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Almgren's frequency formula for an extension problem related to the anisotropic fractional Laplacian

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Abstract. We consider the anisotropic version of an extension problem studied by Caffarelli and Silvestre. We present the *anisotropic fractional Laplacian* and prove the Almgren's frequency formula obtained by Caffarelli and Silvestre for the anisotropic case.

1. Introduction

Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a smooth bounded function, let $0 < d < 1$, and let $u\colon \mathbb{R}^n \times [0,+\infty) \to \mathbb{R}$ be a solution to the extension problem

$$\begin{cases} u(x,0) = f(x), & \text{in } \mathbb{R}^n, \\ \Delta_x u + \dfrac{d}{y} u_y + u_{yy} = 0 & \text{in } \mathbb{R}^n \times (0,+\infty). \end{cases} \tag{1.1}$$

In [3], an interesting relation was established:

$$C(-\Delta)^s f(x) = -\lim_{y \to 0^+} y^d u_y(x,y), \tag{1.2}$$

where $(-\Delta)^s$ represents the fractional Laplacian, $s = (1-d)/2$, and $C > 0$ is a constant that depends on n and s. The relation (1.2) extends the classical version

$$C(-\Delta)^{1/2} f(x) = -\lim_{y \to 0^+} u_y(x,y) \tag{1.3}$$

and has many applications:

1. It allows us to use EDP (local) methods to prove Harnack's inequality and Harnack's inequality up to the boundary for Δ^s-harmonic solutions.

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2. Almgren's frequency formula for the problem:

$$\begin{cases} \Delta_x u + \dfrac{d}{y} u_y + u_{yy} = 0, & \text{in } B_1^+ := B_1 \cap \{y > 0\}, \\ \lim\limits_{y \to 0} y^d u_y(x, y) = 0, & \text{in } \{|x| \le 1\}. \end{cases} \tag{1.4}$$

In this work we study the anisotropic version of problem (1.1). Precisely, we will analyze an extension problem related to the *anisotropic fractional Laplacian*

$$(-\Delta)^{\beta,d} f(x) = C_{\beta,d} \int_{\mathbb{R}^n} \frac{f(\varsigma) - f(x)}{\left(\sum_{i=1}^n |\varsigma_i - x_i|^{b_i}\right)^{(c+1-d)/2}} \, d\varsigma, \tag{1.5}$$

where $\beta = \big(b_1, \dots, b_n\big) \in \mathbb{R}^n$, $b_i > 0$, $0 < d < 1$, $c = \sum_{i=1}^n 2/b_i$ and $C_{\beta,d} > 0$ is a normalization constant. An important example of *anisotropic fractional Laplacian* was studied in [1]. In fact, if

$$b_i = n + \sigma_i \quad \text{and} \quad d = c - 1, \tag{1.6}$$

where $\sigma_i \in (0, 2)$, we have

$$(-\Delta)^\sigma f(x) = C_{n,\sigma} \int_{\mathbb{R}^n} \frac{f(\varsigma) - f(x)}{\sum_{i=1}^n |\varsigma_i - x_i|^{n+\sigma_i}} \, d\varsigma,$$

for $\sigma = \big(\sigma_1, \dots, \sigma_n\big)$. The first step to find an extension problem related to the *anisotropic fractional Laplacian* is to obtain an equation such that the fundamental solution at the origin has the following form:

$$\frac{1}{\left(\sum_{i=1}^n |\varsigma_i - x_i|^{b_i}\right)^\kappa}$$

for a suitable constant $\kappa > 0$. Naturally, we find a divergence equation (see next section)

$$\operatorname{div}(A(x)\nabla u) = 0, \quad \text{in } \mathbb{R}^n \setminus \mathcal{S}, \tag{1.7}$$

where the matrix $A = (a_{ij})$ is defined by

$$a_{ij} := \begin{cases} \dfrac{4}{b_i^2} |x_i|^{2-b_i}, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases}$$

and $\mathcal{S} = \bigcup_{i=1}^n \{x_i = 0\}$. As in [3], the second step is to think that the equation of the anisotropic extension problem is a harmonic extension ($b_i = 2$, for $i = n+1, \dots, n+1+d$) of equation (1.7) in $1 + d$ dimensions. Thus, we obtain the extension problem

$$\begin{cases} u(x, 0) = f(x), & \text{in } \mathbb{R}_+^n, \\ \operatorname{div}_x(A(x)\nabla u) + \dfrac{d}{y} u_y + u_{yy} = 0, & \text{in } \mathbb{R}_+^n \times \big(0, +\infty\big), \end{cases} \tag{1.8}$$

where $\mathbb{R}^n_+ = \{(x_1,\dots,x_n)\mathbb{R}^n : x_i > 0\}$. In order to extend the problem (1.8) to $\mathbb{R}^n \times [0,+\infty)$ we need some boundary conditions (see Proposition 2.6):

$$\lim_{x_i\to 0^+} x_i^{2-b_i}\,\partial_i u(x,y) = 0, \tag{1.9}$$

where $\partial_i u$ represents the i-th partial derivative of u and $i = 1,\dots,n$. The equation in (1.8) can also be written as

$$\operatorname{div}(A(x,y)\nabla u) = 0, \tag{1.10}$$

where the matrix $A(x,y) = (a_{ij}(x,y))$ is given by

$$a_{ij}(x,y) := \begin{cases} y^d\,\dfrac{4}{b_i^2}\,|x_i|^{2-b_i}, & \text{if } i = j < n+1,\\ y^d, & \text{if } i = j = n+1,\\ 0, & \text{if } i \neq j. \end{cases}$$

Furthermore, with the change of variables $z = (y/(1-d))^{1-d}$ and $\tau = -2d/(1-d)$ in (1.8) we find

$$\operatorname{div}_x(A(x)\nabla u) + z^\tau u_{zz} = 0. \tag{1.11}$$

We also obtain the following relation up to a multiplicative constant (see Section 3):

$$(-\Delta)^{\beta,d} f(x) = -\lim_{y\to 0^+} y^d \partial_y u(x,y) = -u_z(x,0). \tag{1.12}$$

Since the equation (1.10) represents a harmonic function in some metric g on $\mathbb{R}^n_+ \times \big(0,\infty\big)$, it is to be expected an Almgren's frequency formula for solutions to (1.10). The main difficulty we come across to obtain an anisotropic version of the Almgren frequency formula obtained in [3] is to find the suitable geometry to generalize the arguments used in [3]. A careful analysis of the metric determined by the matrix $A(x,y)$ allows us to conclude that the appropriate geometry is the geometry determined by the level sets of the kernel that governs the operator $(-\Delta)^{\beta,d}$. Namely, we should choose anisotropic balls (ellipses in the appropriate metric)

$$\Theta_r = \Big\{(x_1,\dots,x_n,y) \in \mathbb{R}^{n+1} : \sum_{i=1}^n |x_i|^{b_i} + y^2 < r^2\Big\}$$

in our Almgren's frequency formula. More precisely, we consider a solution u to the equation

$$\operatorname{div}_x(A(x)\nabla u) + \frac{d}{y}u_y + u_{yy} = 0, \quad \text{in } \Theta_1^+ = \Theta_1 \cap (\mathbb{R}^n_+ \times \big(0,+\infty\big)), \tag{1.13}$$

where u satisfies (1.9) and

$$\lim_{y\to 0} y^d\,\partial_y u(x,y) = 0, \tag{1.14}$$

for all $(x,0) \in \Theta_1^+$. We will prove that

$$\Psi(r) = r\,\frac{\int_{\Theta_r^+} \langle A_0(x)\nabla u, \nabla u\rangle\, y^d\, \mu(x)\, dM}{\int_{\partial\Theta_r^+} |u|^2\, y^d\, \mu(x)\, \partial M} \tag{1.15}$$

is a monotone nondecreasing function. Here $M = (\Theta_1^+, g)$, where g is a appropriate metric on Θ_1^+, $\mu(x)$ is the natural weight from the metric g, and $A_0(x) = y^{-d}A(x,y)$. We can summarize what was discussed above as follows.

