



Automorphy of m -fold tensor products of $\mathrm{GL}(2)$

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Abstract. We prove that for any $m > 1$, given any m -tuple of Hecke eigenforms f_i of level 1 whose weights satisfy the usual regularity condition, there is a self-dual cuspidal automorphic form π of $\mathrm{GL}_{2m}(\mathbb{Q})$ corresponding to their tensor product, i.e., such that the system of Galois representations attached to π agrees with the tensor product of the ones attached to the cuspforms f_i .

– Eh bien, Axel, me dit mon oncle, cela va, et le plus difficile est fait.

– Comment, le plus difficile? m'écriai-je.

– Sans doute, nous n'avons plus qu'à descendre!

– Si vous le prenez ainsi, vous avez raison; mais enfin, après avoir descendu, il faudra remonter, j'imagine?

– Oh! cela ne m'inquiète guère!

“Voyage au centre de la terre”, Jules Verne

1. Introduction

This paper should be considered as a sequel to [10] since not only we follow the strategy developed in [9] and [10] of propagating automorphy through suitable “safe chains” but also the skeleton of this paper is the safe chain of 11 steps constructed in [10]. Since we are going to reuse that chain (with just one minor modification), the reader is warmly recommended to read its construction in section 3 of [10], (and also in the Corrigenda to [10]), before reading this paper.

Our goal is to prove automorphy of m -fold tensor products of Galois representations attached to level 1 cuspforms, assuming that such tensor product representations are regular (i.e., that they have 2^m different Hodge–Tate numbers). We are going to use the 11 steps long chain from our previous paper (plus 4 extra moves at the end) in the first component (or factor) of the m -fold tensor product in order to safely connect it with a CM form of prime power level. At this point, automorphy of the tensor product will be reduced to a similar automorphy statement for

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the $(m - 1)$ -fold tensor product involving all but the first component. Thus, the result will be proved by induction on m . This induction can of course start at the trivial case $m = 1$, or it can start at $m = 2$ since for the case of two newforms automorphy of the tensor product was proved by D. Ramakrishnan (see [20]).

There are, however, several technical problems that do not allow us to do this in such a simple and straightforward way. Assuming that in the process of moving the first component through the safe chain built in our previous paper (while leaving the other components unchanged) the residual images in the m -fold tensor product representations involved in each congruence are sufficiently large for some *automorphy lifting theorem* (ALT) to apply; which is something relatively easy to guarantee by introducing good-dihedral primes in all components, using information proved in [10] about the residual images in the first component and applying some results about adequacy of tensor products taken from [17]; there still remain two main issues tricky enough to keep us busy for a while:

- the m -fold tensor products appearing in each congruence will not necessarily be regular,
- even if they were regular, the local conditions required to apply some available ALT would fail to hold oftentimes, for example, the chain built in [10] requires working with small primes p , such as 7 or 11, and for large m and small p , assuming that the m representations in the tensor product are crystalline at p , it is impossible to have 2^m different Hodge–Tate weights and to be in a Fontaine–Laffaille situation at p , thus a priori it is by no means clear that any ALT can be applied.

In order to make the argument work in spite of the two serious difficulties just described, we are going to develop two different kind of tools: first, we are going to prove two variants of the ALT in [2] and in [11]. These will be ALT for the special case in which the two Galois representations involved are tensor decomposable, one of them deals with the “mixed case” of tensor products where one factor is ordinary (not necessarily potentially crystalline) and the others are potentially diagonalizable, and the other is just an straightforward application of Harris’ trick to the case where one is just moving the weight of one component (assuming potential diagonalizability in this component), leaving the others unchanged (this kind of situation will appear very often in the proof). Both theorems will be proved by following the arguments in section 4 of [2].

Secondly, as the reader may have guessed at this point, we are going to be forced to make some non-trivial changes in the other components (some “genetic manipulations”), from the second to the last, of the tensor product.

In an analogy with Rubik’s cube, you can think of this as if our goal was to complete one face of the cube, yet in order to do so you are forced to make some moves involving exclusively the other faces.

The congruences that we are going to apply on these other components are thus not directed to link them to any sort of base case, they are just meant to make all local conditions (plus regularity) to hold in the m -fold tensor products while moving the first component through the safe chain built in our previous paper. These moves on the other components will thus be interspersed with the moves on

the first component, whenever required. Needless to say, in any of these auxiliary congruences we must also make sure that some ALT applies allowing automorphy to propagate.

The key tool that we will use in this process, in order to manipulate the weights and behavior at p of these other components, is a result from [1] on existence of potentially diagonalizable crystalline lifts of arbitrary large weight for mod p modular Galois representations.

This is a good point to mention that we will introduce two notions on the weights of the modular forms appearing in the tensor products, those of “spread” and “ C -very spread” (where C is a positive integer). The latter being stronger than the former, both notions imply regularity of the tensor product. We will use the tool taken from [1] also to, at the very beginning, manipulate the weights in order to reduce them to a situation where they are “ C -very spread” for a suitable value of C . This will be key to guarantee regularity during the rest of the proof, in fact after this step the weights will always be “spread”.

To summarize, our main task is to prove two useful variants of ALT, and to explain the tricks in the just described manipulations that are needed to make things work. There are other issues, such as largeness and adequacy of residual images; invariance of automorphy under twisting and Galois conjugation; making sure that induction on m work by, after having reduced the problem to an $(m - 1)$ -fold tensor product, rewinding the process to observe that such a tensor product is safely connected to the initial one (with the first component dropped) whose components come from $m - 1$ level 1 modular forms; etc. These and other issues will be addressed on the go, we believe that none of them involves any serious difficulty, they just need to be taken care of.

Let us record the main result of this paper.

Theorem 1.1. *Let f_1, f_2, \dots, f_m be an m -tuple of level 1 cuspforms, whose weights we will denote k_1, \dots, k_m . We make the standard regularity assumption that the m integers $u_i = k_i - 1$ not only are all different but also the 2^m partial sums that one can form with them are all different (we set 0 as the sum of the elements in the empty set).*

Under this assumption, the m -fold tensor product of the ℓ -adic representations attached to the f_i is automorphic, and the m -fold product L -function for the f_i has analytic continuation to the whole complex plane.

Remarks: we are not going to insist in this paper on “corollaries” of this result (combined with the base change results from [9] and [10]), we leave it as an exercise to show that for any totally real number field F unramified at 3 one can lift this m -fold tensor product to an automorphic representation on $GL_{2^m}(F)$. Moreover, changing the CM cuspform to be used as base case, one may relax the assumption on the field F .

Also, as happened in previous papers, conditional to some slight strengthening of the available ALT, the theorem above (and its base changed version) should also hold for newforms of higher level, at least if the levels are odd (one can try to use weight reduction to prove this unconditionally, but as far as we know, not being

allowed to work in characteristic 2, we can reduce the weight up to $k = 4$, while it would be required to reduce it up to $k = 2$).

Finally, for odd p , one can easily deduce that irreducible m -fold tensor products of 2-dimensional odd mod p representations ramifying only at p are automorphic (in the sense that they admit an automorphic p -adic lift) by combining the above theorem with Serre's conjecture. Incidentally, observe that in this mod p situation, the number of components m must be bounded as a function of p in order to have irreducibility for the tensor product (because there is only a finite number of level 1 mod p modular Galois representations).

