Extreme Khovanov spectra

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Abstract. We prove that the spectrum constructed by González-Meneses, Manchón and the second author is stably homotopy equivalent to the Khovanov spectrum of Lipshitz and Sarkar at its extreme quantum grading.

1. Introduction

Khovanov homology is a powerful link invariant introduced by Mikhail Khovanov in [3] as a categorification of the Jones polynomial. More precisely, given an oriented diagram D representing a link L, he constructed a finite $\mathbb{Z}$-graded family of chain complexes

$$\ldots \longrightarrow C^{i,j}(D) \xrightarrow{d_i} C^{i+1,j}(D) \xrightarrow{d_{i+1}} C^{i+2,j}(D) \longrightarrow \ldots$$

whose bigraded homology groups, $Kh^{i,j}(D)$, are link invariants satisfying

$$J(L)(q) = \sum_{ij} q^j (-1)^i \operatorname{rank}(Kh^{i,j}(L)),$$

where $J(L)$ is the Jones polynomial of L. The groups $Kh^{i,j}(L)$ are known as the *Khovanov homology groups* of L, and the indexes i and j as *homological* and *quantum gradings*, respectively.

A decade later, Lipshitz and Sarkar [6] constructed a $\mathbb{Z}$-graded family of spectra $\mathcal{X}^j(D)$ associated to a link diagram D, and they proved that

for each $j \in \mathbb{Z}$, the spectrum $\mathcal{X}^j(D)$ is a link invariant up to homotopy and there is a canonical isomorphism $H^*(\mathcal{X}^j(D)) \cong Kh^{*,j}(D)$.

The construction of these spectra was later simplified in [4] and [5], where it was shown that each spectrum $\mathcal{X}^j$ can be obtained as the suspension spectrum of the realisation of a certain cubical functor on pointed topological spaces.

For a given link diagram D, the Khovanov chain complex is trivial for all but finitely many j's. Let $j_{\min}(D)$ be the minimal quantum grading such that the complex $\{C^{i,j}(D), d_i\}$ is non-trivial. In [1] González-Meneses, Manchón and Silvero introduced a simplicial complex X_D satisfying the following:

> There is a canonical isomorphism $H^{*+n_- -1}(X_D) \cong Kh^{*,j_{\min}}(D)$, where n_- is the number of negative crossings of D.

In this paper we show that, for the minimal quantum grading, both constructions are stably homotopy equivalent. We present now some remarks and consequences of this result:

(1) Observe that the minimal quantum grading $j_{\min}(D)$ depends on the diagram, and may differ for two different diagrams representing the same link. Given an oriented link diagram L, write $\underline{j}(L)$ for the lowest value of j such that $Kh^{i,j}(L)$ is non-trivial for at least one value of i. Then, $\underline{j}(L)$ is a link invariant and $j_{\min}(D) \leqslant \underline{j}(L)$, for every diagram D representing L. Actually, the equality holds if and only if the simplicial complex X_D is not contractible, and therefore the result in [1] can be rephrased as

> If X_D is not contractible, there is a canonical isomorphism $H^{*+n_- -1}(X_D) \cong Kh^{*,\underline{j}}(L)$, where n_- is the number of negative crossings of D.

(2) Lipshitz and Sarkar have proven that the Khovanov spectra are invariants of links up to stable homotopy. The equivalence result given in this paper implies that X_D, if not contractible, is a link invariant up to stable homotopy.

(3) The Khovanov spectra give a strict refinement of Khovanov homology, since they are not always a wedge of Moore spaces [7]. In [9] Przytycki and the second author conjectured that the simplicial complex X_D is homotopy equivalent to a wedge of spheres. The result of the present paper translates this question to the extreme Khovanov spectra of Lipshitz and Sarkar.

(4) The above conjecture would imply that the extreme Khovanov homology of any link diagram is torsion-free. In particular, it would give a negative answer to the question on whether every oriented link L can be represented by a diagram D so that $j_{\min}(D) = \underline{j}(L)$, as there exist links whose real-extreme Khovanov homology $Kh^{*,\underline{j}}(L)$ contains torsion (the torus knot $T(5,6)$ is such an example).