Theorem 1.1. *If u is a solution to (1.13) in $\Theta_{r_0}^+$ such that, for any $(x,0) \in \Theta_{r_0}^+$, u satisfies (1.9) and (1.14), then*

$$\Psi(r) = r\,\frac{\int_{\Theta_r^+} \langle A_0(x)\nabla u, \nabla u\rangle\, y^d\, \mu(x)\, dM}{\int_{\partial\Theta_r^+} |u|^2\, y^d\, \mu(x)\, \partial M}$$

is a monotone nondecreasing function in $0 < r < r_0 \leq 1$.

In [4], an Almgren's frequency formula was obtained for solutions to (1.10) if the matrix $A(x,y)$ is assumed to be Lipschitz and elliptic. We emphasize that our matrix $A(x,y)$ is not elliptic. Thus, Theorem 1.1 is not a particular case of the result achieved in [4]. Since monotonicity formulas are crucial to exploit the local properties of the equations by giving information about the blowup configurations, we point out that Theorem 1.1 is an important tool in the study of the free boundary of obstacle problems in the anisotropic case, see [2].

The paper is organized as follows. In Section 2 we obtain some basic results: fundamental solution, conjugate equation, Poisson formula and reflection extensions. In Section 3 we present the a relation between the operator $(-\Delta)^{\beta,d}$ and the extension problem (1.8). Section 4 is devoted to the proof of Theorem 1.1.

2. Preliminaries

In this section we gather anisotropic versions of some results obtained in [3]. We begin with the fundamental solution at the origin to (1.7).

Lemma 2.1 (Fundamental solution). *Given $(b_1, \ldots, b_n) \in \mathbb{R}^n$, let $c = \sum_{i=1}^n \frac{2}{b_i}$. Then the function $\Gamma_c \colon \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ defined by*

$$\Gamma_c(x) = \frac{1}{\left(\sum_{i=1}^n |x_i|^{b_i}\right)^{(c-2)/2}}$$

is a solution to equation (1.7).

Proof. Consider the function $h \colon \mathbb{R}^n \to \mathbb{R}$ defined by

$$h(x) = \sum_{i=1}^n |x_i|^{b_i}.$$

A computation yields

$$\partial_i h = b_i\,|x_i|^{b_i-2}\,x_i, \tag{2.1}$$

and

$$\partial^2_{ii} h = b_i\big(b_i-2\big)\,|x_i|^{b_i-4}\,x_i^2 + b_i\,|x_i|^{b_i-2}. \tag{2.2}$$

Thus, if $r = (h(x))^{1/2}$, we find

$$\partial_i r = \frac{1}{2}\,r^{-1}\,\partial_i h \tag{2.3}$$

and

$$\partial^2_{ii} r = -\frac{1}{4}\,r^{-3}\,(\partial_i h)^2 + \frac{1}{2}\,r^{-1}\,\partial^2_{ii} h. \tag{2.4}$$

We attempt to find a radially anisotropic solution $u(x) = u(r)$ to (1.7). Notice that

$$\partial_i u(x) = u'(r)\,\partial_i r, \quad \text{and} \quad \partial^2_{ii} u(x) = u''(r)\,(\partial_i r)^2 + u'(r)\,\partial^2_{ii} r. \tag{2.5}$$

Moreover, we have

$$\partial_i a_{ii} = \frac{1}{b_i^2}\,(2-b_i)\,|x_i|^{-b_i}\,x_i. \tag{2.6}$$

Then, it follows from (2.1), (2.3), (2.5) and (2.6) that

$$\sum_{i=1}^n \partial_i a_{ii}\,\partial_i h = \sum_{i=1}^n \Big(\frac{4}{b_i^2}\,(2-b_i)\,|x_i|^{-b_i} x_i\Big)\,\big(b_i\,|x_i|^{b_i-2}\,x_i\big) = 4\sum_{i=1}^n \frac{2-b_i}{b_i}$$

and

$$\begin{aligned}\sum_{i=1}^n \partial_i a_{ii}\,\partial_i u &= \sum_{i=1}^n \Big(\frac{4}{b_i^2}\,(2-b_i)\,|x_i|^{-b_i}\,x_i\Big)\,\Big(u'(r)\,\frac{r^{-1}}{2}\,b_i\,|x_i|^{b_i-2}\,x_i\Big) \\ &= 2u'(r)\,r^{-1}\sum_{i=1}^n \frac{2-b_i}{b_i}.\end{aligned}$$

We also obtain

$$\sum_{i=1}^n a_{ii}(\partial_i h)^2 = \sum_{i=1}^n \Big(\frac{4}{b_i^2}\,|x_i|^{2-b_i}\Big)\,\big(b_i^2\,|x_i|^{2(b_i-2)}\,x_i^2\big) = 4\,r^2.$$

On the other hand, from (2.2), (2.4) and (2.5) we find

$$\sum_{i=1}^n a_{ii}\,\partial^2_{ii} h = \sum_{i=1}^n \Big(\frac{4}{b_i^2}\,|x_i|^{2-b_i}\Big)\,\big\{[b_i(b_i-2)+b_i]\,|x_i|^{b_i-2}\big\} = 4\sum_{i=1}^n \frac{b_i-1}{b_i},$$

and so

$$\begin{aligned}\sum_{i=1}^n a_{ii}\partial_{ii}^2 u &= \sum_{i=1}^n a_{ii}\big[u''(r)\,(\partial_i r)^2 + u'(r)\,\partial_{ii}^2 r\big]\\
&= u''(r)\sum_{i=1}^n a_{ii}(\partial_i r)^2 + u'(r)\sum_{i=1}^n a_{ii}\,\partial_{ii}^2 r\\
&= u''(r)\sum_{i=1}^n a_{ii}\,\frac{1}{4}\,r^{-2}(\partial_i h)^2 + u'(r)\sum_{i=1}^n a_{ii}\Big[-\frac{1}{4}\,r^{-3}(\partial_i h)^2 + \frac{1}{2}\,r^{-1}\partial_{ii}^2 h\Big]\\
&= u''(r)\,\frac{1}{4}\,r^{-2}\sum_{i=1}^n a_{ii}(\partial_i h)^2\\
&\quad + u'(r)\Big[-\frac{1}{4}\,r^{-3}\sum_{i=1}^n a_{ii}\,(\partial_i h)^2 + \frac{1}{2}\,r^{-1}\sum_{i=1}^n a_{ii}\,\partial_{ii}^2 h\Big]\\
&= u''(r) + \frac{u'(r)}{r}\Big(-1 + 2\sum_{i=1}^n \frac{b_i - 1}{b_i}\Big).\end{aligned}$$

Hence, we have $\operatorname{div}(A(x)\nabla u) = 0$ if and only if

$$u''(r) + \frac{c-1}{r}\,u'(r) = 0.$$

Then, if $u'(r) \neq 0$, we obtain

$$u(x) = \frac{c_1}{\big(\sum_{i=1}^n |x_i|^{b_i}\big)^{(c-2)/2}} + c_2. \qquad \square$$

As in [3] we also have a conjugate equation:

Lemma 2.2 (Conjugate equation)**.** *If* $u\colon \mathbb{R}^n \times \big(0,+\infty\big) \to \mathbb{R}$ *is a solution to the equation*

$$\operatorname{div}_x(A(x)\nabla u) + \frac{d}{y}\,u_y + u_{yy} = 0, \tag{2.7}$$

then $v = y^d u_y$ *is a classical solution to the equation*

$$\operatorname{div}_x(A(x)\nabla u) - \frac{d}{y}\,u_y + u_{yy} = 0. \tag{2.8}$$