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2. In which two variants of the ALT in [2] are proved

Let us begin with the most difficult one, which is an ALT for a “mixed” case of an ordinary representation tensored with the tensor product of some potentially diagonalizable ones. The proof does not have any original idea in it, we are just going to follow closely the proofs of Theorems 4.1.1 and 4.2.1 in [2] but with the particular feature that all the important things, i.e., the construction of a congruence “against” an ordinary lift of large weights and the application of Harris' trick to make weights (and irreducible components at ℓ) match, will be applied just to the potentially diagonalizable components, and by just plugging in the ordinary component in the two sides of the congruence we get what we want. Since we need our result for “small ℓ ” we have to work as in [11] in a way that avoids the condition $\ell \geq 2(n + 1)$, where n is the dimension of the tensor product. Since it is good enough for our applications, we will work for simplicity under the condition $\ell \geq 2(n_i + 1)$ for all i , where the n_i denote the dimensions of the potentially diagonalizable components, but see the remark after the proof of the theorem on how to relax this condition. The theorem that we can prove, in a fairly general version good enough for our purposes, is the following (as usual, ALT are stated over imaginary CM fields, but a standard quadratic base change argument provides a version of them over totally real fields).

Given any ℓ -adic representation x , we denote by \bar{x} the semi-simplification of the corresponding residual representation.

Theorem 2.1. *Let F be an imaginary CM field with maximal real subfield F^+ . Let ℓ be an odd prime, and let*

$$(R, \mu_0), (S_i, \mu_i), i = 1, \dots, t$$

be $t + 1$ regular algebraic, irreducible, n_i -dimensional (with $n_0 = \dim(R)$) polarized ℓ -adic representations of G_F , with

$$R \otimes \bigotimes_i S_i$$

regular and irreducible.

Suppose that we are also given another $(t + 1)$ -tuple of ℓ -adic representations of G_F with the same dimensions n_i , all them regular algebraic, irreducible and polarized; let us call them

$$(R', \mu'_0), (S'_i, \mu'_i), \quad i = 1, \dots, t.$$

On the level of residual representations, we suppose that (R, μ_0) and (R', μ'_0) are congruent mod ℓ , and similarly, for every i , we assume that (S_i, μ_i) and (S'_i, μ'_i) are congruent mod ℓ .

We also assume that the tensor product

$$R' \otimes \bigotimes_i S'_i$$

is automorphic, regular algebraic, irreducible and polarized. We make the following technical assumptions:

- (1) For every place v of F dividing ℓ , $R|_{G_{F_v}}$ and $R'|_{G_{F_v}}$ are ordinary and $S_i|_{G_{F_v}}$ and $S'_i|_{G_{F_v}}$ are potentially diagonalizable, for every i .
- (2) $\overline{R \otimes \bigotimes_i S_i}(G_{F(\zeta_\ell)})$ is adequate, and $\zeta_\ell \notin F$.
- (3) For every $i > 0$, $\ell \geq 2(n_i + 1)$.

Then,

$$R \otimes \bigotimes_i S_i$$

is automorphic.

Proof. The proof consists on applying the same procedure used in Theorems 4.1.1 and 4.2.1 in [2], ignoring for the most part the ordinary component R (or R'), just plugging in R (R' , respectively) after Harris' trick having been applied to the remaining components: this works fine because the "connectedness" relation from [2] (at any place v) behaves well under tensor products, and a representation is clearly connected to itself. Finally, for the use of Geraghty's theorem, tensoring with an ordinary representation of sufficiently spread weights preserves ordinarity, as we will see later in this proof, thus we will be comparing ordinary representations at the key step of changing weights and types, as required. We limit ourselves to mention the few differences, but we are not going to reproduce here the application of Harris' trick since it is to be applied exactly as in Theorem 4.1.1 of op. cit.

It is important to remark that condition (2) implies that

$$(2.1) \quad \text{for every } i > 0, \quad \overline{S_i}((G_{F(\zeta_\ell)})) \text{ is irreducible.}$$

Construction of the ordinary lifts: as in Theorem 4.2.1 of op. cit., this has to be applied twice, one of each side, so to ease notation let us do it for the S_i (a similar procedure applies to the S'_i). As a general rule, when we go to a solvable extension, even if the construction only involves the S_i , we are going to put the extra condition that such an extension is linearly disjoint from the fixed field of $\ker \overline{R}$, so that property (2) is preserved on such an extension. With just this extra

restriction in mind, here we proceed as in op. cit.: we take a solvable extension F' of F where locally at all places above ℓ all the \bar{S}_i have trivial image, and there by Theorem 3.2.1 of op. cit. (which can be applied thanks to condition (3) above and the irreducibility condition (2.1)) we can construct a crystalline (at each place above ℓ) ordinary lift of \bar{S}_i of arbitrary regular Hodge–Tate weights, such that it is connected to S_i locally at every place v not dividing ℓ . If we call W_i these ordinary lifts, in order to have regularity later when applying Harris’ trick we need the following to hold:

$$(2.2) \quad R \otimes \bigotimes_i S_i \otimes \bigotimes_i W_i$$

is regular. Let us show how to pick the weights of each W_i in order that this condition is satisfied, using an iterated version of what is done in op. cit., Theorem 4.2.1. For the ordinary lift W_1 , we pick an integer t_1 such that it is bigger than $|h - h'|$ for any pair h, h' of different Hodge–Tate weights of $R \otimes \bigotimes_i S_i$, and we choose as weights: $\{0, t_1, \dots, (n_1 - 1)t_1\}$.

Having done this for all W_i for $i < j \leq t$, for the ordinary lift W_j we pick an integer t_j such that it is bigger than $|h - h'|$ for any pair h, h' of different Hodge–Tate weights of

$$R \otimes \bigotimes_i S_i \otimes \bigotimes_{i < j} W_i$$

and we choose as weights: $\{0, t_j, \dots, (n_j - 1)t_j\}$

This concludes the construction of the ordinary lift of the tensor product of the S_i , which is the tensor product of the W_i . Moreover, and this will be key later in order to apply Geraghty’s theorem, $R \otimes \bigotimes_i W_i$ is ordinary.

At the referee’s request, let us expand on this: tensoring two ordinary representations, even when we know that the tensor product is regular, is not enough to guarantee that the tensor product is ordinary. The condition that may fail is that the Hodge–Tate weights shall be increasing as we move through the graded pieces corresponding to the filtration by Galois invariant subspaces (see for example [2] for the precise definition of ordinariness). But an easy computation shows inductively that in the tensor product under consideration the Hodge–Tate weights of each W_i have been chosen to be sufficiently spread so that not only regularity but also ordinariness of the tensor product holds. In fact, the easier way to perform this computation is by considering the tensor products in the opposite sense: $W_1 \otimes R$, $W_2 \otimes W_1 \otimes R$, etc. The required condition on the increasing of the Hodge–Tate weights along the graded pieces follows then automatically from the inequalities imposed on the t_i (and, of course, from the assumption that each representation in the tensor product is itself ordinary). For example, $W_1 \otimes R$ is ordinary because the difference of two consecutive weights of W_1 is t_1 , and by construction t_1 is larger than the difference between the largest and the smallest weight of R . This proves that the tensor product is ordinary, by just observing that in an irreducible tensor product permuting the order of the factors gives always the same representation, up to isomorphism (alternatively, one can prove first the ALT with R and R' as last factors on both sides, and again because permuting the order of the factors in

a tensor product gives isomorphic representations the result follows as stated since automorphy of a Galois representation is preserved by an isomorphism).