(5) Lipshitz and Sarkar have given formulas to compute the first two Steenrod operations in the Khovanov spectra they constructed [7]. If the previous conjecture were false, the equivalence result presented in this paper would provide a new approach to compute higher Steenrod operations in the extreme Khovanov spectra, by applying the classical formulas for Steenrod operations in a simplicial complex.
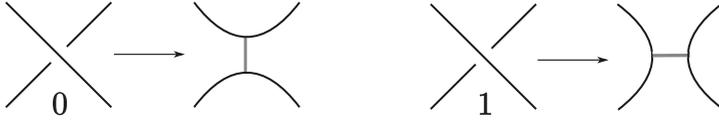
FIGURE 1. The smoothing of a crossing according to its 0 or 1 label.

2. A stable homotopy equivalence

2.1. States and enhancements

Let $\mathbf{2^n}$ be the poset $\{1 \to 0\}^n$, which has an initial element $\vec{1} = (1, 1, \ldots, 1)$ and a terminal element $\vec{0} = (0, 0, \ldots, 0)$, and write $|v| = \sum_{i=1}^n v_i$. Endow the vertices of the cube $\mathbf{2^n}$ with the standard partial order so that $u < v$ if $u_i \leqslant v_i$ for all i. Note that this order is opposite to the natural order in the poset $\mathbf{2^n}$.

Let D be an oriented link diagram with n ordered crossings, where n_+ (n_-) of them are positive (negative). A state is an assignation of a label, 0 or 1, to each crossing in D. There is a bijection between the set $\mathcal{S}$ of states of D and the elements of $\mathbf{2^n}$ by considering $v \in \mathbf{2^n}$ as the state that assigns the ith coordinate of v to the ith crossing of D. Write $D(v)$ for the set of topological circles and chords obtained after smoothing each crossing of D according to its label by following Figure 1.

An enhancement of a state v is a map x assigning a sign ± 1 to each of the $|D(v)|$ circles in $D(v)$. Write $\tau(v, x) = \sum x(c)$ where c ranges over all circles in $D(v)$, and define, for the enhanced state (v, x), the integers

$$h(v, x) = h(v) = -n_- + |v|, \quad q(v, x) = n_+ - 2n_- + |v| + \tau(v, x).$$

Let $j_{\min} = \min\{q(v, x) \mid (v, x) \text{ is an enhanced state of } D\}$, and for any state v write x_- for the constant enhancement of v with value -1.

Proposition 1 (Proposition 4.1 in [1])**.** *In this setting, $j_{\min} = q(\vec{0}, x_-)$ and $q(v, x) = j_{\min}$ if and only if $(v, x) \in \mathcal{S}_{\min}$, where*

$$\mathcal{S}_{\min} = \{\text{enhanced states } (v, x) \text{ such that } |D(v)| = |D(\vec{0})| + |v| \text{ and } x = x_-\}.$$

In particular, $j_{\min} = n_+ - 2n_- - |D(\vec{0})|$.

Let $\mathcal{S}'_{\min} \subset \mathbf{2^n}$ be the subposet of those states v such that $(v, x_-) \in \mathcal{S}_{\min}$, that is, the states v satisfying $|D(v)| = |D(\vec{0})| + |v|$.

Proposition 2. *Let $v \in \mathcal{S}'_{\min}$. If $u < v$ then $u \in \mathcal{S}'_{\min}$.*

Proof. Note that if two states differ just in one coordinate, then one resolution is obtained from the other by splitting one circle into two or merging two circles into one. As $|D(v)| = |D(\vec{0})| + |v|$, necessarily v is obtained from $\vec{0}$ by performing $|v|$ splittings in the crossings corresponding to non-zero coordinates of v. Hence, if u and v differ on k coordinates, v is obtained from u by performing k splittings, that is, $|D(u)| = |D(v)| - k = |D(\vec{0})| + |u|$. $\qquad\square$
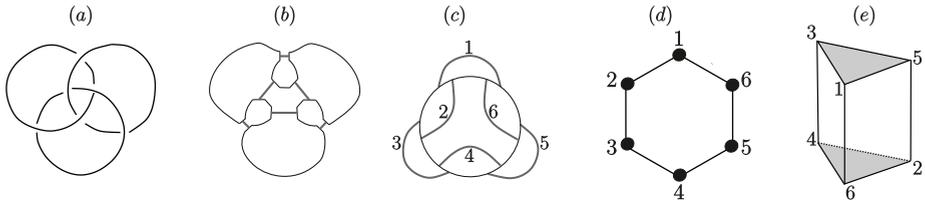
FIGURE 2. (a) A link diagram D; (b) and (c) are two versions of $D(\vec{0})$; (d) is the associated Lando graph G_D; (e) is the geometric realisation of the simplicial complex X_D.

