2u(x)=f^{N-2}(x_1)\cdot M\left(f''(x_1)f(x_1),(f'(x_1))^2\right),$$

where the matrix M is given by (2.3). Using Lemma 1 with $a=f''(x_1)f(x_1)$ and $b=(f'(x_1))^2\geq 0$, we have

$$\mathcal{P}_1^+(D^2u(x))=f^{N-2}(x_1)\cdot\left(f''(x_1)f(x_1)+(N-1)(f'(x_1))^2\right) \quad \text{for } x\in\mathcal{D}.$$

In particular,

$$\mathcal{P}_1^+(D^2u(x))+\mu u(x)=0 \quad \text{for } x\in\mathcal{D} \tag{3.3}$$

if and only if

$$\begin{cases} f''(t)f(t)+(N-1)(f'(t))^2+\mu f^2(t)=0, & t\in(-R,R),\\ f(-R)=f(R)=0. \end{cases} \tag{3.4}$$

This is equivalent to

$$\begin{cases} (f^N)''(t)+N\mu f^N(t)=0, & t\in(-R,R),\\ f(-R)=f(R)=0, \end{cases}$$

and

$$\mu=\frac{1}{N}\left(\frac{\pi}{2R}\right)^2, \qquad f(t)=\sqrt[N]{\cos\left(\frac{\pi}{2R}\,t\right)}. \tag{3.5}$$

Now we need to prove that for such f, the function u given by (3.2) is in turn a solution in the whole cube Q_{2R}. By means of the representation formula (2.1), this is equivalent to show that

$$\max_{v\in\mathbb{R}^N\setminus\{0\}}\frac{\langle D^2u(x)v,v\rangle}{|v|^2}=-\mu u(x) \qquad \forall x\in Q_{2R}. \tag{3.6}$$

Let $x\in Q_{2R}$ and let $v=(f(x_1)\xi_1,\dots,f(x_N)\xi_N)^{\mathsf{T}}$, with $\xi_1,\dots,\xi_N\in\mathbb{R}$ and such that $|v|\neq 0$. Then

$$\langle D^2u(x)v,v\rangle=\Big[\sum_{i=1}^N f''(x_i)\,f(x_i)\,\xi_i^2+2\sum_{i>j}f'(x_i)\,f'(x_j)\,\xi_i\,\xi_j\Big]\,u(x)$$

and using (3.4),

$$
\begin{aligned}
&\langle D^2u(x)v,v\rangle\\
(3.7)\quad &= -\Big[\mu\sum_{i=1}^N f^2(x_i)\,\xi_i^2 + (N-1)\sum_{i=1}^N (f'(x_i))^2\,\xi_i^2 - 2\sum_{i>j} f'(x_i)\,f'(x_j)\,\xi_i\,\xi_j\Big]u(x)\\
&= -\Big[\mu|v|^2 + \sum_{i>j}\big(f'(x_i)\,\xi_i - f'(x_j)\,\xi_j\big)^2\Big]u(x) \le -\mu\,u(x)\,|v|^2.
\end{aligned}
$$

Taking the supremum over $(\xi_1,\dots,\xi_N)\neq(0,\dots,0)$, we deduce that

$$\max_{v\in\mathbb{R}^N\setminus\{0\}} \frac{\langle D^2u(x)v,v\rangle}{|v|^2} \le -\mu\,u(x).$$

Let

$$\tilde D = \Big\{x\in Q_{2R}:\ \exists\ i_0\in\{1,\dots,N\} \text{ such that } \prod\nolimits_{j\neq i_0} x_j\neq 0\Big\}. \tag{3.8}$$

Setting $\xi_i=\prod_{j\neq i} f'(x_j)$, let

$$\hat v = \Big(f(x_1)\prod_{j\neq 1} f'(x_j),\dots,f(x_N)\prod_{j\neq N} f'(x_j)\Big)^{\mathsf{T}}.$$

For $x\in\tilde D$, we have $\hat v\cdot e_{i_0}\neq 0$ since the only zero of $f'(t)$ in $(-R,R)$ is $t=0$. In this way $|\hat v|>0$ and

$$\langle D^2u(x)\hat v,\hat v\rangle = -\mu\,u(x)\,|\hat v|^2. \tag{3.9}$$

In view of (3.7), (3.9), for every $x\in\tilde D$,

$$\mathcal{P}_1^+(D^2u(x)) + \mu\,u(x) = 0\,. \tag{3.10}$$

By continuity, see (2.2), equality (3.10) continues to be true in the whole cube. Summing up we have obtained the following.