Here, as in Theorem 4.1.1 of *op. cit.*, Harris' trick is applied to “force” $\bigotimes_i S_i$ and $\bigotimes_i W_i$ to connect also locally at places dividing ℓ by tensoring with certain well-constructed monomial representations, both with the same residual image. Observe that this makes sense because we have assumption (1) implying that $\bigotimes_i S_i$ is potentially diagonalizable. At this point, we tensor by R both sides, something that preserves connectedness at every finite place. Regularity is ensured in this process (equivalently, (2.2) is regular) thanks to our choice of the weights of the ordinary lifts W_i . As for the images: we stress once again that in all auxiliary fields, including the field M where the characters giving rise to the monomial representations in Harris' trick are defined, we have to incorporate the extra condition that they should be linearly disjoint from the fixed field of the kernel of \bar{R} . This being said, we have from assumption (2) that the residual images of the tensor product of the R and the S_i 's will be adequate (even when restricted to the cyclotomic extension) and by applying Lemma A.3.1 in [1] we see that tensoring with a well-suited monomial representation (i.e., one satisfying the linear disjointness condition just discussed) will preserve this adequacy of the residual images.

After playing Harris' trick, we obtain two tensor product representations and as in section 4 of [2], we would like to apply Thorne's ALT (Theorem 2.3.1 in *op. cit.*) to conclude this step.

From what we have done we see that all conditions needed to apply Thorne's theorem are satisfied, except for the fact that we may be dealing with representations that are not potentially crystalline, since R may not be so.

Luckily, as observed by Calegari in [6], Thorne's theorem easily extends to the case of potentially semistable representations. When written, this observation in [6] required two technical conditions (smoothness of the point in the local deformation ring at places above ℓ , and Shin-regularity of the weights), but since then local-to-global compatibility at $\ell = p$ has been fully established by Caraiani in [7], and it is easy to see that as a consequence the variant of Calegari to Thorne's result holds in general (the smoothness condition is always satisfied as follows from the proof of Lemma 1.3.2 of [2], see [6], Remark 2.8; and the Shin-regularity, only needed in [6] to use a geometric model of the Galois representations attached to automorphic forms, which in turn is just used to deduce finiteness of the number of possible types at p that show up during the Taylor–Wiles process, is not necessary now since it follows clearly from local-to-global compatibility that by fixing the level at p the types locally at p of the p -adic Galois representations that occur are finite in number, a finite set of types independent of the auxiliary primes introduced to the level of the automorphic forms appearing in the Taylor–Wiles construction).

This being said, we apply this variant of Thorne's theorem and the outcome is that the automorphy of $R \otimes \bigotimes_i S_i$ follows from that of $R \otimes \bigotimes_i W_i$. Here we are hiding the core idea of Harris' trick, namely, that automorphy behaves well, in both directions, under tensoring by a monomial representation (see [2] for details).

The rest of the argument is verbatim as in Theorem 4.2.1 of [2]: all that remains is a second application of the above procedure, on the other side, and

then an application of Geraghty's theorem ensuring that automorphy propagates well among ordinary representations, which we can apply due to assumption (2), and the care we have had in choosing the solvable extensions. This concludes the proof. \square

Remark. Condition (3) can be replaced by the condition:

(3') For every i , there is a finite extension F'_i of F such that after restriction to $G_{F'_i}$ the residual representation \bar{S}_i is potentially diagonalizably automorphic and $\bar{S}_i(G_{F'_i(\zeta_\ell)})$ is adequate.

In fact, using Theorem A.4.1 in [1] instead of Theorem 3.2.1 in [2], the proof follows with the same arguments.

Condition (3') is strictly weaker than (3) when combined with the other conditions in the theorem: assuming (3), since we also have condition (2) implying that $\bar{S}_i(G_{F(\zeta_\ell)})$ is irreducible for every $i > 0$, we can apply Corollary 4.5.3 of [2] together with the main result in [17] to deduce that (3') also holds.

The second ALT that we propose is a variant of Theorem 4.1.1 in [2] with the addendum that tensor product by a common representation (or by two representations that are connected locally everywhere, one on each side) has been taken. The statement is the following.

Theorem 2.2. *Let F be an imaginary CM field with maximal real subfield F^+ .*

Let ℓ be an odd prime, and let

$$(R, \mu_0), (S, \mu_1)$$

be two regular algebraic, irreducible, polarized ℓ -adic representations of G_F , with $R \otimes S$ regular and irreducible.

Suppose that we are also given another pair of ℓ -adic representations of G_F ,

$$(R', \mu'_0), (S', \mu'_1),$$

of the same dimension than R and S , respectively, both of them regular algebraic, irreducible and polarized, and such that $R' \otimes S'$ is regular and irreducible,

On the level of residual representations, we suppose that (R, μ_0) and (R', μ'_0) are congruent mod ℓ , and that (S, μ_1) and (S', μ'_1) are congruent mod ℓ .

We also assume that the tensor product

$$R' \otimes S'$$

is automorphic. We make the following technical assumptions:

- (1) *For every place v of F dividing ℓ , $R|_{G_{F_v}}$ and $R'|_{G_{F_v}}$ are potentially diagonalizable, and for every v not dividing ℓ , $R|_{G_{F_v}} \sim R'|_{G_{F_v}}$.*
- (2) *S and S' , locally at any finite place v , are connected.*
- (3) *$\overline{R \otimes S}(G_{F(\zeta)})$ is adequate.*
- (4) *$R \otimes R' \otimes S$ is regular.*

Then, $R \otimes S$ is automorphic.

The proof is a straightforward adaptation of the proof of Theorem 4.1.1 in [2], and the modifications required are included in what we explained during the proof of the previous theorem. For the reader convenience, let us repeat the relevant arguments: we proceed as in Theorem 4.1.1 of op. cit., applying Harris' trick to R and R' . Through the construction of a suitable pair of monomial representations, over a suitable extension, and tensoring with them R and, respectively, R' , we manage to obtain a congruence between representations that now connect locally at every place. This is possible thanks to assumption (1). If we now tensor one side by S and the other by S' , due to assumption (2), we still obtain a congruence between two representations that connect locally at every place. At this point, we want to apply Thorne's theorem (2.3.1 in op. cit.), or maybe Calegari's variant to it (in case S is not potentially crystalline at some v dividing ℓ) recalled during the proof of the previous theorem, to these two triple tensor products. So it remains to check that the conditions required to apply this ALT hold. As in the course of the previous proof, one should take the extra care during the applications of Harris' trick to choose all solvable extensions and auxiliary number fields to be linearly disjoint from the fixed field of the kernel of \bar{S} . In particular, if the image of \bar{S} is monomial, we make sure that the monomial representations constructed in Harris' trick are induced from characters of a field that is linearly disjoint from the one from which \bar{S} is induced. This being said, condition (3) together with Lemma A.3.1 in [1] guarantee the required adequacy condition on residual images. Regularity is ensured by condition (4), which is the natural generalization of a similar condition in Theorem 4.1.1 of [2]. This implies that the ALT of Thorne can be applied and we conclude as in Harris' trick that $R \otimes S$ is automorphic because automorphy behaves well under tensor products with monomial representations.