2.2. The simplicial complex for extreme Khovanov homology

Let D be an oriented link diagram. In [1], a simplicial complex X_D was constructed, whose simplicial cochain complex is canonically isomorphic to the extreme Khovanov complex $\{C^{i,j_{\min}}(D), d_i\}$ shifted by $n_- - 1$. Next, we review the construction of X_D (cf. Figure 2).

The Lando graph G_D associated to D consists of a vertex for each chord in $D(\vec{0})$ having both endpoints in the same circle, and an edge between two vertices if the endpoints of the corresponding chords alternate in the same circle. The simplicial complex X_D is defined as the independence complex of the graph G_D; in other words, the simplices of X_D are the subsets of pairwise non-adjacent vertices of G_D. Alternatively, it is the clique complex of the complement graph of G_D.

2.3. Functors to the Burnside category

Let $\mathbf{Top}_\bullet$ be the category of pointed topological spaces with basepoint $*$. Let $\mathbf{Set}_\bullet$ be the category of pointed finite sets, which we view as the full subcategory of $\mathbf{Top}_\bullet$ whose objects are finite discrete pointed spaces. Let $\mathcal{B}$ be the Burnside 2-category for the trivial group, whose objects are finite sets, morphisms are correspondences and 2-morphisms are morphisms of correspondences. We will freely refer to the results and notation of [4] and [5] in what follows (see also the survey [8]).

We say that a correspondence $X \xleftarrow{s} Q \xrightarrow{t} Y$ is *free* if the source map s is injective. Note that there is at most one 2-morphism between any pair of free correspondences. We identify the subcategory $\mathcal{B}_{\text{inj}} \subset \mathcal{B}$, whose objects are finite sets and whose morphisms are free correspondences, with the category $\mathbf{Set}_\bullet$ via the equivalence of categories $\mathbf{Set}_\bullet \simeq \mathcal{B}_{\text{inj}}$ that sends a pointed set A to $A \smallsetminus \{*\}$, and a morphism $f \colon A \to B$ to the correspondence

$$(2.1) \qquad A \smallsetminus \{*\} \leftarrow A \smallsetminus f^{-1}(*) \xrightarrow{f} B \smallsetminus \{*\}.$$

Given $f \colon A \to B$ a function between sets, we write $A_+ := A \cup \{*\}$ and $f_+ \colon A_+ \to B_+$ for the map induced by f. If X is a pointed space, we write $\Sigma^k X$ for the k-fold reduced suspension of X.

A *k-dimensional spatial refinement* (Section 7 in [5], see also Definition 5.1 in [4]) of a functor $F \colon \mathbf{2^n} \to \mathcal{B}$ is a homotopy coherent diagram $G \colon \mathbf{2^n} \to \mathbf{Top}_\bullet$ such that:

(1) For each vertex $u \in \mathbf{2^n}$,

$$G(u) = (F(u) \times [0,1]^k)/(F(u) \times \partial([0,1]^k)) \simeq \bigvee_{x \in F(u)} S^k.$$

(2) For each $u > v$ whose image under F is the correspondence

$$F(u) \xleftarrow{s} F(u > v) \xrightarrow{t} F(v),$$

there exists some embedding

$$\Phi_{u,v} \colon F(u > v) \times [0,1]^k \longrightarrow F(u) \times [0,1]^k$$

making the following diagram commute:

(2.2)
$$
\begin{array}{ccc}
F(u > v) \times [0,1]^k & \xhookrightarrow{\Phi_{u,v}} & F(u) \times [0,1]^k \\
\downarrow & & \downarrow \\
F(u > v) & \xrightarrow{s} & F(u),
\end{array}
$$

and sending $(a, (x_1, \ldots, x_k))$ to $(s(a), (c_1 + d_1 x_1, \ldots, c_k + d_k x_k))$, for some non-negative constants $c_1, d_1, \ldots, c_k, d_k$. Then $G(u > v)$ is the composition