Proof. Notice that

$$\partial_i v = y^d\,\partial_y(\partial_i u) \quad \text{and} \quad \partial_{ii} v = y^d\,\partial_y(\partial_{ii} u) \tag{2.9}$$

for $i = 1, \dots, n$. We also have

$$\partial_y v = d\,y^{d-1}\,\partial_y u + y^d\,\partial_{yy}^2 u \tag{2.10}$$

and

$$\partial_{yy} v = d\,(d-1)\,y^{d-2}\,\partial_y u + y^d\,\partial_{yyy}^2 u. \tag{2.11}$$

Then, it follows from (2.9), (2.10) and (2.11) that

$$\begin{aligned}\operatorname{div}_x(A(x)\nabla v) &- \frac{d}{y}v_y + v_{yy}\\ &= y^d\,\partial_y(\operatorname{div}_x(A(x)\nabla u)) - d^2\,y^{d-2}\,\partial_y u - d\,y^{d-1}\,\partial^2_{yy}u\\ &\quad + d\,(d-1)\,y^{d-2}\,\partial_y u + y^d\,\partial^2_{yyy}u\\ &= y^d\,\partial_y(\operatorname{div}_x(A(x)\nabla u)) - d\,y^{d-1}\,\partial^2_{yy}u - d\,y^{d-2}\,\partial_y u + y^d\,\partial^2_{yyy}u\\ &= y^d\,\partial_y(\operatorname{div}_x(A(x)\nabla u) + \frac{d}{y}\,u_y + u_{yy}) = 0. \end{aligned}$$

□

Remark 2.3. In the next proposition, we will use an interesting fact: if τ_1 is the topology generated by Euclidean balls B_r, and τ_2 is the topology generated by anisotropic balls Θ_r, then $\tau_1 = \tau_2$. In fact, if $0 < r < 1$ and $x = (x_1, \ldots, x_n) \in B_r$ we have

$$\sum_{i=1}^n |x_i|^{b_i} = \sum_{i=1}^n (x_i^2)^{b_i/2} \le \sum_{i=1}^n (r^2)^{b_i/2} \le n\,r^{b_{\min}}.$$

Thus, we obtain $B_r \subset \Theta_{(nr^{b_{\min}})^{1/2}}$. On the other hand, if $x = (x_1, \ldots, x_n) \in \Theta_r$ we find

$$\sum_{i=1}^n x_i^2 = \sum_{i=1}^n (|x_i|^{b_i})^{2/b_i} \le \sum_{i=1}^n (r^2)^{2/b_i} \le n\,r^{4/b_{\max}}$$

and so $\Theta_r \subset B_{(n\,r^{4/b_{\max}})^{1/2}}$. Given $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ denote

$$\|x\| = \Big(\sum_{i=1}^n |x_i|^{b_i}\Big)^{1/2}. \tag{2.12}$$

We will also use the following estimate:

$$\begin{aligned}\|x+y\|^2 &= \sum_{i=1}^n |x_i + y_i|^{b_i} \le \sum_{i=1}^n 2^{b_i}(|x_i|^{b_i} + |y_i|^{b_i})\\ &\le 2^{b_{\max}}\Big(\sum_{i=1}^n |x_i|^{b_i} + \sum_{i=1}^n |y_i|^{b_i}\Big) \le 2^{b_{\max}}\big(\|x\| + \|y\|\big)^2,\end{aligned}$$

for all $x, y \in \mathbb{R}^n$. Hence, we obtain

$$\|x+y\| \le 2^{b_{\max}/2}\big(\|x\| + \|y\|\big).$$

Proposition 2.4 (Poisson formula)**.** *The function $u\colon \mathbb{R}^n_+ \times \big[0,+\infty\big) \to \mathbb{R}$ defined by*

$$u(x,y) = \int_{\mathbb{R}^n} \mathcal{P}(x-\varsigma, y)\,f(\varsigma)\,d\varsigma \tag{2.13}$$

is a solution to the problem

$$\begin{cases} u(x,0) = f(x), & \text{in } \mathbb{R}^n_+,\\ \operatorname{div}_x(A(x)\nabla u) + \dfrac{d}{y}\,u_y + u_{yy} = 0, & \text{in } \mathbb{R}^n_+ \times \big(0,+\infty\big), \end{cases} \tag{2.14}$$

where $\mathcal{P}(x,y)$ *is defined by*

$$\mathcal{P}(x,y) = C_{n,\beta,d}\,\frac{y^{1-d}}{\left(\sum_{i=1}^{n}|x_i|^{b_i}+|y|^2\right)^{(c+1-d)/2}},$$

and $C_{n,\beta,d}>0$ *is a normalization constant.*

Proof. Notice that $\mathcal{P}(x,y) = -y^d\,\partial_y\Gamma_{-d}(x,y)$. Thus, using Lemma 2.2 we conclude that $\mathcal{P}$ is a solution to (2.8) in $\mathbb{R}^n_+\times\big(0,+\infty\big)$. Moreover, by Remark 2.3, given $x_0\in\mathbb{R}^n$ and $\varepsilon>0$ we can choose $\delta>0$ such that

$$|f(\varsigma)-f(x_0)| < \frac{\varepsilon}{2}\quad \text{if } \|\varsigma - x_0\|<\delta \tag{2.15}$$

for all $\varsigma\in\mathbb{R}^n$. Then, if $(x,y)\in\mathbb{R}^n_+\times(0,+\infty)$ and $\|\varsigma-x_0\|<\delta$, we estimate

$$\begin{aligned}|u(x,y)-f(x_0)| &= \Big|\int_{\mathbb{R}^n}\mathcal{P}(x-\varsigma,y)\,(f(\varsigma)-f(x_0))\,d\varsigma\Big|\\ &\le \int_{\Theta_\delta(x_0)}\mathcal{P}(x-\varsigma,y)\,|(f(\varsigma)-f(x_0))|\,d\varsigma\\ &\quad+\int_{\mathbb{R}^n\setminus\Theta_\delta(x_0)}\mathcal{P}(x-\varsigma,y)\,|(f(\varsigma)-f(x_0))|\,d\varsigma.\end{aligned}$$

From (2.15) we obtain

$$\int_{\Theta_\delta(x_0)}\mathcal{P}(x-\varsigma,y)\,|(f(\varsigma)-f(x_0))|\,d\varsigma\le\frac{\varepsilon}{2}.$$

Now if $\|(x,y)-(x_0,0)\|\le\dfrac{\delta}{2^{(1+b_{\max}/2)}}$ and $\|\varsigma-x_0\|\ge\delta$ we find

$$\|\varsigma-x_0\|\le 2^{b_{\max}/2}\,\|\varsigma-(x,y)\|+\frac{\delta}{2}\le 2^{b_{\max}/2}\,\|\varsigma-(x,y)\|+\frac{\|\varsigma-x_0\|}{2}.$$