Remark. As in [11], we have applied results from [1] to be able to prove a theorem valid also for “small” primes.

3. In which some important features of the safe chain from [10] are recalled, and a little extra effort is done to end the chain in a CM form

We begin by reproducing the description taken from section 2 of [10] (modified according to the Corrigenda to that paper) of the 11 steps long safe chain. In all the congruences appearing in the following 11 steps residual images are “large”, as defined in the Introduction to op. cit. As proved in [16] this implies that these images are adequate, even after restriction to any solvable extension. Recall also that all these congruences are in characteristic larger than 5.

We start with a cuspform f of level 1 and weight $k \geq 12$:

- (1) Introduce a good-dihedral prime (as in [18]): level raises to q^2 .
- (2) Weight reduction via Galois conjugation: weight reduces to $2 < k \leq 14$ (k even).

- (3) *Ad hoc* tricks to make the weight equal to 16 (Sophie Germain primes, Khare’s weight reduction, etc): end up with a newform of weight 16 and level q^2 (see Corrigenda to [10]).
- (4) Introduce MGD prime 43: end up with newform of weight 2 and level $43^2 \cdot q^2$ (see Corrigenda to [10]).
- (5) Remove the good-dihedral prime (in two moves): end up with a newform of weight $q + 1$ and level 43^2 .
- (6) Again weight reduction via Galois conjugation but this time “highly improved”, because we need to ensure large residual image at each step using just the MGD prime. Weight reduces to $2 < k \leq 14$ (k even).
- (7) (Similar to Step 3) *Ad hoc* tricks to make the weight smaller than 17 and divisible by 4: end up with a newform of weight 16 and level 43^2 .
- (8) Introduce nebentypus at 17 of order 8: end up with a newform of weight 2, level $43^2 \cdot 17$, with nebentypus.
- (9) Remove the MGD prime modulo 11 via an *ad hoc* lemma to ensure residual irreducibility: get congruence (maybe using level-raising) with a newform Steinberg at 43, of weight 2 and level $43 \cdot 17$, with nebentypus of order 8 at 17.
- (10) Move from weight 2 to weight 44 by reducing modulo 43: irreducibility checked by hand.
- (11) As predicted by the generalized Maeda conjecture in weight 44, level 17, nebentypus of order 8: check that this space has a unique orbit.

For what we will do later, it is easier to introduce a simplification in Step 1: in [10], before performing the level raising at q , we first make a move to reduce to a weight 2 situation. Here we prefer to do it directly in the given weight: if f is a level 1 newform of weight k , we select a bound B larger than 68 and larger than $2k$ (these two inequalities appear also in Steps 1 and 2 of op. cit., they are required at some steps), and by taking a prime t larger than B and congruent to 1 mod 4 such that the image of the mod t Galois representation attached to f is $\mathrm{GL}(2, \mathbb{F}_t)$ one can through level raising add a good-dihedral prime q (good-dihedral with respect to the bound B , cf. [10], page 626), a supercuspidal prime in the level, satisfying $t \mid q + 1$, $q \equiv 1 \pmod{8}$, and $q \equiv 1 \pmod{p}$ for every $p \leq B$. The reader should consult [12] where introduction of good-dihedral primes (called there “tamely dihedral”) for newforms of weight $k > 2$ is discussed in full detail. Ramification at q of the level-raised Galois representation is given by a character of order t of the quadratic unramified extension of \mathbb{Q}_q . The key point of a good-dihedral prime is that as long as characteristics t and q are avoided, the residual images will be absolutely irreducible, and as long as one works in characteristic $p \leq B$, or any characteristic p such that q is a square mod p (which, due to quadratic reciprocity, is equivalent to require that p is a square mod q) the residual images will be large.

In fact, if we rename now f to f_1 and k, B, t, q to k_1, B_1, t_1, q_1 , where f_1 is the newform of level 1 of smallest weight k_1 in a regular (in the sense used in

Theorem 1.1) m -tuple of newforms f_i of weights k_i (we suppose for simplicity that the sequence k_i is increasing), the very first move that we will do to prove Theorem 1.1 will be this level raising in the first component f_1 modulo t_1 . Thus, it is very convenient to impose the following extra conditions: when choosing B_1 , take it also to be larger than all k_i , for all i , thus since $t_1 > B_1$ when reducing modulo t_1 the m residual representations will all be Fontaine–Laffaille, thus potentially diagonalizable, at t_1 . Also, take t_1 so that the m components have all large, therefore adequate, residual images (this is possible due to the large images theorem of Ribet, see [21]). It is worth stressing that the definition of large (see [10], Introduction) implies that the projectivized residual images will be (non-abelian) almost simple groups. Finally, observe that since we are in a Fontaine–Laffaille situation and all weights are different, it is easy to see by looking at the action of the inertia group at t_1 that the m residual representations are not isomorphic to each other, not even up to twist, and this implies from the results in [17] that the tensor product of the t_i -adic Galois representations attached to the f_i has residually adequate image¹. The same is true if we restrict to any solvable extension, since largeness is preserved by doing such thing. Therefore, we see that we have a congruence modulo t_1 where the good-dihedral prime q_1 is being introduced to the first component, and the other components are left unchanged, such that Theorem 4.2.1 in [2] can be applied to show that automorphy of one side is equivalent to automorphy of the other.

After this simplified version of Step 1, the 11 steps go exactly as in Section 3 of [10]. This is what will happen in the first component f_1 , we have seen in the previous paragraph how Step 1 can be done in such a way that an ALT applies to the m -fold tensor product of the f_i , the rest of this paper will be devoted to either just observe or make suitable manipulations in the other components to make this happen, that we can go along the 11 steps in the first component in such a way that some ALT can be applied at any of the involved congruences between m -fold tensor products.

But before getting into this, it remains to complement the chain (we want it to end in a CM point): as it is it ends, in Step 11 above, in a space of newforms of “small” level and weight with a single orbit of conjugated newforms, we will call this space the “bottom space”. We can try to argue as in the proof of base change in [10] that we can use the exact same strategy to connect an unspecified CM form to this bottom space, but there we were working exclusively in dimension 2, where very powerful ALT of Kisin made the level-reduction possible. To remedy this we will show by direct computations that a suitable CM form can be connected to the bottom space in just four moves, and the local conditions showing up in each of these four congruences will be similar to those already appearing in the 11 steps above, so it is a safe way to complement our chain. We will explain the four

¹Let us elaborate on this: if we take the definition of adequacy from [1], Appendix A, we see trivially that conditions (1) and (2) are satisfied, condition (3) follows from the fact that the components satisfy this condition as proved in [17], Lemma 2.(ii), and finally for condition (4) applying Künneth’s formula one can see again that it is satisfied because it is satisfied by the components (just write the adjoint representation as the tensor product of the representation with its dual). We thank R. Guralnick for providing this proof.

moves, just remember that they have to be applied in reverse order when linking this to our chain, starting from a newform in the bottom space, and ending in the CM form.