(2.3) $$F(u) \times [0,1]^k/\partial \longrightarrow F(u > v) \times [0,1]^k/\partial \longrightarrow F(v) \times [0,1]^k/\partial,$$

where the ∂ symbol denotes the unions of the boundaries of the cubes, the first map sends a point of the form $\Phi_{u,v}(a,x)$ to $(a,x) \in F(u > v) \times [0,1]^k$ and all other points to the collapsed boundary, and the second map sends (a,x) to $(t(a),x)$.

(3) For each $u > v > w$, the coherent homotopy from $G(v > w) \circ G(u > v)$ to $G(u > w)$ is obtained from an isotopy of embeddings of disjoint unions of k-dimensional intervals.

Lemma 3. *A functor $F \colon \mathbf{2^n} \to \mathcal{B}$ has a 0-dimensional spatial refinement if and only if F factors as $\mathbf{2^n} \to \mathbf{Set_\bullet} \hookrightarrow \mathcal{B}$. If this is the case, the refinement is naturally isomorphic to $\tilde{F} \colon \mathbf{2^n} \to \mathbf{Set_\bullet} \subset \mathbf{Top_\bullet}$.*

Proof. Suppose first that F has a 0-dimensional spatial refinement. Then, $[0,1]^k$ is a point, and therefore the vertical maps in (2.2) are bijections. This implies that for every $u > v$ the source map s of the correspondence $F(u > v)$ is injective, and therefore $F(u > v)$ is free.

Suppose now that F factors through some functor $\tilde{F} \colon \mathbf{2^n} \to \mathbf{Set_\bullet}$. By (2.1), we have that $\tilde{F}(u) = F(u)_+$ and that $\tilde{F}(u > v) \colon F(u)_+ \to F(v)_+$ sends $a \in F(u)$ to $t \circ s^{-1}(a)$ if a is in the image of s and to the basepoint $*$ otherwise. Let us construct an arbitrary 0-dimensional spatial refinement G of F. By definition, for every $u \in \mathbf{2^n}$,

$$G(u) = (F(u) \times [0,1]^0)/(F(u) \times \partial[0,1]^0) = (F(u) \times \{0\})_+.$$

Next, for every $u > v$, any map

$$\Phi_{u,v} \colon F(u > v) \times [0,1]^0 \longrightarrow F(u) \times [0,1]^0$$

making diagram (2.2) commute sends $(a,0)$ to $(s(a),0)$. Moreover, this is an embedding because, as F factors through $\mathbf{Set_\bullet}$, the source map s of the correspondence $F(u > v)$ is injective. The resulting map $G(u > v)$ obtained following (2.3) sends a pair $(a,0)$ to $(\tilde{F}(u > v)(a),0)$. As $k = 0$, there is only one available coherent homotopy from $G(v > w) \circ G(u > v)$ to $G(u > w)$ which is the trivial one, since the space of embeddings between disjoint unions of 0-dimensional intervals is discrete. Therefore, G is well-defined if and only if it commutes on the nose, in which case it is unique. Observe now that there is a natural isomorphism $G \to \tilde{F}$ whose value on a vertex u is the isomorphism $G(u) \to F(u)_+$ sending $(a,0) \in (F(u) \times \{0\})_+$ to $a \in F(u)_+$. Since $\tilde{F}$ commutes on the nose, we deduce that G also does. As a consequence, G is a 0-dimensional spatial refinement naturally isomorphic to $\tilde{F}$. $\square$

