



The Lie group of automorphisms of a Courant algebroid and the moduli space of generalized metrics

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Abstract. We endow the group of automorphisms of an exact Courant algebroid over a compact manifold with an infinite dimensional Lie group structure modeled on the inverse limit of Hilbert spaces (ILH). We prove a slice theorem for the action of this Lie group on the space of generalized metrics. As an application, we show that the moduli space of generalized metrics is stratified by ILH submanifolds and relate it to the moduli space of usual metrics. Finally, we extend these results to odd exact Courant algebroids.

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1. Introduction

This paper lays the foundations for a programme concerning the action of symmetries, automorphisms of a Courant algebroid, on geometric structures in generalized geometry. In the study of moduli spaces of differential geometric structures, the knowledge of infinite-dimensional Lie groups plays a fundamental role. In this work, we focus on the Lie group structure of the group of automorphisms of an

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exact or odd exact Courant algebroid over a compact manifold, and study the action of this group on the space of generalized metrics.

The bundle $TM+T^*M$, the Whitney sum of the tangent and cotangent bundles of a manifold, is canonically endowed with a pairing and a projection to the tangent bundle. Further equipped with the Dorfman bracket on its sections, it provides not only the frame for the theory of Dirac structures [3] but also the motivating example for the definition of a Courant algebroid [18].

Generalized geometry, as initiated by Hitchin [14], and more specifically generalized complex [11] and Kähler [12] geometry, were firstly developed on $TM+T^*M$ and its twisted versions, known as exact Courant algebroids (see Section 3.1 for definitions). Generalized geometry is concerned with the definition and study of geometric structures on $TM+T^*M$ or a more general Courant algebroid. The automorphisms of this Courant algebroid become the symmetries of the theory: the generalized diffeomorphisms GDiff . A remarkable feature of this group is that it includes new symmetries: the B -fields.

One of the first examples of a generalized geometric structure is that of a generalized metric, a maximally positive-definite subbundle. Generalized metrics were introduced by Gualtieri [10] in the context of generalized Kähler geometry, and have been recently used to restate some systems of coupled equations in a simpler fashion. For instance, field equations originating in supergravity are interpreted as the vanishing of a suitable generalized Ricci tensor [7]. On the other hand, the Strominger system, whose solutions provide, as proposed by Yau [30], a generalization of Calabi–Yau structures, is reformulated as generalized Killing spinor equations [9]. Very recently, special holonomy metrics with torsion have also been described by Killing spinors in generalized geometry [8]. All these applications arise on Courant algebroids of the form $TM + \text{ad } P + T^*M$, where P is a principal G -bundle over the manifold and G is a compact Lie group. Odd exact Courant algebroids correspond to $G = \text{S}^1$ and their study in this work provides a toy model for future applications.

Motivated by the applications of the generalized Ricci tensor and generalized Killing spinors, and the fact that generalized Riemannian quantities are natural under the GDiff -action, one is lead to consider the moduli of generalized metrics. In this context, the present paper presents two main results. The first one concerns the symmetries of the theory.

Theorem 1.1. *The group GDiff of automorphisms of an exact or odd exact Courant algebroid over a compact manifold carries a strong ILH Lie group structure, i.e., it is modelled on an inverse limit of Hilbert manifolds.*

There are many notions of infinite-dimensional Lie group depending on the model space for the underlying manifold. Choosing a particular Banach space as a local model for the charts leads to the notion of a Banach Lie group. Despite enjoying very nice analytical properties, Banach Lie groups turn out to be too restrictive: a Banach Lie group acting effectively and transitively on a finite-dimensional compact smooth manifold must be finite-dimensional [23]. On the other hand, Fréchet Lie groups give a too large category, a reason being the ab-

sense of fundamental results such as the local inverse theorem, Frobenius' theorem or the existence of local solutions to ODEs. These considerations motivated the introduction of *strong ILH Lie groups* by Omori [21] in his study of the group of diffeomorphisms of a compact manifold. The existence of a Frobenius' theorem in this category makes strong ILH Lie groups suitable for the study of automorphisms of Courant algebroids.

In the classical setting, the existence of a strong ILH Lie group structure on the group of diffeomorphisms of a compact manifold was proved by Omori [21], following the work by Ebin [4]. As a corollary, this result lead to interesting applications in the context of hydrodynamics [5]. Indeed, the geodesics in the group of diffeomorphisms fixing a prescribed volume form give solutions to the Euler equation of motion of a perfect fluid. Theorem 1.1 could thus be of interest in a different context, as supergravity theories.

As an application of Theorem 1.1, we initiate the study of the moduli space of generalized metrics on a fixed exact or odd exact Courant algebroid E under the action of GDiff . Recall that a motivation to study the moduli space of Riemannian metrics came from general relativity [6]. Indeed, an interesting class of solutions to the Einstein equations on a space-time $M^3 \times \mathbb{R}$ are interpreted as a particular class of geodesics in the space of Riemannian structures on M^3 . A good account of these facts is found in [19].

In this work, we extend a result by Bourguignon [2] showing that the moduli space of Riemannian structures is stratified. Namely, there exists a partition into topological subspaces, or strata, in such a way that if one stratum intersects the closure of another stratum, the first is contained in the closure of the second. We prove the following:

Theorem 1.2. *The moduli space of generalized metrics on E is stratified by ILH manifolds, where strata are labelled by the set of conjugacy classes of generalized isometry groups of generalized metrics. Moreover, the moduli space of generalized metrics projects onto the moduli space of metrics, the preimage of a stratum being a union of strata.*

The precise relation between the two set of strata is described in Section 5.4.

The proof of Theorem 1.2 relies on a description of generalized isometry groups and a slice result for the GDiff action on the space of generalized metrics \mathcal{GM} . To state this construction, denote by $\rho_{\mathcal{GM}}: \text{GDiff} \times \mathcal{GM} \rightarrow \mathcal{GM}$ the action of GDiff on \mathcal{GM} , and, for each generalized metric $V_+ \in \mathcal{GM}$, denote by $\text{Isom}(V_+)$ its isotropy group for $\rho_{\mathcal{GM}}$.

Theorem 1.3. *Let V_+ be a generalized metric on E . There exists an ILH submanifold \mathcal{S} of \mathcal{GM} such that:*

- a) *For all $F \in \text{Isom}(V_+)$, $\rho_{\mathcal{GM}}(F, \mathcal{S}) = \mathcal{S}$.*
- b) *For all $F \in \text{GDiff}$, if $\rho_{\mathcal{GM}}(F, \mathcal{S}) \cap \mathcal{S} \neq \emptyset$, then $F \in \text{Isom}(V_+)$.*
- c) *There is a local cross-section χ of the map $F \mapsto \rho_{\mathcal{GM}}(F, V_+)$ on a neighbourhood \mathcal{U} of V_+ in $\rho_{\mathcal{GM}}(\text{GDiff}, V_+)$ such that the map from $\mathcal{U} \times \mathcal{S}$ to \mathcal{GM} given by $(V_1, V_2) \mapsto \rho_{\mathcal{GM}}(\chi(V_1), V_2)$ is a homeomorphism onto its image.*

The ideas for the proof of Theorem 1.3 go back to Ebin [4], who first obtained a similar result for the action of diffeomorphisms on Riemannian metrics on a given compact manifold. However, we should emphasize that, unlike the Lie algebra of vector fields, the Lie algebra of GDiff is not described as the space of smooth sections for a vector bundle on M , but rather by a subspace of sections satisfying a differential relation. For this reason, standard elliptic operator theory, as used in [4], cannot be applied directly. Instead, we need to establish comparison results on operators in the ILH category, which allow us to use an ILH Frobenius' theorem [24]. It is, to our knowledge, the first example of a slice construction under these constraints.

A natural sequel to our work is the study of subspaces of the moduli of generalized metrics given by the vanishing of natural curvature quantities, such as the generalized Ricci tensor or the generalized scalar curvature (see e.g. [8]). On the other hand, note that we have focused on the exact and odd exact cases for the sake of simplicity. We believe the same results hold for Courant algebroids of the form $TM + \text{ad } P + T^*M$. The proof of this fact and its applications are work in progress.

The plan of the paper is as follows. In Section 2 we introduce the basics of ILH geometry and gather the results that we will need throughout the paper. In Section 3 we review the definitions of Courant algebroid and generalized diffeomorphisms, and prove that the group of generalized diffeomorphisms GDiff is a strong ILH Lie group (Theorem 3.8). In Section 4 we introduce generalized metrics and prove a slice theorem for the action of GDiff (Theorem 4.7). In Section 5 we use the slice theorem to describe the ILH stratification of the orbit space \mathcal{GR} of generalized metrics under the GDiff -action (Theorem 5.2). In Section 6, we state the main results for odd exact Courant algebroids, focusing on the main differences with the exact case. Finally, the appendix contains technical results on elliptic operator theory and ILH chains that are used in the proofs of Section 4.

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2. Review on ILH geometry

In this section, we review the basics of ILH geometry and state some results that will be used throughout the paper. We refer to [24] for an extensive introduction to the subject and the proofs of the results.

2.1. ILH calculus

We introduce inverse limits of Hilbert spaces as our model space for infinite dimensional manifolds. Denote by $\mathbb{N}(d)$ the set of integers $k \geq d$ for an integer d .

A set of complete locally convex topological vector spaces

$$\{\mathbf{E}, E^k \mid k \in \mathbb{N}(d)\}$$

is called an *ILH chain* if, for each $k \in \mathbb{N}(d)$,

- E^k is a Hilbert space with norm $\|\cdot\|_k$,
- E^{k+1} embeds continuously in E^k with dense image,
- and $\mathbf{E} = \bigcap_{k \in \mathbb{N}(d)} E^k$, endowed with the inverse limit topology.

Let $\{\mathbf{F}, F^k \mid k \in \mathbb{N}(d)\}$ be another ILH chain. A linear map $f: \mathbf{E} \rightarrow \mathbf{F}$ is called an *ILH map* if, for each $k \in \mathbb{N}(d)$, the map f extends to a continuous linear map $E^k \rightarrow F^k$.

To talk about differentiability, consider an open subset U of E^d . A map

$$f: U \cap \mathbf{E} \rightarrow \mathbf{F}$$

is called a C^r *ILH map*, for $r \in \mathbb{N}$, if for each $k \in \mathbb{N}(d)$, the map f extends to a C^r map from $U \cap E^k \rightarrow F^k$, also denoted by f . In order to have an implicit function theorem, we will need some uniform control on the norms of the derivatives

$$df: (U \cap E^k) \times E^k \rightarrow F^k,$$

as we now explain. We say that a C^∞ ILH map f is a $C^{\infty,r}$ *ILH normal map* if, for every $x \in U \cap \mathbf{E}$, there is a convex neighbourhood $W \subset U$ of x , a constant C and polynomials P_k , for each $k \in \mathbb{N}(d)$, with the property that if $y \in \mathbf{E}$ is such that $x + y \in W$, then, for every $k \in \mathbb{N}(d)$ and every integer $1 \leq s \leq r$, we have

$$\begin{aligned} \|(d^s f)_{x+y}(v_1, \dots, v_s)\|_k &\leq C \|y\|_k \|v_1\|_d \cdots \|v_s\|_d \\ &\quad + C \sum_{i=1}^s \|v_1\|_d \cdots \|v_{i-1}\|_d \|v_i\|_k \|v_{i+1}\|_d \cdots \|v_s\|_d \\ (2.1) \quad &\quad + P_k(\|y\|_{k-1}) \|v_1\|_{k-1} \cdots \|v_s\|_{k-1}. \end{aligned}$$

Note that if $f: U \cap \mathbf{E} \rightarrow \mathbf{F}$ is a $C^{\infty,r}$ ILH normal map, then, for every $x \in U \cap \mathbf{E}$, there are constants C and D_k , for each $k \in \mathbb{N}(d)$, such that the map $\mathbf{A} := df_x$ satisfies

$$(2.2) \quad \|\mathbf{A}(x)\|_k \leq C \|x\|_k + D_k \|x\|_{k-1}.$$

A map \mathbf{A} satisfying (2.2) is said to be a *linear ILH normal map*, and estimates like (2.1) and (2.2) are called *linear estimates*.

The implicit function theorem for ILH calculus reads as follows.

Theorem 2.1 ([24], I, Theorem 6.9). *Let U and V be open neighbourhoods of zero in E^d and F^d , respectively, and let*

$$\Phi: U \cap \mathbf{E} \rightarrow V \cap \mathbf{F}$$

be a $C^{\infty,r}$ ILH normal map for $r \geq 2$. Assume that $\mathbf{A} = d\Phi_0$ has a right inverse \mathbf{B} that is a linear ILH normal map.

Set

$$(2.3) \quad E_1 = \ker(\mathbf{A} : E \rightarrow F), \quad E_1^k = \ker(\mathbf{A} : E^k \rightarrow F^k).$$

Then, $\{E_1, E_1^k, k \in \mathbb{N}(d)\}$ is an ILH chain with

$$(2.4) \quad E = E_1 \oplus \mathbf{B}F, \quad E^k = E_1^k \oplus \mathbf{B}F^k,$$

and there are neighbourhoods W_1 and V' of zero in E_1^d and F^d , respectively, as well as a $\mathcal{C}^{\infty, r}$ ILH normal map

$$\Psi : (W_1 \cap E_1) \times (V' \cap F) \rightarrow \mathbf{B}F,$$

such that

$$\Phi(u, \Psi(u, v)) = v.$$

2.2. ILH manifolds, groups and actions

Let M be a manifold modelled on a locally convex topological space E . We call M an *ILH manifold* modelled on the ILH chain $\{E, E^k \mid k \in \mathbb{N}(d)\}$ when the following are satisfied:

- a) The manifold M is the inverse limit of smooth Hilbert manifolds M^k modelled on E^k such that $M^l \subset M^k$ for all $l \geq k$.
- b) For all $x \in M$, there exist open charts (U_k, ϕ_k) of M^k containing x such that if $l \geq k$, $U_l \subset U_k$ and $(\phi_k)|_{U_l} = \phi_l$.

Moreover, if the inverse limit U_∞ of $(U_k)_{k \in \mathbb{N}(d)}$ in b) is an open neighbourhood of x in M , then M is called a *strong ILH manifold*.

A sufficient condition for two different sets of open charts to define the same ILH manifold structure is the following: for each $x \in M$ with local open charts $\{(U_k, \phi_k)\}, \{(V_k, \psi_k)\}$, the inverse limit of the maps $\psi_k^{-1} \circ \phi_k, \phi_k^{-1} \circ \psi_k$, which are, by strongness, maps on the neighbourhood $U_\infty \cap V_\infty$,

$$(2.5) \quad \psi_\infty^{-1} \circ \phi_\infty, \phi_\infty^{-1} \circ \psi_\infty : U_\infty \cap V_\infty \rightarrow V_\infty \cap U_\infty,$$

are $\mathcal{C}^{\infty, \infty}$ ILH normal.

A map $\Phi : M \rightarrow N$ between ILH manifolds is a \mathcal{C}^r *ILH map* (of order j) if there is $j \in \mathbb{N}$ such that for all suitable k , the map Φ extends to a \mathcal{C}^r map $\Phi_k : M^{j+k} \rightarrow N^k$ satisfying $(\Phi_k)|_{M^{j+k+1}} = \Phi_{k+1}$. If Φ is a \mathcal{C}^r ILH map for all r , then Φ is a *smooth ILH map*.

Remark 2.2. Note that in order to recover the notion of a \mathcal{C}^r ILH map for ILH chains from a \mathcal{C}^r ILH map between ILH manifolds, one needs to shift (by j) the indices of the ILH chains modelling the ILH manifolds.

Example 2.3. Let (E, h) be a smooth Riemannian vector bundle over a compact Riemannian manifold (M, g) . Assume that (E, h) admits a metric-compatible connection ∇ . Combining g and (h, ∇) , we build metrics $(\cdot, \cdot)_h$ on the bundles $(T^*M)^{\otimes p} \otimes E$, as well as compatible connections.

We define the norms, for $u \in \Gamma(E)$,

$$(2.6) \quad \|u\|_k := \left(\sum_{i=0}^k \int_M (\nabla^i u, \nabla^i u)_h \, \text{dvol}_g \right)^{1/2},$$

where dvol_g is the volume element of g and ∇^i the i th covariant derivative associated to the extended connection on E . As an example, given (M, g) , we can endow the spaces of forms Ω^i or of symmetric tensors $\Gamma(S^2 T^* M)$ with Hilbert norms. The space of Riemannian metrics on a compact manifold is then an example of ILH manifold, modelled on the space of symmetric 2-tensors endowed with these norms.

A strong ILH Lie group G is a topological group with the structure of a strong ILH manifold such that the group operations are smooth ILH maps. For later purposes, we recall the formal definition from [24], formulated in terms of what happens around the identity at the infinitesimal level.

Definition 2.4. A *strong ILH Lie group* modelled on an ILH chain $\{E, E^k \mid k \in \mathbb{N}(d)\}$ is a topological group G such that there exist open neighbourhoods $0 \in U \subset E^d$, $e \in \tilde{U} \subset G$, an homeomorphism called *chart at the origin* $\xi: U \cap E \rightarrow \tilde{U}$ with $\xi(0) = e$, and an open neighbourhood V of 0 in E^d such that

$$\xi(V \cap E)^2 \subset \xi(U \cap E) \quad \text{and} \quad \xi(V \cap E)^{-1} = \xi(V \cap E),$$

satisfying that, for any $u, v \in V \cap E$, any $k \in \mathbb{N}(d)$ and any $l \geq 0$,

- a) The local product $\eta(u, v) = \xi^{-1}(\xi(u) \cdot \xi(v))$ extends to a C^l map $V \cap E^{k+l} \times V \cap E^k \rightarrow U \cap E^k$.
- b) The local right translation $\eta_v(u) = \eta(u, v)$ is a C^∞ map $V \cap E^k \rightarrow U \cap E^k$.
- c) Its differential $\theta(w, u, v) = (d\eta_v)_u w$ extends to a C^l map $E^{k+l} \times V \cap E^{k+l} \times V \cap E^k \rightarrow E^k$.
- d) The local inverse $\iota(u) = \xi^{-1}(\xi(u)^{-1})$ extends to a C^l map $V \cap E^{k+l} \rightarrow V \cap E^{k+l}$.
- e) For each $g \in G$, the local conjugation $A_g(u) = \xi^{-1}(g^{-1}\xi(u)g)$, defined on a neighbourhood W of $0 \in E^d$ such that $g^{-1}\xi(W \cap E)g \subset \xi(U \cap E)$, extends to a C^∞ map $W \cap E^k \rightarrow U \cap E^k$.

Given a strong ILH Lie group G and $g \in G$, consider the map $\xi_g(u) = \xi(u)g$ for $u \in V$, then

$$\{(W \cap E, \xi_g), g \in G, W \text{ open in } V\}$$

is an atlas for G . One can define Hilbert manifolds G^k , with the structure of a topological group, covered by the atlas

$$\{(W \cap E^k, \xi_g), g \in G, W \text{ open in } V\}$$

such that G is the inverse limit of the topological groups G^k and the axioms of the definition can be stated in terms of the G^k s (see [24], III, Theorem 3.7). We will refer to G as $\{G, G^k \mid k \in \mathbb{N}(d)\}$ when necessary.

Hilbert Lie groups, or finite-dimensional Lie groups, are strong ILH Lie groups. As in these cases, the spaces E and E^k , $k \in \mathbb{N}(d)$, are endowed with Lie algebra structures. We will denote them by \mathfrak{g} and \mathfrak{g}^k , which are, respectively the Lie algebras of G and G^k for all $k \in \mathbb{N}(d)$.

Example 2.5. A typical example of a strong ILH Lie group is Diff , the space of diffeomorphisms of an n -dimensional compact manifold M (see, e.g., [24]). This Lie group is modelled on $\{\Gamma(TM), \Gamma(TM)^k, k \geq n+5\}$, where $\Gamma(TM)$ is the space of smooth vector fields on M , and $\Gamma(TM)^k$ is its completion with respect to the norm defined by (2.6). The choice of $k \geq n+5$ is made to make use of Sobolev embedding theorems and deal with regularity issues [24], Ch. V. Choose a Riemannian metric g_0 on M . The chart at the origin (ξ, U) is built using the exponential map associated to the geodesics provided by g_0 :

$$(2.7) \quad \begin{aligned} \xi : U \cap \Gamma(TM) &\rightarrow \text{Diff} \\ u &\mapsto (x \mapsto \exp_x(u(x))), \end{aligned}$$

where $t \mapsto \exp_x(tu(x))$ is the geodesic starting from $(x, u(x))$, and U is a small neighbourhood of 0 in $\Gamma(TM)^{n+5}$. Note that by the Sobolev embedding theorem, ξ is well defined. Moreover, the differentiable structure just defined does not depend on the choice of Riemannian metric ([17], Rem. 4).

We will also need the notion of action in this category.

Definition 2.6. An *ILH (right) action* of a strong ILH Lie group $\{G, G^k \mid k \in \mathbb{N}(d)\}$ on an ILH manifold M is a map

$$\rho : M \times G \rightarrow M$$

such that:

- a) For any $v \in M$, $g, h \in G$, $\rho(\rho(v, g), h) = \rho(v, gh)$ and $\rho(v, e) = v$.
- b) For every $k \in \mathbb{N}(d)$ and $l \geq 0$, ρ extends to a C^l map from $M^{k+l} \times G^k$ to M^k .

If $\{M, M^k \mid k \in \mathbb{N}(d)\}$ is simply an ILH chain, and if ρ is linear with respect to v , ρ is called an ILH representation.

A fundamental example of an ILH representation of Diff is given by its action on differential forms. Consider the ILH chain $\{\Omega^j, \Omega^{j,k}, k \geq n+5\}$, where $\Omega^{j,k}$ is the completion of the space of smooth j -forms Ω^j on a compact manifold M with respect to the norm $\|\cdot\|_k$ defined as in equation (2.6). Consider the pullback map

$$(2.8) \quad \begin{aligned} \rho : \Omega^* \times \text{Diff} &\rightarrow \Omega^* \\ (\omega, g) &\mapsto g^*\omega. \end{aligned}$$

Notice the shift in indices in the ILH chain for Diff in the following proposition.

Proposition 2.7 ([24], VI, Theorem 6.1, Corollary 6.3). *The map ρ gives a representation of the strong ILH group $\{\text{Diff}, \text{Diff}^{k+1}, k \geq n+5\}$ on $\{\Omega^*, \Omega^{*,k}, k \geq n+5\}$. Moreover, for every fixed $\omega \in \Omega^j$, the map*

$$\Psi_\omega : U \cap \Gamma(TM) \rightarrow \Omega^j$$

defined by $\Psi(u) = \rho(\omega, \xi(u))$ is a $C^{\infty, \infty}$ ILH normal map.