The CM form g_1 that we consider has weight 2 and level 27, it corresponds to a CM elliptic curve. We check that to this newform, the prime 43 can be added as a Steinberg prime to the level modulo 13, in fact its eigenvalue satisfies $a_{43} = 8 \equiv \pm 44 \pmod{13}$ (i.e., the condition under which Ribet proved level-raising holds). Moreover, using Magma we do a few computations in the space of newforms of weight 2 and level $43 \cdot 27$ and we find the newform it is congruent to: a non-CM newform g_2 whose field of coefficients \mathbb{Q}_{g_2} has degree 10 having a dihedral mod 13 Galois representation (dihedral but no bad-dihedral, since it is induced from $\mathbb{Q}(\sqrt{-3})$).

The 3-adic Galois representations attached to g_2 are potentially Barsotti–Tate (thus potentially diagonalizable, as proved in [15]) and there is a prime dividing 3 in \mathbb{Q}_{g_2} whose inertial degree is 3, and we compute the corresponding residual representation and see that its image contains $\mathrm{SL}_2(\mathbb{F}_{27})$ (in fact the projective image is $\mathrm{PGL}_2(\mathbb{F}_{27})$) and it is thus large and adequate (see [16]). We check that, for the chosen prime above 3 and residual representation, there are no congruences (not even up to twist) with newforms of level 43 and weight 2, and there is a congruence with one of level 43 and weight 4, which we call g_3 . This newform has a field of coefficients of degree 6.

Since the mod 3 congruence between g_2 and g_3 will be part of our chain, we need to know more about g_3 locally at 3. It is crystalline, because 3 is not in the level of g_3 , but it fails to be Fontaine–Laffaille. Fortunately, since we know (by computations) that there is no congruence with a newform of weight 2 and level 43, this implies that g_3 must be ordinary at 3 (to see this, combine Corollary 4.1.3(1) in [3] with the strong form of Serre’s conjecture). Thus it is ordinary and crystalline, hence potentially diagonalizable.

Now we do the last two moves, which are standard: we reduce mod 17 and take a weight 2 lift, whose nebentypus is ω^2 , of order 8, and then, as in Step 10 above, reduce mod 43 to end up in the bottom space. The last step is thus exactly as Step 10 and irreducibility and largeness of the residual image follow as in that step (see [10]). In the previous step, working modulo 17, with weight 4 and level 43, we have a congruence between a Fontaine–Laffaille representation and a potentially Barsotti–Tate one, and the only problem is that we should check that the residual image is large. But this follows from the results in [4] (see “Theorem for square-free level case”, in the Introduction of that paper) since $17 \nmid 43^4 - 1$ and $17 > 4k - 3 = 13$. This completes the four moves required to go from the bottom space to a CM form, or viceversa.

Definition. The safe chain that we have just recalled, and complemented, starting on the system of Galois representations attached to the level 1 newform f_1 and finishing in the system attached to a CM form of weight 2 and level 27 will be called the *skeleton chain*.

4. In which spread and C -very spread weights are defined, and the proof is reduced to a C -very spread case

Recall that, by assumption, in Theorem 1.1 we have m newforms f_i whose weights k_i are such that the tensor-product Galois representation is regular. In such a case, we say that the m -tuple of weights $0 < k_1 < k_2 < \cdots < k_m$ is *regular*.

We make two more restrictive definitions on an m -tuple of weights:

Definition. Given an m -tuple of weights k_1, k_2, \dots, k_m , we say that it is *spread* if the following inequalities hold:

$$k_2 > k_1, \quad k_3 > k_1 + k_2.$$

And, in general, for every $2 \leq i \leq m$,

$$k_i > k_1 + k_2 + \cdots + k_{i-1}.$$

Definition. Suppose that we have an m -tuple of weights $k_1 < k_2 < \cdots < k_m$ such that the first weight k_1 is a variable (i.e., it is not constant), and it is constrained to move in the interval

$$(4.1) \quad 2 \leq k_1 \leq C$$

for certain constant $C > 2$. Then, we say that the m -tuple is *C -very spread* if, together with the condition (4.1) in the varying first weight, the other weights are constant and satisfy

$$k_2 > 2C, \quad k_3 > 2C + k_2.$$

And, in general, for every $2 \leq i \leq m$,

$$k_i > 2C + k_2 + \cdots + k_{i-1}.$$

Obviously, if k_1 is fixed and satisfies (4.1) for some C , being C -very spread implies being spread. Also, spread implies regular, as it is easy to check.

If we forget for one second about the factor 2 in the $2C$ appearing in the definition of C -spread, the reason for this notion is clear: we will be moving the first weight in a certain range, bounded above by certain C , and we want the set of weights to be spread along the process, therefore regular. If we know the value of C and if we assume that the weights k_2, \dots, k_m are as in the definition of C -spread, then this means that the weights will be spread even in the worst case where $k_1 = C$, thus we win: we have regularity through the whole process.

The reason for the 2 multiplying C is that sometimes we will be moving k_1 (the weight of a newform showing up in the first component of our m -fold tensor product) always taking values bounded by some C , and thus we will have a congruence, in the first component, between two newforms f_1 and f'_1 of different weights k_1 and k'_1 , both bounded by C . Then, as it will happen many many times in our proof, suppose that we want to apply Theorem 2.2 to this congruence of m -fold

tensor products, i.e., with the Galois representations attached to f_1 and f'_1 playing the role of R and R' and with $S = S'$ corresponding to the Galois representation which is the $(m - 1)$ -fold tensor product of the other components. If you assume that all conditions except number (4) of that theorem are satisfied (in particular, this means that we must be in a situation where both f_1 and f'_1 are potentially diagonalizable), still in order to apply it we have to make sure that the “extra-regularity” condition (4) is satisfied, i.e., if we say (just to ease notation) that $k_1 < k'_1$ we want the $(m + 1)$ -tuple of weights:

$$(k_1, k'_1, k_2, k_3, \dots, k_m)$$

to be regular. But $k_1 < k'_1 \leq C$ together with the other inequalities in the definition of C -very spread imply that this $(m + 1)$ -tuple is spread, thus regular, so this extra-regularity condition is satisfied.

This is perhaps a good moment to reveal what will the value of C be in our proof: as some readers may have already guessed, we will take $C = q_1 + 1$, where q_1 is the good-dihedral prime already introduced to the first component. Looking at the skeleton chain from section 3, it is easy to see that as the first component moves along this chain, its weight k_1 satisfies $k_1 \leq q_1 + 1$. From now on, we fix: $C := q_1 + 1$.

But the starting situation is that of a regular m -tuple of weights, and after adding the extra ramification at q_1 to the first component (at this step we have seen, via Theorem 4.2.1 of [2], that automorphy propagates well) we still are in a regular situation. We will apply a trick based on results from [1] to reduce to a C -very spread situation.