Let $\mathbf{2}^{\mathbf{n}}_+$ be the poset obtained as follows: take a second copy of $\mathbf{2}^{\mathbf{n}}$, and rename its terminal object $\vec{0}$ as $\circ$. The poset $\mathbf{2}^{\mathbf{n}}_+$ is the union of both copies along the subposet $\mathbf{2}^{\mathbf{n}} \smallsetminus \{\vec{0}\}$. Alternatively, it is the result of adding two cones to $\mathbf{2}^{\mathbf{n}} \smallsetminus \{\vec{0}\}$ with apices $\vec{0}$ and $\circ$. If $G \colon \mathbf{2}^{\mathbf{n}} \to \mathbf{Top_\bullet}$ is a k-dimensional spatial refinement, then its totalisation is defined as follows: extend G to a functor $G_+ \colon \mathbf{2}^{\mathbf{n}}_+ \to \mathbf{Top_\bullet}$ by declaring $G_+(\circ) = *$ and define

$$\mathrm{Tot}\, G = \mathrm{hocolim}\, G_+ \in \mathbf{Top_\bullet}.$$

2.4. Khovanov spectra

Fix a link diagram D and let $\mathcal{F} \colon \mathbf{2}^{\mathbf{n}} \to \mathcal{B}$ be the functor constructed in Proposition 6.1 of [5] whose value at a vertex v is the set of all possible enhancements associated to the state v. Let $\mathcal{F}^j$ be the subfunctor whose values are those enhancements with quantum grading j. If $\mathcal{G}^j$ is a k-dimensional spatial refinement of $\mathcal{F}^j$, then the Khovanov spectrum of Lipshitz and Sarkar in quantum grading j is (Theorem 3 in [4])

$$(2.4) \qquad\qquad \mathcal{X}^j \simeq \Sigma^{-k-n_-} \Sigma^\infty \mathrm{Tot}\, \mathcal{G}^j.$$

Setting $j = j_{\min}$, we can restate Propositions 1 and 2 in the following way.

Proposition 4. *The value of $\mathcal{F}^{j_{\min}}$ at a vertex $v \in \mathbf{2}^{\mathbf{n}}$ is either the singleton x_- for the case when $v \in \mathcal{S}'_{\min}$, or empty otherwise. Moreover, the value of $\mathcal{F}^{j_{\min}}$ at an arrow $v > u$ is, depending on the values of $\mathcal{F}^{j_{\min}}(u)$ and $\mathcal{F}^{j_{\min}}(v)$,*

	$\mathcal{F}^{j_{\min}}(v) = \emptyset$	$\mathcal{F}^{j_{\min}}(v) = x_-$
$\mathcal{F}^{j_{\min}}(u) = \emptyset$	$\emptyset \leftarrow \emptyset \to \emptyset$	$\nexists$
$\mathcal{F}^{j_{\min}}(u) = x_-$	$\emptyset \leftarrow \emptyset \to \mathcal{F}^{j_{\min}}(u)$	$\mathcal{F}^{j_{\min}}(v) \leftarrow \mathcal{F}^{j_{\min}}(v) \to \mathcal{F}^{j_{\min}}(u)$

In particular, we obtain the following corollary.

Corollary 5. $\mathcal{F}^{j\min}$ *factors through* **Set.** *and therefore the factorisation* $\tilde{\mathcal{F}}^{j\min}$ *is naturally isomorphic to the 0-dimensional spatial refinement of* $\mathcal{F}^{j\min}$. *In fact, it further factors through the inclusion* **Set** $\subset$ **Set.** *sending a set* A *to the pointed set* $A \cup \{*\}$. *If we write* $\hat{\mathcal{F}}^{j\min}$ *for the latter factorisation, we get*

$$
\begin{array}{ccc}
\mathbf{2^n} & \xrightarrow{\mathcal{F}^{j\min}} & \mathcal{B} \\
\Big\downarrow{\scriptstyle\hat{\mathcal{F}}^{j\min}} & \nearrow{\scriptstyle\tilde{\mathcal{F}}^{j\min}} & \Big\uparrow \\
\mathbf{Set} & \hookrightarrow & \mathbf{Set.}
\end{array}
$$

Explicitly, the value of $\hat{\mathcal{F}}^{j\min}$ at a vertex v is either x_- or empty, depending on whether $v \in \mathcal{S}'_{\min}$ or not, and its value at a morphism $v > u$ is

	$\hat{\mathcal{F}}^{j\min}(v) = \emptyset$	$\hat{\mathcal{F}}^{j\min}(v) = x_-$
$\hat{\mathcal{F}}^{j\min}(u) = \emptyset$	$\mathrm{Id}_\emptyset$	$\nexists$
$\hat{\mathcal{F}}^{j\min}(u) = x_-$	$\emptyset \to \hat{\mathcal{F}}^{j\min}(u)$	$\hat{\mathcal{F}}^{j\min}(v) \cong \hat{\mathcal{F}}^{j\min}(u)$