We state another result related to the action of Diff on forms that will be used in Section 3. First, for any $u \in U \cap \Gamma(TM)$ denote by $D\xi(u)$ the differential of the diffeomorphism $\xi(u)$. For any $x \in M$, set $\tau(\exp_x u(x))$ to be the parallel transport on $\Lambda^j T^*M$ along the geodesic $t \mapsto \exp_x(tu(x))$, from $t = 0$ to $t = 1$, with respect to the metric induced by g . Then define the map

$$\Psi_{-1} : \Omega^j \times U \cap \Gamma(TM) \rightarrow \Omega^j$$

$$(2.9) \quad (\omega, u) \mapsto \omega(D(\xi(u))^{-1}\tau(\exp_x u(x))\cdot, \dots, D(\xi(u))^{-1}\tau(\exp_x u(x))\cdot)$$

where more precisely $\Psi_{-1}(\omega, u)$ is the form

$$(x, (V_1, \dots, V_j)) \mapsto \omega(x)(D(\xi(u))^{-1}\tau(\exp_x u(x))V_1, \dots, D(\xi(u))^{-1}\tau(\exp_x u(x))V_j).$$

Lemma 2.8. *The map Ψ_{-1} is a $C^{\infty, \infty}$ ILH normal map from $\{\Omega^j, \Omega^{j,k}, k \geq n + 5\} \times \{\Gamma(TM), \Gamma^{k+1}, k \geq n + 5\}$ to $\{\Omega^j, \Omega^{j,k}, k \geq n + 5\}$.*

Proof. The proof of Lemma 2.8 follows from the proof of [24], VI, Lemma 6.2 and is an application of [24], V, Thm. 3.1. The operator Ψ_{-1} here is slightly different from the one denoted Ψ in Omori's book, but the argument works equally as the inverse map $D\phi \mapsto (D\phi)^{-1}$ is defined by a smooth fibre preserving bundle map. \square

Lastly, a subgroup $\mathbf{H} \subset \mathbf{G}$ is called a *strong ILH Lie subgroup* of $\{\mathbf{G}, G^k \mid k \in \mathbb{N}(d)\}$ if, for a chart at the origin (ξ, U) ,

- a) there is a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ which extends to $\mathfrak{g}^k = \mathfrak{h}^k \oplus \mathfrak{m}^k$ for every $k \in \mathbb{N}(d)$, where $\mathfrak{h}, \mathfrak{h}^k$ and $\mathfrak{m}, \mathfrak{m}^k$ are closed subspaces of the spaces E, E^k .
- b) There is a neighbourhood V of 0 in \mathfrak{g}^d and a map $\xi' : V \cap \mathfrak{g} \rightarrow \mathbf{G}$ such that $\xi'\xi^{-1}$ is a C^∞ ILH diffeomorphism on a neighbourhood of zero and $\xi'(V \cap \mathfrak{h}) \subset \mathbf{H}$.
- c) The pair $(\xi'_{V \cap \mathfrak{h}}, V \cap \mathfrak{h})$ is a chart at the origin for \mathbf{H} .

As explained in [24], examples of strong ILH Lie subgroups of Diff are the group of symplectomorphisms of a given symplectic structure, the subgroup of hamiltonian symplectomorphisms, or the group of volume preserving diffeomorphisms for a given volume form.

2.3. ILH Frobenius' theorem

In order to build strong ILH Lie subgroups, we will need to integrate distributions on strong ILH Lie groups. This can be done by means of a Frobenius' theorem in this category. We first define vector bundles on strong ILH Lie groups. The definition is based on the properties of the tangent bundle TG of a strong ILH Lie group G . As it happens for Lie groups, TG is a trivial bundle, thanks to the map

$$dR : \mathfrak{g} \times G \rightarrow TG$$

defined by $dR(u, g) = d(R_g)_e u$, where $R_g(h) = hg$ is the right translation. Considering generalisations of the differential of the right translation map leads to a definition of vector bundles that will be trivial over the ILH Lie group G and carry a suitable ILH structure [24], Section IX.1.

Definition 2.9. An *ILH vector bundle* $B(\mathbb{F}, \mathbb{G}, \tilde{T})$ over a strong *ILH Lie group* $\{\mathbb{G}, G^k \mid k \in \mathbb{N}(d)\}$ consists of an *ILH chain* $\{\mathbb{F}, F^k \mid k \in \mathbb{N}(d)\}$ and a *defining map*

$$\begin{aligned} \tilde{T} : \mathbb{F} \times (\tilde{U} \cap \mathbb{G}) \times (\tilde{U} \cap \mathbb{G}) &\rightarrow \mathbb{F} \\ (u, g, h) &\mapsto \hat{T}(g, h)u \end{aligned}$$

for \tilde{U} a neighbourhood of e in \mathbb{G} , linear on F and satisfying, for any $k \in \mathbb{N}(d)$, for any $l \geq 0$ and $(g, h, h') \in G^3$:

- a) $\hat{T}(g, e) = \text{Id}$.
- b) $\hat{T}(gh, h')\hat{T}(g, h) = \hat{T}(g, hh')$.
- c) \tilde{T} extends to a \mathcal{C}^l map $F^{k+l} \times \tilde{U} \cap G^{k+l} \times \tilde{U} \cap G^k \rightarrow F^k$.
- d) $(u, g) \mapsto \tilde{T}(u, g, h)$ is a \mathcal{C}^∞ map $F^k \times \tilde{U} \cap G^k \rightarrow F^k$.

Remark 2.10. This definition deserves some explanation. Fix a triple $(\mathbb{F}, \mathbb{G}, \tilde{T})$ as in Definition 2.9, an open neighbourhood V of 0 in E^d , as in Definition 2.4, and set $\tilde{V} = \xi(V)$. One can define transition functions from the chart $(\tilde{V} \cap \mathbb{G})g$ to the chart $(\tilde{V} \cap \mathbb{G})h$, via the maps

$$(2.10) \quad \begin{aligned} t_{h,g} : (\tilde{V} \cap \mathbb{G})g \times \mathbb{F} &\rightarrow \mathbb{F} \\ (xg, u) &\mapsto \hat{T}(x, gh^{-1})u \end{aligned}$$

One can check that the family $\{t_{h,g}, (g, h) \in G^2\}$ defines a \mathcal{C}^∞ vector bundle $B(F^k, G^k, \tilde{T})$ on G^k for all $k \in \mathbb{N}(d)$, with fibre F^k (in the Hilbert category). Then $B(\mathbb{F}, \mathbb{G}, \tilde{T})$ is the projective limit of these bundles.

As an example, recall the maps θ and ξ from the definition of a strong ILH Lie group \mathbb{G} . Then

$$(2.11) \quad \tilde{T}_\theta(u, g, h) = \theta(u, \xi^{-1}(g), \xi^{-1}(h))$$

satisfies the above axioms, and $B(\mathbb{E}, \mathbb{G}, \tilde{T}_\theta)$ is the tangent bundle of \mathbb{G} .

For any $g \in \mathbb{G}$, the right translation map R_g on \mathbb{G} extends to a fibre preserving map \tilde{R}_g on $B(\mathbb{F}, \mathbb{G}, \tilde{T})$ (see [24], IX). Consider another ILH vector bundle $B(\mathbb{F}', \mathbb{G}, \tilde{T}')$ for an ILH chain $\{\mathbb{F}', F_2^k, k \in \mathbb{N}(d)\}$. Then any linear operator $\mathbf{A} : \mathbb{F} \rightarrow \mathbb{F}'$ can be extended to an ILH bundle homomorphism

$$\tilde{\mathbf{A}} : B(\mathbb{F}, \mathbb{G}, \tilde{T}) \rightarrow B(\mathbb{F}', \mathbb{G}, \tilde{T}')$$

by setting $\tilde{\mathbf{A}} = \tilde{R}_g \mathbf{A} \tilde{R}_g^{-1}$ on the fibre over $g \in \mathbb{G}$. However, even if \mathbf{A} is continuous, $\tilde{\mathbf{A}}$ may have little regularity. The regularity of $\tilde{\mathbf{A}}$ is checked by means of its *local expression*. By definition, if (U, ξ_G) is a chart at the origin for \mathbb{G} , the local expression of $\tilde{\mathbf{A}}$ is the map

$$(2.12) \quad \begin{aligned} \Phi_{\mathbf{A}} : U \cap \mathfrak{g} \times \mathbb{F} &\rightarrow \mathbb{F}' \\ (u, w) &\mapsto \hat{T}'(e, \xi_G(u)) \circ \tilde{\mathbf{A}} \circ \hat{T}(e, \xi_G(u))^{-1}w. \end{aligned}$$

If $\Phi_{\mathbf{A}}$ satisfies some linear estimates, when taking derivatives with respect to its first variable, we say that $\tilde{\mathbf{A}}$ is a $\mathcal{C}^{\infty, r}$ *ILH normal bundle homomorphism*. We refer to [24], IX, Def. 1.3, for the precise definition.

Frobenius' theorem in this context is given by the following result.

Theorem 2.11 (Frobenius' theorem, [24], IX, Theorem 3.4). *Let G be a strong ILH Lie group, with tangent bundle $B(\mathfrak{g}, G, \tilde{T}_\theta)$. Let $B(F, G, \tilde{T}')$ be another ILH-vector bundle over G and let*

$$\tilde{\mathbf{A}} : B(\mathfrak{g}, G, \tilde{T}_\theta) \rightarrow B(F, G, \tilde{T}')$$

be a right invariant $C^{\infty, r}$ ILH normal bundle homomorphism, with $r \geq 1$. Suppose that the restriction $\mathbf{A} : \mathfrak{g} \rightarrow F$ of $\tilde{\mathbf{A}}$ to the fibre at the identity satisfies, for all $k \in \mathbb{N}(d)$:

- a) The subspace $\mathfrak{h} = \ker \mathbf{A}$ is a Lie subalgebra of \mathfrak{g} , and $\mathbf{A}\mathfrak{g}$ is a closed subspace of F .
- b) There are closed subspace E_2 and F_2 such that $\mathfrak{g} = \mathfrak{h} \oplus E_2$ and $F = \mathbf{A}\mathfrak{g} \oplus F_2$.
- c) There exist constants C and D_k , for $k \in \mathbb{N}(d)$ with $D_d = 0$, such that, for all $u \in E_2$, $\|\mathbf{A}u\|_k \geq C\|u\|_k - D_k\|u\|_{k-1}$.
- d) There exist constants C' and D'_k , for $k \in \mathbb{N}(d)$ with $D'_d = 0$, such that, the projection $p : F \rightarrow \mathbf{A}\mathfrak{g}$ with respect to the decomposition in b) satisfies, for all $v \in F$, $\|pv\|_k \leq C'\|v\|_k + D'_k\|v\|_{k-1}$.

Then, there are neighbourhoods $0 \in V_1 \subset \mathfrak{h}^d$, $0 \in V_2 \subset E_2^d$, $e \in W' \subset G^d$ and a C^∞ diffeomorphism $\xi' : V_1 \cap \mathfrak{h} \times V_2 \cap E_2 \rightarrow W' \cap G$ satisfying, for all $k \in \mathbb{N}(d)$:

- a) The map ξ' extends to a C^∞ diffeomorphism $(V_1 \cap \mathfrak{h}^k) \times (V_2 \cap E_2^k) \rightarrow W' \cap G^k$.
- b) For every $w \in V_2 \cap E_2^k$, $\xi'((V_1 \cap \mathfrak{h}^k) \times \{w\})$ is an integral submanifold of the involutive subbundle $\mathfrak{h}^k = \ker(\tilde{\mathbf{A}} : B(\mathfrak{g}^k, G^k, \tilde{T}_\theta) \rightarrow B(F^k, G^k, \tilde{T}'))$.
- c) The kernel $\tilde{\mathfrak{h}} = \ker \tilde{\mathbf{A}}$ is an ILH subbundle of $B(\mathfrak{g}, G, \tilde{T}_\theta)$, and the maximal integral submanifold H of $\tilde{\mathfrak{h}}$ through e is a strong ILH Lie subgroup of G .

Corollary 2.12. *In the notation of Theorem 2.11, $\xi'(\{0\} \times (V_2 \cap E_2))$ provides a slice (at the origin) to the integral submanifolds of \mathfrak{h} . Thus the quotient G/H naturally inherits an ILH manifold structure.*

Remark 2.13. Note that in the case of the trivial bundle $\mathbb{R}^d \times G$, hypothesis d) of Theorem 2.11 is trivially satisfied, as well as the fact that $\mathbf{A}\mathfrak{g}$ is closed and admits a closed complement.

Remark 2.14. Let G be a Banach Lie group and K a subbundle of TG . To prove Frobenius' theorem in the Banach setting, one describes the subbundle K of TG in a local chart $x \in V \subset T_x$ by

$$K_y = \{w + J(y)w, w \in K_x\}, \quad y \in V,$$

with $y \mapsto J(y) \in \mathcal{L}(K_x, K'_x)$ a smooth map, where K'_x is a closed complement of K_x . In the setting of Theorem 2.11, the map J is given by $J(u)w = -\chi(u) \circ p \circ \Phi(u)w$, with Φ the local expression of \tilde{A} at e and $\chi(u)$ the inverse for $p \circ \Phi(u) : E_2 \rightarrow \text{Im}(\mathbf{A})$. The hypotheses of the theorem ensure that the map J is $C^{\infty, r}$ normal. One can apply Frobenius' theorem for the groups G^k for each k , and obtain a

local integration of $\ker A_k$. The normality condition ensures that this local result remains in the inverse limit procedure.

Next, we settle some examples of bundles and right-invariant $C^{\infty,r}$ ILH normal bundle homomorphisms that will be used in Section 3. We consider Diff endowed with its strong ILH Lie group structure built by means of the exponential map. Recall the open set $V \subset \Gamma(T)^d$, the maps ξ, η as in Definition 2.4, and set $\tilde{V} = \xi(V)$. Let F be a smooth Riemannian bundle over M , endowed with a metric compatible connection. With these data, we can consider the ILH chain $\{\Gamma(F), \Gamma(F)^k, k \geq n+5\}$ where $\Gamma(F)^k$ is the completion of $\Gamma(F)$ with respect to the norm defined as in equation (2.6).

Definition 2.15. The vector bundle $B(\Gamma(F), \text{Diff}, \tilde{T}_F)$ is defined to be the bundle over Diff , with fibre $\Gamma(F)$ and function \tilde{T}_F :

$$(2.13) \quad \tilde{T}_F : \Gamma(F) \times (\tilde{V} \cap \text{Diff}) \times (\tilde{V} \cap \text{Diff}) \rightarrow \Gamma(F)$$

with

$$\begin{aligned} \tilde{T}_F(w, \xi(u), \xi(v))(x) \\ = \tau(\exp_x \eta(u, v)(x))^{-1} \tau(\exp_{\exp_x(v(x))}(u(\exp_x v(x))))w(\exp_x(v(x))); \end{aligned}$$

here $\tau(\exp_y u(y))$ denotes the parallel transport along the geodesic $t \mapsto \exp_y(tu(y))$ from $t = 0$ to $t = 1$.

Let F' be another Riemannian bundle. Then from any continuous linear operator

$$\mathbf{A} : \Gamma(F) \rightarrow \Gamma(F'),$$

one can build a right-invariant bundle morphism

$$\tilde{\mathbf{A}} : B(\Gamma(F), \text{Diff}, \tilde{T}_F) \rightarrow B(\Gamma(F'), \text{Diff}, \tilde{T}_{F'})$$

by setting $\tilde{\mathbf{A}} = \tilde{R}_g \mathbf{A} \tilde{R}_g^{-1}$ on the fibre over $g \in \text{Diff}$. We gather the following results from [24], IX, Theorems 4.3 and 5.6, and Proposition 6.1.

Theorem 2.16. *Let \mathbf{A} , F and F' be as before. Then,*

- a) *If there is a bundle homomorphism γ such that $(\mathbf{A}u)(x) = \gamma(u(x))$ for any $u \in \Gamma(F)$, then $\tilde{\mathbf{A}}$ is a $C^{\infty,\infty}$ ILH normal bundle homomorphism.*
- b) *If \mathbf{A} is a differential operator of order r , then, up to a shift of indices, $\tilde{\mathbf{A}}$ is a $C^{\infty,\infty}$ ILH normal bundle homomorphism. More precisely, $\tilde{\mathbf{A}}$ extends as a smooth bundle homomorphism from $B(\Gamma^{k+r}(F), \text{Diff}^{k+r}, \tilde{T}_F)$ to $B(\Gamma^k(F'), \text{Diff}^{k+r}, \tilde{T}_{F'})$.*
- c) *Let $h \in \Gamma(F)$ and assume that*

$$\mathbf{A}(u) = \langle u, h \rangle_{L_g^2}.$$

Then $\tilde{\mathbf{A}}$ is a $C^{\infty,\infty}$ ILH normal bundle homomorphism from $B(\Gamma(F), \text{Diff}, \tilde{T}_F)$ to the trivial bundle $\text{Diff} \times \mathbb{R}$.

3. The automorphism group of an exact Courant algebroid

In this section, we show that the group of automorphisms $\text{Aut}(E)$ of an exact Courant algebroid E over a compact n -dimensional manifold M carries the structure of a strong ILH Lie group. We will also consider the subgroup of exact automorphisms.

3.1. Courant algebroids, automorphisms and splittings

A *Courant algebroid* over a manifold M is a tuple $(E, \langle \cdot, \cdot \rangle, [\cdot, \cdot], \pi)$ consisting of a vector bundle $E \rightarrow M$ together with a non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ on E , a Dorfman bracket $[\cdot, \cdot]$ on the sections $\Gamma(E)$ and a bundle map $\pi: E \rightarrow TM$ such that the following properties are satisfied for any $e, e', e'' \in \Gamma(E)$:

$$(C1): [e, [e', e'']] = [[e, e'], e''] + [e', [e, e'']],$$

$$(C2): \pi(e)\langle e', e'' \rangle = \langle [e, e'], e'' \rangle + \langle e', [e, e''] \rangle,$$

$$(C3): [e, e] = D\langle e, e \rangle, \text{ or, equivalently, } [e, e'] + [e', e] = 2D\langle e, e' \rangle,$$

with $D: C^\infty(M) \rightarrow \Gamma(E)$ defined, for $\phi \in C^\infty(M)$, by $D\phi = \pi^*d\phi$, where we use $2\langle \cdot, \cdot \rangle$ to identify E^* and E . Note that, as a consequence of (C2), we also have the properties ([29])

$$(C4): [e, \phi e'] = \phi[e, e'] + (\pi(e)\phi)e',$$

$$(C5): \pi([e, e']) = [\pi(e), \pi(e')].$$

Example 3.1. The best-known example of a Courant algebroid is, for a closed 3-form $H \in \Omega_{\text{cl}}^3$, the tuple

$$(3.1) \quad (TM + T^*M)_H := (TM + T^*M, \langle \cdot, \cdot \rangle, [\cdot, \cdot]_H, \pi),$$

where the pairing is given by $\langle u + \alpha, v + \beta \rangle = i_u\beta - i_v\alpha$, the anchor map π is the projection to TM , and the Dorfman bracket is given by

$$[u + \alpha, v + \beta]_H = [u, v] + \mathcal{L}_u\beta - i_v d\alpha + i_u i_v H.$$

These Courant algebroids are the framework for Dirac structures, which encompass presymplectic and (possibly twisted) Poisson geometry.

Remark 3.2. Note that, for the sake of simplicity, we use the notation $TM + T^*M$ for the Whitney sum $TM \oplus T^*M$.

Denote by $O(E)$ the group consisting of smooth bundle maps $F: E \rightarrow E$, covering a diffeomorphism $\phi: M \rightarrow M$, such that, for $u, v \in \Gamma(E)$,

$$\langle F(u), F(v) \rangle = \phi_*\langle u, v \rangle.$$

The subgroup of anchor-preserving orthogonal maps is given by

$$O_\pi(E) = \{F \in O(E) \mid \pi(F(u)) = \phi_*\pi(u) \text{ for } u \in \Gamma(E)\}.$$

The *automorphism group* $\text{Aut}(E)$ of the Courant algebroid E consists of those maps that are moreover bracket-preserving:

$$\text{Aut}(E) = \{F \in \text{O}(E) \mid \pi(F(u)) = f_*\pi(u), [F(u), F(v)] = F([u, v]), \text{ for } u, v \in \Gamma(E)\}.$$

Finally, we denote by $\mathcal{G}(E)$ the subgroup of $\text{Aut}(E)$ covering the identity diffeomorphism.

Example 3.3. In Example 3.1, we have $\text{O}_\pi((TM + T^*M)_H) = \text{Diff} \times \Omega^2$ for any H . For $H = 0$, we have $\text{Aut}(TM + T^*M) = \text{Diff} \times \Omega_{\text{cl}}^2$, and $\mathcal{G}(TM + T^*M) = \Omega_{\text{cl}}^2$, where Ω_{cl}^2 denotes the closed 2-forms. A diffeomorphism $\phi \in \text{Diff}$ acts by pushforward ϕ_* both in vector fields and forms, and $B \in \Omega_{\text{cl}}^2$, known as a B -field, acts by

$$u + \alpha \rightarrow u + \alpha + i_u B.$$

The group product in $\text{O}_\pi(TM + T^*M)$ and $\text{Aut}(TM + T^*M)$ is given by

$$(3.2) \quad (\phi, B)(\psi, B') = (\phi \circ \psi, \psi^* B + B'),$$

whereas $\mathcal{G}(TM + T^*M)$ is the usual abelian additive group of 2-forms.

We say that two Courant algebroids $(E, \langle \cdot, \cdot \rangle, [\cdot, \cdot], \pi)$ and $(E', \langle \cdot, \cdot \rangle', [\cdot, \cdot]', \pi')$ are isomorphic when there exists an isomorphism of vector bundles $F: E \rightarrow E'$ satisfying, for $e_1, e_2 \in \Gamma(E)$,

$$\pi' \circ F = \pi, \quad \langle Fe_1, Fe_2 \rangle' = \langle e_1, e_2 \rangle, \quad [Fe_1, Fe_2]' = F[e_1, e_2].$$

By considering $\pi^*: T^*M \rightarrow E^*$ and identifying $E^* \cong E$ using the non-degenerate pairing, we always have a sequence

$$T^*M \rightarrow E \rightarrow TM$$

associated to any Courant algebroid. We say that E is an *exact Courant algebroid* when this sequence is exact, i.e.,

$$(3.3) \quad 0 \rightarrow T^*M \rightarrow E \rightarrow TM \rightarrow 0.$$

It is always possible to split the sequence (3.3) as a sequence of vector bundles. The splitting $\lambda: TM \rightarrow E$ can be chosen to be isotropic, meaning that $\lambda(TM) \subset E$ is an isotropic subbundle. Indeed, for any splitting λ' , the splitting $\lambda: u \mapsto \lambda'(u) - \frac{1}{2}\pi^*\langle \lambda'(u), \cdot \rangle$ is isotropic. The *space of isotropic splittings* of a Courant algebroid

$$(3.4) \quad \Lambda := \{\lambda: TM \rightarrow E \mid \lambda \text{ is injective and } \lambda(TM) \subset E \text{ is isotropic}\}$$

is an affine space modelled on Ω^2 , i.e., an Ω^2 -torsor, where the action, for $B \in \Omega^2$, which we simply denote by $\lambda + B$, is given by

$$(\lambda + B): u \mapsto \lambda(u) + \pi^*(i_u B).$$

Choosing an isotropic splitting $\lambda: TM \rightarrow E$ helps us know E better, as it provides a Courant algebroid isomorphism

$$(3.5) \quad \begin{aligned} E &\simeq_\lambda (TM + T^*M)_H \\ \lambda(u) + \alpha &\leftrightarrow u + \alpha, \end{aligned}$$

where the 3-form H of Example 3.1 is given, for $u, v, w \in \Gamma(TM)$, by

$$H(u, v, w) = \langle [\lambda(u), \lambda(v)], \lambda(w) \rangle.$$

By the isomorphism (3.5) and Example 3.3, the group $O_\pi(E)$ is always isomorphic to $\text{Diff} \times \Omega^2$. The subgroup Ω^2 sits naturally inside $O_\pi(E)$, acting by $C(e) = \pi^*(i_{\pi(e)}C)$ for $C \in \Omega^2$, $e \in \Gamma(E)$, whereas we have a natural projection to the diffeomorphisms, yielding the sequence

$$(3.6) \quad 0 \rightarrow \Omega^2 \rightarrow O_\pi(E) \rightarrow \text{Diff} \rightarrow 0.$$

The way we regard the diffeomorphisms inside $O_\pi(E)$ depends on the splitting of E : if, under a splitting λ , the element $F \in O_\pi(E)$ corresponds to the diffeomorphism $(\phi, 0)$, under a different splitting $\lambda + C$, the element F corresponds, for the product rule given in (3.2), to

$$(3.7) \quad (\text{Id}, -C)(\phi, 0)(\text{Id}, C) = (\phi, C - \phi^*C).$$

To describe $\text{Aut}(E)$, we look first at $\text{Aut}((TM + T^*M)_H)$, which we denote by GDiff_H and whose elements will be referred to as generalized diffeomorphisms. We have

$$(3.8) \quad \text{GDiff}_H = \{(\phi, B) \in \text{Diff} \times \Omega^2 \mid \phi^*H - H = dB\},$$

whose elements act on the left on $\Gamma((TM + T^*M)_H)$ by

$$u + \alpha \mapsto \phi_*u + \phi_*(\alpha + i_u B).$$

By (3.8), the map ϕ belongs to the diffeomorphisms preserving the cohomology class $[H]$, which we denote by $\text{Diff}_{[H]}$, and the elements covering the identity are just closed forms. The automorphism group $\text{Aut}(E) \subset O_\pi(E)$ is thus an extension

$$0 \rightarrow \Omega_{\text{cl}}^2 \rightarrow \text{Aut}(E) \rightarrow \text{Diff}_{[H]} \rightarrow 0,$$

which, unlike (3.6), is not a semidirect product (unless $[H] = 0$).