Theorem 2. *The principal eigenvalue of $\mathcal{P}_1^+$ in the cube Q_{2R} is*

$$\mu_1^+ = \frac{1}{N}\Big(\frac{\pi}{2R}\Big)^2 \tag{3.11}$$

and a corresponding principal eigenfunction is given by

$$u(x) = \prod_{i=1}^N \sqrt[N]{\cos\Big(\frac{\pi}{2R}x_i\Big)}.$$

For the function u in the theorem above, the functions tu, with any $t>0$, are principal eigenfunctions of $\mathcal{P}_1^+$. It is not known if these are the only principal eigenfunctions.

Conversely to what one could expect, the only cases in which the eigenvalue problem (3.1) with $\Omega=Q_{2R}$ has a solution of type (3.2) are $k=1$ and $k=N$. This is proved in the following:

Theorem 3. *Let $2 \le k \le N-1$ and let us assume that u is a solution of (3.1). Then there are no functions $f \in C^2(-R,R)$ such that $u(x) = \prod_{i=1}^N f(x_i)$.*

Proof. By contradiction let us assume that $u(x) = \prod_{i=1}^N f(x_i)$ is a solution of (3.1). Arguing as in the proof of Theorem 2, we discover that f must satisfy

$$kf''(t)f(t) + (N-k)(f'(t))^2 + \mu f^2(t) = 0 \quad \text{for } t \in (-R,R). \tag{3.12}$$

Hence

$$f(t) = \Big(\cos\Big(\frac{\pi}{2R}t\Big)\Big)^{k/N} \quad \text{and} \quad \mu = \frac{1}{N}\Big(\frac{k\pi}{2R}\Big)^2.$$

We claim that the function

$$u(x) = \prod_{i=1}^N \Big(\cos\Big(\frac{\pi}{2R}x_i\Big)\Big)^{k/N}$$

fails to be a solution of

$$\mathcal{P}_k^+(D^2u(x)) + \mu u(x) = 0$$

for some $x \in Q_{2R}\backslash\mathcal{D}$. Let $v_1,\dots,v_k \in \mathbb{R}^N$ be such that $v_i \cdot v_j = \delta_{ij}$ for any $i,j = 1,\dots,k$.

Using equation (3.12), for $x \in Q_{2R}$,

$$\begin{aligned} &\sum_{i=1}^k \langle D^2u(x)v_i, v_i\rangle \\ &= \sum_{i=1}^k \Big[-\frac{\mu}{k} - \frac{N-k}{k}\sum_{l=1}^N \frac{(f'(x_l))^2}{f^2(x_l)}(v_i)_l^2 + 2\sum_{l>m}\frac{f'(x_l)}{f(x_l)}\frac{f'(x_m)}{f(x_m)}(v_i)_l(v_i)_m\Big]u(x) \\ &= -\Big[\mu + \sum_{i=1}^k \Big\langle M\Big(\frac{N-k}{k},-1\Big)w_i, w_i\Big\rangle\Big]u \end{aligned} \tag{3.13}$$

where

$$w_i = \Big(\frac{f'(x_1)}{f(x_1)}(v_i)_1,\dots,\frac{f'(x_N)}{f(x_N)}(v_i)_N\Big)^{\mathsf{T}}$$

and $M((N-k)/k,-1)$, see Lemma 1, has eigenvalues $\frac{1-k}{k}N < 0$, which is simple, and N/k with multiplicity $N-1$. Now the idea is to choose $x \in Q_{2R}$ and $v_1,\dots,v_k$ such that w_1 is in the eigenspace relative to $-\frac{k-1}{k}N$ and $w_2,\dots,w_k$ are in the orthogonal eigenspace.