Thus, we have $\|\varsigma-x_0\|/2^{(1+b_{\max}/2)}\le\|\varsigma-(x,y)\|$ and we estimate

$$\begin{aligned}&\int_{\mathbb{R}^n\setminus\Theta_\delta(x_0)}\mathcal{P}(x-\varsigma,y)\big|\big(f(\varsigma)-f(x_0)\big)\big|\,d\varsigma\\ &\quad\le C\,y^{1-d}\int_{\mathbb{R}^n\setminus\Theta_\delta(x_0)}\mathcal{P}(x-\varsigma,y)\,d\varsigma\le C\,y^{1-d}\int_{\mathbb{R}^n\setminus\Theta_\delta(x_0)}\frac{1}{\|\varsigma-x_0\|^{(c+1-d)}}\,d\varsigma,\end{aligned}$$

where $C=C(n,d,\beta,\|f\|_\infty)>0$ is a constant. Furthermore, using polar coordinates (see Lemma 4.2), we have

$$\begin{aligned}\int_{\mathbb{R}^n\setminus\Theta_\delta(x_0)}\|\varsigma-x_0\|^{-(c+1-d)}\,d\varsigma &= \int_\delta^{+\infty}\int_{\Theta_1}r^{-(c+1-d)}\,r^{c-1}\,d\sigma\,dr\\ &=\Big(\int_\delta^{+\infty}r^{-(2-d)}\,dr\Big)\,C(\beta)=\frac{C(\beta)\,\delta^{-(1-d)}}{1-d}.\end{aligned}$$

Taking $y \to 0^+$ we deduce

$$|u(x,y) - f(x_0)| < \varepsilon$$

provided $\|(x,y) - (x_0,0)\|$ is sufficiently small. □

Remark 2.5. By a change variables we obtain a Poisson kernel for (1.11). Precisely, we have

$$\mathcal{P}(x,z) = C_{n,\beta,d} \frac{z}{\left(\sum_{i=1}^n |x_i|^{b_i} + (1-d)^2 |z|^{2/(1-d)}\right)^{(c+1-d)/2}}.$$

The next result provides a reflection extensions for a solution of (1.7) across

$$\left(\bigcup_{i=1}^n \{x_i = 0\} \right) \cup \{y = 0\}$$

in a suitable sense.

Proposition 2.6 (Reflection extensions)**.** *If $u\colon \mathbb{R}^n_+ \times [0,+\infty) \to \mathbb{R}$ is a solution to the problem (1.7) and satisfies (1.9) and (1.14), then there exists an extension v to the whole space such that*

$$\operatorname{div}\left(|y|^d A(x,y)\nabla v\right) = 0, \tag{2.16}$$

in the weak sense in the $(n+1)$-dimensional ball B_r.

Proof. Inductively, we can extend u to $\mathbb{R}^n \times \left[0,+\infty\right)$ (using reflection extensions). Now we consider $v\colon \mathbb{R}^{n+1} \to \mathbb{R}$ defined by

$$v(x,y) = \begin{cases} u(x,y), & \text{if } y \geq 0, \\ u(x,-y), & \text{if } y < 0. \end{cases} \tag{2.17}$$

We will show that

$$\int_{B_r} \langle A(x)\nabla v, \nabla \xi \,\rangle |y|^d \, dx = 0,$$

for all $\xi \in C_0^\infty(B_r)$. In fact, given $\varepsilon > 0$, consider the set

$$\mathcal{R}_\varepsilon = \left(\bigcup_{i=1}^n \{|x_i| \leq \varepsilon\} \right) \cup \{|y| \leq \varepsilon\}.$$

Since $\operatorname{div}(y^a A(x)\nabla v) = 0$, we have

$$\operatorname{div}(y^a \xi A(x)\nabla v) = \langle \nabla \xi, y^a A(x)\nabla v \rangle + \xi \ \operatorname{div}(y^a \xi A(x)\nabla v) = \langle \nabla \xi, y^a A(x)\nabla v \rangle,$$

in $B_r \setminus \mathcal{R}_\varepsilon$. Then, using the divergence theorem, we find

$$\begin{aligned}\int_{B_r} \langle A(x)\nabla v, \nabla\xi\rangle |y|^d dx &= \int_{B_r\setminus\mathcal{R}_\varepsilon} \langle A(x)\nabla v, \nabla\xi\rangle |y|^d dx + \int_{\mathcal{R}_\varepsilon} \langle A(x)\nabla v, \nabla\xi\rangle |y|^d dx \\ &= \int_{B_r\setminus\mathcal{R}_\varepsilon} \operatorname{div}\left(|y|^d \xi\, A(x)\nabla v\right) dx + \int_{\mathcal{R}_\varepsilon} \langle A(x)\nabla v, \nabla\xi\rangle |y|^d dx \\ &= \sum_{i=1}^n \int_{\{|x_i|=\varepsilon\}} \frac{4}{b_i^2}\varepsilon^{2-b_i}\xi\,\partial_i v|y|^d dx + \int_{\{|y|=\varepsilon\}} \xi\,\partial_y v|y|^d dx \\ &\quad + \int_{\mathcal{R}_\varepsilon} \langle A(x)\nabla v, \nabla\xi\rangle |y|^d dx.\end{aligned}$$

Since $|\langle A(x)\nabla v, \nabla v\rangle|\,|y|^d$ is locally integrable the third term above goes to zero when we let $\varepsilon \to 0$. Moreover, u satisfies (1.9) and (1.14). Thus, the remaining terms also go to zero when we let $\varepsilon \to 0$. □

3. Relation with anisotropic fractional Laplacian

As in [3], in this section we will establish the relation between the equation (2.16) and the operator $\Delta^{\beta,d}$. Precisely, we have

$$\lim_{y\to 0^+} y^d \partial_y u(x,y) = u_z(x,0) = -C_{n,\beta,d}(-\Delta)^{\beta,d} f(x).$$

In fact, from Proposition 2.4 it follows that

$$\begin{aligned} u_z(x,0) &= \lim_{z\to 0^+} \frac{u(x,z)-u(x,0)}{z} = \lim_{z\to 0^+}\left[\frac{1}{z}\int_{\mathbb{R}^n} \mathcal{P}(x-\varsigma,z)(f(\varsigma)-f(x))\,d\varsigma\right] \\ &= \lim_{z\to 0^+}\int_{\mathbb{R}^n} C_{n,\beta,d}\frac{f(\varsigma)-f(x)}{\left(\sum_{i=1}^n|\varsigma_i-x_i|^{b_i}+(1-d)^2|z|^{2/(1-d)}\right)^{(c+1-d)/2}}\,d\varsigma \\ &= C_{n,\beta,d}\int_{\mathbb{R}^n} \frac{f(\varsigma)-f(x)}{\left(\sum_{i=1}^n|\varsigma_i-x_i|^{b_i}\right)^{(c+1-d)/2}}\,d\varsigma = -C_{n,\beta,d}(-\Delta)^{\beta,d}f(x), \end{aligned} \tag{3.1}$$

where we obtain (3.1) if f is regular enough.

4. Almgren's frequency formula

This section is devoted to the proof of Theorem 1.1 which assures an Almgren's frequency formula for a solution of the equation (1.13).