Remark 3.4. Note that a different splitting $\lambda + B$ yields, via (3.5), the 3-form $H + dB$. Indeed, equivalence classes of exact Courant algebroids up to isomorphism are parametrized by $[H] \in H^3(M, \mathbb{R})$, known as the Ševera class of the Courant algebroid.

3.2. The ILH Lie group structure

We can endow the groups $O_\pi(E)$ and $\text{Aut}(E)$ with the C^∞ -Whitney topology, and we will show that they carry strong ILH Lie group structures. First, we show that $O_\pi(E)$ carries such a structure and then we will use the implicit function Theorem 2.1 for GDiff_H , in a similar fashion as one does for the group of symplectomorphisms [24].

Remark 3.5. We should mention other important categories of infinite dimensional Lie groups that could have been used in the context of generalized geometry. Tamed Fréchet Lie groups were introduced by Hamilton [13]. These groups enjoy the Nash–Moser inverse function theorem. However this theorem requires the existence of a local tamed family of inverses for the differential of a map to be invertible. In the ILH category, one only needs to invert the differential of a map at a single point to apply an inverse function theorem. Another category is the one of convenient Lie groups, as developed by Michor and Kriegl [16]. This category enables one to deal with non-compact situations. However, to our knowledge, there is no general Frobenius'-type theorem for these groups. We refer to [20] for an introduction to infinite-dimensional Lie groups.

Throughout this section, we fix a Riemannian metric g_0 on M , giving the chart at the origin (ξ, U) for the group Diff , as in Example 2.5.

Proposition 3.6. *The group $O_\pi(E) \simeq_\lambda \text{Diff} \times \Omega^2$ carries a strong ILH Lie group structure modelled on $\{\Gamma(TM) \times \Omega^2, \Gamma(TM)^{k+1} \times \Omega^{2,k}, k \geq n+5\}$. A chart at the origin (ξ', U') on $U' = U \times \Omega^{2,n+5}$ is given by*

$$(3.9) \quad \begin{aligned} \xi' : U' \cap (\Gamma(TM) \times \Omega^2) &\rightarrow \text{Diff} \times \Omega^2 \\ (u, b) &\mapsto (\xi(u), b). \end{aligned}$$

Moreover, the ILH structure is independent of the choice of splitting λ , and its Lie algebra is given by $\Gamma(TM) \times \Omega^2$ with the bracket, for $(X, b), (Y, c) \in \Gamma(TM) \times \Omega^2$,

$$[(X, b), (Y, c)] = ([X, Y], \mathcal{L}_X c - \mathcal{L}_Y b).$$

Proof. The proof of the first part of the proposition follows directly from Proposition 2.7 and the fact that Diff is a strong ILH Lie group. For the independence of the choice of splitting, consider, at a point x , charts $\{U_k, \phi_k\}, \{V_k, \psi_k\}$ for the splittings λ and $\lambda + C$. The map $\phi_k^{-1} \circ \psi_k$ and $\psi_k^{-1} \circ \phi_k$ in (2.5) are given by conjugation by C and $-C$,

$$(3.10) \quad (\text{Id}, -C)(\phi, B)(\text{Id}, C) = (\phi, B + C - \phi^* C),$$

which are $\mathcal{C}^{\infty, \infty}$ ILH normal by Proposition 2.7, and hence the ILH structure does not depend on the splitting. The expression for the bracket follows from the action

$$(X, b)(Y + \eta) = \mathcal{L}_X(Y + \eta) - i_Y b. \quad \square$$

As we have shown the independence from the choice of splitting, in what follows we will work with $(TM + T^*M)_H$ and GDiff_H . In the remaining of this section, the ILH chains associated to the spaces $\Omega^2, \Gamma(TM)$ and Ω^3 are $\{\Omega^2, \Omega^{2,k}, k \geq n+5\}, \{\Gamma(TM), \Gamma^{k+1}, k \geq n+5\}$ and $\{\Omega^3, \Omega^{3,k-1}, k \geq n+5\}$.

We will build the strong ILH Lie group GDiff_H as a strong ILH Lie subgroup of $\text{Diff} \times \Omega^2$. Introduce the map

$$(3.11) \quad \begin{aligned} \tilde{\rho} : \Omega^3 \times (\text{Diff} \times \Omega^2) &\rightarrow \Omega^3 \\ (H', (\phi, B)) &\mapsto \phi^* H' - dB. \end{aligned}$$

Then $\text{GDiff}_H = \tilde{\rho}(H, \cdot)^{-1}(H)$. Define, using (ξ', U') from Proposition 3.6,

$$\begin{aligned} \Phi : U' \cap (\text{Diff} \times \Omega^2) &\rightarrow \Omega^3 \\ (u, b) &\mapsto \tilde{\rho}(H, \xi'(u, b)). \end{aligned}$$

from the ILH chain $\{\Gamma(TM) \times \Omega^2, \Gamma^{k+1} \times \Omega^{2,k}, k \geq n+5\}$ to $\{\Omega^3, \Omega^{3,k-1}, k \geq n+5\}$.

From Proposition 2.7, the map Φ is a $C^{\infty, \infty}$ ILH normal map from $U \cap \text{Diff} \times \Omega^2$ to Ω^3 , with respect to these ILH structures. Using $(d\xi)_0 = \text{Id}$, the derivative of Φ at zero is

$$(3.12) \quad d\Phi_0(u, b) = d(\iota_u H - b).$$

In order to apply the implicit function theorem, we need a right inverse for $d\Phi_0$ in the category of $C^{\infty, 2}$ ILH normal maps. Consider the map

$$(3.13) \quad \begin{aligned} \mathbf{B} : \Omega^3 &\rightarrow \Gamma(TM) \times \Omega^2 \\ h &\mapsto (0, -d^* \mathbb{G}h), \end{aligned}$$

where d^* is the adjoint of d with respect to the L^2 inner pairing on forms provided by g_0 , and \mathbb{G} is the Green operator for the Hodge Laplacian. In the following lemma, we consider the space of exact 3-forms $d\Omega^2$ as a subspace of Ω^3 endowed with the ILH chain structure $\{\Omega^3, \Omega^{3,k-1}, k \geq n+5\}$.

Lemma 3.7. *The restriction of the operator \mathbf{B} to the space of exact 3-forms $d\Omega^2$ is a linear ILH normal map that is a right-inverse for the map $d\Phi_0$.*

For a proof of this lemma, in particular the linear estimates, see [24], VII, Lemma 5.10. We can then apply the implicit function theorem, Theorem 2.1, and we have thus proved the following.

Theorem 3.8. *The group $\text{Aut}(E)$ is a strong ILH Lie subgroup of $\text{O}_\pi(E)$, or, equivalently in terms of a splitting, the group GDiff_H is a strong ILH Lie subgroup of $\text{Diff} \times \Omega^2$.*

The Lie algebra of GDiff_H , or algebra of derivations of $(TM + T^*M)_H$, coincides with the kernel of $d\Phi_0$ in (3.12) and is explicitly given by

$$\mathfrak{gdiff}_H = \{(u, b) \in \Gamma(TM) \times \Omega^2 \mid d(\iota_u H - b) = 0\}.$$

The subalgebra $\mathfrak{gdiff}_H^e \subset \mathfrak{gdiff}_H$ of exact derivations is given by

$$(3.14) \quad \mathfrak{gdiff}_H^e = \{(u, b) \in \Gamma(TM) \times \Omega^2 \mid \iota_u H - b = da \text{ for some } a \in \Omega^1\}.$$

This subalgebra integrates to the subgroup of exact generalized diffeomorphisms GDiff_H^e .

If we start by an exact Courant algebroid E with two different splittings λ , $\lambda + C$, so that

$$\text{GDiff}_H \simeq_\lambda \text{Aut}(E) \simeq_{\lambda+C} \text{GDiff}_{H+dC},$$

the Lie algebras of the first and third groups are isomorphic by $(u, b) \mapsto (u, b + i_u dC)$, which interchanges exact derivations. Hence, the Lie algebra of exact derivations $\text{Der}^e(E)$ and the subgroup $\text{Aut}^e(E)$ of exact automorphisms are well defined.

We will show that $\text{Aut}^e(E)$ is a strong ILH Lie subgroup of $\text{Aut}(E)$ by fixing a splitting λ , so that we work with GDiff_H , using the simpler version of Frobenius' Theorem 2.11, stated in Remark 2.13, and following the ideas in [24] for the group of exact symplectomorphisms.

As a subalgebra of \mathfrak{gdiff}_H , the algebra \mathfrak{gdiff}_H^e can be described as the space of pairs (u, b) such that $\iota_u H - b$ is orthogonal to the harmonic 2-forms for the fixed metric g_0 on M . Let $s = \dim H^2(M)$, and set $\{e_i\}_{1 \leq i \leq s}$ a basis of the harmonic 2-forms. Define a map:

$$(3.15) \quad \begin{aligned} I : \Omega^2 &\rightarrow \mathbb{R}^s \\ \omega &\mapsto \langle \omega, e_i \rangle_{L^2_{g_0}} \end{aligned}$$

where $\langle \cdot, \cdot \rangle_{L^2_{g_0}}$ denotes the L^2 inner product on forms defined by g_0 . Set also

$$(3.16) \quad \begin{aligned} \kappa : \mathfrak{gdiff}_H &\rightarrow \Omega^2 \\ (u, b) &\mapsto \iota_u H - b. \end{aligned}$$

Note that $\mathfrak{gdiff}_H^e = (I \circ \kappa)^{-1}(0)$. We will consider an extension of $I \circ \kappa$ to $T\text{GDiff}_H$ and show that this extension is a $\mathcal{C}^{\infty, \infty}$ ILH normal homomorphism to the trivial bundle over GDiff_H . Let V be an open neighbourhood of zero in $\Gamma(TM)^{n+5}$ as in Definition 2.4 and set $\tilde{V} = \xi(V)$. In order to obtain a simple description of $T\text{GDiff}_H$, and in view of the composition law (3.2), we define the map

$$(3.17) \quad \begin{aligned} T_{\Omega^2} : \Omega^2 \times (\tilde{V} \cap \text{Diff}) \times (\tilde{V} \cap \text{Diff}) &\rightarrow \Omega^2 \\ (b, \xi(u), \xi(v)) &\mapsto \xi(v)^* b. \end{aligned}$$

It is straightforward to check that T_{Ω^2} satisfies the requirements of Definition 2.9, so we have introduced the bundle $B(\Omega^2, \text{Diff}, T_{\Omega^2})$ over Diff .

Lemma 3.9. *The tangent bundle $T\text{GDiff}_H$ is the pullback of the Whitney sum*

$$B(\Omega^2, \text{Diff}, T_{\Omega^2}) \oplus T\text{Diff}$$

by the projection map $\pi : \text{GDiff}_H \rightarrow \text{Diff}$.

Proof. The proof of this lemma follows directly from the expression of the defining maps for the bundles that are considered. Let T_θ be a defining map for $T\text{Diff}$ and T_{θ_H} be a defining map for $T\text{GDiff}_H$. Then for (u, b) in \mathfrak{gdiff}_H and $(v, B), (w, B')$ in $U' \cap \text{GDiff}_H$, a direct computation using (2.11) and (3.2) leads to

$$T_{\theta_H}((u, b), (\xi(v), B), (\xi(w), B')) = (T_\theta(u, \xi(v), \xi(w)), \xi(w)^* b). \quad \square$$

Next, consider the right-invariant extensions of I and κ :

$$(3.18) \quad B(\Omega^2, \text{Diff}, T_{\Omega^2}) \oplus T\text{Diff} \xrightarrow{\tilde{\kappa}} B(\Omega^2, \text{Diff}, \tilde{T}_{\Omega^2}) \xrightarrow{\tilde{I}} \text{Diff} \times \mathbb{R}^d$$

defined by setting $\tilde{\kappa} = R_\phi \circ \kappa \circ R_\phi^{-1}$ and $\tilde{I} = R_\phi \circ I \circ R_\phi^{-1}$ on the fibre over $\phi \in \text{Diff}$, and where \tilde{T}_{Ω^2} is defined as in Definition 2.15 (the Riemannian structure on $\Lambda^2 T^*M$ being induced by the fixed metric g_0).

Proposition 3.10. *The maps \tilde{I} and \tilde{k} are $\mathcal{C}^{\infty, \infty}$ ILH normal bundle homomorphisms.*

Proof. For \tilde{I} , the result follows from c) in Theorem 2.16. For \tilde{k} , we need to consider the local expression, say $\Phi_{\tilde{k}}$, of this bundle homomorphism. By definition, this is given, for $u \in U \cap \text{Diff}$ and $(v, b) \in \Gamma(TM) \times \Omega^2$, by

$$\Phi_{\tilde{k}}(u)(v, b)(x) = \tau(\exp_x(u(x)))^{-1}((\iota_{v'}H) - (\xi(u)^{-1})^*b)(\exp_x(u(x))),$$

where $v' = T_{\theta}(\xi(u), e)^{-1}(v)$. By definition,

$$(u, v) \mapsto \tau(\exp_x(u(x)))^{-1}(\iota_{v'}H)(\exp_x u(x))$$

is the local expression for the right-invariant extension from $T\text{Diff}$ to $B(\Omega^2, \text{Diff}, \tilde{T}_{\Omega^2})$ of the homomorphism $v \mapsto \iota_v H$. Thus, by a) in Theorem 2.16, this component of \tilde{k} is $\mathcal{C}^{\infty, \infty}$ ILH normal. The map

$$(u, b) \mapsto \tau(\exp_x(u(x)))^{-1}(\xi(u)^{-1})^*b$$

can be rewritten as

$$\tau(\exp_x(u(x)))^{-1}(\xi(u)^{-1})^*b(\exp_x u(x)) = \Psi_{-1}(u, b)(x)$$

with Ψ_{-1} defined in (2.9). By Lemma 2.8, this is a $\mathcal{C}^{\infty, \infty}$ ILH normal homomorphism. Thus, Φ satisfies the required estimates and the result follows. \square

Finally, we are ready to prove the last result of this section.

Proposition 3.11. *The group $\text{Aut}^e(E)$ is a strong ILH Lie subgroup of $\text{Aut}(E)$, or, equivalently in terms of a splitting, GDiff_H^e is a strong ILH Lie subgroup of GDiff_H .*

Proof. From Proposition 3.10, the operator $\tilde{I} \circ \tilde{k}$ is $\mathcal{C}^{\infty, \infty}$ ILH normal. We can extend $I \circ \kappa$ to a right-invariant $\mathcal{C}^{\infty, \infty}$ ILH normal bundle homomorphism

$$\pi^*(B(\Omega^2, \text{Diff}, T_{\Omega^2}) \oplus T\text{Diff}) \rightarrow \text{TGDiff}_H \times \mathbb{R}^d.$$

By Lemma 3.9, this gives an extension from TGDiff_H to a trivial bundle. As $\ker(I \circ \kappa) = \mathfrak{gdiff}_H^e$, the result follows from Remark 2.13. \square

4. A slice theorem for generalized metrics

4.1. Generalized metrics and statement of the slice theorem

Let E be an exact Courant algebroid. We define a generalized metric as follows.

Definition 4.1. A generalized metric on an exact Courant algebroid E over an n -dimensional manifold M is a rank n subbundle

$$V_+ \subset E$$

such that $\langle \cdot, \cdot \rangle|_{V_+}$ is positive definite.

Example 4.2. Consider $E = (TM + T^*M)_H$ for any $H \in \Omega_{\text{cl}}^3$. A usual metric g on M defines a generalized metric on E by its graph $V_+ = \{u + i_u g \mid u \in TM\}$.

For any generalized metric $V_+ \subset E$, we have that the projection $\pi_{V_+} : V_+ \rightarrow TM$ is an isomorphism and induces a metric g on TM by

$$g(u, v) = \langle \pi_{V_+}^{-1}(u), \pi_{V_+}^{-1}(v) \rangle.$$

Moreover, the map $\lambda : TM \rightarrow E$ given by

$$\lambda : u \mapsto \pi_{V_+}^{-1}(u) - \iota_u g$$

defines an isotropic splitting of E . Conversely, a pair (g, λ) consisting of a metric and an isotropic splitting defines a generalized metric by the subbundle

$$V_+ = \{\lambda(u) + \iota_u g \mid u \in TM\} \subset E.$$

Let \mathcal{GM} denote the set of generalized metrics on E . With the notation of (3.4) for the space of isotropic splittings Λ , and the notation

$$\mathcal{M} := \{g \in \Gamma(S^2 T^* M) \mid g \text{ is positive definite}\}$$

for the space of metrics, the argument above is summed up in the isomorphism

$$\mathcal{GM} \cong \mathcal{M} \times \Lambda.$$

By choosing any splitting $\lambda \in \Lambda$, we have an isomorphism $\Lambda \simeq_\lambda \Omega^2$, which shows that the space \mathcal{GM} is an ILH manifold modelled on the ILH chain

$$\{\Omega^2 \times \Gamma(S^2 T^* M), \Omega^{2,k} \times \Gamma(S^2 T^* M)^k, k \geq n + 5\},$$

as \mathcal{GM} is regarded, indeed, as an open subspace of $\Omega^2 \times \Gamma(S^2 T^* M)$.

Since E is fixed and no confusion is possible, we denote $O_\pi(E)$ by O_π . The strong ILH Lie group O_π preserves the pairing $\langle \cdot, \cdot \rangle$ and thus acts on the right on the ILH manifold \mathcal{GM} , with ILH action, by pull-back:

$$(4.1) \quad \begin{aligned} \rho_{\mathcal{GM}} : O_\pi(E) \times \mathcal{GM} &\rightarrow \mathcal{GM} \\ (F, V_+) &\mapsto F^{-1}(V_+). \end{aligned}$$

Likewise, we denote the groups of automorphisms $\text{Aut}^e(E)$ and $\text{Aut}(E)$ by GDiff^e and GDiff . The restriction of (4.1) defines ILH actions of GDiff and GDiff^e on \mathcal{GM} . For any $F \in \text{GDiff}$, if S is a subset of \mathcal{GM} , we will denote by $F \cdot S$ the image of S by the action of F , namely $\rho_{\mathcal{GM}}(F, S)$.

By taking a splitting $\lambda \in \Lambda$, we have isomorphisms

$$(4.2) \quad O_\pi \simeq_\lambda \text{Diff} \times \Omega^2, \quad \mathcal{GM} \simeq_\lambda \mathcal{M} \times \Omega^2, \quad V_+ \simeq_\lambda \{u + i_u g + i_u \omega \mid u \in TM\}.$$

The action then reads

$$\begin{aligned} \rho_{\mathcal{GM}} : (\text{Diff} \times \Omega^2) \times \mathcal{GM} &\rightarrow \mathcal{GM} \\ ((\phi, B), (g, \omega)) &\mapsto (\phi^* g, \phi^* \omega - B), \end{aligned}$$

which restricts to actions of GDiff_H and GDiff_H^e on \mathcal{GM} .

In this section we prove a slice theorem for these actions, inspired by the classical result of Ebin [4]. The main difference in the proof is that we need to use the abstract Frobenius' theorem (Theorem 2.11) in order to endow the orbits under the action of GDiff^e with an ILH manifold structure. We will only sketch a proof of the result, focusing on the differences with [4]. The strategy of the proof is as in the finite-dimensional case. We fix a generalized metric V_+ on E , and let $\text{Isom}^e(V_+)$ be its isotropy subgroup under the GDiff^e action. The ILH Frobenius' theorem enables us to endow the space $\text{GDiff}^e/\text{Isom}^e(V_+)$ with an ILH structure. The orbit $\text{GDiff}^e \cdot V_+$ is homeomorphic to this quotient, and carries an ILH structure. Then, by means of an invariant metric on \mathcal{GM} , we can consider the normal bundle to $\text{GDiff}^e \cdot V_+$, as well as an exponential map on \mathcal{GM} . The slice is then obtained by exponentiating small vectors on the normal bundle to the orbit at V_+ .

Remark 4.3. The space of generalized metrics on a given M is a fairly complicated object. The slice result enables us to have a better description of this space. It would be interesting to have a further decomposition of the space of generalized metrics as in [15]. This would rely on a solution to the Yamabe problem for the generalized scalar curvature in a generalized conformal class of metrics.

In all this section, we fix an auxiliary metric g_0 on M , and thus endow the spaces of sections $\Gamma(TM^{\otimes p} \otimes (T^*M)^{\otimes q})$ with $L^{2,k}$ inner products as defined in (2.6).

Definition 4.4. The *group of generalized isometries* (resp., exact generalized isometries) of $V_+ \in \mathcal{GM}$, denoted $\text{Isom}(V_+)$ (resp., $\text{Isom}^e(V_+)$), is the isotropy group of V_+ under the GDiff action (resp., under the GDiff^e action). When a splitting λ is chosen, so that

$$(4.3) \quad V_+ \simeq_\lambda (g, \omega), \quad \text{GDiff} \simeq_\lambda \text{GDiff}_H, \quad \text{GDiff}^e \simeq_\lambda \text{GDiff}_H^e,$$

we will refer to it as $\text{Isom}_H(g, \omega)$ (resp., $\text{Isom}_H^e(g, \omega)$).