Remark 6. One could ask whether the Khovanov functor $\mathcal{F}^j$ has a 0-dimensional spatial refinement when considering a non-minimal quantum grading j. A careful look at the construction of Lawson, Lipshitz and Sarkar reveals that this is the case if and only if for every enhanced state (u, x_u) with quantum grading j either every 0-chord of $D(u)$ has both endpoints in the same circle or x_u is constant (i.e., x_u assigns the same sign to every circle in $D(u)$). This situation appears in the almost-extreme quantum degree of Khovanov homology of semiadequate links, which has been studied in [10]. An extension of this case by introducing a new functor equivalent to the one defined in [4] will appear in a forthcoming paper by the authors.

2.5. A homotopy equivalence

Let **Poset** be the category of posets, and let **SComp** be the category of simplicial complexes. There are functors

$$
\mathbf{Poset} \underset{\mathscr{X}}{\overset{\mathscr{K}}{\rightleftarrows}} \mathbf{SComp} \xrightarrow{\;|\cdot|\;} \mathbf{Top},
$$

where $\mathscr{X}$ takes a simplicial complex to its poset of non-empty faces, $\mathscr{K}$ takes a poset P to the simplicial complex whose 0-simplices are the elements of P, and whose i-simplices are ascending chains of $i+1$ elements in P. The functor $|\cdot|$ takes a simplicial complex to its realisation. The composition $\mathscr{K} \circ \mathscr{X}$ takes a simplicial complex Y to its barycentric subdivision $\mathrm{sd}(Y)$. We will denote the composition $|\mathscr{K}(\cdot)|$ by $\|\cdot\|$. If P is a poset and $F \colon P \to \mathbf{Top}$ is a functor taking every element of P to a singleton, then (see, for example, [2], pp. 117–118)

$$
(2.5) \qquad\qquad\qquad \mathrm{hocolim}\, F \simeq \|P\|.
$$

The poset $\mathbf{2^n}$ can be identified with the poset of (possibly empty) faces of the standard $(n-1)$-dimensional simplex with the arrows reversed, where we identify $\vec{0}$ with the empty face and $\vec{1}$ with the top-dimensional face. Under this identification, the poset of non-empty faces of X_D becomes precisely $\mathcal{S}'_{\min} \smallsetminus \{\vec{0}\}$ ([1], Proposition 4.3). In summary,

$$(2.6) \qquad \|\mathcal{S}'_{\min} \smallsetminus \{\vec{0}\}\| = |\mathrm{sd}(X_D)| \cong |X_D|.$$

Theorem 7. *There is a homotopy equivalence*

$$\mathcal{X}^{j_{\min}} \simeq \Sigma^{1-n_-} \Sigma^{\infty} |X_D|.$$

Proof. From (2.4) and Corollary 5, we have that

$$\mathcal{X}^{j_{\min}} \simeq \Sigma^{-n_-} \Sigma^{\infty} \operatorname{hocolim} \tilde{\mathcal{F}}^{j_{\min}}_{+}.$$

We will prove that

$$\operatorname{hocolim} \tilde{\mathcal{F}}^{j_{\min}}_{+} \simeq \Sigma \|\mathcal{S}'_{\min} \smallsetminus \{\vec{0}\}\|,$$

and the result will follow from the homeomorphism $\|\mathcal{S}'_{\min} \smallsetminus \{\vec{0}\}\| \cong |X_D|$ in (2.6).

As $\mathbf{2^n_+}$ is constructed as the pushout of two cubes, there is a pushout diagram

$$
\begin{array}{ccc}
\operatorname{hocolim} \tilde{\mathcal{F}}^{j_{\min}}_{+}|_{\mathbf{2^n} \smallsetminus \{\vec{0}\}} & \longrightarrow & \operatorname{hocolim} \tilde{\mathcal{F}}^{j_{\min}}_{+}|_{\mathbf{2^n}} \\
\downarrow & & \downarrow \\
\operatorname{hocolim} \tilde{\mathcal{F}}^{j_{\min}}_{+}|_{\mathbf{2^n_+} \smallsetminus \{\vec{0}\}} & \longrightarrow & \operatorname{hocolim} \tilde{\mathcal{F}}^{j_{\min}}_{+},
\end{array}
$$