Consider on $M = \Theta_1$ the Riemannian metric g defined by the inner product

$$\langle X_p, Y_p\rangle_M = \langle A_0^{-1}(p)X_p, Y_p\rangle,$$

where $A_0 = y^{-d}A(p)$, $X_p, Y_p \in T_pM$ and T_pM is the tangent space to M at p. Using the coordinates system $\Psi\colon \Theta_1 \to \Theta_1$ defined by $\Psi(x,y) = (x,y)$, we find the coefficients

$$g_{ij}(x,y) := \begin{cases} \left(\frac{2}{b_i}\right)^{-2}|x_i|^{b_i-2}, & \text{if } i = j < n+1, \\ 1, & \text{if } i = j = n+1, \\ 0, & \text{if } i \neq j. \end{cases}$$

Furthermore, if $0 < r < 1$ we have

$$\Theta_r = \left\{(x,y) \in \Theta_1 : \langle T(x,y),(x,y)\rangle_M < r^2\right\},$$

where $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is the anisotropic map

$$Te_i = \begin{cases} \dfrac{4}{b_i^2}\, e_i, & \text{if } i = 1,\dots,n, \\ e_{n+1}, & \text{if } i = n+1. \end{cases}$$

With the notations $\nabla_M u$ and $\operatorname{div}_M W$ we denote, respectively, the intrinsic gradient of a function u and the intrinsic divergence of a field $W = \sum_{i=1}^{n+1} W_i$ in Θ_1. We have

$$\nabla_M u = \sum_{i,j=1}^{n} g^{ij}\, \partial_i u\, \partial_i = A_0(x,y)\nabla u$$

and

$$\operatorname{div}_M W = \frac{1}{\sqrt{g_0}} \sum_{i=1}^{n} \partial_i(\sqrt{g_0}\, W_i)$$

where $g_0 = |\det(g_{ij})|$. Thus, u satisfies

$$\operatorname{div}_x(A(x)\nabla u) + \frac{d}{y} u_y + u_{yy} = 0 \quad \text{in } \Theta_1^+$$

if and only if

$$\operatorname{div}_M(y^d \mu(x)\nabla_M u) = 0 \quad \text{in } \Theta_1^+,$$

where

$$\mu(x) = (\sqrt{g_0})^{-1} = \prod_{i=1}^{n} \frac{2}{b_i}\, |x_i|^{(2-b_i)/2}.$$

Moreover, if $0 < r < 1$ and $\gamma\colon (-\varepsilon,\varepsilon) \to \partial\Theta_r$ is a smooth curve with $\gamma(0) = p$ and $\gamma'(0) = X_p$ (since $\gamma((-\varepsilon,\varepsilon)) \subset \partial\Theta_r$), we have

$$r^2 = \langle T\gamma, \gamma\rangle_M = \sum_{i=1}^{n} |\gamma_i|^{b_i} + \gamma_{n+1}^2. \tag{4.1}$$

From (4.1), we find

$$0 = \sum_{i=1}^{n} b_i\,|\gamma_i|^{b_i-2}\,\gamma_i\,\gamma_i' + 2\,\gamma_{n+1}\,\gamma_{n+1}' \tag{4.2}$$
$$= \sum_{i=1}^{n} \left[b_i\left(\frac{2}{b_i}\right)^2\right]\left[\left(\frac{2}{b_i}\right)^{-2}|\gamma_i|^{b_i-2}\right]\gamma_i\,\gamma_i' + 2\,\gamma_{n+1}\,\gamma_{n+1}'$$
$$= 2\left\{\sum_{i=1}^{n} g_{ii}\,\gamma_i\left[\frac{2}{b_i}\,\gamma_i\right]\gamma_i' + \gamma_{n+1}\,\gamma_{n+1}'\right\}.$$

Thus, we obtain the following outward unit normal vector field along $\partial\Theta_r$:

$$N(x,y) = \frac{T_\beta(x,y)}{r}, \tag{4.3}$$

where $T_\beta\colon \mathbb{R}^n \to \mathbb{R}^n$ is the anisotropic map

$$T_\beta e_i = \begin{cases} \dfrac{2}{b_i}\,e_i, & \text{if } i = 1,\dots,n,\\ e_{n+1}, & \text{if } i = n+1,\end{cases}$$

and we use that

$$|T_\beta(x,y)|^2 = \left(\sum_{i=1}^{n} g_{ii}\left(\frac{2}{b_i}\right)^2 x_i^2\right) + y^2 = \left(\sum_{i=1}^{n} |x_i|^{b_i}\right) + y^2 = r^2,$$

on $\partial\Theta_r$. In the sequel we will show some fundamental tools in our strategy to obtain the proof of Theorem 1.1.

Lemma 4.1. *If u is a solution to (1.13) in $\Theta_{r_0}^+$ such that, for any $x \in \Theta_{r_0}^+$, u satisfies (1.9), (1.14) and $y^d\partial_y u(x,y)$ is bounded, then the following identity holds for any $0 < r < r_0$:*

$$\int_{\partial\Theta_r^+} \left(|u_T|^2 - |u_N|^2\right) y^d \mu(x)\,\partial M = \frac{(c+d-1)}{r}\int_{\Theta_r^+} \langle A_0(x,y)\nabla u, \nabla u\rangle\, y^d\mu(x)\,dM,$$

where u_T stands for the gradient of u tangential to $\partial\Theta_r^+$.

Proof. If we write

$$D(x,y) = \operatorname{div}_M\left(y^d\mu(x)\,\frac{|\nabla_M u|^2}{2}\,T_\beta(x,y) - y^d\mu(x)\langle T_\beta(x,y), \nabla_M u\rangle_M \nabla_M u\right),$$

we have

$$D(x,y) = \frac{1}{\sqrt{g}}\operatorname{div}\left(y^d\,\frac{\langle A(x,y)\nabla u, \nabla u\rangle}{2}\,T_\beta(x,y) - y^d\langle T_\beta(x,y), \nabla u\rangle A_0(x,y)\nabla u\right)$$

for all $(x,y) \in \Theta_r^+$. Taking into account that

$$\operatorname{div}(fW) = \langle \nabla f, W\rangle + f\operatorname{div}(W)$$

for all function $f \in C^\infty(\Theta_r^+)$ and for all vector field W along Θ_r^+, we obtain

$$
\begin{aligned}
\sqrt{g}D(x,y) = &\frac{1}{2}\langle \nabla(\langle A_0(x,y)\nabla u, \nabla u\rangle), y^d T_\beta(x,y)\rangle \\
&+ \frac{1}{2}\langle A_0(x,y)\nabla u, \nabla u\rangle \operatorname{div}(y^d T_\beta(x,y)) \\
&- \langle \nabla(\langle T_\beta(x,y), \nabla u\rangle), y^d A_0(x,y)\nabla u\rangle \\
&- \langle T_\beta(x,y), \nabla u\rangle \operatorname{div}(y^d A_0(x,y)\nabla u).
\end{aligned} \tag{4.4}
$$

Since

$$
\langle A_0(x,y)\nabla u, \nabla u\rangle = \sum_{j=1}^n a_{jj}(\partial_j u)^2 + (\partial_y u)^2 \quad \text{and} \quad \langle (x,y), \nabla u\rangle = \sum_{j=1}^n x_j \partial_j u + y\partial_y u,
$$

a computation yields

$$
\partial_i(\langle A_0(x,y)\nabla u, \nabla u\rangle) = \partial_i a_{ii}\,(\partial_i u)^2 + 2\sum_{j=1}^n a_{jj}\,\partial_j u\,\partial_{ij} u \tag{4.5}
$$

and

$$
\partial_i(\langle T_\beta(x,y), \nabla u\rangle) = \frac{2}{b_i}\,\partial_i u + \sum_{j=1}^n \frac{2}{b_j}\, x_j\,\partial_{ij} u, \tag{4.6}
$$

for $i = 1, \ldots, n$. From (4.5), we obtain

$$
\begin{aligned}
&\langle \nabla(\langle A_0(x,y)\nabla u, \nabla u\rangle), T_\beta(x,y)\rangle \\
&= \sum_{i=1}^n \left[\frac{2}{b_i}\,\partial_i a_{ii}\, x_i\,(\partial_i u)^2\right] + 2y\,\partial_y u\,\partial_{yy} u + 2\sum_{i,j=1}^n \frac{2}{b_i}\, a_{jj}\, x_i \partial_j u\,\partial_{ij} u \\
&= \sum_{i=1}^n \left[\frac{2(2-b_i)}{b_i} a_{ii}(\partial_i u)^2\right] + 2y\,\partial_y u\,\partial_{yy} u + 2\sum_{i,j=1}^n \frac{2}{b_j}\, a_{ii} x_j\,\partial_i u\,\partial_{ij} u.
\end{aligned} \tag{4.7}
$$