Let $\beta > \alpha > 0$ be real fixed number. Let $x \in Q_{2R}$ be such that $x_1 = \dots = x_{N-1}$ and

$$\frac{f'(x_1)}{f(x_1)} = \alpha, \quad \frac{f'(x_N)}{f(x_N)} = \beta.$$

Note that such choice is possible since

$$\frac{f'(t)}{f(t)} = -\frac{k\pi}{2RN}\tan\Big(\frac{\pi}{2R}t\Big)$$

maps the interval $(-R, R)$ onto $\mathbb{R}$. Setting

$$\gamma^2 = \frac{(\alpha\beta)^2}{(N-1)\beta^2 + \alpha^2},$$

we define

$$v_1 = \gamma\Big(\frac{1}{\alpha}, \dots, \frac{1}{\alpha}, \frac{1}{\beta}\Big)^{\mathsf{T}}, \tag{3.14}$$

so that $w_1 = \gamma(1, \dots, 1)^{\mathsf{T}}$ and

$$\langle M w_1, w_1\rangle = -\frac{k-1}{k}\, N^2\gamma^2. \tag{3.15}$$

Now we consider $k-1$ orthonormal vectors $v_2, \dots, v_k$ of the $(N-2)$-dimensional subspace of $\mathbb{R}^N$

$$V = \big\{v \in \mathbb{R}^N \,:\, (v)_1 + \cdots + (v)_{N-1} = 0,\ (v)_N = 0\big\}\,.$$

In this way $w_2, \dots, w_k$ belong to the eigenspace relative to N/k and

$$\sum_{i=2}^{k} \langle M w_i, w_i\rangle = \frac{N(k-1)}{k}\, \alpha^2. \tag{3.16}$$

Since by construction $\langle v_i, v_j\rangle = \delta_{ij}$ for any $i, j = 1, \dots, k$, we can use (3.15)–(3.16) in (3.13) to discover that

$$\begin{aligned}\mathcal{P}_k^+(D^2u(x)) &= \max_{\langle v_i, v_j\rangle=\delta_{ij}} \sum_{i=1}^{k}\big\langle D^2u(x)v_i, v_i\big\rangle = -\Big[\mu + \min_{\langle v_i, v_j\rangle=\delta_{ij}} \sum_{i=1}^{k}\langle M w_i, w_i\rangle\Big]u(x)\\ &\geq -\Big[\mu + \frac{N(k-1)}{k}\,\big(\alpha^2 - N\gamma^2\big)\Big]u(x),\end{aligned}$$

and $\big(\alpha^2 - N\gamma^2\big)$ is strictly negative by the choice $\beta > \alpha > 0$. This contradicts the fact that u is a solution of (3.1) in the whole square. □

3.2. Reversed baby Faber–Krahn inequality

Let $R > 0$ and let $\alpha = (\alpha_1, \dots, \alpha_N)$ be such that $\alpha_i > 0$ for any $i = 1, \dots, N$. We consider the N-dimensional open rectangle with center 0 and side lengths $2\alpha_i^{-1}R$, i.e.,

$$\mathrm{Rect}(\alpha) = \prod_{i=1}^{N}(-\alpha_i^{-1}R, \alpha_i^{-1}R).$$

Note that

$$|\mathrm{Rect}(\alpha)| = (2R)^N \prod_{i=1}^{N}\frac{1}{\alpha_i} = |Q_{2R}|$$

if and only if $\prod_{i=1}^{N}\alpha_1 = 1$. We are going to show that

(FK) *"The cube has the largest μ_1^+ among all rectangles with a given measure".*

Theorem 4. *Let $\alpha = (\alpha_1, \dots, \alpha_N)$ be such that $\alpha_i > 0$ for $i = 1, \dots, N$. Then the principal eigenvalue of $\mathcal{P}_1^+$ in* $\mathrm{Rect}(\alpha)$ *is*

$$\mu_1^+ = \frac{1}{1/\alpha_1^2 + \cdots + 1/\alpha_N^2} \left(\frac{\pi}{2R}\right)^2. \tag{3.17}$$

Moreover, there exists $p = (p_1, \dots, p_N)$, $p_i > -1$ for any $i = 1, \dots, N$, such that

$$u(x) = \prod_{i=1}^{N} \left(\cos\left(\frac{\pi}{2R}\,\alpha_i\, x_i\right)\right)^{1/(p_i+1)}$$

is a principal eigenfunction.

Before giving the proof of the theorem, let us explicitly remark that, in view of (3.11) and (3.17), the statement (FK) reduces to the well-known inequality between the harmonic mean and the geometric mean:

$$\frac{N}{1/\alpha_1^2 + \cdots + 1/\alpha_N^2} \le \sqrt[N]{\alpha_1^2 \cdots \alpha_N^2}\,.$$

The equality occurs if and only if the rectangle is a cube. Moreover, it is worth to point out that from (3.17) we immediately deduce that the infimum of μ_1^+ among all domains with fixed measure is zero.