We choose a prime p larger than all weights, larger than t_1 and q_1 , and such that the m residual mod p representations have all large images, are not isomorphic to each other not even up to twist (this, as in the previous section, can be done due to the main theorem of [21], and regularity plus Fontaine–Laffaille theory for the non-isomorphic condition). Recall that in this situation, it follows from results of [17] that the residual image of the tensor product is adequate, and a similar statement also holds after restriction to any solvable extension. We consider the mod p residual representations attached to f_2, f_3, \dots, f_m and, applying Hida’s theory if we are in a residually ordinary locally at p case (which implies p -adic ordinarity, since we are in a Fontaine–Laffaille situation) or Lemma 4.1.19 in [1] combined with Theorems 4.3.1 and 4.2.1 in [2] in case the residual representation is irreducible locally at p , we construct a modular crystalline potentially diagonalizable lift of each of these mod p representations having arbitrarily large weight k'_i . We explain this in more detail: in the ordinary case, we consider the Hida family containing the p -adic representation attached to f_i and we know that it contains crystalline members of arbitrarily large weight. Being ordinary and crystalline, any such member is potentially diagonalizable at p . In the residually irreducible (locally at p) case, the result from [1] is local, ensuring the existence of a potentially diagonalizable crystalline lift of arbitrarily large weight of the local at p residual representation, but then Theorem 4.3.1 in [2] ensures the existence of a

global lift with this local behavior², and moreover, and this is also important, it can be taken of the same type as the p -adic representation attached to f_i at all other places. Finally, the ALT 4.2.1 of op. cit. allows us to deduce modularity of the lift just constructed from the modularity of the given representation, the one attached to f_i , since the latter is known to be Fontaine–Laffaille, thus potentially diagonalizable at p .

Let us record what we have just done in a lemma, which we will state in sufficient generality for our future uses of it. For some of the theorems we have mentioned in the above paragraph, some conditions on the size of the residual image are required, and for simplicity (maybe this condition can be relaxed) we also assume that $p > 5$ (as required in Theorem 4.3.1 of [2]):

Lemma 4.1. *Let f be a newform of weight k and let $p > 5$ be a prime such that the p -adic Galois representation attached to f is crystalline and Fontaine–Laffaille (in particular, it can be Barsotti–Tate) locally at p . In other words, $p > k$ and p is not in the level of f . Suppose that the image of the mod p Galois representation attached to f is large. Let us call \bar{r} this residual representation. Then, for any given integer E , there is an integer $k' > E$ such that \bar{r} has a p -adic lift which is modular, associated to a modular form f' of weight k' , and this p -adic lift is crystalline and potentially diagonalizable at p (in particular, p is not in the level of f'). Moreover, we can choose f' such that the p -adic Galois representations attached to f and f' connect locally at any prime $\ell \neq p$ (they have the same types).*

Since we can take them to be arbitrarily large, we declare that the “new” weights k'_2, \dots, k'_m , which are chosen one at a time, are going to satisfy the inequalities in the definition of C -very spread. Thus we have reduced to a situation where the weights are C -very spread (we recall that we have fixed $C = q_1 + 1$).

We have to check that, in the above process, while moving all components from the second to the last through suitable congruences, some ALT can be applied to the m -fold tensor products guaranteeing that automorphy of one side is equivalent to automorphy of the other. But we have Fontaine–Laffaille (thus potentially diagonalizable) components on one side, and we have a C -very spread set of po-

²Actually, to apply Theorem 4.3.1 we should replace \mathbb{Q} by a quadratic imaginary field F where p and all primes in the level are split, but there is a standard argument to show that, once the lift is constructed over F , then there also exists a lift over \mathbb{Q} : the idea is that for the restriction of the given representation to G_F one can prove, as a consequence of automorphy and properties of Hecke algebras, that the universal deformation ring R'_S with the given local properties is a finite module over the ring of integers of certain extension of \mathbb{Q}_p , and by comparing this ring with the corresponding universal deformation ring R_S for the given representation of $G_{\mathbb{Q}}$, one deduces that the same property holds for the latter. On the other hand, by computing the local deformation rings plus some cohomological calculations, one can show that the Krull dimension of R_S is at least 1, and this together with finiteness imply the existence of a $\bar{\mathbb{Q}}_p$ point in this ring, giving the lift we wanted. All this procedure holds in certain generality, including the case where the local conditions correspond to fixing irreducible components of the local universal deformation rings, thus covering our case. For references, see section 2 in [8], section 4 in [19], lecture 5 in [5], and section 7 in [14]. In a primitive form, the trick of comparing R'_S to R_S originated during the proof of Serre’s conjecture, and, previous to that, the lower bound for the Krull dimension of the ring in works of Böckle.

tentially diagonalizable components on the other side and we also have already remarked that the adequacy condition on the residual image of the tensor product is satisfied, thus Theorem 4.2.1 in *op. cit.* (or, if we do not want to assume that p is larger than $2(2^m + 1)$, we can use the improvement in [11]) can be applied and we are fine.

To ease notation, let us call these new weights again k_2, \dots, k_m , in the understanding that we now know that, together with k_1 , this is a C -very spread m -tuple of weights.

5. In which the other good-dihedral primes enter the picture

Before going on, and in order to guarantee that the components from the second to the last will also have large residual images in what follows, we are going to add good-dihedral primes to all of them. This is done exactly as we did for the first component in section 3, with the following conditions.

We apply this procedure one component at a time. Suppose we have applied it to the components from the second to the $(j - 1)$ -th one, then for the j -th component we choose a bound $B_j > C = q_1 + 1$, and we pick a prime $t_j > B_j$ such that the required conditions on all residual mod t_j images are satisfied (see section 3 for the case of the first component and do exactly the same), larger than the m weights k_i and different from any of the t_i, q_i $i = 1, \dots, j - 1$, already introduced, and modulo this prime t_j introduce a good-dihedral prime q_j such that it is different to all the q_i and t_i , $i = 1, 2, \dots, j - 1$, and such that it satisfies the definition of good-dihedral prime with respect to the bound B_j .

Since all the bounds B_j , $j > 1$, are larger than C , as long as we work in characteristic $p \leq C$, as we would do while moving through the skeleton chain in the first component, we know that the residual images of all but the first component will be large (and the residual image of the first component is also large³ because of how the skeleton chain was built, see section 3 and [10]). This is even true if the characteristic was equal to 3 or 5 (see the Introduction of [10] for the definition of large in this case) because we have an element of large order t_i in the projective residual image.

Observe also that the residual representations from now on will not be isomorphic to each other, not even up to twist, because of their differences in ramification, this is why we insisted to pick all the q_i different, and this remains true until the end since the good-dihedral primes q_2, \dots, q_m will never be removed.

This being said, and recalling once again that largeness is preserved by restriction to solvable extensions, we see (this follows from [17], see footnote in section 3) that as long as we work in characteristics $p \leq C$ the condition of adequacy of the residual image of the m -fold tensor product after restriction to the cyclotomic extension, required for ALT to work, will hold.

We will work in characteristic $p \leq C$ most of the time, but at a few steps we will also need to work in larger characteristics: we will choose them different from

³With the exception of the very last move, which will be discussed in Section 9.

the t_i, q_i , so at least the local behavior at q_i will not be altered. When working in such a large characteristic p , we will have some freedom to choose it, thus largeness of the residual images will again be easily established, usually through Ribet's large image theorem.

6. In which we show that at least in most of the steps things work well

We have reduced the proof of the main theorem to a situation in which the m components have each its own good-dihedral prime q_i , the weights are C -very spread, and now we want to keep moving the first component through the skeleton chain (the first step, which is the addition of a good-dihedral prime, has already been done). We are not going to explain this one move at a time, we are just going to classify the three possible type of congruences showing up in the skeleton chain, and we will see how to proceed in each case. We begin in this section, with the most common type of congruence (this one occurs in all but a few moves inside the skeleton chain). Let us define it.

Definition. We say that a mod p congruence between two cuspforms is of type (A) if both of the forms are potentially diagonalizable at p , of different weights, and they are of the same type at any other prime.