Proposition 4.5. *Let $V_+ \in \mathcal{GM}$ with induced metric $g \in \mathcal{M}$. The groups $\text{Isom}(V_+)$ and $\text{Isom}^e(V_+)$ are isomorphic to compact subgroups of $\text{Isom}(g)$.*

Proof. Choose a splitting λ so that we have $V_+ \simeq_\lambda (g, \omega)$. The isotropy subgroup $\widetilde{\text{Isom}}_H(g, \omega)$ of (g, ω) under the action of $\text{Diff} \ltimes \Omega^2$ is isomorphic, as a topological group, to $\text{Isom}(g)$ via the map $(\phi, B) \mapsto \phi$. Indeed, (ϕ, B) fixes (g, ω) if and only if $\phi^*g = g$, i.e., $g \in \text{Isom}(g)$, and B is uniquely determined by $B = \phi^*\omega - \omega$. The images under this map of the closed subgroups $\widetilde{\text{Isom}}_H(g, \omega) \cap \text{GDiff}_H$ and $\widetilde{\text{Isom}}_H^e(g, \omega) \cap \text{GDiff}_H^e$ give the result. \square

The main result of this section is the following.

Theorem 4.6. *Let V_+ be a generalized metric on E . There exists an ILH submanifold \mathcal{S} of \mathcal{GM} such that:*

- a) For all $F \in \text{Isom}^e(V_+)$, $F \cdot \mathcal{S} = \mathcal{S}$.
- b) For all $F \in \text{GDiff}^e$, if $(F \cdot \mathcal{S}) \cap \mathcal{S} \neq \emptyset$, then $F \in \text{Isom}^e(V_+)$.

- c) *There is a local cross-section χ of the map $F \mapsto \rho_{\mathcal{GM}}(F, V_+)$ on a neighbourhood \mathcal{U} of V_+ in $\text{GDiff}^e \cdot V_+$ such that the map from $\mathcal{U} \times \mathcal{S}$ to \mathcal{GM} given by $(V_1, V_2) \mapsto \rho_{\mathcal{GM}}(\chi(V_1), V_2)$ is a homeomorphism onto its image.*

A similar statement holds for the full group of generalized diffeomorphisms:

Theorem 4.7. *Let V_+ be a generalized metric on E . There exists an ILH submanifold \mathcal{S} of \mathcal{GM} such that:*

- a) *For all $F \in \text{Isom}(V_+)$, $F \cdot \mathcal{S} = \mathcal{S}$.*
 b) *For all $F \in \text{GDiff}$, if $(F \cdot \mathcal{S}) \cap \mathcal{S} \neq \emptyset$, then $F \in \text{Isom}(V_+)$.*
 c) *There is a local cross-section χ of the map $F \mapsto \rho_{\mathcal{GM}}(F, V_+)$ on a neighbourhood \mathcal{U} of V_+ in $\text{GDiff} \cdot V_+$ such that the map from $\mathcal{U} \times \mathcal{S}$ to \mathcal{GM} given by $(V_1, V_2) \mapsto \rho_{\mathcal{GM}}(\chi(V_1), V_2)$ is a homeomorphism onto its image.*

These theorems provide slices to the actions of $\text{GDiff}^e(E)$ and $\text{GDiff}(E)$ on \mathcal{GM} , thus generalizing the results of Ebin [4].

4.2. Proof of the slice theorem

This section will consist in proving Theorems 4.6 and 4.7, for which we choose a splitting λ in order to have

$$E \simeq_\lambda (TM + T^*M)_H$$

and the identifications (4.2) and (4.3). For simplicity, we use the notation $\text{Isom}_H(V_+)$ for $\text{Isom}_H(g, \omega)$ when $V_+ \simeq_\lambda (g, \omega)$, and similarly for exact isometries.

The first step will be to build an ILH structure on the orbits. For a technical reason, we will first address the case of exact automorphisms of E .

Proposition 4.8. *Let $V_+ \in \mathcal{GM}$. The group $\text{Isom}_H^e(V_+)$ is a strong ILH Lie subgroup of GDiff_H^e . Moreover, the quotient space $\text{GDiff}_H^e / \text{Isom}_H^e(V_+)$ carries an ILH manifold structure.*

Proof. We will apply Frobenius' theorem (Theorem 2.11) to show that $\text{Isom}_H^e(V_+)$ is an ILH Lie subgroup. The ILH manifold structure in the quotient will then be a direct consequence of Corollary 2.12. Consider the map

$$(4.4) \quad \begin{aligned} \Phi : \text{GDiff}_H^e &\rightarrow \Gamma(S^2T^*M) \times \Omega^2 \\ (\phi, B) &\mapsto \rho_{\mathcal{GM}}((\phi, B), (g, \omega)), \end{aligned}$$

for which $\Phi^{-1}(V_+) = \text{Isom}_H^e(V_+)$. We have that the differential of Φ defines a right-invariant $\mathcal{C}^{\infty, \infty}$ ILH normal bundle homomorphism from $T\text{GDiff}_H^e$ to the bundle $B(\Gamma(S^2T^*M) \times \Omega^2, \text{GDiff}_H^e, T_{gm})$, where T_{gm} is defined by

$$(4.5) \quad T_{gm}((\dot{g}, \dot{\omega}), (\xi(u), B), (\xi(v), B)) = \xi(v)^*(\dot{g}, \dot{\omega}),$$

for (ξ, U) a chart at the origin for Diff . Denote by \mathbf{A}' the differential of Φ at the origin:

$$\begin{aligned} \mathbf{A}' : \mathfrak{gdiff}_H^e &\rightarrow \Gamma(S^2T^*M) \times \Omega^2 \\ (u, b) &\mapsto (\mathcal{L}_u g, \mathcal{L}_u \omega - b). \end{aligned}$$

The proof will consist in checking the hypotheses of Theorem 2.11 for \mathbf{A}' . For some technical results regarding elliptic operator theory we will refer to Appendix A.

In order to apply elliptic operator theory, we parametrize $\mathfrak{g}\mathfrak{d}\mathfrak{i}\mathfrak{f}\mathfrak{f}_H^e$ by $\Gamma(TM + T^*M)$. This space of sections has an ILH structure by the ILH chain $\Gamma(TM + T^*M) := \{\Gamma(TM) \times \Omega^1, \Gamma(TM)^{k+1} \times \Omega^{1,k+1}, k \geq n+5\}$. Consider the surjective morphism

$$\begin{aligned} \iota_e : \Gamma(TM + T^*M) &\rightarrow \mathfrak{g}\mathfrak{d}\mathfrak{i}\mathfrak{f}\mathfrak{f}_H^e \\ u + \alpha &\mapsto (u, \iota_u H - d\alpha) \end{aligned}$$

Set $\mathbf{A} := \mathbf{A}' \circ \iota_e$, which is a first-order differential operator from $\Gamma(TM) \times \Omega^1$ to $\Gamma(S^2T^*M) \times \Omega^2$:

$$(4.6) \quad \begin{aligned} \mathbf{A} : \Gamma(TM + T^*M) &\rightarrow \Gamma(S^2T^*M) \times \Omega^2 \\ u + \alpha &\mapsto (\mathcal{L}_u g, \mathcal{L}_u \omega - \iota_u H + d\alpha) \end{aligned}$$

We then define the first-order operator \mathbf{B} on the ILH chain $\{\Omega^0, \Omega^{0,k+2}, k \geq n+5\}$ by

$$\begin{aligned} \mathbf{B} : \Omega^0 &\rightarrow \Gamma(TM + T^*M) \\ f &\mapsto (0, df). \end{aligned}$$

As $u \mapsto \mathcal{L}_u g$ has injective symbol, the sequence

$$(4.7) \quad \Omega^0 \xrightarrow{\mathbf{B}} \Gamma(TM + T^*M) \xrightarrow{\mathbf{A}} \Gamma(S^2T^*M) \times \Omega^2$$

is elliptic in the middle, that is the range of the principal symbol of \mathbf{B} equals the kernel of the principal symbol of \mathbf{A} . We can then use elliptic operator theory to prove that the operator \mathbf{A} satisfies the properties a), b), c) and d) of Proposition A.2 (compare with the hypotheses of Theorem 2.11).

Now we check the hypotheses for \mathbf{A}' . From the surjectivity of ι_e , we deduce

$$\mathfrak{g}\mathfrak{d}\mathfrak{i}\mathfrak{f}\mathfrak{f}_H^e = \ker \mathbf{A}' \oplus \iota_e(\text{Im} \mathbf{A}^*), \quad \Gamma(S^2T^*M) \times \Omega^2 = \text{Im} \mathbf{A}' \oplus \ker \mathbf{A}^*,$$

where \mathbf{A}^* denotes the adjoint operator of \mathbf{A} for the $L^{2,0}$ pairing (see Appendix A). Note that ι_e admits a continuous right inverse onto a complementary subspace of $\ker \iota_e$:

$$\iota_e^{-1} : (u, b) \mapsto u - \mathbb{G}d^*(b - \iota_u H).$$

Using ι_e^{-1} , the closedness of $\text{Im} \mathbf{A}^*$, and the continuity of ι_e , we deduce that $\iota_e(\text{Im} \mathbf{A}^*)$ is a closed subspace of $\mathfrak{g}\mathfrak{d}\mathfrak{i}\mathfrak{f}\mathfrak{f}_H^e$, and a) in Theorem 2.11 is satisfied. Since $\text{Im} \mathbf{A}' = \text{Im} \mathbf{A}$, and as $\text{Im} \mathbf{A}$ is closed, we have that $\text{Im} \mathbf{A}'$ is a closed subspace of $\Gamma(S^2T^*M) \times \Omega^2$, so b) is satisfied. To conclude, as \mathbf{A} satisfies the hypotheses c) and d) in Proposition A.2, we find the estimates for \mathbf{A}' (hypotheses c) and d) in Theorem 2.11) by using the ILH linear normal maps ι_e and ι_e^{-1} . \square

Now we can prove:

Proposition 4.9. *Let $V_+ \in \mathcal{GM}$. Then $\text{Isom}_H(V_+)$ is a strong ILH Lie subgroup of GDiff_H . Moreover, the quotient space $\text{GDiff}_H/\text{Isom}_H(V_+)$ carries an ILH manifold structure.*

Proof. We use the proof of Proposition 4.8. Consider this time the differential of

$$(4.8) \quad \begin{aligned} \Phi : \text{GDiff}_H &\rightarrow \Gamma(S^2T^*M) \times \Omega^2 \\ (\phi, B) &\mapsto \rho_{\mathcal{G}\mathcal{M}}((\phi, B), (g, \omega)), \end{aligned}$$

which defines a right-invariant $C^{\infty, \infty}$ ILH normal bundle homomorphism from $T\text{GDiff}_H$ to $B(\Gamma(S^2T^*M) \times \Omega^2, \text{GDiff}_H, T_{gm})$, where T_{gm} is defined as in (4.5). Denote by \mathbf{A}' the differential of Φ at the origin:

$$\begin{aligned} \mathbf{A}' : \mathfrak{gdiff}_H &\rightarrow \Gamma(S^2T^*M) \times \Omega^2 \\ (u, b) &\mapsto (\mathcal{L}_u g, \mathcal{L}_u \omega - b) \end{aligned}$$

As in Proposition 4.8, the proof consists in checking the hypotheses of Theorem 2.11 for \mathbf{A}' and we rely on Appendix A for elliptic operator theory.

Consider the map

$$\begin{aligned} \Psi : \mathfrak{gdiff}_H &\rightarrow \Gamma(TM) \times (d\Omega^1 \oplus \mathcal{H}^2) \\ (u, b) &\mapsto (u, \iota_u H - b), \end{aligned}$$

where \mathcal{H}^2 is the space of harmonic 2-forms with respect to the metric g .

Then Ψ is a linear ILH continuous isomorphism, with continuous inverse given by the map $(u, \beta) \mapsto (u, \iota_u H - \beta)$ for $u \in \Gamma(TM)$ and $\beta \in d\Omega^1 \oplus \mathcal{H}^2$. Thus

$$\mathfrak{gdiff}_H = \Psi^{-1}(\Gamma(TM) \times d\Omega^1) \oplus \Psi^{-1}(\mathcal{H}^2).$$

But $\Psi^{-1}(\Gamma(TM) \times d\Omega^1) = \mathfrak{gdiff}_H^e$, and $\Psi^{-1}(\mathcal{H}^2) = \mathcal{H}^2$, so

$$\mathfrak{gdiff}_H = \mathfrak{gdiff}_H^e \oplus \mathcal{H}^2.$$

We are in the setting of Section A.2, with $E = \mathfrak{gdiff}_H$, $E_0 = \mathfrak{gdiff}_H^e$ and $\mathcal{H} = \mathcal{H}^2$, as, by the proof of Proposition 4.8, the restriction of \mathbf{A}' to \mathfrak{gdiff}_H^e satisfies the hypotheses of Theorem 2.11.

By Lemma A.3, the subspace $\ker \mathbf{A}'$ admits a complementary closed subspace E_2' . We want to apply Lemma A.4 to \mathbf{A}' . First, note that, by definition, $\|\mathbf{A}'h\|_k = \|h\|_k$ for all $h \in \mathcal{H}^2$. We consider the norm $\|\cdot\|_0$ on the space \mathfrak{gdiff}_H (this makes sense even if the ILH chain defining \mathfrak{gdiff}_H starts at $n+5$). Let $z \in \mathfrak{gdiff}_H$, which we decompose as $z = (u, \iota_u H - da) + (0, h)$ in $\mathfrak{gdiff}_H^e \oplus \mathcal{H}^2$. Then, by Hodge decomposition into orthogonal components,

$$\|z\|_0^2 \geq \|h + h_u\|_0^2$$

where h_u denotes the harmonic part of $\iota_u H$. Using the Cauchy–Schwarz inequality,

$$\|h\|_0^2 \leq \|z\|_0^2 + 2\|h\|_0 \|h_u\|_0.$$

There is a uniform constant C depending on H such that $\|h_u\|_0 \leq \|\iota_u H\|_0 \leq C\|u\|_0$. As $\|u\|_0 \leq \|z\|_0$, we have $\|h\|_0^2 \leq \|z\|_0^2 + 2C\|h\|_0 \|z\|_0$, and thus,

$$\|h\|_0 \leq (1 + 2C)\|z\|_0.$$

We can now apply Lemma A.4 and get the linear estimates c) of Theorem 2.11 for \mathbf{A}' . By Lemma A.5, we have an orthogonal decomposition

$$\Gamma(TM) \times \Omega^2 = \text{Im}\mathbf{A}' \oplus \mathbb{F}_3$$

into closed subspaces, so a) and b) are satisfied. Finally, by Lemma A.6, \mathbf{A}' satisfies d). \square

One of the main ingredients in the proof of Theorem 4.6 is a GDiff_H -invariant metric on the space \mathcal{GM} . Take a splitting λ so that we have $\mathcal{GM} \simeq_\lambda \mathcal{M} \times \Omega^2$ and define, for $(g, \omega) \in \mathcal{GM}$ and $(\dot{g}, \dot{\omega}) \in T_{(g, \omega)}\mathcal{GM}$, the pairing

$$(4.9) \quad \langle (\dot{\omega}, \dot{g}), (\dot{\omega}, \dot{g}) \rangle_{g, \omega} = \int_M \langle \dot{\omega}, \dot{\omega} \rangle_g \text{dvol}_g + \int_M \langle \dot{g}, \dot{g} \rangle_g \text{dvol}_g$$

We thus have a smooth GDiff_H -invariant (weak) Riemannian metric on \mathcal{GM} .

Remark 4.10. This metric only gives the (non-complete) L^2 -topology on the tangent bundle of \mathcal{GM} , hence the denomination *weak* metric. As in [4], Sec. 4, $(g, \omega) \mapsto \langle \cdot, \cdot \rangle_{g, \omega}$ is a smooth metric. It can be extended to a smooth metric on the spaces \mathcal{GM}^k for all $k \geq n + 5$. Moreover, each of these extended metrics admits a Levi-Civita connection and we can define the associated exponential maps $\exp_{\mathcal{GM}^k}^k$. Note that the definition of the metric only involves g , thus the proofs of these facts follow readily from Ebin’s work. An important issue is the existence of an exponential map for $(\mathcal{GM}, \langle \cdot, \cdot \rangle)$. The proof in [4] enables to define a Levi-Civita connection for $\langle \cdot, \cdot \rangle$ on \mathcal{GM} , whose extension to the spaces \mathcal{GM}^k gives the Levi-Civita connections of the extended metrics. On the spaces \mathcal{GM}^k , the exponential is everywhere a local diffeomorphism from $T\mathcal{GM}^k$ to \mathcal{GM}^k , by the implicit function theorem in Hilbert spaces. However, this does not ensure the existence of an exponential map on \mathcal{GM} . Fixing $x \in \mathcal{GM}$, for each k we have a neighbourhood of zero \mathcal{V}^k in $T_x\mathcal{GM}^k$ such that the exponential map is a diffeomorphism from \mathcal{V}^k onto its image. However, the inverse limit of $(\mathcal{V}^k)_{k \in \mathbb{N}(d)}$ could shrink to zero. We will see in the proof of Theorem 4.6 how to overcome this difficulty.

We are now ready to prove Theorems 4.6 and 4.7, following Theorems 7.1 and 7.4 in [4].

Proof of Theorem 4.6. The proof is organized in three steps.

First step: ILH structure on the orbit.

The action $\rho_{\mathcal{GM}}$ of GDiff_H^e on $V_+ \in \mathcal{GM}$ induces a map

$$\rho_{V_+} : \text{GDiff}_H^e / \text{Isom}_H^e(V_+) \rightarrow \mathcal{O}_{V_+},$$

where $\mathcal{O}_{V_+} := \text{GDiff}_H^e \cdot V_+$ denotes the orbit. The map ρ_{V_+} is injective and an immersion by the proof of Proposition 4.8. We will prove that ρ_{V_+} is an homeomorphism onto \mathcal{O}_{V_+} by showing that its image is closed. Recall that $\rho_{\mathcal{GM}}(\phi, B) = (\phi^*g, \phi^*\omega - B)$. From Ebin’s work, we know that the map $\phi \mapsto \phi^*g$ has a closed image and is an homeomorphism from $\text{Diff}/\text{Isom}(g)$ onto its image. Thus, if a sequence $(\phi_j^*g, \phi_j^*\omega - B_j)$ converges to $(g_\infty, \omega_\infty)$, we can find $\phi_\infty \in \text{Diff}$ such that (ϕ_j) converges to ϕ_∞ . But then $(\phi_j^*\omega)$ converges to $\phi_\infty^*\omega$, and there exists a 2-form B_∞

such that (B_j) converges to B_∞ and $\omega_\infty = \phi_\infty^* \omega - B_\infty$. As GDiff_H^e is a closed subgroup of $\text{Diff} \times \Omega^2$, the map ρ_{V_+} is a homeomorphism onto \mathcal{O}_{V_+} , which thus becomes a closed ILH submanifold of \mathcal{GM} .

Second step: construction of the normal bundle to the orbit.

We define the normal bundle ν of the submanifold \mathcal{O}_{V_+} in $T\mathcal{GM}$ to be the orthogonal to $T\mathcal{O}_{V_+}$ with respect to the invariant metric (4.9). As the metric is only a weak one, ν might not be a smooth subbundle of $T\mathcal{GM}|_{\mathcal{O}_{V_+}}$. To show that ν is indeed a smooth subbundle, we will prove that it is the kernel of a smooth map P of bundles in the ILH category. Define a surjective bundle map

$$P : T\mathcal{GM}|_{\mathcal{O}_{V_+}} \rightarrow T\mathcal{O}_{V_+}$$

by transporting along the orbit the map $\mathbf{A} \circ \mathbb{G} \circ \mathbf{A}^*$, where \mathbf{A} is the operator in (4.6), and \mathbb{G} is the Green operator of the complex (4.7). We want to show that P is smooth. Fix $k \geq n + 5$ and work in the Hilbert category on \mathcal{GM}^k , considering the $\text{GDiff}_H^{e,k}$ orbit $\mathcal{O}_{V_+}^k$ of V_+ . The superscript k will refer to the Hilbert completion of the object in this category. The map P extends to a map

$$P^k : T\mathcal{GM}^k|_{\mathcal{O}_{V_+}} \rightarrow T\mathcal{O}_{V_+}^k.$$

If $\eta \in \text{GDiff}_H^{e,k}$, as in [4], proof of Thm. 7.1, we have that $P_{\eta \cdot V_+}^k$ equals $\mathbf{A}_\eta \circ (\mathbf{A}_\eta^* \circ \mathbf{A}_\eta)^{-1} \circ \mathbf{A}_\eta^*$ where $\mathbf{A}_\eta = \eta^* \circ \mathbf{A} \circ (dR_\eta)^{-1}$ and R_η is the right translation. The operator \mathbf{A}_η is the infinitesimal action of $\text{GDiff}_H^{e,k}$ at $\eta \cdot V_+$. As the action of $\text{GDiff}_H^{e,k}$ on \mathcal{GM}^k is smooth, the map $\eta \mapsto \mathbf{A}_\eta$ is smooth. If $\eta \mapsto \mathbf{A}_\eta^*$ is smooth, then $\eta \mapsto \mathbf{A}_\eta^* \circ \mathbf{A}_\eta$ will be smooth and so will be P as the inverse map is smooth. We now compute the adjoint of \mathbf{A} . For $(u, \alpha) \in \Gamma(TM)^{k+1} \times \Omega^{1,k+1}$ and $(\dot{g}, \dot{\omega}) \in \Gamma(S^2T^*M)^k \times \Omega^{2,k}$, if $\langle \cdot, \cdot \rangle_{L_g^2}$ denotes the L^2 inner product induced by g ,

$$\begin{aligned} \langle \mathbf{A}(u, \alpha), (\dot{g}, \dot{\omega}) \rangle_{g, \omega} &= \langle \mathcal{L}_u \omega - \iota_u H + d\alpha, \dot{\omega} \rangle_{L_g^2} + \langle \mathcal{L}_u g, \dot{g} \rangle_{L_g^2} \\ &= \langle \iota_u (d\omega - H), \dot{\omega} \rangle_{L_g^2} + \langle d\iota_u \omega, \dot{\omega} \rangle_{L_g^2} + \langle \mathcal{L}_u g, \dot{g} \rangle_{L_g^2} + \langle d\alpha, \dot{\omega} \rangle_{L_g^2} \\ &= \langle u, F_1(\dot{\omega}) \rangle_{L_g^2} + \langle u, F_2(d^* \dot{\omega}) \rangle_{L_g^2} + \langle u, F_3(\dot{g}) \rangle_{L_g^2} + \langle \alpha, d^* \dot{\omega} \rangle_{L_g^2}, \end{aligned}$$

where F_1 and F_2 are zero order differential operators whose coefficients are rational functions of $(g, \omega, d\omega, H)$ and F_3 is the adjoint of $u \mapsto \mathcal{L}_u g$. Thus the adjoint of \mathbf{A} can be written

$$\mathbf{A}^*(\dot{g}, \dot{\omega}) = (F_1(d^* \dot{\omega}) + F_2(\dot{\omega}) + F_3(\dot{g}), d^* \dot{\omega}).$$

The smoothness of \mathbf{A}_η^* follows from its local expression as in [4], Thm. 7.1, using [4], Lemma 3.2, Thm. 3.3. Then, by definition of P and invariance of the metric (4.9) on \mathcal{GM} , ν is the kernel of P , and thus it is a smooth subbundle of $T\mathcal{GM}|_{\mathcal{O}_{V_+}}$.