Proof of Theorem 4. For $p = (p_1, \dots, p_N)$ to be fixed and $i = 1, \dots, N$, let us consider the functions

$$f_i(t) = \left(\cos\left(\frac{\pi}{2R}\, t\right)\right)^{1/(p_i+1)}, \quad t \in (-R, R).$$

Note that

$$(f_i^{p_i+1})''(t) + \left(\frac{\pi}{2R}\right)^2 f^{p_i+1}(t) = 0, \quad t \in (-R, R),$$

which yield

$$(p_i + 1)\, f_i(t)\, f_i''(t) + p_i\,(p_i + 1)\,(f_i'(t))^2 + \left(\frac{\pi}{2R}\right)^2 f_i^2(t) = 0. \tag{3.18}$$

Set

$$u(x) = \prod_{i=1}^{N} f_i(\alpha_i x_i), \quad x \in \mathrm{Rect}(\alpha).$$

Hence

$$\begin{aligned} \partial_{ii} u(x) &= \alpha_i^2\, f_i''(\alpha_i x_i) \prod_{k \ne i} f_k(\alpha_k x_k) && \text{for } i = 1, \dots, N \\ \partial_{ij} u(x) &= \alpha_i\, \alpha_j\, f_i'(\alpha_i x_i)\, f_j'(\alpha_j x_j) \prod_{k \ne i,j} f_k(\alpha_k x_k) && \text{for } i, j = 1, \dots, N \text{ and } i \ne j. \end{aligned}$$

For $x \in \mathrm{Rect}(\alpha)$ and $v = (v_1, \dots, v_N)^\mathsf{T}$ such that $|v| \neq 0$, one has

$$\langle D^2 u(x)v, v\rangle = \sum_{i=1}^{N} \alpha_i^2 f_i''(\alpha_i x_i) \prod_{k\neq i} f_k(\alpha_k x_k) v_i^2 + 2\sum_{i>j} \alpha_i \alpha_j f_i'(\alpha_i x_i) f_j'(\alpha_j x_j) \prod_{k\neq i,j} f_k(\alpha_k x_k) v_i v_j.$$

Setting $v_i = f_i(\alpha_i x_i)\xi_i$, $\xi_i \in \mathbb{R}$ for $i = 1, \dots, N$, the previous equality reads as

$$\langle D^2 u(x)v, v\rangle = \Big[\sum_{i=1}^{N} \alpha_i^2 f_i''(\alpha_i x_i) f_i(\alpha_i x_i)\xi_i^2 + 2\sum_{i>j} \alpha_i\alpha_j f_i'(\alpha_i x_i) f_j'(\alpha_j x_j)\xi_i \xi_j\Big] u(x).$$

Now, using (3.18), we obtain

$$\begin{aligned}\langle D^2 u(x)v, v\rangle &= -\Big[\Big(\frac{\pi}{2R}\Big)^2 \sum_{i=1}^{N} \frac{\alpha_i^2}{p_i+1} v_i^2 + \sum_{i=1}^{N} p_i (f_i'(\alpha_i x_i)\alpha_i \xi_i)^2 \\ &\qquad - 2\sum_{i>j} \alpha_i \alpha_j f_i'(\alpha_i x_i) f_j'(\alpha_j x_j)\xi_i \xi_j\Big] u(x) \\ &= -\Big[\Big(\frac{\pi}{2R}\Big)^2 \sum_{i=1}^{N} \frac{\alpha_i^2}{p_i+1} v_i^2 + \langle Mw, w\rangle\Big] u(x),\end{aligned} \tag{3.19}$$

where

$$w = (f_1'(\alpha_1 x_1)\alpha_1 \xi_1, \dots, f_N'(\alpha_N x_N)\alpha_N \xi_N)^\mathsf{T} \tag{3.20}$$

and

$$M = \begin{pmatrix} p_1 & -1 & \cdots & -1 \\ -1 & p_2 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & \cdots & -1 & p_N \end{pmatrix}.$$

Our aim is now to prove that there exist $p_1, \dots, p_N$ and a positive constant κ such that

$$\begin{gathered}\frac{\alpha_i^2}{p_i+1} = \frac{1}{\kappa} \quad \text{for } i = 1, \dots, N \\ \lambda_1(M) = 0 \le \lambda_2(M) \le \cdots \le \lambda_N(M).\end{gathered} \tag{3.21}$$