Typically, type (A) in the skeleton chain appears in congruences between Fontaine–Laffaille and potentially Barsotti–Tate representations. But there are other cases, like in one of the final four moves (see section 3).

Let us see that if the congruence in the first component is of type (A), while the other $m - 1$ components are left unchanged, then Theorem 2.2 can be applied to show that automorphy propagates well through this congruence of m -fold tensor products. Here we are taking the 2-dimensional Galois representations in the first component as R and R' , and $S = S'$ to be the $(m - 1)$ -fold tensor product of the other components. Since we have that the two weights k_1 and k'_1 involved in the congruence in the first component are both bounded by C , and moreover that the weights are C -very spread, we see that the extra-regularity condition (4) in Theorem 2.2 is satisfied, as explained after the definition of C -very spread (see section 4).

As for the other three conditions in Theorem 2.2, condition (1) is satisfied since we are in a type (A) case, condition (2) is automatic since $S = S'$, and for condition (3) this follows from known properties of the skeleton chain in the first component plus the good-dihedral primes q_i , $i > 1$, in the other components, as explained at the end of section 5.

Thus, we conclude that automorphy propagates well if the congruence in the first component is of type (A).

Remark. There is one and only one congruence in the skeleton chain with residual characteristic smaller than 7, and it is the mod 3 congruence in one of the last four

moves introduced in Section 3. As we explained there, this congruence is between potentially diagonalizable representations of different weights, and the type at primes other than 3 is not altered, so it is of type (A). The residual image is large because it contains $\mathrm{SL}_2(\mathbb{F}_{27})$, and for the components from the second to the last as we have already remarked the existence of good-dihedral primes also forces the residual images to be large (recall that large in characteristic 3 means that it must contain $\mathrm{SL}_2(k)$ with k a finite extension of \mathbb{F}_3 different from \mathbb{F}_3). Thus Theorem 2.2 applies in this congruence.

7. What do to in cases where no ALT would apply: the “genetic manipulation” of the other components

There are other two kind of congruences showing up while moving along the skeleton chain, we define them now.

Definition. We say that a mod p congruence between two cuspforms is of type (B) if both of the forms are ordinary at p , of different weights, and they are of the same type at any other prime.

Definition. We say that a mod p congruence between two cuspforms is of type (C) if both of the forms are Barsotti–Tate at p , and they are of different type at some prime $\ell \neq p$.

Remarks. (1) As the attentive reader may have observed, the congruence in Step 1, as done in section 3, to add a good-dihedral prime, is not of any of the three types defined. We simply exclude it from our considerations, since we have already seen that an ALT was applicable there (alternatively, we could have given a slightly more general definition of type (C)).

(2) In type (B), we can assume that at least one of the two ordinary representations is not potentially crystalline, else this will be type (A).

(3) Whenever a congruence of type (B) or (C) occurs in the skeleton chain, we have $p \geq 7$ (see the remark at the end of section 6).

The problem is that if the congruence in the first component is of any of these two types, and the other components are not known to be potentially diagonalizable (the typical situation is that the prime p is smaller than some of the weights, and the representations are crystalline but not Fontaine–Laffaille), no ALT can be applied to propagate automorphy. So, what can we do to keep moving forward in our proof?

What we propose is to change, through suitable congruences (where suitable ALT will guarantee that automorphy propagates well) all components from the second to the last one, one at a time, in order to make them potentially diagonalizable at p . This is what we informally call a “genetic manipulation”. *After finishing with this procedure*, we will be able to go through the move in the skeleton chain, i.e., in the first component, either involving type (B) or type (C), as we will explain later.

First of all let us stress the key idea that has allowed us to do this: assuming that we are in the i -th component, $i > 1$, and the p -adic Galois representation there is in a Fontaine–Laffaille situation (we will show how to reduce to such a case, *one component at a time*: the pace is key, else this would be impossible if p is smaller, say, than m , because of regularity), we can consider, using Lemma 4.1, a congruence with another modular form of arbitrarily large weight k'_i such that it is crystalline and potentially diagonalizable at p .

The procedure is a bit painful, since we need to make sure that suitable ALT do apply at each move performed during the manipulation (and this implies keeping track of local conditions, plus regularity, plus residual images), and we also want to make sure that when we finish we have not altered a fundamental fact: we start with an m -tuple of weights k_1, k_2, \dots, k_m which is C -very spread, and the output must be another m -tuple of weights (we do not change the first one) $k_1, k'_2, k'_3, \dots, k'_m$ also C -very spread. In this procedure, we will also need to introduce auxiliary primes (larger than the bounds B_i) in the level of all the components from the second to the last one: this implies in particular that we will have to ensure that this does not alter the fact that their good-dihedral primes q_i remain “good-dihedral primes with respect to the bound B_i ” after this extra ramification has been introduced (i.e., that the local ramification at q_i is enough to guarantee large residual images for the compatible systems of Galois representations attached to these level-raised modular forms, at any characteristic bounded by B_i).

So, here is where the hard work begins. We will divide the algorithm in two cases, not depending on whether we are acting “just before” a congruence of type (B) or of type (C) in the skeleton chain, but depending on the following fact: Let k_1 be the input weight in the skeleton chain, i.e., we have a newform of weight k_1 and we want to move through a congruence with another newform, possibly of different output weight. Then we will consider separately the case $k_1 = 2$ and the case $2 < k_1 \leq C$.

Remark. To avoid any confusion with the notions of input and output weights, we insist that we are moving along the skeleton chain starting on some level 1 newform f_1 and finishing in a small level CM form. Thus, if we were to draw this chain from left to right, the leftmost point will correspond to f_1 , and the input of a congruence is just at the left of its output. In the end, automorphy propagates just the other way round, from right to left, this is why I wanted to clarify these somehow arbitrary notions.

We will assume first that the input weight of the first component is not 2, the easiest case, even if this case rarely occurs. Let us see how to transform, one at a time, all other components into representations that not only still have very large weights but also such that they are potentially diagonalizable at p . This is done simply, for example to the second component, by first taking a weight 2 minimal lift modulo any sufficiently large prime p_2 , i.e., a congruence with another newform of weight 2 and the same level except for the extra prime p_2 in its level. This prime must be taken to be, and this we have seen already a couple of times how can be done with standard techniques, such that the residual image of the m -fold

tensor product satisfies the usual adequacy condition (after restriction to the cyclotomic field) and larger than all weights, in order to be in a Fontaine–Laffaille situation (and having in the second component a congruence with a potentially Barsotti–Tate, thus potentially diagonalizable, representation). We also take it to be different from any other prime ever considered thus far in this paper. Observe also that after this step the m -tuple of weights become (in increasing order):

$$(2, k_1, k_3, \dots, k_m)$$

which is clearly spread since $k_1 > 2$. Automorphy propagates well through this congruence, due to Theorem 4.2.1 of [2], since both potentially Barsotti–Tate and Fontaine–Laffaille cases are potentially diagonalizable.

Just one important point before moving on: we must choose the prime p_2 to be a square mod q_2 , so that q_2 is a square mod p_2 . This way, even if we have introduced extra ramification at p_2 in the second component, the good-dihedral prime q_2 will still be enough to ensure large residual images for the second component, in characteristics up to B_2 , as can be easily checked (using the usual method for proving that good-dihedral primes give large residual images).