Third step: construction of the slice.

Here, we fix $k \geq n + 5$ and work in the Hilbert category on \mathcal{GM}^k , considering the $\text{GDiff}_H^{e,k}$ orbit $\mathcal{O}_{V_+}^k$ of V_+ . We will use the invariant exponential map $\exp_{\mathcal{GM}}^k$ associated to the weak metric (4.9).

We have to find small neighbourhoods \mathcal{V}^k of zero in the fibre $\nu_{V_+}^k$ of ν^k over V_+ and \mathcal{U}^k of V_+ in $\mathcal{O}_{V_+}^k$ with a cross-section $\chi: \mathcal{U}^k \rightarrow \text{GDiff}_H^{e,k}$ such that

$$(4.10) \quad \mathcal{W}^k := \{d_2\rho_{\mathcal{G}\mathcal{M}}(\phi, x), \phi \in \chi(\mathcal{U}^k) \text{ and } x \in \mathcal{V}^k\} \subset \nu^k,$$

where $d_2\rho_{\mathcal{G}\mathcal{M}}$ is the differential of $\rho_{\mathcal{G}\mathcal{M}}$ with respect to the second variable. Then, the restriction of $\exp_{\mathcal{G}\mathcal{M}}^k$ to \mathcal{W}^k defines a diffeomorphism onto a small neighbourhood of $\mathcal{O}_{V_+}^k$ in $\mathcal{G}\mathcal{M}^k$, and the slice will be $\exp_{\mathcal{G}\mathcal{M}}(\mathcal{V}^k) \cap \mathcal{G}\mathcal{M}$. Note that this last step needs a little explanation to be adapted from Ebin's proof. Indeed, the fact that $\exp_{\mathcal{G}\mathcal{M}}^k(\mathcal{V}^k) \cap \mathcal{G}\mathcal{M}$ is homeomorphic to $\exp_{\mathcal{G}\mathcal{M}}^k(\mathcal{V}^k \cap T\mathcal{G}\mathcal{M})$ is not straightforward as we work in the category of ILH manifolds. The solution to this problem in Ebin's work is provided by [4], Theorem 7.5, Corollary 7.6.

We adapt here the argument. The construction of the neighbourhoods \mathcal{V}^k , \mathcal{U}^k and \mathcal{W}^k satisfying (4.10) is done exactly as in [4].

One needs to show that

$$\exp_{\mathcal{G}\mathcal{M}}^k : \mathcal{W}^k \cap T\mathcal{G}\mathcal{M} \rightarrow \exp_{\mathcal{G}\mathcal{M}}(\mathcal{W}^k) \cap \mathcal{G}\mathcal{M}$$

is well defined and a homeomorphism (for the smooth topology). Elements in GDiff_H^e act on $T\mathcal{G}\mathcal{M}^k$ and $\mathcal{G}\mathcal{M}^k$. Denote by $\mathcal{L}_{(u,b)}$ the infinitesimal action, or Lie derivative. We restrict ourselves to the action of elements of the form $(u, \iota_u H) \in \mathfrak{gdiff}_H^e$, for $u \in \Gamma(TM)$. The action at $(g, \omega) \in \mathcal{G}\mathcal{M}^k$ reads

$$\mathcal{L}_{u, \iota_u H}(g, \omega) = (\mathcal{L}_u g, \iota_u H + \mathcal{L}_u \omega).$$

As H is smooth, we deduce that (g, ω) is in $\mathcal{G}\mathcal{M}$ if and only if all its iterated Lie derivatives with respect to elements of the form $(u, \iota_u H) \in \mathfrak{gdiff}_H^e$ exist. As $\exp_{\mathcal{G}\mathcal{M}}^k$ is GDiff_H^e -invariant, it commutes with these Lie derivatives. Hence, $\exp_{\mathcal{G}\mathcal{M}}$ sends smooth elements to smooth elements. Moreover, a sequence (g_n, ω_n) converges to (g, ω) in $\mathcal{G}\mathcal{M}$ if and only if all its Lie derivatives do. As \mathcal{W}^k is stable under the action of small elements in GDiff_H^e , $\exp_{\mathcal{G}\mathcal{M}}^k$ is a continuous map from $\mathcal{W}^k \cap T\mathcal{G}\mathcal{M}$ to $\exp_{\mathcal{G}\mathcal{M}}^k(\mathcal{W}^k) \cap \mathcal{G}\mathcal{M}$. The same argument applied to the inverse map $(\exp_{\mathcal{G}\mathcal{M}}^k)^{-1}$ implies surjectivity of $\exp_{\mathcal{G}\mathcal{M}}^k$ onto smooth elements and that $\exp_{\mathcal{G}\mathcal{M}}^k$ is an homeomorphism from $\mathcal{W}^k \cap T\mathcal{G}\mathcal{M}$ to $\exp_{\mathcal{G}\mathcal{M}}^k(\mathcal{W}^k) \cap \mathcal{G}\mathcal{M}$.

Then, one can set $\mathcal{S} = \exp_{\mathcal{G}\mathcal{M}}^k(\mathcal{V}^k) \cap \mathcal{G}\mathcal{M}$. The required properties for the slice follow from the invariance of the metric under the GDiff_H^e -action, and the regularities of the action and the exponential map. \square

Proof of Theorem 4.7. The proof follows the proof of Theorem 4.6. The only step which deserves some clarification is the second step: the construction of the normal bundle to the orbit as a smooth subbundle of $T\mathcal{G}\mathcal{M}$. We use the notations from the proofs of Propositions 4.8 and 4.9. Denote by ν' the normal bundle (with respect to the invariant metric) to the GDiff_H -orbit \mathcal{O}_{V_+} . First, we proceed as for the case of exact diffeomorphisms, and consider the orthogonal splitting

$$T_{V_+}\mathcal{G}\mathcal{M} = \text{Im}\mathbf{A} \oplus \ker \mathbf{A}^*$$

where \mathbf{A} is the operator of Proposition 4.8. Transport equivariantly along the GDiff_H -orbit the operator $\mathbf{A} \circ \mathbb{G} \circ \mathbf{A}^*$, where \mathbb{G} is the Green operator of the com-

plex (4.7). We thus obtain a smooth bundle homomorphism from $T\mathcal{GM}|_{\mathcal{O}_{V_+}}$ to $T\mathcal{O}_{V_+}$, whose kernel defines a smooth ILH bundle ν over the orbit.

From the proof of Proposition 4.9 and by Lemma A.5, we have the orthogonal decompositions

$$T_{V_+}\mathcal{GM} = \text{Im}\mathbf{A}' \oplus \mathbb{F}_3 = \text{Im}\mathbf{A} \oplus \ker \mathbf{A}^*, \quad \text{Im}\mathbf{A}' = \text{Im}\mathbf{A} \oplus \mathcal{F},$$

where \mathbb{F}_3 is the kernel of the orthogonal projection $p_0: \ker \mathbf{A}^* \rightarrow \mathcal{F}$ onto a finite-dimensional subspace $\mathcal{F} \subset \ker \mathbf{A}^*$ defined as follows. Let (h_i) be an orthonormal basis of \mathcal{H}^2 . Decompose $\mathbf{A}'h_i = \mathbf{A}'x_i \oplus f_i$ in the direct sum decomposition $\Gamma(TM) \times \Omega^2 = \mathbf{A}'\mathfrak{g}\text{diff}_H^e \oplus \ker \mathbf{A}^*$, and define \mathcal{F} to be the span of the f_i . As $\text{Im}\mathbf{A}$ is orthogonal to $\ker \mathbf{A}^*$, and as for all $h \in \mathcal{H}^2$, we have $\mathbf{A}'h = (h, 0)$, the projection operator can be written $p_0(x) = \sum_i \langle x, h_i \rangle_0 h_i$. This is nothing but the projection onto the space of harmonic 2-forms.

Note that, by construction, $\nu_{V_+} = \ker \mathbf{A}^* = \mathbb{F}_3 \oplus \mathcal{F}$, and $\nu'_{V_+} = \mathbb{F}_3 = \ker p_0$.

We can extend equivariantly the operator I_0 along the orbit \mathcal{O}_{V_+} and define a bundle homomorphism \tilde{I}_0 from ν to the trivial bundle $\mathbb{R}^{b_2(M)}$ over \mathcal{O}_{V_+} . As the kernel of the Laplacian operator on 2-forms varies smoothly with the metric, \tilde{I}_0 is a smooth ILH bundle homomorphism. Then, the kernel of \tilde{I}_0 is by construction the normal bundle ν' to the orbit, and this defines a smooth ILH bundle.

The other steps of the proof are analogous to Theorem 4.6. \square

Remark 4.11. The space \mathcal{GM} is a strong ILH manifold, but Ebin's proof does not allow us to conclude that the slices we built are strong ILH submanifolds. So far, we have proved that they are ILH submanifolds of \mathcal{GM} . The problem to obtain a strong ILH manifold structure is related to the regularity of the exponential map $\exp_{\mathcal{GM}}$. To obtain a stronger structure for \mathcal{S} , one would need linear estimates on the connection related to the weak inner product in order to ensure regularity of geodesics, as in [22] or [24], Sec. VI.1.

5. The ILH stratification of the moduli space

Let $E \rightarrow M$ be an exact Courant algebroid. Its automorphism group, or group of generalized diffeomorphisms GDiff , acts on the set of generalized metrics \mathcal{GM} by inverse image, as in (4.1). The *moduli space of generalized metrics* \mathcal{GR} is defined by

$$\mathcal{GR} := \mathcal{GM}/\text{GDiff}.$$

In this section we will exhibit an ILH stratification for \mathcal{GR} .

Definition 5.1. Let \mathcal{T} be a topological space and A be a countable partially ordered set. A partition $(\mathcal{T}_\alpha)_{\alpha \in A}$ of \mathcal{T} is an *ILH stratification* of \mathcal{T} if

- 1) For each $\alpha \in A$, the space \mathcal{T}_α is a strong ILH manifold whose topology is induced by the topology on \mathcal{T} .
- 2) Whenever $\mathcal{T}_\alpha \cap \overline{\mathcal{T}_\beta} \neq \emptyset$, then $\beta < \alpha$ and $\mathcal{T}_\alpha \subset \overline{\mathcal{T}_\beta}$.

The spaces \mathcal{T}_α are called the strata of the stratification.

For any compact subgroup $G \subset \text{GDiff}$, denote by \mathcal{GM}_G the space of generalized metrics whose generalized isometry group is G and by N^G the normalizer of G in GDiff . There is an action of $G \backslash N^G$ on \mathcal{GM}_G , whose quotient space we denote by $\mathcal{GR}_{(G)}$. This is the space of generalized metrics whose isotropy group is conjugated to G . We will show the following theorem.

Theorem 5.2. *The space \mathcal{GR} admits an ILH stratification $(\mathcal{GR}_{(G)})_{(G) \in \mathcal{A}}$, where \mathcal{A} is the set of conjugacy classes of generalized isometry groups of E . Moreover, if $(G) \subsetneq (G')$, then $\mathcal{GR}_{(G')} \subset \overline{\mathcal{GR}_{(G)}}$, i.e., the strata intersect as much as possible.*

Again, in order to provide the proof for this theorem, we choose a splitting λ so that

$$E \simeq_\lambda (TM + T^*M)_H$$

and we have the identifications (4.2) and (4.3).

5.1. A description of generalized isometry groups

Following [2], to prove Theorem 5.2, we first describe the generalized isometry groups of E . Denote by Π the projection

$$\Pi : \text{Diff} \times \Omega^2 \rightarrow \text{Diff}.$$

Note that $\text{Diff}_{[H]}$, the space of diffeomorphisms that preserve the cohomology class $[H]$ is a strong ILH Lie subgroup of Diff (these two groups share the same Lie algebra, $\text{Diff}_{[H]}$ contains the connected component of identity in Diff). Recall the following, which follows implicitly from Proposition 4.8 and recalls Proposition 4.5.

Lemma 5.3. *Let $(g, \omega) \in \mathcal{GM}$. Set $G = \text{Isom}_H(g, \omega)$. Then G , respectively $\Pi(G)$, is a compact Lie subgroup of GDiff_H , respectively $\text{Diff}_{[H]}$. The restriction of Π to G is an isomorphism of Lie groups and its inverse reads*

$$(5.1) \quad \Pi^{-1}(\phi) = (\phi, \omega - \phi^*\omega).$$

Proof. The fact that Π is an isomorphism of groups from G to its image follows from the definition of GDiff_H and its action on \mathcal{GM} . Then $\Pi(G)$ is a compact subgroup of $\text{Diff}_{[H]}$, thus a Lie subgroup of $\text{Diff}_{[H]}$ by a theorem of Myers and Steenrod, see [2], Proposition III.2. \square

Corollary 5.4. *The set \mathcal{A} of conjugacy classes of generalized isometry groups is countable.*

Proof. It follows as in Lemme VI.3 of [2] from the classification of reductive Lie groups and their actions [25]. \square

In the reverse direction, we have the following partial result.

Lemma 5.5. *For any compact subgroup G of GDiff_H , there exists $(g, \omega) \in \mathcal{GM}$ such that $G \subset \text{Isom}_H(g, \omega)$.*

Proof. Average any generalized metric V_+ with respect to the Haar measure $d\mu_G$ on G ,

$$V_+^a := \int_{\psi \in G} \psi \cdot V_+ d\mu_G(\psi),$$

so that the averaged metric V_+^a is given by a pair (g, ω) such that $G \subset \text{Isom}_H(g, \omega)$. \square

We will obtain a more precise statement describing the groups G that are generalized isometry groups for a generalized metric, and not just subgroups of generalized isometry groups.

Let G be any compact subgroup of GDiff_H , isomorphic to $\Pi(G)$. The group G is then a compact Lie group and acts on M via the (left) action of $\Pi(G)$ on M :

$$\begin{aligned} G \times M &\rightarrow M \\ (\psi, m) &\mapsto \Pi(\psi)(m). \end{aligned}$$

Remark 5.6. Note that all the results from [2] on compact Lie groups acting on compact manifolds extend to G , and hence to any generalized isometry group, by Lemma 5.3.

For any $m \in M$, denote by G_m the stabilizer of m under the G -action. A *principal orbit* for the G -action is the orbit of a point $m \in M$ such that the image of G_m in $GL(T_m M / T_m(G \cdot m))$ is the identity (this property does not depend on the choice of the point in the orbit).

The group G_m acts on $S^2 T_m^* M \times \Lambda^2 T_m^* M$ with twisted action:

$$(5.2) \quad \begin{aligned} G_m \times (S^2 T_m^* M \times \Lambda^2 T_m^* M) &\rightarrow S^2 T_m^* M \times \Lambda^2 T_m^* M \\ ((\phi, B), g_m, \omega_m) &\mapsto (\phi \cdot g_m, \phi \cdot \omega_m + B(m)). \end{aligned}$$

where $\phi \cdot g_m = g_m((D\phi)_{m \cdot}, (D\phi)_{m \cdot})$ and $\phi \cdot \omega_m = \omega_m((D\phi)_{m \cdot}, (D\phi)_{m \cdot})$ come from the action by pullback on tensors. We can then define $(S^2 T_m^* M \times \Lambda^2 T_m^* M)_{G_m}$ to be the space of invariants (fixed elements) under the G_m -action.

We will need the following proposition, which is an adaptation of Proposition III.12 in [2].

Proposition 5.7. *Consider $G' = \text{Isom}_H(g, \omega)$ for some generalized metric (g, ω) , and $G \subset G'$ a compact subgroup. The following are equivalent:*

- (i) *There exists a generalized metric $(g, \omega) \in \mathcal{GM}$ that is not invariant under the action of G' or one of its conjugate but that is G -invariant.*
- (ii) *There is a G -orbit (for the left action on M) that is not a G' -orbit or, there is a point m of a principal orbit for G such that $(\Lambda^2 T_m^* M \times S^2 T_m^* M)_{G_m} \neq (\Lambda^2 T_m^* M \times S^2 T_m^* M)_{G'_m}$.*

Proof. The proof follows the proof of Proposition III.12 in [2], considering elements of $S^2 T^* M \times \Omega^2$ instead of $S^2 T^* M$, and using the twisted action (5.2).

The proofs of [2], Prop. III.12, (i) \implies (ii) and [2], Prop. III.12, (ii) \implies (i) in the case of G and G' having the same orbits extend directly to our setting.

We shall give some details to extend [2], Prop.III.12, (ii) \implies (i) in the case when G' and G have different orbits. In this situation, as in the proof of [2], Prop.III.12, (ii) \implies (i), consider a perturbation of (g, ω) of the form $(g, \omega) + t(h, \omega_h)$ for small t . Using the slice result for generalized metrics, a necessary condition for $\text{Isom}_H(g + th, \omega + t\omega_h)$ to be conjugated to G' is that

$$(5.3) \quad \exists u \in \Gamma(TM) \text{ such that } \phi^*h - h = \mathcal{L}_u g$$

for some element $(\phi, B) \in G'$. To see this, just restrict to the metric component in the Lie derivative considerations. Thus, if we can find (h, ω_h) that is $\Pi(G)$ -invariant, and $(\phi, B) \in G'$ such that (5.3) is not satisfied, then $G \subset \text{Isom}_H(g + th, \omega + t\omega_h)$ for small t , $\text{Isom}_H(g + th, \omega + t\omega_h)$ is not conjugated to G' , and we are done.

Note that $\Pi(G) \subset \text{Isom}_H(g) \subset \text{Isom}(g)$. Moreover, as G and G' have different orbits on M , so do $\Pi(G)$ and $\text{Isom}(g)$. As in the proof of [2], Prop.III.12, there exists $h \in S^2T^*M$ and $\phi \in \text{Isom}(g)$ such that h is $\Pi(G)$ -invariant and (5.3) is not satisfied. As $\Pi(G') \subset \text{Isom}(g)$ and $\Pi(G)$ and $\Pi(G')$ have different orbits, we can actually take $\phi \in \Pi(G')$. Using the Haar measure on G , and the average trick, we can find a $\Pi(G)$ -invariant pair (h, ω_h) . Setting $(\phi, \omega - \phi^*\omega) \in G'$, the equation (5.3) is not satisfied and the perturbation $(g + th, \omega + t\omega_h)$ gives the required generalized metric.

The rest of the proof readily follows Bourguignon's arguments. □

Remark 5.8. In the proof of Proposition 5.7, for (ii) \implies (i), one starts with a metric (g', ω') such that $G \subset \text{Isom}_H(g', \omega')$. Then the metric (g, ω) such that $\text{Isom}_H(g, \omega) = G$ can be constructed as close as we want to (g', ω') in \mathcal{GM} .

We are now ready to give a characterization of the compact subgroups G of GDiff_H such that there is a generalized metric (g, ω) with $G = \text{Isom}_H(g, \omega)$:

Proposition 5.9. *Let G be a compact subgroup of GDiff_H . The following are equivalent:*

- (i) *There exists a generalized metric $(g, \omega) \in \mathcal{GM}$ such that $G = \text{Isom}_H(g, \omega)$.*
- (ii) *For any compact subgroup G' of GDiff_H such that $G \subsetneq G'$ and having the same orbits as G , there is a point m of a principal orbit for G such that $(\Lambda^2 T_m^* M \times S^2 T_m^* M)_{G_m} \neq (\Lambda^2 T_m^* M \times S^2 T_m^* M)_{G'_m}$.*

Proof. The proof follows the proof of Theorem III.23 in [2]. □

5.2. Centralizer and normalizer of a generalized isometry group

In this subsection we show that the spaces $\mathcal{R}_{(G)}$ are ILH manifolds. First, we need a strong ILH Lie group structure on the groups acting on \mathcal{GM}_G . Set $G = \text{Isom}_H(g, \omega)$ for some generalized metric (g, ω) . We begin with the study of the centralizer GDiff_H^G of G in GDiff_H .

Lemma 5.10. *The centralizer GDiff_H^G of G in GDiff_H is a strong ILH Lie subgroup of GDiff_H .*

Proof. We first reduce to the case where $\omega = 0$. To do this consider the map

$$(5.4) \quad \begin{aligned} \mathcal{A}_\omega : \text{Diff} \times \Omega^2 &\rightarrow \text{Diff} \times \Omega^2 \\ (\phi, B) &\mapsto (\phi, B - \omega + \phi^*\omega), \end{aligned}$$

which is just the conjugation by the B -field $(\text{Id}, -\omega)$. The maps \mathcal{A}_ω and its inverse $\mathcal{A}_{-\omega}$ are C^∞ ILH normal diffeomorphisms by Proposition 2.7. Moreover, they are topological group isomorphisms. We deduce that the precomposition by \mathcal{A}_ω or its inverse preserve the strong ILH Lie group structure of $\text{Diff}^G \times (\Omega^2)^G$ (that is the regularity of product and inverse in local charts at the origin). Note that under \mathcal{A}_ω , the group GDiff_H is sent to $\text{GDiff}_{H+d\omega}$ and the group $\text{Isom}_H(g, \omega)$ is sent to $\text{Isom}_{H+d\omega}(g, 0)$. Similarly, the centralizer of $G = \text{Isom}_H(g, \omega)$ in $\text{Diff}^G \times (\Omega^2)^G$ is sent to the centralizer of $\text{Isom}_H(g, 0)$ in $\text{Diff}^G \times (\Omega^2)^G$. Then, we can assume without loss of generality that $\omega = 0$ and $G = \text{Isom}_H(g, 0)$.

Consider now the centralizer of G in $\text{Diff}^G \times (\Omega^2)^G$, denoted $(\Omega^2)^G \rtimes \text{Diff}^G$. This is the semi-direct product of G -invariant 2-forms in Ω^2 and the centralizer of $\Pi(G)$ in Diff . By Proposition V.8 in [2], Diff^G is a strong ILH Lie sub-group of Diff , and its connected component of identity is totally geodesic in Diff with respect to an L^2 metric induced by g . Similarly, using the projection operator

$$(5.5) \quad \begin{aligned} \Omega^2 &\rightarrow \Omega^2 \\ B &\mapsto \int_G \phi^* B d\mu_G((0, \phi)), \end{aligned}$$

where $d\mu_G$ denotes the Haar measure on G , the group $(\Omega^2)^G$ is a totally geodesic strong ILH Lie subgroup of Ω^2 . Using the chart at the identity (3.9), we see that $(\Omega^2)^G \rtimes \text{Diff}^G$ is a (totally geodesic) strong ILH Lie-subgroup of $\text{Diff}^G \times (\Omega^2)^G$.

Finally, we can use the same argument as in the proof of Theorem 3.8 to obtain that GDiff_H^G is a strong ILH Lie subgroup of $\text{Diff}^G \times (\Omega^2)^G$. As the change of chart to obtain GDiff_H^G as a strong ILH Lie subgroup of $\text{Diff}^G \times (\Omega^2)^G$ and the change of chart to obtain GDiff_H as a strong ILH Lie subgroup of $\text{Diff}^G \times (\Omega^2)^G$ are both obtained by applying the implicit function theorem to the same map, we conclude that GDiff_H^G is a strong ILH Lie subgroup of GDiff_H . \square

Remark 5.11. In the proof of Theorem 5.2, we do not really need the fact that GDiff_H^G is a strong ILH Lie subgroup of GDiff_H , we just need the fact that this is a strong ILH Lie subgroup of $\text{Diff}^G \times (\Omega^2)^G$.