Since $p_i = \kappa\alpha_i^2 - 1$, we obtain

$$M = -\begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix} + \kappa\,\mathrm{diag}(\alpha_1^2, \dots, \alpha_N^2).$$

Hence for any $w = (w_1, \dots, w_N)^\mathsf{T}$,

$$\langle Mw, w\rangle = -(w_1 + \cdots + w_N)^2 + \kappa(\alpha_1^2 w_1^2 + \cdots + \alpha_N^2 w_N^2)$$

and (3.21) follows by taking

$$\kappa = \max_{|w|\neq 0} \frac{(w_1 + \cdots + w_N)^2}{\alpha_1^2 w_1^2 + \cdots + \alpha_N^2 w_N^2} = \frac{1}{\alpha_1^2} + \cdots + \frac{1}{\alpha_N^2}. \tag{3.22}$$

Coming back now to (3.19), we deduce that

$$\begin{aligned}\left\langle D^2u(x)v, v\right\rangle &= -\left[\left(\frac{\pi}{2R}\right)^2 \frac{1}{\kappa}\,|v|^2 + \langle Mw, w\rangle\right]u(x)\\ &\leq -\left(\frac{\pi}{2R}\right)^2 \frac{1}{\kappa}\,|v|^2 u(x) \quad \text{for any } v \in \mathbb{R}^N.\end{aligned}$$

Moreover, the equality

$$\left\langle D^2u(x)v, v\right\rangle = -\left(\frac{\pi}{2R}\right)^2 \frac{1}{\kappa}\,|v|^2$$

is achieved if w, which is given by (3.20), realize the maximum in (3.22). Then, if $f_i'(\alpha_i x_i) \neq 0$, it is sufficient to take $\xi_1, \dots, \xi_N$ such that $f_i'(\alpha_i x_i)\alpha_i \xi_i = 1/\alpha_i^2$. Since $f_i'(t) = 0$ implies $t = 0$, we deduce that

$$\begin{aligned}\mathcal{P}_1^+(D^2u(x)) &= \max_{v\in\mathbb{R}^N\setminus\{0\}} \frac{\left\langle D^2u(x)v, v\right\rangle^2}{|v|}\\ &= -\frac{1}{1/\alpha_1^2 + \cdots + 1/\alpha_N^2}\left(\frac{\pi}{2R}\right)^2 u(x) \quad \text{if } \prod_{i=1}^N x_i \neq 0.\end{aligned} \tag{3.23}$$

By continuity, the equality (3.23) still holds in the whole of $\mathrm{Rect}(\alpha)$. □

The previous results show that the behavior of the principal eigenvalues $\mu(\Delta)$ of the Laplacian Δ and μ_1^+ of $\mathcal{P}_1^+$ is opposite with respect to the symmetry of the domain, at least for square type domains. Note that in $\mathrm{Rect}(\alpha)$ one has $\mu(\Delta) = (\frac{\pi}{2R})^2 \sum_{i=1}^N \alpha_i^2$, while $\mu_1^+ = (\frac{\pi}{2R})^2\big(\sum_{i=1}^N \frac{1}{\alpha_i^2}\big)^{-1}$. This surprising feature can be further strengthened:

(FK2) *"The ball has a larger principal eigenvalue than the cube having the same measure".*

Let us consider the ball B_ρ of radius $\rho > 0$. We know, by (1.5), that

$$\mu_1^+(B_\rho) = \left(\frac{\pi}{2\rho}\right)^2,$$

with $u(x) = \cos(\frac{\pi}{2\rho}|x|)$ as principal eigenfunction. Now if we fix the measure, say equals to $(2R)^N$, and we take $\rho = 2R\omega_N^{-1/N}$, being ω_N the measure of the unit ball in $\mathbb{R}^N$, then

$$|B_\rho| = |Q_{2R}|$$

and

$$\mu_1^+(B_\rho) = \left(\frac{\pi}{2R}\right)^2 \frac{\omega_N^{2/N}}{4} > \frac{1}{N}\left(\frac{\pi}{2R}\right)^2 = \mu_1^+(Q_{2R}).$$

This proves (FK2).