Moreover, it follows from (4.6) that

$$
\begin{aligned}
&\langle \nabla(\langle T_\beta(x,y), \nabla u\rangle), A(x,y)\nabla u\rangle \\
&= \sum_{i=1}^n \frac{2}{b_i}\, a_{ii}(\partial_i u)^2 + \sum_{i,j=1}^n \frac{2}{b_j}\, a_{ii}\, x_j\,\partial_i u\,\partial_{ij} u + (\partial_y u + y\,\partial_{yy} u)\,\partial_y u \\
&= \sum_{i=1}^n \frac{2}{b_i}\, a_{ii}\,(\partial_i u)^2 + (\partial_y u)^2 + \sum_{i,j=1}^n \frac{2}{b_i}\, a_{ii}\, x_j\,\partial_i u\,\partial_{ij} u + y\,\partial_y u\,\partial_{yy} u.
\end{aligned} \tag{4.8}
$$

We also have

$$
\frac{1}{2}(2-b_i)\,\frac{2}{b_i} - \frac{2}{b_i} = -1 \quad \text{and} \quad \operatorname{div}(y^d T_\beta X) = y^d(c+d+1). \tag{4.9}
$$

Hence, combining (4.4), (4.7), (4.8) and (4.9) we find

$$D(x,y)=\frac{(c+d+1)\langle A_0(x,y)\nabla u,\nabla u\rangle y^d}{2\sqrt{g}}=\frac{(c+d+1)\langle A_0(x,y)\nabla u,\nabla u\rangle}{2}y^d\mu(x).$$

Hereafter, we denote $X=(x,y)$ and $A_0=A_0(x,y)$. Applying the divergence theorem in the set $\Theta_r^+\setminus\mathcal{R}_\varepsilon$, we obtain

$$\begin{aligned}\int_{\Theta_r^+\setminus\mathcal{S}_\varepsilon}\frac{m\langle A_0\nabla u,\nabla u\rangle}{2}\,y^d\,\mu(x)\,dM&=\int_{\partial(\Theta_r^+\setminus\mathcal{S}_\varepsilon)}\frac{\langle A_0\nabla u,\nabla u\rangle}{2}\langle T_\beta X,N\rangle_M y^d\mu(x)\partial M\\&\quad-\int_{\partial(\Theta_r^+\setminus\mathcal{S}_\varepsilon)}\langle T_\beta X,\nabla u\rangle\langle A_0\nabla u,N\rangle_M\,y^d\,\mu(x)\,\partial M\\&=I_1+I_2\end{aligned}$$

where $m=c+d+1$, $\mathcal{S}_\varepsilon=\left(\bigcup_{i=1}^n\{x_i=\varepsilon\}\right)\cup\{y=\varepsilon\}$ and we denote

$$\begin{aligned}I_1=&\int_{\partial\Theta_r^+\setminus\mathcal{S}_\varepsilon}\frac{\langle A_0\nabla u,\nabla u\rangle}{2}\,\langle T_\beta X,N\rangle_M\,y^d\,\mu(x)\,\partial M\\&-\int_{\partial\Theta_r^+\setminus\mathcal{S}_\varepsilon}\langle T_\beta X,\nabla u\rangle\langle A_0\nabla u,N\rangle_M\,y^d\,\mu(x)\,\partial M\end{aligned}$$

and

$$\begin{aligned}I_2=&\int_{\mathcal{S}_\varepsilon}\frac{\langle A_0\nabla u,\nabla u\rangle}{2}\,\langle T_\beta X,N\rangle_M y^d\,\mu(x)\,\partial M\\&-\int_{\mathcal{S}_\varepsilon}\langle T_\beta X,\nabla u\rangle\langle A_0\nabla u,N\rangle_M\,y^d\,\mu(x)\,\partial M.\end{aligned}$$

From (4.3) we find

$$\int_{\partial\Theta_r^+\setminus\mathcal{S}_\varepsilon}\frac{\langle A_0\nabla u,\nabla u\rangle}{2}\langle T_\beta X,N\rangle_M y^d\mu(x)\partial M=\int_{\partial\Theta_r^+\setminus\mathcal{S}_\varepsilon}\frac{|\nabla_M u|^2}{2}y^d\mu(x)\partial M \tag{4.10}$$

and we also have

$$\begin{aligned}&\int_{\partial\Theta_r^+\setminus\mathcal{S}_\varepsilon}\langle T_\beta X,\nabla u\rangle\langle A_0\nabla u,N\rangle_M\,y^d\,\mu(x)\,\partial M\\&\qquad=\int_{\partial\Theta_r^+\setminus\mathcal{S}_\varepsilon}\langle\nabla_M u,N\rangle_M^2\,y^d\,\mu(x)\,\partial M.\end{aligned}\tag{4.11}$$

Combining (4.10) and (4.11) we obtain

$$\begin{aligned}&\int_{\Theta_r^+\setminus\mathcal{S}_\varepsilon}\langle(c+d+1)\,A_0\nabla u,\nabla u\rangle\,y^d\,\mu(x)\,dM\\&\qquad=r\int_{\partial\Theta_r^+\setminus\mathcal{S}_\varepsilon}(|\nabla_M u|^2-2|u_N|^2)\,y^d\,\mu(x)\,\partial M+2\,I_2\\&\qquad=r\int_{\partial\Theta_r^+\setminus\mathcal{S}_\varepsilon}(|u_T|^2-|u_N|^2)\,y^d\,\mu(x)\,\partial M+2\,I_2.\end{aligned}$$

On the other hand, we have

$$\int_{\mathcal{S}_\varepsilon} \frac{\langle A_0 \nabla u, \nabla u\rangle}{2} \langle T_\beta X, N\rangle_M \, y^d \, \mu(x) \, \partial M$$
$$= \sum_{i=1}^n \int_{\{x_i=\varepsilon\}} \frac{\langle A_0 \nabla u, \nabla u\rangle}{2} \langle A_0^{-1} T_\beta X, N_i\rangle \, y^d \, dx + \int_{\{y=\varepsilon\}} \frac{\langle A_0 \nabla u, \nabla u\rangle}{2} \langle A_0^{-1} T_\beta X, e_n\rangle \, \varepsilon^d \, dx,$$

where N_i is the outward unit normal vector field along $\{x_i = \varepsilon\}$. Using the coordinates system $\Psi_i \colon \mathbb{R}^n \to \{x_i = \varepsilon\}$ defined by

$$\Psi_i(x,y) = (x_1, \dots, \varepsilon, \dots, x_n, y), \quad \text{where } x \in \mathbb{R}^{n-1},$$

we get the following outward unit normal vector field along $\{x_i = \varepsilon\}$:

$$N_i(x,y) = -\frac{2\, e_i}{b_i \, \varepsilon^{(b_i-2)/2}}.$$

Furthemore, we have

$$\partial M = \mu^{-1}(x_1, \dots, \varepsilon, \dots, x_n) \, \frac{2}{b_i \, \varepsilon^{(b_i-2)/2}} \, dx.$$

Hence, we obtain

$$\sum_{i=1}^n \int_{\{x_i=\varepsilon\}} \frac{\langle A_0 \nabla u, \nabla u\rangle}{2} \langle A_0^{-1} T_\beta X, N_i\rangle \, y^d \, dx = -\varepsilon \sum_{i=1}^n \int_{\{x_i=\varepsilon\}} \frac{\langle A_0 \nabla u, \nabla u\rangle}{b_i} \, y^d \, dx.$$

Moreover, we find

$$\int_{\{y=\varepsilon\}} \frac{\langle A_0 \nabla u, \nabla u\rangle}{2} \langle A_0^{-1} T_\beta X, e_n\rangle \, \varepsilon^d \, dx = \int_{\{y=\varepsilon\}} \frac{\langle A_0 \nabla u, \nabla u\rangle}{2} \, \varepsilon^{d+1} \, dx$$

and

$$\int_{\mathcal{S}_\varepsilon} \langle T_\beta X, \nabla u\rangle \langle A_0 \nabla u, N\rangle_M y^d \mu(x) \partial M = \int_{\{y=\varepsilon\}} y^d \langle T_\beta X, \nabla u\rangle \partial_y u \, dx - \sum_{i=1}^n \int_{\{x_i=\varepsilon\}} y^d \langle T_\beta X, \nabla u\rangle \tilde{b}_i \varepsilon^{2-b_i} \partial_i u \, dx,$$

where $\tilde{b}_i = (4/b_i^2)$. Then, since u satisfies (1.9) and (1.14), the term I_2 goes to zero when we let $\varepsilon \to 0$. The proof of the lemma is concluded. □

Our second tool is the expression of the volume form ∂M in polar coordinates.