Let N_G be the normalizer of G in GDiff_H .

Corollary 5.12. *The quotient $G \backslash N_G$ is an ILH Lie group whose topology is induced by the inclusion $G \backslash N_G \rightarrow G \backslash \text{GDiff}_G$.*

Proof. This follows as in Theorem V.22 in [2], and the discussion on abstract Lie groups in [2], Section V. \square

5.3. The ILH stratification

Let $G = \text{Isom}_H(g, \omega)$ as before. We turn now to the quotient space $\mathcal{GR}_{(G)}$.

Proposition 5.13. *The space $\mathcal{GR}_{(G)}$ is a strong ILH manifold. Moreover, its topology coincides with the topology induced by the inclusion in \mathcal{GR} .*

Proof. We use the slice result (Proposition II.16 in [2]) for the action of $G \backslash N_G$ on \mathcal{GM}_G . The proof follows as in [2], Theorem V.24. \square

Now we can prove Theorem 5.2:

Proof of Theorem 5.2. The proof follows the one of Theorem VI.4 in [2].

From Corollary 5.4, the set \mathcal{A} of conjugacy classes of generalized isometry groups is countable. It is partially ordered by inclusion. It is also clear that $(\mathcal{GR}_{(G)})_{(G) \in \mathcal{A}}$ is a partition of \mathcal{GR} .

From Proposition 5.13, the spaces $\mathcal{GR}_{(G)}$ are strong ILH manifolds whose topology coincides with the induced topology from \mathcal{GR} .

Then, one has to show the intersection properties of the strata. Let (G) and (G') be two conjugacy classes in \mathcal{A} . Assume that $\mathcal{GR}_{(G')} \cap \overline{\mathcal{GR}_{(G)}} \neq \emptyset$. Let V'_+ be a generalized metric whose class is in $\mathcal{GR}_{(G')} \cap \overline{\mathcal{GR}_{(G)}}$, with isometry group G' . Applying the slice Theorem 4.7 at V'_+ , we find a slice \mathcal{S} parametrizing generalized metrics nearby V'_+ . Then, any element in this slice has generalised isometry group conjugated to a subgroup of G' . From this we deduce that G is conjugated to a subgroup of G' , hence $(G) \subsetneq (G')$.

Last, let (G) and (G') be two conjugacy classes in \mathcal{A} such that $(G) \subsetneq (G')$. Then G' is the isometry group of some generalized metric V'_+ . We can assume up to conjugation that G is a subgroup of G' and the generalized isometry group of another generalized metric. To conclude, one needs to construct a generalized metric V_ε , arbitrarily close to V'_+ , such that the generalized isometry group of V_ε is G . This is done by following the argument of the proof of Proposition 5.9. \square

As a corollary, we also obtain generalized versions of other results in [2].

Corollary 5.14. *We have the following.*

- 1) *The space \mathcal{GM}_G is an open dense subset of \mathcal{GM}_G^1 , where \mathcal{GM}_G^1 is the space of generalized metrics whose generalized isometry group contains G .*
- 2) *The space $\mathcal{GM}_{\{1\}}$ is an open dense subset of \mathcal{GM} as long as $\dim M \geq 2$.*
- 3) *In a neighbourhood of a generalized metric with non-trivial generalized isometry group, \mathcal{GM} is not an ILH manifold.*
- 4) *In each homotopy class of a loop in \mathcal{GR} , there is a loop that can be lifted to \mathcal{GM} .*

5.4. Relation between the strata of the moduli spaces

In this section we relate the moduli space of generalized metrics of an exact Courant algebroid, $\mathcal{GR} = \mathcal{GM}/\text{GDiff}$, with the moduli space of usual metrics, or Riemannian structures, of the base manifold, $\mathcal{R} = \mathcal{M}/\text{Diff}$, and other moduli spaces naturally arising. In particular, we study the relation between the isometries of a generalized metric and the isometries of the usual metric to which it projects, and use this to compare the strata of the moduli spaces.

Since the definition of a generalized metric on a Courant algebroid does not depend on the bracket, we start by introducing the orbit space for the action of the group O_π ,

$$\mathcal{GR}^\pi := \mathcal{GM}/O_\pi.$$

Following Sections 4 and 5, one can show that \mathcal{GR}^π admits an ILH stratification

$$\mathcal{GR}^\pi = \bigcup_{(G) \in \mathcal{A}_\pi} (\mathcal{GR}^\pi)_{(G)},$$

where $(G) \in \mathcal{A}_\pi$ runs through all the conjugacy classes of isotropy groups, $\text{Isom}(V_+)$, for the O_π -action in \mathcal{GM} . Again, we choose a splitting λ , in such a way that

$$E \simeq_\lambda (TM + T^*M)_H, \quad V_+ \simeq_\lambda (g, \omega),$$

to provide proofs of the results. Recall that g does not depend on the splitting.

Proposition 5.15. *The projection $\mathcal{GR}^\pi \rightarrow \mathcal{R}$ defined by $[(g, \omega)] \mapsto [g]$ is an isomorphism of ILH stratified topological spaces.*

Proof. This map and its inverse $[g] \mapsto [(g, \omega)]$ are well defined and continuous. On the other hand, an element $(\phi, B) \in O_\pi$ stabilizing $(g, \omega) \in \mathcal{GM}$ should satisfy $\phi^*g = g$, i.e., ϕ is an isometry of g , and $B = \phi^*\omega - \omega$, i.e., B is completely determined by ω and ϕ . We thus have that $\text{Isom}_H(g, \omega) \cong \text{Isom}(g) \subset \text{Diff}_{[H]}$, so the ILH stratification is preserved. \square

We are mainly concerned with the moduli space of generalized metrics

$$\mathcal{GR}^H := \frac{\mathcal{GM}}{\text{GDiff}_H},$$

which, as GDiff_H is a subgroup of O_π , projects onto $\mathcal{GR}^\pi \cong \mathcal{R}$.

We have another natural projection of \mathcal{GR}^H . Consider the action of the ILH Lie group $\text{Diff}_{[H]}$ on the space of metrics \mathcal{M} . Its orbit space is, by following [2] and Sections 4 and 5, an ILH stratified space

$$\mathcal{R}^{[H]} := \mathcal{M}/\text{Diff}_{[H]} = \bigcup_{(G) \in \mathcal{B}_{[H]}} \mathcal{R}_{(G)}^{[H]}$$

where $\mathcal{B}_{[H]}$ is the set of conjugacy classes of isometry groups in $\text{Diff}_{[H]}$. Again, the projection

$$\pi_H : \mathcal{GR}^H \rightarrow \mathcal{R}^{[H]}$$

is well defined and maps strata inside strata. In order to express more precisely the relation between the two stratifications, we look at the isometry groups under the conjugation by O_π .

Proposition 5.16. *For $(g, \omega) \in \mathcal{GM}$ and $(\psi, C) \in O_\pi$ we have*

$$(5.6) \quad (\psi, C)^{-1} \text{Isom}_H((g, \omega))(\psi, C) = \text{Isom}_{\psi^*(H+dC)}((\psi^*g, \psi^*\omega - C)),$$

where both isotropy groups are subgroups of O_π .

Proof. On the one hand, (ϕ, B) fixes (g, ω) if and only if $(\psi, C)^{-1}(\phi, B)(\psi, C)$ fixes $(\psi, C) \cdot (g, \omega)$, as the action is on the right. Note that

$$(\psi, C)^{-1}(\phi, B)(\psi, C) = (\psi^{-1}\phi\psi, \psi^*(B - \phi^*C + C)),$$

On the other hand, the condition $\phi^*H - H = dB$ is equivalent to

$$(\psi^{-1}\phi\psi)^*(\psi^*(H + dC)) - (\psi^*(H + dC)) = d(\psi^*(B + \phi^*C - C)). \quad \square$$

From the projection $\pi_H([(g, \omega)]) = [g]$, we can express the preimage of a stratum $\mathcal{R}_{(G)}^{[H]}$ in $\mathcal{R}^{[H]}$, for $G = \text{Isom}_{[H]}(g) \subset \text{Diff}_{[H]}$, as

$$\pi_H^{-1}(\mathcal{R}_{(G)}^{[H]}) = \bigcup_{\omega \in \Omega^2} \mathcal{GR}_{(\text{Isom}_H(g, \omega))}^H.$$

The following proposition uses Hodge theory to give a more concrete description. Note first that

$$\text{Isom}_{[H]}(g, \omega) = \{(\phi, \phi^*\omega - \omega) \mid \phi \in \text{Isom}_{[H]}(g), \phi^*(H - d\omega) = H - d\omega\}.$$

Proposition 5.17. *Given $g \in \mathcal{M}$, there exists $C \in \Omega^2$ such that*

$$\text{Isom}_H(g, \omega) = (\text{Id}, -C)G(\omega - C)(\text{Id}, C),$$

where we set $G(\omega) := \{\phi \in G \mid \phi^*\omega = \omega\}$, and identify $G(\omega) \subset \text{Diff}_{[H]}$ with $G(\omega) \times \{0\} \subset \text{GDiff}_H$. As a consequence, we have the description

$$\pi_H^{-1}(\mathcal{R}_{(G)}^{[H]}) = \bigcup_{\omega \in \Omega^2} \mathcal{GR}_{((\text{Id}, -C)G(\omega)(\text{Id}, C))}^H.$$

Proof. We will use the conjugation in (5.6). We average H under the $\text{Isom}_{[H]}(g)$ -action to obtain H' , which belongs to the same cohomology class of H and hence there is $C_1 \in \Omega_2$ such that $H' = H + dC_1$. The conjugation of $\text{Isom}_H(g, \omega)$ by (Id, C_1) is then

$$\text{Isom}_{H'}(g, \omega - C_1).$$

The metric g being fixed, we have a Hodge decomposition on forms

$$\Omega^2 = d^*\Omega^3 \oplus \ker(d).$$

Let C_2 be the projection of $\omega - C_1$ onto $\ker(d)$. The conjugation of $\text{Isom}_{H'}(g, \omega - C_1)$ by (Id, C_2) is $\text{Isom}_{H'}(g, \omega')$, where $\omega' = \omega - C_1 - C_2$ belongs to $d^*\Omega^3$ and can be written as $\omega' = d^*d\tilde{\omega}$ for $\tilde{\omega} \in \Omega^2$. We now have the group

$$\text{Isom}_{H'}(g, \omega') = \{(\phi, \phi^*\omega' - \omega') \mid \phi \in \text{Isom}_{[H]}(g), \phi^*(d\omega') = d\omega'\}.$$

Take $\phi \in G$. The condition $\phi^*d\omega' = d\omega'$ becomes $\phi^*\Delta_g d\tilde{\omega}' = \Delta_g d\tilde{\omega}'$. As $\phi \in \text{Isom}(g)$, we deduce $\Delta_g \phi^*d\tilde{\omega}' = \Delta_g d\tilde{\omega}'$. The laplacian being an isomorphism on the orthogonal of harmonic forms, $\phi^*d\tilde{\omega}' = d\tilde{\omega}'$. But ϕ commutes with the Hodge dual for g , so $\phi^*\omega' = \omega'$. Thus,

$$\text{Isom}_{H'}(g, \omega') = \{(\phi, 0) \mid \phi \in \text{Isom}_{[H]}(g), \phi^*\omega' = \omega'\} = G(\omega') \times \{0\},$$

and $C = C_1 + C_2$ gives the first part of theorem. As ω running through Ω^2 corresponds to $\omega - C$ running through Ω^2 , the second part follows. \square

Remark 5.18. Consider $g \in \mathcal{M}$ and $G \subset \text{Isom}_{[H]}(g)$. By adapting the arguments in [2], Proposition III.12, there exists $\omega \in \Omega^2$ such that $G = G(\omega)$ if and only if, for all compact subgroups $G' \subset \text{Isom}_{[H]}$ containing G and having the same orbits as G , there is a point m of a principal orbit for G such that $(\Lambda^2 T_m^* M)_{G_m} \neq (\Lambda^2 T_m^* M)_{G'_m}$.

The relation between \mathcal{GR}^H and \mathcal{R} , can be described by the sequence

$$\mathcal{GR}^H \rightarrow \mathcal{R}^{[H]} \rightarrow \mathcal{R}$$

where the last arrow is the projection $\mathcal{M}/\text{Diff}_{[H]} \rightarrow \mathcal{M}/\text{Diff}$, whose fibre is isomorphic to the *mapping class type* group $\text{Diff}/\text{Diff}_{[H]}$. The preimage of a strata $\mathcal{R}_{(G)} \subset \mathcal{R}$ consists of

$$\bigcup_{(G') \in \mathcal{A}_{[\mathcal{H}]}^G} \mathcal{R}_{(G')}^{[H]}$$

where $\mathcal{A}_{[\mathcal{H}]}^G$ is the set of $\text{Diff}_{[H]}$ -conjugacy classes for the groups in the Diff-conjugacy class (G) .

Summarizing, we have described the relation between the strata of the following spaces

$$\mathcal{GR}^H \rightarrow \mathcal{R}^{[H]} \rightarrow \mathcal{R} \cong \mathcal{GR}^\pi.$$

6. Odd exact Courant algebroids

We have focused so far on exact Courant algebroids, as this is the best-known case. A future aim of our work is the exploitation of these results for the class of transitive Courant algebroids, those of the form $TM + \text{ad } P + T^*M$ for P a principal G -bundle. The simplest instance in this class is the bundle $TM + 1 + T^*M$, where 1 denotes a trivial line bundle over M . Indeed, as $TM + \text{ad } P + T^*M$ is related to solutions of the Strominger system [9], $TM + 1 + T^*M$ corresponds to the abelian case $G = S^1$.

We briefly introduce the generalized geometry arising from $TM + 1 + T^*M$, and state the main results, drawing the analogies with the exact case and stressing the relevant technical or conceptual differences.

6.1. Basics on B_n -generalized geometry

The vector bundle $TM + 1 + T^*M$, whose sections we denote by $u + f + \alpha$ and $v + g + \beta$, has a Courant algebroid structure consisting of the pairing

$$\langle u + f + \alpha, v + g + \beta \rangle = i_u \beta + f^2,$$

the projection π to TM as the anchor map, and the Dorfman bracket

$$(6.1) \quad [u + f + \alpha, v + g + \beta] = [u, v] + u(g) - v(f) + L_u \beta - i_v d\alpha + 2gdf.$$

The study of the generalized geometry on this bundle and its twisted versions is called generalized geometry of type B_n , as the pairing has signature $(n + 1, n)$,

whose group of special orthogonal transformations is $\text{SO}(n+1, n)$, a real form of a complex group of type B_n . This geometry was defined in [1] and developed in [26], [27] and [28].

An *odd exact Courant algebroid* [28] is a transitive Courant algebroid E , i.e., π is surjective, such that $\text{rk } E = 2 \dim M + 1$ is satisfied¹. They fit into the diagram

$$(6.2) \quad \begin{array}{ccccc} & & T^*M & & \\ & & \downarrow & & \\ & & Q^* & \longrightarrow & E \xrightarrow{\pi} TM, \\ & & \downarrow & & \\ & & 1 & & \end{array}$$

where Q is the Lie algebroid E/T^*M and the row and the column are exact. Choosing an isotropic splitting $\lambda: T \rightarrow E$ determines a unique splitting $1 \rightarrow Q^* \rightarrow E \rightarrow TM$ whose image is orthogonal to $\lambda(TM)$, and hence gives an isomorphism of vector bundles $E \simeq_\lambda TM + 1 + T^*M$. The resulting bracket on $TM + 1 + T^*M$ is the bracket in (6.1) twisted, by $H \in \Omega^3$ and $F \in \Omega_2^{cl}$ such that $dH + F \wedge F = 0$, as follows:

$$\begin{aligned} [u + f + \alpha, v + g + \beta]_{H,F} &= [u, v] + u(g) - v(f) + i_u i_v F + L_u \beta - i_v d\alpha \\ &\quad + 2g df + 2(g i_u F - f i_v F) + i_u i_v H. \end{aligned}$$

We refer to this Courant algebroid as $(TM + 1 + T^*M)_{H,F}$.

6.2. The ILH Lie group structure of the automorphism group

One of the main novelties of B_n -geometry is the appearance, apart from B -fields, of A -fields, which show non-abelianness. Define Ω^{2+1} to be the space $\Omega_2 \times \Omega_1$. When considered as a Lie group, the product in Ω^{2+1} is

$$(B, A)(B', A') = (B + B' + A \wedge A', A + A').$$

The group of orthogonal transformations of an odd exact Courant algebroid commuting with the anchor are then described, by choosing a splitting, as

$$\text{O}_\pi(E) \simeq_\lambda \text{Diff} \times \Omega^{2+1}.$$

Analogously, the automorphism group $\text{Aut}(E)$ of an odd exact Courant algebroid is isomorphic to the automorphism group of $(TM + 1 + T^*M)_{H,F}$, which is denoted by $\text{GDiff}_{H,F}$ and given by

$$\{\psi \times (B, A) \in \text{Diff} \times \Omega^{2+1} \mid \psi^* H - H = dB - A \wedge (2F + dA), \psi^* F - F = dA\},$$

¹An exact Courant algebroid (Section 3) is equivalently defined as a transitive Courant algebroid E such that $\text{rk } E = 2 \dim M$.

with product

$$(6.3) \quad (\psi \times (B, A))(\psi' \times (B', A')) = \psi\psi' \times (\psi'^*B + B' + \psi'^*A \wedge A', \psi'^*A + A'),$$

and Lie algebra $\mathfrak{gdiff}_{H,F}$ given by

$$\{(u, (b, a)) \in \Gamma(TM) \times \Omega^{2+1} \mid d(\iota_u H - b) + 2(\iota_u F + a) \wedge F = 0, d(\iota_u F - a) = 0\}.$$

Analogously to Proposition 3.6, the group $O_\pi(E) \simeq_\lambda \text{Diff} \times \Omega^{2+1}$ carries a strong ILH Lie group structure, independent of the choice of splitting λ , modelled on

$$\{\Gamma(TM) \times \Omega^{2+1}, \Gamma(TM)^{k+1} \times \Omega^{2+1,k}, k \geq n+5\}.$$

A chart at the origin (ξ', U') on $U' = U \times \Omega^{2+1, n+5}$, with (ξ, U) a chart of Diff , is given by

$$(6.4) \quad \begin{aligned} \xi' : U' \cap (\Gamma(TM) \times \Omega^{2+1}) &\rightarrow \text{Diff} \times \Omega^{2+1} \\ (u, (b, a)) &\mapsto (\xi(u), (b, a)). \end{aligned}$$

Denote the space $\Omega^3 \times \Omega^2$ by Ω^{3+2} . In the remaining of this section, the ILH chains associated to the spaces $\Gamma(TM)$, Ω^{2+1} and Ω^{3+2} are $\{\Gamma(TM), \Gamma^{k+1}, k \geq n+5\}$, $\{\Omega^{2+1}, \Omega^{2+1,k}, k \geq n+5\}$ and $\{\Omega^{3+2}, \Omega^{3+2,k-1}, k \geq n+5\}$. Moreover, we fix a twist $(H, F) \in \Omega^{3+2}$ with $dF = 0$ and $dH + F \wedge F = 0$.

We describe $\text{GDiff}_{H,F}$ as $\tilde{\rho}((H, F), \cdot)^{-1}(H, F)$ for the map

$$(6.5) \quad \begin{aligned} \tilde{\rho} : \Omega^{3+2} \times (\text{Diff} \times \Omega^{2+1}) &\rightarrow \Omega^{3+2} \\ ((H', F'), (\phi, (B, A))) &\mapsto (\phi^*H' - dB + A \wedge (2F' + dA), \phi^*F' - dA). \end{aligned}$$

In order to endow $\text{GDiff}_{H,F}$ with an ILH Lie group structure, we use the implicit function theorem. First, we look at the map $\tilde{\rho}((H, F), \cdot)$ using the chart (ξ', U') . Define

$$\begin{aligned} \Phi : U' \cap (\text{Diff} \times \Omega^{2+1}) &\rightarrow \Omega^{3+2} \\ (u, (b, a)) &\mapsto \tilde{\rho}((H, F), \xi'(u, (b, a))), \end{aligned}$$

whose derivative at zero is given, as $(d\xi)_0 = \text{Id}$, by

$$(6.6) \quad d\Phi_0(u, (b, a)) = (d(\iota_u H - b) + 2(\iota_u F + A) \wedge F, d(\iota_u F - A)).$$

The map Φ is a $\mathcal{C}^{\infty, \infty}$ ILH normal map, as in Proposition 2.7. A right inverse for $d\Phi_0$ in the category of $\mathcal{C}^{\infty, 2}$ ILH normal maps is given, using the notation of (3.13), by the restriction of the following operator to the image of $d\Phi_0$:

$$\begin{aligned} \mathbf{B} : \Omega^{3+2} &\rightarrow \Gamma(TM) \times \Omega^{2+1} \\ (h, f) &\mapsto (0, -d^*\mathbb{G}(h + 2(d^*\mathbb{G}f) \wedge F), -d^*\mathbb{G}f). \end{aligned}$$

From the implicit function Theorem 2.1, $\text{GDiff}_{H,F}$ is a strong ILH (sub)manifold of $\text{Diff} \times \Omega^{2+1}$. As we have proved that $\text{Diff} \times \Omega^{2+1}$ is a strong ILH Lie group, the regularity of the local inverse, product, etc. follow on $\text{GDiff}_{H,F}$, thus obtaining the following.

Proposition 6.1. *The group $\text{GDiff}_{H,F}$ is a strong ILH Lie subgroup of $\text{Diff} \times \Omega^{2+1}$.*

Remark 6.2. Implicitly, we have used here the fact that the complex

$$\begin{aligned} d_F : \Omega^{i+1,i} &\rightarrow \Omega^{i+2,i+1} \\ (b, a) &\mapsto (db - 2(-1)^i a \wedge F, da) \end{aligned}$$

is an elliptic complex of differential operators. This complex encodes some variations of Courant algebroids structures on $TM + 1 + T^*$ modulo symmetries fixing the anchor.