3.3. A corollary of Theorems 2 and 4 on the intersection of rectangles

As was said in the introduction, in [17] Lieb showed that if $A, B \subset \mathbb{R}^N$ are two bounded domains, then

$$\inf_{x\in\mathbb{R}^N} \mu\,(\Delta, A \cap B_x) < \mu(\Delta, A) + \mu(\Delta, B). \tag{3.24}$$

We now show that the inequality (3.24) is not true in general for μ_1^+ , actually it is reversed if A and B are some specific rectangles. Let $0 < \alpha_1 \leq \cdots \leq \alpha_N$ and let

$$A = \prod_{i=1}^N \Big(-\frac{R}{\alpha_i}, \frac{R}{\alpha_i}\Big), \quad B = \Big(-\frac{R}{\alpha_2}, \frac{R}{\alpha_2}\Big) \times \Big(-\frac{R}{\alpha_1}, \frac{R}{\alpha_1}\Big) \times \prod_{i=3}^N \Big(-\frac{R}{\alpha_i}, \frac{R}{\alpha_i}\Big).$$

If $N = 2$, then $B = (-R/\alpha_2, R/\alpha_2) \times (-R/\alpha_1, R/\alpha_1)$.

Corollary 5. *If*

$$\frac{1}{\alpha_1^2} > \frac{3}{\alpha_2^2} + \sum_{i=3}^N \frac{1}{\alpha_i^2}$$

then

$$\inf_{x\in\mathbb{R}^N} \mu_1^+(A \cap B_x) > \mu_1^+(A) + \mu_1^+(B). \tag{3.25}$$

Proof. Let

$$C = \Big(-\frac{R}{\alpha_2}, \frac{R}{\alpha_2}\Big) \times \Big(-\frac{R}{\alpha_2}, \frac{R}{\alpha_2}\Big) \times \prod_{i=3}^N \Big(-\frac{R}{\alpha_i}, \frac{R}{\alpha_i}\Big).$$

By using Theorem 4 and (3.17),

$$\inf_{x\in\mathbb{R}^N} \mu_1^+(A \cap B_x) = \mu_1^+\,(C) = \frac{1}{\frac{2}{\alpha_2^2} + \sum_{i=3}^N \frac{1}{\alpha_i^2}} \left(\frac{\pi}{2R}\right)^2,$$

whereas

$$\mu_1^+(A) = \mu_1^+(B) = \frac{1}{\sum_{i=1}^N \frac{1}{\alpha_i^2}} \left(\frac{\pi}{2R}\right)^2,$$

and so (3.25) is satisfied by choosing $0 < \alpha_1 \leq \cdots \leq \alpha_N$ in such a way

$$\frac{1}{\alpha_1^2} > \frac{3}{\alpha_2^2} + \sum_{i=3}^N \frac{1}{\alpha_i^2}. \qquad \square$$

4. Application: Hölder continuity in convex domains

We study the global Hölder continuity of viscosity solutions of

$$\begin{cases} \mathcal{P}_1^+(D^2u) = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded convex domain and f is a continuous and bounded function in Ω.

For notational simplicity, let $Q \equiv Q_\pi$ and $Q(y)$ be respectively the N-dimensional open cubes with centers 0 and $y \in \mathbb{R}^N$ and side length π, i.e.,

$$Q = \Big(-\frac{\pi}{2}, \frac{\pi}{2}\Big)^N, \quad Q(y) = \prod_{i=1}^N \Big(y_i - \frac{\pi}{2}, y_i + \frac{\pi}{2}\Big).$$

By convexity and rescaling, the domain Ω may be expressed as "intersection of cubes" of side length π: there exist a subset $\mathcal{C}$ of $Y \times \mathcal{O}$, $\mathcal{O}$ being the set of $N \times N$ orthogonal matrices and $Y \subset \mathbb{R}^N$, such that

$$\Omega = \bigcap_{(y,O)\in\mathcal{C}} OQ(y), \tag{4.2}$$

where $OQ(y) = \{Ox : x \in Q(y)\}$.