Lemma 4.2 (Polar coordinates)**.** *Given $r > 0$, there exists a coordinates system $\tilde{\varphi}\colon (0, r] \times U \to \overline{\Theta_r}$ such that*

$$\partial M = \sqrt{g(r,\theta)}\, d\theta_1 \cdots d\theta_n,$$

where $U \subset \mathbb{R}^n$ is an open set and $(r, \theta) = (r, \theta_1, \dots, \theta_n)$. Precisely, we have

$$g(r,\theta) = r^{2n} g(1,\theta).$$

Proof. Consider the polar coordinates system $\varphi\colon (0, r] \times U \to \overline{B_r}$ defined by

$$\varphi(r, \theta_1, \dots, \theta_n) = (r\,\phi_1(\theta_1, \dots, \theta_n), \dots, r\,\phi_{n+1}(\theta_1, \dots, \theta_n)).$$

Notice that

$$\tilde{\varphi} = \big(r^{2/b_1}\,\Phi_1, \dots, r^{2/b_n}\,\Phi_n, r\,\phi_{n+1}\big),$$

where $\Phi_i(\theta_1, \dots, \theta_n) = (\phi_i(\theta_1, \dots, \theta_n))^{2/b_i}$, is a polar coordinates system on the closed ball $\overline{\Theta_1}$. Moreover, we find

$$\partial_r \tilde{\varphi} = \big((2/b_1)\, r^{2/b_1 - 1}\,\Phi_1, \dots, (2/b_n)\, r^{2/b_n - 1}\,\Phi_n, \phi_{n+1}\big)$$

and

$$\partial_{\theta_j} \tilde{\varphi} = \big(r^{2/b_1}\,\partial_{\theta_j}\Phi_1, \dots, r^{2/b_n}\,\partial_{\theta_j}\Phi_n, r\partial_{\theta_j}\phi_{n+1}\big).$$

Next we will calculate the metric coefficients in the polar coordinates system (r, θ). A computation yields

$$\begin{aligned}
\langle \partial_r, \partial_r \rangle_M &= \sum_{i=1}^{n} \left(\frac{2}{b_i}\right)^{-2} \left| r^{2/b_i}\,\Phi_i \right|^{b_i - 2} \left(\frac{2}{b_i}\right) r^{4/b_i - 2}\,\Phi_i^2 + \phi_{n+1}^2 \\
&= \sum_{i=1}^{n} \phi_i^2 + \phi_{n+1}^2 = 1,
\end{aligned}$$

and

$$\begin{aligned}
\langle \partial_{\theta_j}, \partial_{\theta_k} \rangle_M &= \sum_{i=1}^{n} \left(\frac{2}{b_i}\right)^{-2} \left| r^{2/b_i}\,\Phi_i \right|^{b_i - 2} r^{4/b_i}\,(\partial_{\theta_j}\Phi_i \partial_{\theta_k}\Phi_i) + r^2(\partial_{\theta_j}\phi_i \partial_{\theta_k}\phi_i) \\
&= r^2 \Big[\sum_{i=1}^{n} \left(\frac{2}{b_i}\right)^{-2} (\partial_{\theta_j}\Phi_i \partial_{\theta_k}\Phi_i)\,\Phi_i^{b_i - 2} + \partial_{\theta_j}\phi_{n+1}\,\partial_{\theta_k}\phi_{n+1} \Big].
\end{aligned}$$

Furthermore, we find

$$\begin{aligned}
\langle \partial_r, \partial_{\theta_j} \rangle_M &= \sum_{i=1}^{n} \left(\frac{2}{b_i}\right)^{-2} \left| r^{2/b_i}\,\Phi_i \right|^{b_i - 2} \left(\frac{2}{b_i}\right) r^{4/b_i - 1}\,\Phi_i\,\partial_{\theta_j}\Phi_i + r\,\phi_{n+1}\,\partial_{\theta_j}\phi_{n+1} \\
&= r \sum_{i=1}^{n} \left(\frac{2}{b_i}\right)^{-1} \Phi_i^{b_i - 1}\,\partial_{\theta_j}\Phi_i + r\,\phi_{n+1}\,\partial_{\theta_j}\phi_{n+1} \\
&= r \left(\sum_{i=1}^{n} \phi_i\,\partial_{\theta_j}\phi_i + \phi_{n+1}\,\partial_{\theta_j}\phi_{n+1} \right) = 0.
\end{aligned}$$

Then we have

$$\partial M = \sqrt{g(r,\theta)} N_r \, d\theta_1 \cdots d\theta_n \quad \text{and} \quad g(r,\theta) = r^{2n} g(1,\theta),$$

where $N(X) = T_\beta X / \|T_\beta X\|$ is the outward unit normal vector field along $\partial\Theta_r$ and

$$N = N_r \partial_r + \sum_{i=1}^n N_i \partial_{\theta_i}.$$

Thus, we find $N_r = \langle N, \partial_r \rangle_M$. Moreover, we have

$$\begin{aligned} \langle T_\beta \tilde\varphi, T_\beta \tilde\varphi \rangle_M &= \sum_{i=1}^n \left(\frac{2}{b_i}\right)^{-2} \left| r^{2/b_i} \Phi_i \right|^{b_i - 2} \left(\frac{2}{b_i}\right)^2 r^{4/b_i} \Phi_i^2 + r^2 \phi_{n+1}^2 \\ &= r^2 \Big(\sum_{i=1}^n \phi_i^2 + \phi_{n+1}^2 \Big) = r^2 \end{aligned}$$

and

$$\begin{aligned} \langle T_\beta \tilde\varphi, \partial_r \rangle_M &= \sum_{i=1}^n \left(\frac{2}{b_i}\right)^{-2} \left| r^{4/b_i} \Phi_i \right|^{b_i - 2} \left(\frac{2}{b_i}\right)^2 r^{4/b_i - 1} \Phi_i^2 + r \phi_{n+1}^2 \\ &= r \Big(\sum_{i=1}^n \phi_i^2 + \phi_{n+1}^2 \Big) = r. \end{aligned}$$

Finally, we obtain

$$\langle N, \partial_r \rangle_M = \frac{\langle T_\beta \tilde\varphi, \partial_r \rangle_M}{\langle T_\beta \tilde\varphi, T_\beta \tilde\varphi \rangle_M^{1/2}} = 1. \qquad \square$$

We have now gathered all the tools and ingredients we need to establish our Almgren's frequency formula.