and as the two cubes have final elements $\vec{0}$ and $\circ$, we have

$$\operatorname{hocolim} \tilde{\mathcal{F}}^{j_{\min}}_{+}|_{\mathbf{2^n}} \simeq \tilde{\mathcal{F}}^{j_{\min}}_{+}(\vec{0}) = \{x_-, *\}, \quad \operatorname{hocolim} \tilde{\mathcal{F}}^{j_{\min}}_{+}|_{\mathbf{2^n_+} \smallsetminus \{\vec{0}\}} \simeq \tilde{\mathcal{F}}^{j_{\min}}_{+}(\circ) = *.$$

We now proceed to the computation of the upper left term in the diagram. Recall from the second part of Corollary 5 that $\tilde{\mathcal{F}}^{j_{\min}}$ factors as $\hat{\mathcal{F}}^{j_{\min}} : \mathbf{2^n} \to \mathbf{Set} \subset \mathbf{Set_\bullet}$. Since the inclusion $\mathbf{Top} \subset \mathbf{Top_\bullet}$ is a left adjoint, it preserves colimits, and therefore

$$\operatorname{hocolim} \tilde{\mathcal{F}}^{j_{\min}}_{+}|_{\mathbf{2^n} \smallsetminus \{\vec{0}\}} = \operatorname{hocolim} \hat{\mathcal{F}}^{j_{\min}}|_{\mathbf{2^n} \smallsetminus \{\vec{0}\}} \cup \{*\}$$

Now, from Proposition 4, it follows that $\hat{\mathcal{F}}^{j_{\min}}(u)$ is either the singleton x_- or empty depending on whether u belongs to $\mathcal{S}'_{\min}$ or not; therefore

$$\operatorname{hocolim} \hat{\mathcal{F}}^{j_{\min}}|_{\mathbf{2^n} \smallsetminus \{\vec{0}\}} = \operatorname{hocolim} \hat{\mathcal{F}}^{j_{\min}}|_{\mathcal{S}'_{\min} \smallsetminus \{\vec{0}\}},$$

and the homotopy equivalence (2.5) leads to

$$\operatorname{hocolim} \hat{\mathcal{F}}^{j_{\min}}|_{\mathcal{S}'_{\min} \smallsetminus \{\vec{0}\}} \simeq \|\mathcal{S}'_{\min} \smallsetminus \{\vec{0}\}\|.$$

Finally, we face again the original pushout diagram in **Top**$_\bullet$:

$$
\begin{array}{ccc}
\|\mathcal{S}'_{\min} \smallsetminus \{\vec{0}\}\| \cup \{*\} & \longrightarrow & \{x_-, *\} \\
\downarrow & & \downarrow \\
* & \longrightarrow & \operatorname{hocolim} \tilde{\mathcal{F}}_+^{j_{\min}},
\end{array}
$$

where the upper horizontal map collapses $\|\mathcal{S}'_{\min} \smallsetminus \{\vec{0}\}\|$ to $\{x_-\}$. Replacing $\{x_-, *\}$ by $\operatorname{Cone}(\|\mathcal{S}'_{\min} \smallsetminus \{\vec{0}\}\|) \cup \{*\}$ and the $*$ in the lower left corner by $\operatorname{Cone}(\|\mathcal{S}'_{\min} \smallsetminus \{\vec{0}\}\|)$ with basepoint the cone point, we obtain a homotopy equivalent cofibrant pushout diagram, whose colimit is the (unreduced) suspension of $\|\mathcal{S}'_{\min} \smallsetminus \{\vec{0}\}\|$. □

Remark 8. One can similarly define a maximal quantum grading $j_{\max}$ and define a simplicial complex Y_D as the Alexander dual of X_{D^*} where D^* is the mirror image of D (cf. Theorem 7.4 in [9]). The fact that the Khovanov spectrum of a link diagram is the Spanier–Whitehead dual of the Khovanov spectrum of its mirror image, immediately implies that $\mathcal{X}^{j_{\max}} \simeq \Sigma^{n_+ - 1} \Sigma^\infty Y_D$.

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