Let us denote by

$$\phi(x) = \prod_{i=1}^N \sqrt[N]{\cos x_i}$$

the eigenfunction provided by Theorem 2. Note that for any $(y, O) \in \mathcal{C}$, the function $\phi_{y,O}(x) = \phi(O^\mathsf{T} x - y)$ solves

$$\begin{cases} \mathcal{P}_1^+(D^2\phi_{y,O}) + \frac{1}{N}\phi_{y,O} = 0 & \text{in } OQ(y), \\ \phi_{y,O} = 0 & \text{on } \partial(OQ(y)). \end{cases} \tag{4.3}$$

Moreover, for any $x, z \in \overline{OQ(y)}$ one has

$$|\phi_{y,O}(x) - \phi_{y,O}(z)| \le \big(\sqrt{N}\,|x - z|\big)^{1/N}. \tag{4.4}$$

Theorem 6 (Hölder). *Let Ω be given by (4.2). If there exist $\alpha > 0$ and $\beta \in (0, 1]$ such that*

$$f(x) \ge -\alpha \Big(\inf_{\mathcal{C}} \phi_{y,O}(x)\Big)^\beta \qquad \forall x \in \Omega, \tag{4.5}$$

then there exists a unique viscosity solution u of (4.1). Moreover, $u \in C^{0,\beta/N}(\overline{\Omega})$, and the Hölder norm of u depends only on α, β, N and the L^∞ norms of u and f.

Proof. The existence and uniqueness of u follows from Perron's method, [12], Theorem 4.1. For this, note that the construction of a continuous subsolution $\underline{u}$ of (4.1), with general f, is standard under the uniform exterior sphere (or cone) condition of Ω. On the other hand, owing to the degeneracy of the operator $\mathcal{P}_1^+$ with respect inf type operations, the equivalent argument used for subsolutions actually fails for the construction of supersolutions null on the boundary. This is the point where the assumption (4.5) is used. For any $x \in \overline{\Omega}$, let

$$\overline{u}(x) = \frac{N\alpha}{\beta} \inf_{\mathcal{C}} \phi_{y,O}^{\beta}(x).$$

Using (4.4), then for any $x, z \in \overline{\Omega}$ one has

$$\begin{aligned} |\overline{u}(x) - \overline{u}(z)| &\le \frac{N\alpha}{\beta} \sup_{\mathcal{C}} \left|\phi_{y,O}^{\beta}(x) - \phi_{y,O}^{\beta}(z)\right| \\ &\le \frac{N\alpha}{\beta} \sup_{\mathcal{C}} |\phi_{y,O}(x) - \phi_{y,O}(z)|^{\beta} \le \frac{N\alpha}{\beta}\big(\sqrt{N}\,|x-z|\big)^{\beta/N}. \end{aligned}$$

Hence $\overline{u} \in C^{0,\beta/N}(\overline{\Omega})$. Moreover, for any $(y, O) \in \mathcal{C}$ and any $x \in \Omega$,

(4.6)
$$\begin{aligned} \mathcal{P}_1^+\Big(D^2 \frac{N\alpha}{\beta}\, \phi_{y,O}^{\beta}(x)\Big) &\le N\alpha(\beta-1)\, \phi_{y,O}^{\beta-2}(x)\, \mathcal{P}_1^-\left(D\phi_{y,O}(x) \otimes D\phi_{y,O}(x)\right) \\ &\quad + N\,\alpha\, \phi_{y,O}^{\beta-1}(x)\, \mathcal{P}_1^+(D^2\phi_{y,O}(x)) \\ &= N\,\alpha\, \phi_{y,O}^{\beta-1}(x)\, \mathcal{P}_1^+(D^2\phi_{y,O}(x)) = -\alpha\, \phi_{y,O}^{\beta}(x) \le f(x). \end{aligned}$$

Then $\overline{u}$, which is the infimum of all $\phi_{y,O}$, is in turn a supersolution of (4.1). Moreover $\overline{u} = 0$ on $\partial\Omega$. Hence the Perron method provides existence and uniqueness for (4.1).

Let us prove now that the solution $u \in C^{0,\beta/N}(\overline{\Omega})$. Without loss of generality we may assume $u \not\equiv 0$.

Let $\Delta_\delta = \{(x, y) \in \Omega \times \Omega \,:\, |x-y| < \delta\}$, where δ is a positive number such that

$$2\,\|u\|_\infty \frac{\beta}{N}\Big(1 - \frac{\beta}{N}\Big)\delta^{-2} > \|f\|_\infty\,. \tag{4.7}$$

Set

$$M = \max\Big(\frac{N^{1+\frac{\beta}{2N}}\alpha}{\beta}, \frac{2\|u\|_\infty}{\delta^{\beta/N}}\Big).$$

We assume by contradiction that

$$0 < \max_{\overline{\Delta}_\delta} \{u(x) - u(y) - M|x-y|^{\beta/N}\} = u(x_0) - u(y_0) - M|x_0 - y_0|^{\beta/N}. \tag{4.8}$$