Denote by $\text{GDiff}_{H,F}^e$ the group of exact generalized diffeomorphisms [27]. We will show that this is a strong ILH Lie subgroup of $\text{GDiff}_{H,F}$ by means of Frobenius' theorem. Its Lie algebra $\mathfrak{gdiff}_{H,F}^e$ is given by

$$\{(u, (b, a)) \in \Gamma(TM) \times \Omega^{2+1} \mid \iota_u H - b = d\xi + 2fF, \iota_u F - a = df \text{ for some } \xi \in \Omega^1, f \in C^\infty(M)\}.$$

As a subalgebra of $\mathfrak{gdiff}_{H,F}$, it can be described as the space of triples $(u, (b, a))$ such that

$$\begin{cases} \iota_u H - b - 2(\mathbb{G}d^*(\iota_u F - a))F \in (\mathcal{H}^2)^\perp \\ \iota_u F - a \in (\mathcal{H}^1)^\perp \end{cases}$$

where \mathcal{H}^i denotes the space of harmonic i -forms. Let $s = h^2(M) + h^1(M)$, where $h^i(M) = \dim H^i(M)$, and take $(e_i)_{1 \leq i \leq h^2}$ and $(f_i)_{1 \leq i \leq h^1}$, bases of the harmonic 2-forms and 1-forms, respectively. Define the map

$$\begin{aligned} I : \Omega^{2+1} &\rightarrow \mathbb{R}^s \\ (\omega_2, \omega_1) &\mapsto (\langle \omega_2, e_i \rangle_{L^2_g}, \langle \omega_1, f_i \rangle_{L^2_g}). \end{aligned}$$

Set also

$$\begin{aligned} \kappa_1 : \mathfrak{gdiff}_{H,F} &\rightarrow \Omega^{2+1} \\ (u, (b, a)) &\mapsto (\iota_u H - b, \iota_u F - a) \end{aligned}$$

and

$$\begin{aligned} \kappa_2 : \Omega^{2+1} &\rightarrow \Omega^{2+1} \\ (\beta, \alpha) &\mapsto (\beta - 2\mathbb{G}d^*(\alpha)F, \alpha). \end{aligned}$$

Note that $\mathfrak{gdiff}_{H,F}^e = (I \circ \kappa_2 \circ \kappa_1)^{-1}(0)$. We will consider an extension of $I \circ \kappa_2 \circ \kappa_1$ to $T\text{GDiff}_{H,F}$, which will be a $C^{\infty, \infty}$ ILH normal homomorphism to the trivial bundle over $\text{GDiff}_{H,F}$.

First, consider the bundle $B(\Omega^{2+1}, \text{Diff}, \tilde{T}_{\Omega^{2+1}})$ over Diff , where $\tilde{T}_{\Omega^{2+1}}$ is defined as in Definition 2.15. Denote by $\pi : \text{Diff} \times \Omega^{2+1} \rightarrow \text{Diff}$ the projection. Then the restriction of $\pi^* \tilde{T}_{\Omega^{2+1}}$ to $\text{GDiff}_{H,F}$ defines an ILH vector bundle with fibre Ω^{2+1} , whose transition functions are defined by parallel transport on M . Extend κ_2 and I to right-invariant bundle homomorphisms

$$\tilde{\kappa}_2 : B(\Omega^{2+1}, \text{GDiff}_{H,F}, \pi^* \tilde{T}_{\Omega^{2+1}}) \rightarrow B(\Omega^{2+1}, \text{GDiff}_{H,F}, \pi^* \tilde{T}_{\Omega^{2+1}})$$

and

$$\tilde{I} : B(\Omega^{2+1}, \text{GDiff}_{H,F}, \pi^* \tilde{T}_{\Omega^{2+1}}) \rightarrow \text{GDiff}_{H,F} \times \mathbb{R}^s,$$

by setting $\tilde{\kappa}_2 = R_\phi \circ \kappa_2 \circ R_\phi^{-1}$ and $\tilde{I} = R_\phi \circ I \circ R_\phi^{-1}$ for $\phi \in \text{GDiff}_{H,F}$.

Lemma 6.3. *The bundle homomorphisms \tilde{I} and $\tilde{\kappa}_2$ are $\mathcal{C}^{\infty, \infty}$ ILH normal.*

Proof. This is a direct consequence of Theorem 2.16 b) for $\tilde{\kappa}_2$ and Theorem 2.16 c) for \tilde{I} . Note that Theorem 2.16 is stated for Diff, but the bundles we consider here are pullbacks of bundles defined on Diff, so the extension results hold in our setting as well. \square

To apply Frobenius' theorem, it remains to extend the operator κ_1 . As before, extend κ_1 to a right-invariant bundle homomorphism

$$\tilde{\kappa}_1 : B(\mathfrak{g}\text{Diff}_{H,F}, \text{GDiff}_{H,F}, T_{\theta_{H,F}}) \rightarrow B(\Omega^{2+1}, \text{GDiff}_{H,F}, \pi^* \tilde{T}_{\Omega^{2+1}}),$$

where $T_{\theta_{H,F}}$ is the defining map for $T\text{GDiff}_{H,F}$.

Lemma 6.4. *The defining map for $T\text{GDiff}_{H,F}$ is given, for $(u, (b, a)) \in \mathfrak{g}\text{Diff}_{H,F}$, $(\xi(v), (B, A))$ and $(\xi(w), (B', A'))$ in $U' \cap \text{GDiff}_{H,F}$, by*

$$\begin{aligned} T_{\theta_{H,F}}((u, (b, a)), (\xi(v), (B, A)), (\xi(w), (B', A'))) \\ = (T_{\theta}(u, \xi(v), \xi(w)), \xi(w)^* b + \xi(w)^* a \wedge A', \xi(w)^* a), \end{aligned}$$

where T_{θ} is the defining map for $T\text{Diff}$, and (ξ, U) is a chart at the origin for Diff.

Proof. Recall the definition of the defining map (2.11) for the tangent bundle. The result follows from a direct computation of θ (see Definition 2.4), using the chart (6.4) and the product (6.3). \square

Lemma 6.5. *The bundle homomorphism $\tilde{\kappa}_1$ is $\mathcal{C}^{\infty, \infty}$ ILH normal.*

Proof. By right invariance, we only need to check regularity of the local expression of $\tilde{\kappa}_1$ at the identity. Set $g = (v, (b, a)) \in U' \cap \mathfrak{g}\text{Diff}_{H,F}$, and $(w, (B, A)) \in \mathfrak{g}\text{Diff}_{H,F}$. By definition, the local expression for $\tilde{\kappa}_1$ at the identity is given by

$$\Phi_{\tilde{\kappa}_1}(g)(w, (B, A)) = (\pi^* \tilde{T}_{\Omega^{2+1}}(e, g)) \circ R_g \circ \kappa_1 \circ R_g^{-1} \circ T_{\theta_{H,F}}(e, g)^{-1}(w, (B, A)).$$

Recall that T_{θ} is the defining map for the tangent bundle of Diff, and $\tau(\exp_x v(x))$ denotes the parallel transport on the bundle $\Lambda^2 T^* M \oplus T^* M$ along the geodesic $t \mapsto \exp_x(tv(x))$, from $t = 0$ to $t = 1$. Setting $w' = T_{\theta}(e, \xi(v))^{-1}w$, we compute using Lemma 6.4:

$$\Phi_{\tilde{\kappa}_1}(g)(w, (B, A)) = (\Phi_1(g) + \Phi_2(g))(w, (B, A))$$

where

$$\Phi_1(g)(w, (B, A))(x) = \tau(\exp_x v(x))^{-1}(\iota_{w'} H, \iota_{w'} F)(\exp_x v(x))$$

and

$$\Phi_2(g)(w, (B, A))(x) = -\tau(\exp_x v(x))^{-1}(\xi(v)^{-1})^*(B - A \wedge a, A)(\exp_x v(x)).$$

The map Φ_1 is the local expression for the (right-invariant pullback of) the right-invariant extension of the map

$$\begin{aligned} \Gamma(TM) &\rightarrow \Omega^{2+1} \\ w &\mapsto (\iota_w H, \iota_w F) \end{aligned}$$

from $B(\Gamma(TM), \text{Diff}, T_\theta)$ to $B(\Omega^{2+1}, \text{Diff}, \tilde{T}_{\Omega^{2+1}})$. By Theorem 2.16 a), this defines a $\mathcal{C}^{\infty, \infty}$ ILH normal map. For Φ_2 , consider, for example, its second component

$$-\tau(\exp_x v(x))^{-1}(\xi(v)^{-1})^* A(\exp_x v(x)) = -A(x)(D(\xi(v)^{-1})\tau(\exp_x v(x)) \cdot).$$

Thus, using Ψ_{-1} as defined in (2.9),

$$\Phi_2(w, (B, A)) = \Psi_{-1}(v, (B - A \wedge a, A)).$$

From Lemma 2.8, the map Φ_2 has the required regularity, which ends the proof. \square

We can then conclude the following.

Proposition 6.6. *The group $\text{GDiff}_{H,F}^e$ is a strong ILH Lie subgroup of $\text{GDiff}_{H,F}$.*

Proof. From Lemmas 6.5 and 6.3, the operator $\tilde{I} \circ \tilde{\kappa}_2 \circ \tilde{\kappa}_1$, right-invariant by construction, is $\mathcal{C}^{\infty, \infty}$ ILH normal. Then the result follows from the lighter version of Frobenius' Theorem 2.11 referred to in Remark 2.13, recalling that $\mathfrak{gdiff}_{H,F}^e = \ker(I \circ \kappa_2 \circ \kappa_1)$. The maps

$$\mathbf{B}_I(t_i, s_i) = \left(\sum t_i e_i, \sum s_i f_i \right), \quad \mathbf{B}_2(\gamma, \delta) = (\gamma, \delta - 2\mathbb{G}d^*\gamma F), \quad \mathbf{B}_1(\beta, \alpha) = (0, (\beta, \alpha))$$

provide $\mathcal{C}^{\infty, \infty}$ ILH normal right inverses for I , κ_2 and κ_1 , respectively. If \mathbf{B} denotes the composition $\mathbf{B}_1 \circ \mathbf{B}_2 \circ I$, then $\mathfrak{gdiff}_{H,F} = \ker(I \circ \kappa_2 \circ \kappa_1) \oplus \mathbf{B}\mathbb{R}^s$. The regularity of \mathbf{B} implies that the hypotheses of Frobenius' theorem are satisfied. Then, $\ker(\tilde{I} \circ \tilde{\kappa}_2 \circ \tilde{\kappa}_1)$ can be integrated to a strong ILH Lie subgroup of $\text{GDiff}_{H,F}$. \square

Remark 6.7. By definition of $\mathfrak{gdiff}_{H,F}$ and $\mathfrak{gdiff}_{H,F}^e$ and by construction of the associated Lie groups, one can check that if $H^1(M, \mathbb{R})$ and $H^2(M, \mathbb{R})$ vanish, then $\text{GDiff}_{H,F}^e = \text{GDiff}_{H,F}$.

6.3. The moduli space of B_n -generalized metrics

A generalized metric on an odd exact Courant algebroid E over an n -dimensional manifold M is given by a rank $n+1$ positive definite subbundle $V_+ \subset E$. We make use now of diagram (6.2). The intersection $U := V_+ \cap Q^*$ must be of constant rank 1 and hence is a line subbundle. Its orthogonal complement $U^\perp \subset V_+$ is positive definite of rank n and projects isomorphically to TM via π_{U^\perp} , thus yielding a usual metric g . The map $\lambda: u \mapsto i_u g - \pi_{U^\perp}^{-1} u \in E$ is an isotropic splitting and the pair (g, λ) determines V_+ completely. In this case, λ determines a compatible splitting $1 \rightarrow Q^*$, which comes from the isomorphism $V_+ \cap Q^* \cong 1$.

The space of isotropic splittings Λ is an Ω^{2+1} -torsor, as the difference of two of them gives a map $TM \rightarrow Q^*$. This map is determined by the projection $TM \rightarrow 1$ and a skew-symmetric map $TM \rightarrow T^*M$, i.e., an element of Ω^{2+1} .

Summarizing, with the notation $\mathcal{M} := \{g \in \Gamma(S^2 T^*M) \mid g \text{ is positive definite}\}$, we have

$$\mathcal{GM} \cong \mathcal{M} \times \Lambda \simeq_\lambda \mathcal{M} \times \Omega^{2+1},$$

showing that \mathcal{GM} is an ILH manifold modelled on the ILH chain

$$\{\Gamma(S^2T^*M) \times \Omega^{2+1}, \Gamma(S^2T^*M)^k \times \Omega^{2+1,k}, k \geq n+5\}.$$

As before, we have a right ILH action $\rho_{\mathcal{GM}}$ of $F \in O_\pi$ on $V_+ \in \mathcal{GM}$ given by $F^{-1}(V_+)$. We fix a splitting λ to write isomorphisms

$$\begin{aligned} O_\pi &\simeq_\lambda \text{Diff} \times \Omega^{2+1}, & \mathcal{GM} &\simeq_\lambda \mathcal{M} \times \Omega^{2+1}, \\ V_+ &\simeq_\lambda \{u + r + \gamma(u) + i_u g + i_u \omega - \gamma(u)\gamma - 2r\gamma \mid X \in TM, r \in 1\}, \end{aligned}$$

where V_+ corresponds to $\exp(\omega, \gamma)\{\iota_X g + r + X \mid X \in TM, r \in 1\}$ with

$$\exp(\omega, \gamma) = \begin{pmatrix} 1 & 0 & 0 \\ \gamma & 1 & 0 \\ \omega - \gamma \otimes \gamma & -2\gamma & 1 \end{pmatrix} \in \Gamma(\text{SO}(TM + 1 + T^*M)).$$

The action $\rho_{\mathcal{GM}}$ then reads

$$\begin{aligned} \rho_{\mathcal{GM}} : (\text{Diff} \times \Omega^{2+1}) \times \mathcal{GM} &\rightarrow \mathcal{GM} \\ ((\phi, (B, A)), (g, (\omega, \gamma))) &\mapsto (\phi^*g, (\phi^*\omega - B - A \wedge \phi^*\gamma, \phi^*\gamma - A)), \end{aligned}$$

which restricts to actions of $\text{GDiff}_{H,F}$ and $\text{GDiff}_{H,F}^e$ on \mathcal{GM} .

We denote the group of *generalized isometries* of $V_+ \simeq_\lambda (g, (\omega, \gamma))$ by

$$\text{Isom}(V_+) \simeq_\lambda \text{Isom}_{H,F}(g, (\omega, \gamma)),$$

and analogously for $\text{Isom}^e(V_+) \simeq_\lambda \text{Isom}_{H,F}^e(g, (\omega, \gamma))$, the group of exact generalized isometries. As in Proposition 4.5, $\text{Isom}(V_+)$ and $\text{Isom}^e(V_+)$ are isomorphic to compact subgroups of $\text{Isom}(g)$.

We will prove the following result.

Theorem 6.8. *Let V_+ be a generalized metric on an odd exact Courant algebroid E . There exists an ILH submanifold \mathcal{S} of \mathcal{GM} such that:*

- a) *For all $\phi \in \text{Isom}^e(V_+)$, $\phi(\mathcal{S}) = \mathcal{S}$.*
- b) *For all $\phi \in \text{GDiff}^e$, if $\phi(\mathcal{S}) \cap \mathcal{S} \neq \emptyset$, then $\phi \in \text{Isom}^e(V_+)$.*
- c) *There is a local cross-section χ of the map $\phi \mapsto \rho_{\mathcal{GM}}(\phi, V_+)$ on a neighbourhood \mathcal{U} of V_+ in $\text{GDiff}^e \cdot V_+$ such that the map from $\mathcal{U} \times \mathcal{S}$ to \mathcal{GM} given by $(V_1, V_2) \mapsto \rho_{\mathcal{GM}}(\chi(V_1), V_2)$ is a homeomorphism onto its image.*

The same result holds for the action of $\text{Isom}(V_+) \subset \text{GDiff}$ on \mathcal{GM} .

The proof of Theorem 6.8 follows Theorems 4.6 and 4.7. We work on a splitting λ , for which $V_+ \simeq_\lambda (g, (\omega, \gamma))$. We need an invariant metric on \mathcal{GM} , which we construct now. First, given a Riemannian metric g on M , consider a metric on the Lie algebra of the group Ω^{2+1} :

$$(b, a) \mapsto \int_M |b|_g^2 + |a|_g^2 \, \text{dvol}_g,$$

where $|b|_g^2 + |a|_g^2$ is the metric on $\Lambda^2 T^*M + T^*M$ induced by g . Transport this metric on $T\Omega^{2+1}$ to obtain a left-invariant metric on Ω^{2+1} . Explicitly, for any $(\omega, \gamma) \in \Omega^{2+1}$ and $(\dot{\omega}, \dot{\gamma}) \in T_{\omega, \gamma}\Omega^{2+1}$, set

$$\|(\dot{\omega}, \dot{\gamma})\|_{(\omega, \gamma), g} = \int_M |\dot{\omega} - \gamma \wedge \dot{\gamma}|_g^2 + |\dot{\gamma}|_g^2 \, \text{dvol}_g.$$

Then, define a metric on \mathcal{GM} for $V_+ = (g, (\omega, \gamma)) \in \mathcal{GM}$ and $(\dot{g}, (\dot{\omega}, \dot{\gamma})) \in T_{(g, (\omega, \gamma))}\mathcal{GM}$:

$$(6.7) \quad \|(\dot{g}, (\dot{\omega}, \dot{\gamma}))\|_{V_+}^2 = \int_M |\dot{\omega} - \gamma \wedge \dot{\gamma}|_g^2 + |\dot{\gamma}|_g^2 + |\dot{g}|_g^2 \, \text{dvol}_g.$$

By construction, this defines a $\text{GDiff}_{H,F}$ -invariant (actually, $\text{Diff} \times \Omega^{2+1}$ -invariant) Riemannian metric on \mathcal{GM} . Just as in [4], this metric is smooth. Even if this is a weak Riemannian metric, it admits a Levi-Civita connection. The proof of this fact causes no difficulties, and can be done following Ebin [4]. Analogously to the exact case, the metric and the connection extend to the spaces \mathcal{GM}^k and we obtain $\text{GDiff}_{H,F}$ -invariant exponential maps on $T\mathcal{GM}^k$. We define invariant strong continuous Riemannian metrics on the spaces \mathcal{GM}^k by

$$(6.8) \quad \|(\dot{g}, (\dot{\omega}, \dot{\gamma}))\|_{L_{V_+}^{2,k}}^2 = \sum_{i=0}^k \int_M |\nabla_g^i(\dot{\omega} - \gamma \wedge \dot{\gamma})|_g^2 + |\nabla_g^i(\dot{\gamma})|_g^2 + |\nabla_g^i(\dot{g})|_g^2 \, \text{dvol}_g.$$

Remark 6.9. For a generalized metric $V_+ = (g, (\omega, \gamma))$, the invariant metrics (6.8) are nothing but the usual $L_g^{2,k}$ metrics defined by equation (2.6) on the space of sections of the Riemannian vector bundle (E, h_{V_+}) where $E = S^2 T^*M \oplus \Lambda^2 T^*M \oplus T^*M$ and the metric h_{V_+} is twisted by (ω, γ) :

$$h_{V_+}(\dot{g}, (\dot{\omega}, \dot{\gamma}))^2 = |\dot{\omega} - \gamma \wedge \dot{\gamma}|_g^2 + |\dot{\gamma}|_g^2 + |\dot{g}|_g^2.$$

We will consider the bundle $\Gamma(S^2 T^*M) \times \Omega^{2+1}$ as the tangent bundle to \mathcal{GM} at a given point V_+ . Then, the underlying Riemannian structure will be given by the twisted metric h_{V_+} . This will be taken into account when computing adjoint of operators that take value into $\Gamma(S^2 T^*M) \times \Omega^{2+1}$.

We denote, for simplicity, $\text{Isom}_{H,F}(g, (\omega, \gamma))$ by $\text{Isom}_{H,F}(V_+)$, and similarly for the exact case.

Proposition 6.10. *Let $V_+ \in \mathcal{GM}$. Then $\text{Isom}_{H,F}^e(V_+)$ is a strong ILH Lie subgroup of $\text{GDiff}_{H,F}^e$. Moreover, the quotient space $\text{GDiff}_{H,F}^e/\text{Isom}_{H,F}^e(V_+)$ carries an ILH manifold structure.*

Proof. The proof follows the outline of the one of Proposition 4.8 and is an application of Theorem 2.11. Consider the map

$$(6.9) \quad \begin{aligned} \Phi : \text{Diff} \times \Omega^{2+1} &\rightarrow \Gamma(S^2 T^*M) \times \Omega^{2+1} \\ (\phi, (B, A)) &\mapsto \rho_{\mathcal{GM}}((\phi, (B, A)), (g, (\omega, \gamma))) \end{aligned}$$

The differential of Φ restricted to $\text{GDiff}_{H,F}^e$ defines a right-invariant $\mathcal{C}^{\infty,\infty}$ ILH normal bundle homomorphism from $T\text{GDiff}_{H,F}^e$ to the bundle $B(\text{GDiff}_{H,F}^e, \Omega^{2+1} \times \Gamma(S^2T^*M), T_{gm})$, with

$$T_{gm}((\dot{g}, (\dot{\omega}, \dot{\gamma})), (\xi(u), (B, A)), (\xi(v), (B', A'))) = \xi(v)^*(\dot{g}, (\dot{\omega}, \dot{\gamma})),$$

where (ξ, U) is a chart at the origin for Diff . Denote by \mathbf{A}_e this differential at the origin:

$$\begin{aligned} \mathbf{A}_e : \mathfrak{gdiff}_{H,F}^e &\rightarrow \Gamma(S^2T^*M) \times \Omega^{2+1} \\ (u, (b, a)) &\mapsto (\mathcal{L}_u g, (\mathcal{L}_u \omega - b - a \wedge \mathcal{L}_u \gamma, \mathcal{L}_u \gamma - a)) \end{aligned}$$

We need to check hypotheses a) to d) of Frobenius' Theorem 2.11 for \mathbf{A}_e . To do this, we consider another parameterization of $\mathfrak{gdiff}_{H,F}^e$. We endow $\Gamma(TM + 1 + T^*M)$ with an ILH structure using the ILH chain

$$\Gamma(E) := \{\Gamma(TM) \times \Omega^{1+0}, \gamma(TM)^{k+1} \times \Omega^{1+0, k+1}, k \geq n+5\},$$

with Ω^{1+0} the space $\Omega^1 \times C^\infty(M)$. By construction of $\text{GDiff}_{H,F}^e$, there is a surjective morphism

$$\begin{aligned} \iota_e : \Gamma(TM + 1 + T^*M) &\rightarrow \mathfrak{gdiff}_{H,F}^e \\ u + f + \alpha &\mapsto (u, (\iota_u H + 2fF - d\alpha, \iota_u F - df)). \end{aligned}$$

Set $\mathbf{A} := \mathbf{A}_e \circ \iota_e$, that is, the first-order differential operator

$$\begin{aligned} \Gamma(TM + 1 + T^*M) &\xrightarrow{\mathbf{A}} \Gamma(S^2T^*M) \times \Omega^{2+1} \\ u + f + \alpha &\mapsto (\mathcal{L}_u g, (\mathcal{L}_u \omega - \iota_u H - 2fF + d\alpha + (\iota_u F - df) \wedge \gamma, \\ &\quad \mathcal{L}_u \gamma - \iota_u F + df)). \end{aligned}$$

The differential $f \mapsto df \in \Gamma(TM + 1 + T^*M)$ defines a first-order operator on the ILH chain $\{\Omega^0, \Omega^{0, k+2}, k \geq n+5\}$. It is routine to check that the sequence

$$\Omega^0 \xrightarrow{d} \Gamma(TM + 1 + T^*M) \xrightarrow{\mathbf{A}} \Gamma(TM) \times \Omega^{2+1}$$

is elliptic at the middle. By Proposition A.2 from the Appendix A, the operator \mathbf{A} satisfies the hypotheses a) to d) of Proposition A.2. Moreover, ι_e admits a continuous right inverse:

$$\iota_e^{-1} : (u, (b, a)) \mapsto u - \mathbb{G}d^*(a - \iota_u F) - \mathbb{G}d^*(b - \iota_u H - 2\mathbb{G}d^*(\iota_u F - a) \wedge F).$$