Proof of Theorem 1.1. It is enough to prove that $\log \Psi(r)$ is nondecreasing. In fact, we have

$$\log \Psi(r) = \log r + \log \int_{\Theta_r^+} \langle A_0 \nabla u, \nabla u \rangle y^d \mu(x) \, dM - \log \int_{\partial\Theta_r^+} |u|^2 y^d \mu(x) \, \partial M. \tag{4.12}$$

For $0 < r \le r_0$ we denote

$$D(r) = \int_{\Theta_r^+} \langle A_0 \nabla u, \nabla u \rangle \, y^d \, \mu(x) \, dM \quad \text{and} \quad F(r) = \int_{\partial\Theta_r^+} |u|^2 \, y^d \, \mu(x) \, \partial M.$$

By Lemma 4.2 we find

$$\int_{\Theta_r^+} \langle A_0 \nabla u, \nabla u \rangle y^d \mu(x) \, dM = \int_0^r \int_U \big(|\nabla_M u|^2) y^d \mu(x) \big) (\tilde\varphi(r,\theta)) \sqrt{g(r,\theta)} \, d\theta \, dr \tag{4.13}$$

and

$$\int_{\partial\Theta_r^+} |u|^2 \, y^d \, \mu(x) \, \partial M = \int_U |u(\tilde\varphi(r,\theta))|^2 \, \mu(\tilde\varphi(r,\theta)) \, \sqrt{g(r,\theta)} \, d\theta_1 \cdots d\theta_n. \tag{4.14}$$

Furthermore, again by Lemma 4.2 we have

$$\mu(\tilde{\varphi}(r,\theta)) = C_{n,\beta}\left(\prod_{i=1}^{n}(r^{2/b_i})^{2-b_i}\,(\Phi_i(\theta))^{2-b_i}\right)^{1/2} = C_{n,\beta}\,r^{c-n}\left(\prod_{i=1}^{n}(\Phi_i(\theta))^{2-b_i}\right)^{1/2} \tag{4.15}$$

and

$$\sqrt{g(r,\theta)} = r^{2n}\sqrt{g(1,\theta)}. \tag{4.16}$$

Hence, from (4.13), (4.14), (4.15) and (4.16) we obtain

$$D'(r) = \int_{\partial\Theta_r^+} \langle A_0\nabla u, \nabla u\rangle\, y^d\,\mu(x)\,\partial M \tag{4.17}$$

and

$$F'(r) = \int_{\partial\Theta_r^+} \left[2\,u\,u_N + r^{-1}\,(c+d)\,|u|^2\right] y^d\,\mu(x)\,\partial M. \tag{4.18}$$

From (4.12), (4.17) and (4.18) we obtain

$$(\log\Psi(r))' = \frac{1}{r} + \frac{\int_{\partial\Theta_r^+}\langle A_0\nabla u, \nabla u\rangle\, y^d\,\mu(x)\,\partial M}{\int_{\Theta_r^+}\langle A_0\nabla u, \nabla u\rangle\, y^d\,\mu(x)\,dM} - \frac{\int_{\partial\Theta_r^+}\left[2\,u\,u_N + r^{-1}\,(c+d)\,|u|^2\right]\, y^d\,\mu(x)\,\partial M}{\int_{\partial\Theta_r^+}|u|^2\,y^d\,\mu(x)\,\partial M}. \tag{4.19}$$

On the other hand, since

$$\mathrm{div}_M(y^d\mu(x)\nabla_M u) = 0,$$

we find

$$\mathrm{div}_M(u\,y^d\,\mu(x)\,\nabla_M u) = |\nabla_M u|^2\,y^d\,\mu(x),$$

which reveals

$$\int_{\Theta^+\setminus\mathcal{S}_\varepsilon}|\nabla_M u|^2\,y^d\,\mu(x)\,dM = \int_{\partial(\Theta_r^+\setminus\mathcal{S}_\varepsilon)} u\,u_N\,y^d\,\mu(x)\,\partial M = I_3 + I_4,$$

where

$$I_3 = \int_{\partial\Theta_r^+\setminus\mathcal{S}_\varepsilon} u\,u_N\,y^d\,\mu(x)\,\partial M \quad \text{and} \quad I_4 = \int_{\mathcal{S}_\varepsilon} u\,u_N\,y^d\,\mu(x)\,\partial M.$$

Notice that

$$\int_{\mathcal{S}_\varepsilon} u\,u_N\,y^d\,\mu(x)\,\partial M = \sum_{i=1}^{n}\int_{\{x_i=\varepsilon\}}\left(\frac{b_i}{2}\,\varepsilon^{2-b_i}\,u\,\partial_i u\right) y^d\,dx + \int_{\{y=\varepsilon\}} u\,\varepsilon^d\,\partial_y u\,dx,$$

and since u satisfies (1.9) and (1.14), the term I_4 goes to zero when we let $\varepsilon \to 0$. Then, we find

$$\int_{\Theta_r^+} \langle A_0 \nabla u, \nabla u \rangle \, y^d \, \mu(x) \, dM = \int_{\partial\Theta_r^+} u \, u_N \, y^d \, \mu(x) \, \partial M, \tag{4.20}$$

and by Lemma 4.1 we have

$$\begin{aligned} \int_{\partial\Theta_r^+} \langle A_0 \nabla u, \nabla u \rangle \, y^d \, \mu(x) \, dM &= \int_{\partial\Theta_r^+} |\nabla_M u|^2 \, y^d \, \mu(x) \, \partial M \\ &= \int_{\partial\Theta_r^+} \left(|u_T|^2 - |u_N|^2 \right) y^d \, \mu(x) \, \partial M \\ &\quad + 2 \int_{\partial\Theta_r^+} |u_N|^2 \, y^d \, \mu(x) \, \partial M \\ &= \frac{(c+d-1)}{r} \int_{\Theta_r^+} \langle A_0 \nabla u, \nabla u \rangle \, y^d \, \mu(x) \, dM \\ &\quad + 2 \int_{\partial\Theta_r^+} |u_N|^2 \, y^d \, \mu(x) \, \partial M. \end{aligned} \tag{4.21}$$

Finally, combining (4.19), (4.20) and (4.21) we obtain

$$\begin{aligned} (\log \Psi(r))' &= \frac{1}{r} + \frac{\frac{(c+d-1)}{r} \int_{\Theta_r^+} \langle A_0 \nabla u, \nabla u \rangle y^d \mu(x) \, dM + 2 \int_{\partial\Theta_r^+} |u_N|^2 y^d \mu(x) \partial M}{\int_{\partial\Theta_r^+} u u_N y^d \mu(x) \partial M} \\ &\quad - \frac{\frac{1}{r} \int_{\partial\Theta_r^+} (c+d) |u|^2 y^d \mu(x) \, dM + 2 \int_{\partial\Theta_r^+} u u_N y^d \mu(x) \partial M}{\int_{\partial\Theta_r^+} |u|^2 y^d \mu(x) \partial M} \\ &= 2 \Big(\frac{\int_{\partial\Theta_r^+} |u_N|^2 \, y^d \, \mu(x) \, \partial M}{\int_{\partial\Theta_r^+} u \, u_N \, y^d \, \mu(x) \, \partial M} - \frac{\int_{\partial\Theta_r^+} u \, u_N \, y^d \, \mu(x) \, \partial M}{\int_{\partial\Theta_r^+} |u|^2 \, y^d \, \mu(x) \, \partial M} \Big) \\ &\quad + \frac{1}{r} \left[1 + (c+d-1) - (c+d) \right] \\ &\geq 0, \end{aligned}$$

where we use the Cauchy–Schwarz inequality to conclude that

$$\begin{aligned} &\Big(\int_{\partial\Theta_r^+} u \, u_N \, y^d \, \mu(x) \, \partial M \Big)^2 \\ &\qquad \leq \Big(\int_{\partial\Theta_r^+} |u_N|^2 \, y^d \, \mu(x) \, \partial M \Big) \Big(\int_{\partial\Theta_r^+} |u|^2 \, y^d \, \mu(x) \, \partial M \Big). \end{aligned}$$

□

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