In particular $x_0 \neq y_0$. If $|x_0 - y_0| = \delta$ then

$$0 < u(x_0) - u(y_0) - M\,|x_0 - y_0|^{\beta/N} \le 2\,\|u\|_\infty - M\,\delta^{\beta/N} \le 0,$$

by the choice of M. If $y_0 \in \partial\Omega$, then there exist $(\tilde{y}_0, \tilde{O}) \in \mathcal{C}$ such that $\Omega \subset \tilde{O}Q(\tilde{y}_0)$ and $y_0 \in \partial(\tilde{O}Q(\tilde{y}_0))$. As in (4.6), the function $\psi(x) = \frac{N\alpha}{\beta}\phi^{\beta}_{\tilde{y}_0,\tilde{O}}(x)$ satisfies in Ω the inequality $\mathcal{P}^+_1(D^2\psi) \le f(x)$. Moreover $\psi \ge 0$ on $\partial\Omega$. By comparison $u \le \psi$ in $\overline{\Omega}$, hence

$$0 < u(x_0) - u(y_0) - M|x_0 - y_0|^{\beta/N} \le \psi(x_0) - \psi(y_0) - M|x_0 - y_0|^{\beta/N} \le 0, \tag{4.9}$$

in view of (4.4) and the choice of M. The above contradictions imply that $(x_0, y_0) \in \Delta_\delta$ or $(x_0, y_0) \in \partial\Omega \times \Omega$. From (4.8) we deduce that $u - \varphi$ has a local minimum at y_0 with $\varphi(y) = -M|y - x_0|^{\beta/N}$. Then

$$f(y_0) \ge \mathcal{P}^+_1(D^2\varphi(y_0)) \ge M\frac{\beta}{N}\Big(1 - \frac{\beta}{N}\Big)\delta^{\beta/N-2} \ge 2\,\|u\|_\infty\,\frac{\beta}{N}\Big(1 - \frac{\beta}{N}\Big)\,\delta^{-2},$$

and this contradicts (4.7). □

Remark 7. Following the argument of the previous proof and looking at (4.9), it is clear that the global $C^{0,\gamma}$ Hölder continuity, $\gamma \in (0,1)$, is still true for any nonpositive supersolutions u of (4.1) without assuming the convexity of Ω (note that $u \le 0$ forces f to be nonnegative somewhere). On the other hand, the global regularity fails if instead we consider nonnegative supersolutions. For example let us consider the nonnegative continuous function

$$u(x) = \begin{cases} \dfrac{1}{\sigma - \sum_{i=1}^N \log(\cos x_i)} & \text{if } x \in Q, \\ 0 & \text{if } x \in \partial Q. \end{cases}$$

We are going to choose σ in such a way that u be concave, in particular $\mathcal{P}^+_1(D^2u) \le 0$ in Q. Let

$$v(x) := \sum_{i=1}^N g(x_i) := \sum_{i=1}^N \log(\cos x_i).$$

Then $\nabla u = \frac{1}{(\sigma - v)^2}\nabla v$, while

$$D^2u = \frac{1}{(\sigma - v)^3}\,(2\nabla v \otimes \nabla v + (\sigma - v)D^2v).$$

And then, since $(D^2v)_{ij} = \delta_{ij}g''(x_i)$, for any $w \in \mathbb{R}^N$,

$$\begin{aligned}
\big\langle D^2u(x)w, w\big\rangle &= \frac{1}{(\sigma - v)^3}\Big(2(\nabla v \cdot w)^2 + (\sigma - v)\sum_{i=1}^N g''(x_i)w_i^2\Big) \\
&= \frac{1}{(\sigma - v)^3}\Big(2\Big(\sum_{i=1}^N \tan(x_i)w_i\Big)^2 - (\sigma - v)\sum_{i=1}^N \frac{1}{\cos^2(x_i)}\,w_i^2\Big) \\
&\le \frac{1}{(\sigma - v)^3}\Big((2N - \sigma)\sum_{i=1}^N \frac{1}{\cos^2(x_i)}\,w_i^2\Big) = 0 \quad \text{if } \sigma = 2N.
\end{aligned}$$

On the other hand, for any $\gamma \in (0,1]$,

$$\sup_{\substack{x,y\in\overline{Q}\\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^\gamma} = +\infty.$$

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