Using the Hodge decomposition, we see that $\text{Im}(\iota_e^{-1})$ is a complementary subspace of $\ker \iota_e$. Then, as in the proof of Proposition 4.8, we deduce the decompositions into closed subspaces

$$\mathfrak{gdiff}_{H,F}^e = \ker \mathbf{A}_e \oplus \iota_e(\text{Im} \mathbf{A}^*), \quad \Gamma(S^2T^*M) \times \Omega^{2+1} = \text{Im} \mathbf{A}_e \oplus \ker \mathbf{A}^*,$$

and that \mathbf{A}_e satisfies hypotheses a) to d) of Theorem 2.11. \square

A similar statement holds for the full group of generalized diffeomorphisms:

Proposition 6.11. *Let $V_+ \in \mathcal{GM}$. Then $\text{Isom}_{H,F}(V_+)$ is a strong ILH Lie subgroup of $\text{GDiff}_{H,F}$. Moreover, the quotient space $\text{GDiff}_{H,F}/\text{Isom}_{H,F}(V_+)$ carries an ILH manifold structure.*

Proof. Restrict the map Φ from (6.9) to $\text{GDiff}_{H,F}$, and denote by \mathbf{A}' its differential at the origin:

$$\begin{aligned} \mathbf{A}' : \mathfrak{gdiff}_{H,F} &\rightarrow \Gamma(S^2T^*M) \times \Omega^{2+1} \\ (u, (b, a)) &\mapsto (\mathcal{L}_u g, (\mathcal{L}_u \omega - b - a \wedge \mathcal{L}_u \gamma, \mathcal{L}_u \gamma - a)). \end{aligned}$$

We will use the elliptic complex from Remark 6.2. We have a Hodge decomposition

$$\Omega^{2+1} = \mathcal{H}_F^2 \oplus d_F \Omega^{1+0} \oplus d_F^* \Omega^{3+2}$$

with \mathcal{H}_F^2 the harmonic elements with respect to the corresponding Laplacian. Consider the map

$$\begin{aligned} \Psi : \mathfrak{gdiff}_{H,F} &\rightarrow \Gamma(TM) \times (d\Omega^{1+0} \oplus \mathcal{H}_F^2) \\ (u, (b, a)) &\mapsto (u, (\iota_u H - b, \iota_u F - a)). \end{aligned}$$

Then Ψ is a linear ILH continuous isomorphism, with continuous inverse, and

$$\mathfrak{gdiff}_{H,F} = \Psi^{-1}(\Gamma(TM) \times d\Omega^{1+0}) \oplus \Psi^{-1}(\mathcal{H}_F^2) = \mathfrak{gdiff}_{H,F}^e \oplus \mathcal{H}_F^2.$$

As in the proof of Proposition 4.9, we conclude the proof using the lemmas of Section A.2. \square

Proof of Theorem 6.8. Fix $V_+ \in \mathcal{GM}$. Then, following Remark 6.9, the decomposition

$$\Gamma(S^2T^*M) \times \Omega^{2+1} = \text{Im} \mathbf{A}_e \oplus \ker \mathbf{A}^*$$

corresponds to an orthogonal decomposition:

$$T_{V_+} \mathcal{GM} = T_{V_+}(\text{GDiff}_{H,F}^e \cdot V_+) \overset{\perp}{\oplus} \nu_{V_+}$$

into the tangent space of the orbit and its the normal bundle (with respect to the metric (6.7)). Using the invariant metrics (6.8) and Proposition 6.10, the proof follows as in Section 4. \square

A corollary of Theorem 6.8, whose proof follows as in Corollary 8.2 of [4], is the following.

Corollary 6.12. *The space of generalized metrics with trivial isometry group is open in \mathcal{GM} .*

With the results above, a generalization of the main result of Section 5 is straightforward. Define the moduli space of generalized metrics \mathcal{GR} of an odd exact Courant algebroid by

$$\mathcal{GR} := \mathcal{GM}/\text{GDiff} \simeq_{\lambda} (\mathcal{M} \times \Omega^{2+1})/\text{GDiff}_{H,F}.$$

Theorem 6.13. *The space \mathcal{GR} admits an ILH stratification $(\mathcal{GR}_{(G)})_{(G) \in \mathcal{A}}$, where \mathcal{A} is the set of conjugacy classes of generalized isometry groups of an odd exact Courant algebroid. Whenever $(G) \subsetneq (G')$, we have $\mathcal{GR}_{(G')} \subset \overline{\mathcal{GR}_{(G)}}$, i.e., the strata intersect as much as possible. Moreover,*

- *the space \mathcal{GM}_G is an open dense subset of \mathcal{GM}_G^1 , where \mathcal{GM}_G^1 is the space of generalized metrics whose generalized isometry group contains G ;*
- *the space $\mathcal{GM}_{\{1\}}$ is an open dense subset of \mathcal{GM} as long as $\dim M \geq 2$;*
- *in a neighbourhood of a generalized metric with non-trivial generalized isometry group, \mathcal{GM} is not an ILH manifold;*
- *in each homotopy class of a loop in \mathcal{GR} , there is a loop that can be lifted to \mathcal{GM} .*

A. Appendix

We give some analytical results on elliptic operator theory and ILH chains which are used to prove Propositions 4.8 and 4.9, leading to the proof of the slice theorem (Theorems 4.6 and 4.7).

A.1. Some results on elliptic complexes

Let $(E_i, h_i), i \in \{0, 1, 2\}$ be smooth Riemannian vector bundles over a compact Riemannian manifold (M, g) , and let

$$(A.1) \quad \Gamma(E_0) \xrightarrow{\mathbf{B}} \Gamma(E_1) \xrightarrow{\mathbf{A}} \Gamma(E_2)$$

be a complex of first-order differential operators. We assume that this complex is elliptic in the middle, that is, at the level of principal symbols, the range of \mathbf{B} equals the kernel of \mathbf{A} (see, e.g., [24], Sec. VII.5). Using the metrics on E_i , and the metric g , we can endow the spaces $\Gamma(E_i)$ with $L^{2,k}$ Hilbert norms $\|\cdot\|_k$ as in (2.6). The completion of $\Gamma(E_i)$ for this $L^{2,k}$ norm will be denoted by $\Gamma(E_i)^k$. We then consider the continuous extensions of \mathbf{B} and \mathbf{A} on these spaces, as well as the ILH chains $\{\Gamma(E_i), \Gamma(E_i)^{k+2-i} \mid k \in \mathbb{N}(d)\}$ for some $d \in \mathbb{N}$, $i \in \{0, 1, 2\}$. Note the shift in the norms in the definition of the ILH chains, so that \mathbf{A} and \mathbf{B} define \mathcal{C}^∞ ILH maps. We denote by \mathbf{A}^* and \mathbf{B}^* the adjoint operators of \mathbf{A} and \mathbf{B} , defined on smooth sections by the $L^{2,0}$ pairing: for $f_i \in \Gamma(E_i)$,

$$\int_M h_2(\mathbf{A}f_1, f_2) \, \text{dvol}_g = \int_M h_1(f_1, \mathbf{A}^*f_2) \, \text{dvol}_g,$$

and similarly for \mathbf{B} . Let $\Delta = \mathbf{A}^*\mathbf{A} + \mathbf{B}\mathbf{B}^*$ be the elliptic operator of the complex (A.1) and denote by \mathbb{G} the associated Green operator. The following lemma follows from the proof of [24], VII, Lemma 5.10.

Lemma A.1. *The operator $\mathbb{G}\mathbf{A}^* : \mathbf{A}\Gamma(E_1) \rightarrow \Gamma(E_1)$ is an ILH linear normal left inverse for $\mathbf{A} : \Gamma(E_1) \rightarrow \mathbf{A}\Gamma(E_1)$ on $\text{Im}\mathbf{A}^*$, and there are constants C and D_k such that, for $w \in \Gamma(E_2)$ and $k \in \mathbb{N}(d)$,*

$$\|\mathbb{G}\mathbf{A}^*w\|_{k+1} \leq C\|w\|_k + D_k\|w\|_{k-1}.$$

Proof. The fact that $\mathbb{G}\mathbf{A}^*$ is a left inverse follows from definition of Δ and \mathbb{G} . For the estimate, first apply the uniform Gårding's inequality ([24], VII, Lemma 5.2) to $\mathbb{G}\mathbf{A}^*w$ to obtain

$$\|\mathbb{G}\mathbf{A}^*w\|_{k+1} \leq C\|\mathbf{A}^*w\|_{k-1} + D_k\|\mathbb{G}\mathbf{A}^*w\|_k.$$

Recall that $\mathbb{G}_k : \mathbf{A}^*\Gamma(E_2)^{k+1} \rightarrow \mathbf{A}^*\Gamma(E_2)^{k+3}$ is continuous. Thus,

$$\|\mathbb{G}\mathbf{A}^*w\|_{k+1} \leq C\|\mathbf{A}^*w\|_{k-1} + D'_k\|\mathbf{A}^*w\|_{k-2},$$

and the result follows, as \mathbf{A}^* is a linear ILH normal operator. \square

We now gather all the results that enable us to use Frobenius' theorem in the ILH category.

Proposition A.2. *The following statements hold.*

- a) *The image $\text{Im}\mathbf{A}^*$ is a closed subspace of $\Gamma(E_1)$ such that $\Gamma(E_1) = \ker \mathbf{A} \oplus \text{Im}\mathbf{A}^*$.*
- b) *The image $\text{Im}\mathbf{A}$ is a closed subspace of $\Gamma(E_2)$ such that $\Gamma(E_2) = \text{Im}\mathbf{A} \oplus \ker \mathbf{A}^*$.*
- c) *There are constants C and D_k , for $k \in \mathbb{N}(d)$ with $D_d = 0$, such that, for $u \in \text{Im}(\mathbf{A}^*)$,*

$$\|\mathbf{A}u\|_k \geq C\|u\|_{k+1} - D_k\|u\|_k.$$

- d) *There are constants C' and D'_k , for $k \in \mathbb{N}(d)$ with $D'_d = 0$, such that the projection $p : \Gamma(E_2) \rightarrow \text{Im}\mathbf{A}$ with respect to the decomposition in b) satisfies, for all $v \in \Gamma(E_2)$,*

$$\|pv\|_k \leq C'\|v\|_k + D'_k\|v\|_{k-1}.$$

Proof. The proof follow from considerations in [24], VII.5 (note the switch of \mathbf{A} and \mathbf{B} in our notation). First-order linear operators are linear ILH normal maps, as follows from [24], V, Theorem 3.1, the fact that the 1-jet map is an ILH linear normal operator, and the stability of ILH linear normal operators by composition. In particular, \mathbf{A} and \mathbf{A}^* are linear normal. From [24], VII.5, we also have linear estimates for \mathbb{G} : for all $v \in \Gamma(E_1)$,

$$\|\mathbb{G}v\|_{k+2} \leq c\|v\|_k + d_k\|v\|_{k-1}.$$

Let \mathbf{A}_k^* and \mathbf{B}_k^* be the extensions of \mathbf{A}^* and \mathbf{B}^* to the Hilbert completions $\Gamma(E_i)^k$, $i \in \{1, 2\}$. The extensions of Δ and \mathbb{G} are denoted by Δ_k and \mathbb{G}_k . Then, for $k \in \mathbb{N}(d)$, we have $\Gamma(E_1)^k = \ker \mathbf{A} \oplus \text{Im}\mathbf{A}_{k+1}^*$. Taking the inverse limit, we obtain the decomposition $\Gamma(E_1) = \ker \mathbf{A} \oplus \text{Im}\mathbf{A}^*$. From $\mathbf{A}^*\mathbf{A}\mathbb{G} = 1$ on $\text{Im}\mathbf{A}^*$, the

fact that \mathbf{A} and \mathbf{A}^* are linear ILH normal and linear estimates for \mathbb{G} , we deduce the estimate

$$\|v\|_k \leq \|\mathbf{A}^* \mathbf{A} \mathbb{G} v\|_k \leq c \|v\|_k + d_k \|v\|_{k-1}$$

for all $v \in \text{Im}(\mathbf{A}^*)$ and for a uniform constant c . Using the fact that $\text{Im} \mathbf{A}_{k+1}^* = \text{Im} \mathbf{A}_{k+1}^* \mathbf{A} \mathbb{G}_k$, we see that $\text{Im} \mathbf{A}^*$ is closed and a) follows.

For b), the decomposition $w = \mathbf{A} \mathbb{G} \mathbf{A}^* w + (1 - \mathbf{A} \mathbb{G} \mathbf{A}^*) w$, for $w \in \Gamma(E_2)$, gives the decomposition $\Gamma(E_2) = \text{Im} \mathbf{A} \oplus \ker \mathbf{A}^*$. To show that $\text{Im} \mathbf{A}$ is closed, we will use the estimate

$$\|w\|_k \leq \|\mathbf{A} \mathbb{G} \mathbf{A}^* w\|_k \leq c \|w\|_k + d_k \|w\|_{k-1}$$

on $\text{Im} \mathbf{A}$, where the first inequality follows from $\mathbf{A} \mathbb{G} \mathbf{A}^* = 1$ on $\text{Im} \mathbf{A}$, and the second one from Lemma A.1 and the fact that \mathbf{A} is an ILH linear normal map.

For c), we use $\mathbb{G} \mathbf{A}^* \mathbf{A} = 1$ on $\text{Im} \mathbf{A}^*$, together with Lemma A.1, to obtain

$$\|u\|_{k+1} = \|\mathbb{G} \mathbf{A}^* \mathbf{A} u\|_{k+1} \leq C \|u\|_k + D_k \|u\|_{k-1}.$$

As \mathbf{A} is linear normal, the estimate in c) follows.

Finally, d) follows from $p = \mathbf{A} \mathbb{G} \mathbf{A}^*$, the fact that \mathbf{A} is linear normal, and Lemma A.1. \square

A.2. On direct sum decompositions of ILH chains

Let \mathbf{A} be a linear normal operator between ILH chains. We assume that the restriction of \mathbf{A} to a closed subspace of its domain satisfies the hypotheses of Theorem 2.11. We consider in this section a special setting in which the hypotheses of Frobenius' theorem (Theorem 2.11) extend to \mathbf{A} .

Let $\{E, E^k \mid k \in \mathbb{N}\}$ and $\{F, F^k, k \in \mathbb{N}\}$ be two ILH chains. We assume in this section that, for each k and all $x \in E$,

$$(A.2) \quad \|x\|_0 \leq \|x\|_k$$

Since in this paper we only consider *graded Fréchet spaces*, satisfying $\|\cdot\|_k \leq \|\cdot\|_{k+1}$, equation (A.2) is always satisfied. Let

$$\mathbf{A} : E \rightarrow F$$

be a linear continuous ILH operator. We assume that

$$E = E_0 \oplus \mathcal{H}$$

where \mathcal{H} and E_0 are closed ILH subspaces of E with \mathcal{H} finite dimensional. Denote by \mathbf{A}_0 the restriction of \mathbf{A} to E_0 . Assume that we have direct sum decompositions into closed subspaces in the ILH category:

$$E_0 = \ker \mathbf{A}_0 \oplus E_2, \quad F = \text{Im} \mathbf{A}_0 \oplus F_2.$$

We assume, moreover, that $\ker \mathbf{A}$ is finite dimensional. We then have the following.

Lemma A.3. *There exists a subspace $\mathcal{H}' \subset \mathcal{H}$ such that the following decomposition into closed subspaces holds in the ILH category:*

$$E = \ker \mathbf{A} \oplus E_2 \oplus \mathcal{H}'.$$

In particular, $E_2 \oplus \mathcal{H}'$ is a closed complementary subspace of $\ker \mathbf{A}$ in E .

Proof. By hypothesis, we have a decomposition into closed ILH subspaces:

$$E = \ker \mathbf{A}_0 \oplus E_2 \oplus \mathcal{H}.$$

As $\ker \mathbf{A}$ is finite dimensional, we can complete a basis of $\ker \mathbf{A}_0$ into a basis of $\ker \mathbf{A}$ with elements $(f_i)_{1 \leq i \leq r}$ of $E_2 \oplus \mathcal{H}$. Write these elements as $f_i = e_i + h_i$ in the decomposition $E_2 \oplus \mathcal{H}$ for $1 \leq i \leq r$. We must have $r \leq \dim \mathcal{H}$, as, otherwise, there would be a non-trivial linear combination $\sum_i \lambda_i h_i = 0$. In that case, $\sum_i \lambda_i f_i = \sum_i \lambda_i e_i \in \ker \mathbf{A} \cap E_2$ and by the decomposition of E_0 , we would have $\sum_i \lambda_i f_i = 0$, which is a contradiction. We can then set \mathcal{H}' to be a complementary subspace of the span of $\{h_i\}_{1 \leq i \leq r}$ in \mathcal{H} . By construction, $E = \ker \mathbf{A} \oplus E_2 \oplus \mathcal{H}'$ and the proof follows. \square

Assume now that \mathbf{A}_0 satisfies the following linear estimate: there are constants C and D_k , for $k \in \mathbb{N}$ with $D_0 = 0$, such that, for $k \in \mathbb{N}$ and $x \in E_2$,

$$(A.3) \quad \|x\|_k \leq C \|\mathbf{A}_0 x\|_k + D_k \|x\|_{k-1}.$$

Lemma A.4. *With the previous notation, assume that, for each $k \in \mathbb{N}$ and all $h \in \mathcal{H}$,*

$$(A.4) \quad \|\mathbf{A}h\|_k = \|h\|_k.$$

Assume, moreover, that there is a constant C such that for any element $x \in E$ written as $x = x_0 + h$ in the decomposition $E_0 \oplus \mathcal{H}$,

$$(A.5) \quad \|h\|_0 \leq C \|x\|_0.$$

Then \mathbf{A} satisfies the following linear estimates: there are constants C' and D'_k , for $k \in \mathbb{N}$ and $D'_0 = 0$, such that, for $k \in \mathbb{N}$ and $x \in E_2 \oplus \mathcal{H}'$,

$$\|x\|_k \leq C' \|\mathbf{A}x\|_k + D'_k \|x\|_{k-1}.$$

Proof. Note that \mathcal{H} is finite dimensional, so there are constants D_k such that for all $h \in \mathcal{H}$,

$$(A.6) \quad \|h\|_k \leq D_k \|h\|_0.$$

Let $x = x_0 + h \in E_2 \oplus \mathcal{H}'$. For the sake of simplicity, we denote by C and D_k positive constants that may vary along the estimates, but always with C independent of k . For each $k \in \mathbb{N}$,

$$\begin{aligned} \|x\|_k &\leq \|x_0\|_k + \|h\|_k \\ &\leq C \|\mathbf{A}x_0\|_k + D_k \|x_0\|_{k-1} + \|h\|_k && \text{by (A.3)} \\ &\leq C \|\mathbf{A}x - \mathbf{A}h\|_k + D_k \|x - h\|_{k-1} + \|h\|_k \\ &\leq C \|\mathbf{A}x\|_k + C \|\mathbf{A}h\|_k + D_k \|x\|_{k-1} + D_k \|h\|_0 && \text{by (A.6)} \\ &\leq C \|\mathbf{A}x\|_k + D_k \|x\|_{k-1} + D_k \|h\|_0 && \text{by (A.4) and (A.6)} \\ &\leq C \|\mathbf{A}x\|_k + D_k \|x\|_{k-1} && \text{by (A.5) and (A.2)}. \quad \square \end{aligned}$$

Recall the decomposition

$$F = \text{Im}\mathbf{A}_0 \oplus F_2.$$

We assume that this decomposition is orthogonal with respect to the inner product in F^0 (this is the case in the applications we consider in this paper, as the decompositions are of the form $F = \text{Im}\mathbf{A} \oplus \ker \mathbf{A}^*$ where the adjoint is computed with respect to an $L^{2,0}$ inner product).

Lemma A.5. *There exist closed subspaces \mathcal{F} and F_3 of F_2 , with \mathcal{F} finite dimensional, such that the following decompositions hold in the ILH category:*

$$F_2 = \mathcal{F} \oplus F_3, \quad \text{Im}\mathbf{A} = \text{Im}\mathbf{A}_0 \oplus \mathcal{F}.$$

In particular, the following is a decomposition into closed subspaces:

$$F = \text{Im}\mathbf{A} \oplus F_3.$$

More precisely, $F_3 = \ker(p_0 : F_2^0 \rightarrow \mathcal{F}) \cap F_2$, where p_0 denotes the orthogonal projection.

Proof. Recall that

$$E = E_0 \oplus \mathcal{H},$$

with \mathcal{H} finite dimensional. For (h_i) a basis of \mathcal{H} , decompose $\mathbf{A}h_i = \mathbf{A}_0 y_i + f_i$ in the direct sum $\text{Im}\mathbf{A}_0 \oplus F_2$. Set \mathcal{F} to be the span of the elements f_i . Then \mathcal{F} is a finite dimensional subspace of $F_2 \cap \text{Im}\mathbf{A}$. Then F_2^0 admits the orthogonal decomposition

$$F_2^0 = \mathcal{F} \oplus \ker(p_0 : F_2^0 \rightarrow \mathcal{F}),$$

where p_0 denotes the orthogonal projection onto \mathcal{F} . It is routine to verify that $F_3 = \ker(p_0 : F_2^0 \rightarrow \mathcal{F}) \cap F_2$ satisfies the required properties. \square

We conclude with linear estimates for the projection onto $\text{Im}\mathbf{A}$. Assume that there are constants C and D_k , for $k \in \mathbb{N}$ and $D_0 = 0$, such that the projection $\pi_0 : F \rightarrow \text{Im}\mathbf{A}_0$ with respect to the decomposition $\text{Im}\mathbf{A}_0 \oplus F_2$ satisfies, for $k \in \mathbb{N}$ and $x \in F$,

$$(A.7) \quad \|\pi_0 x\|_k \leq C \|x\|_k + D_k \|x\|_{k-1}.$$

We then have the following lemma.

Lemma A.6. *Denote by $\pi : F \rightarrow \text{Im}\mathbf{A}$ the projection with respect to the decomposition $F = \text{Im}\mathbf{A} \oplus F_3$. Then there exists C' and D'_k , for $k \in \mathbb{N}$ with $D'_0 = 0$, such that, for $k \in \mathbb{N}$ and $x \in F$,*

$$\|\pi x\|_k \leq C' \|x\|_k + D'_k \|x\|_{k-1}.$$

Proof. Let $x \in F$. From Lemma A.5, we have $\text{Im}\mathbf{A} = \text{Im}\mathbf{A}_0 \oplus \mathcal{F}$ and $F = \text{Im}\mathbf{A}_0 \oplus \mathcal{F} \oplus F_3$. As this decomposition is orthogonal with respect to the inner product on F^0 , we have $\pi x = \pi_0 x + p_0 x$ where p_0 denotes the orthogonal projection from F^0 to \mathcal{H} .

Thus,

$$\|\pi x\|_k \leq \|\pi_0 x\|_k + \|p_0 x\|_k.$$

But \mathcal{H} is finite dimensional, so, if (e_i) denotes an orthonormal basis of \mathcal{H} with respect to $\|\cdot\|_0$,

$$\|p_0 x\|_k \leq D_k \|x\|_0$$

with $D_k = \sum_i \|e_i\|_k$. Using hypothesis (A.7), the result follows. \square

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