Now we are ready for the key step: we have already reduced the second component to a weight 2 situation, so that if we move to characteristic p , p being the prime where the congruence in the skeleton chain will take place, we are in a Barsotti–Tate situation (p was not, and is not, in the level of the newform in the second component). So now we apply Lemma 4.1 to the mod p representation in the second component, since we know that $p > 5$ (see Remark (3) above), and the residual image is large (due to the good-dihedral prime q_2 and $p < C < B_2$) it can be applied and it gives us another modular lift, of weight k'_2 arbitrarily large, crystalline and potentially diagonalizable at p , and preserving the types locally at all other ramified primes.

We pick $k'_2 > 2C + k_3 + \dots + k_m$. In this mod p congruence in the second component, we have as input an m -tuple of weights

$$(2, k_1, k_3, \dots, k_m)$$

And the output weights are

$$(k_1, k_3, \dots, k_m, k'_2)$$

Then, the above inequality is enough to guarantee that the $(m + 1)$ -tuple

$$(2, k_1, k_3, \dots, k_m, k'_2)$$

is spread. Here we are using the fact that the initial weights (k_1, k_2, \dots, k_m) were C -very spread (and in particular $k_1 < C$) and the inequality imposed on k'_2 .

Thus we can apply Theorem 2.2 to see that modularity propagates well through this congruence of m -fold tensor products. We are taking $S = S'$ to be the $(m - 1)$ -fold tensor products of all but the second component, and R and R' the modular Galois representations in the congruence of the second component. We have just

checked that the extra-regularity condition (4) is satisfied, conditions (1) to (3) are easy to check (as explained in section 5, since we are in characteristic $p < C$, the adequacy condition on the residual image holds).

The rest is just an iteration of the same idea: after modifying the $(i - 1)$ -th component, for any $2 < i \leq m$, we manipulate the i -th component in a similar way: we choose as above a sufficiently large prime p_i , larger than all weights (in increasing order)

$$(k_1, k_i, \dots, k_m, k'_2, \dots, k'_{i-1})$$

and satisfying the same conditions we asked to p_2 , but now translated to the i -th component, in particular we ask this prime to be a square mod q_i . We take a weight 2 lift in the i -th component modulo p_i . Here we check again that Theorem 4.2.1 of [2] allows automorphy to propagate. Then, we move to characteristic p , and again via Lemma 4.1 we make a congruence with a modular form of weight k'_i , which we can and will choose to satisfy

$$(7.1) \quad k'_i > 2C + k_{i+1} + \dots + k_m + k'_2 + \dots + k'_{i-1}$$

With this it is easy to recursively show that in this last congruence, the extra-regularity condition (4) in Theorem 2.2 holds, in other words, that the $(m + 1)$ -tuple

$$(2, k_1, k_{i+1}, \dots, k_m, k'_2, \dots, k'_i)$$

is spread. Conditions (1) to (3) of this ALT are again easy to check, thus we conclude that automorphy propagates well.

After applying the algorithm to the m -th component, we end up with an m -tuple of newforms of weights,

$$(k_1, k'_2, \dots, k'_m),$$

such that all components but the first one are crystalline and potentially diagonalizable at p . Moreover, by the way the weights k'_i were chosen it is easy to see that this new set of weights is again C -very spread.

Thus we can proceed and go through the mod p congruence in the first component, either if it is of type (B) or of type (C).

We can apply either Theorem 2.1 (for type (B)) or Theorem 4.2.1 in [2] and its variant in [11] (for type (C)), because we will either be in a “mixed” situation such as the one Theorem 2.1 was meant to deal with, or in a potentially diagonalizable situation. The weights are C -very spread, so no matter the output value of the weight in the first component (it is always bounded by C) we know that we have a congruence between two regular m -fold tensor products, and since $p < C$ we also know that the residual image is sufficiently large.

Now we have to deal with the other case, the case with $k_1 = 2$, which is a bit more complicated. All auxiliary primes considered will be assumed to be different between them and different from any prime previously considered in this paper.

We again have to alter k_2, k_3, \dots, k_m and we will do it one at a time, so to begin with let us “transform” the second component into a representation of some

large weight k'_2 that is potentially diagonalizable at p . The moves are as follows (recall that p is at least 7): starting with weights $(2, k_2, \dots, k_m)$, modulo some prime u_2 larger than all weights such that the first component is ordinary, and residual images are all large we change the weights to $(3, k_2, \dots, k_m)$, using Hida theory and specializing to weight 3. Observe that such a prime exists because of Ribet's large images theorem and the fact that weight 2 newforms have a density one set of ordinary primes (see [13]). Observe also that ALT 4.2.1 in [2] applies to this congruence of m -fold tensor products where we are only changing the first component, because we know that the residual image is sufficiently large, the first components are potentially diagonalizable (the weight 3 modular form is potentially crystalline and ordinary at u_2), the other components are Fontaine–Laffaille, thus also potentially diagonalizable, and of course $3 < C$ so the new set of weights is also C -very spread.

It is perhaps worth remarking at this point that we have violated a sacred rule: we have introduced extra ramification, at u_2 , in the first component, thus altering the skeleton chain. To remedy this, we are going to undo this move as soon as possible, just in two more moves, so to rejoin the skeleton chain after this small detour, safe and sound.

The next move is modulo some large prime v_2 , here we take a weight 2 lift in the second component, i.e., we change weights to $(3, 2, k_3, \dots, k_m)$. This set is easily seen to be spread. This move is as the one we did in some large characteristic while treating the previous case, and again we take this prime to be larger than all weights, so that all residual images are large, and such that v_2 is a square mod q_2 . We easily check that ALT 4.2.1 in [2] applies to this congruence.

Then, modulo some prime w_2 larger than all weights, such that the second component is ordinary, all residual images are large, and such that w_2 is a square mod q_2 (we use again the fact that the set of ordinary primes has density 1 for a weight 2 newform to see that such a prime exists), using Hida theory we change weights to $(3, 6, k_3, \dots, k_m)$, which are spread. As with the prime u_2 , we check that every two-dimensional representation involved in this congruence is potentially diagonalizable at w_2 , and we easily see that ALT 4.2.1 in [2] applies. Then, we move back to characteristic u_2 , and we undo the move on the first component: we obtain weights $(2, 6, k_3, \dots, k_m)$, which are spread. Observe that the residual image of the first component is large for the very simple reason that it was so two moves ago (this was one of the conditions to choose this prime then) and we have not changed the first component since then. In fact, the same applies to the residual images of all but the second component: they were large two moves ago, and the newforms have not changed, so it is clear that their mod u_2 images are still large. But the second component has changed, and thus a priori u_2 could be in the exceptional set for Ribet's theorem for the weight 6 newform that is now in the second component. But we can check using the techniques in [12] that this is not the case, that the residual image in the second component is also large. What we do is to combine the good-dihedral prime with a local analysis at u_2 : because of the good-dihedral prime q_2 , applying its existence in the usual way, we see that the residual image can fail to be large only if it is dihedral and such that the quadratic

field where it becomes reducible is ramified at u_2 . But looking at the action of the tame inertia group at u_2 , since we are in a Fontaine–Laffaille situation, we can apply ideas of Ribet (which are recorded for example as Lemma 3.1 of [10]) to deduce that such a dihedral case can only occur if $u_2 = 2k - 1$ or $u_2 = 2k - 3$, where k is the weight of the given newform. Since in our case $k = 6$ and $u_2 > 12$, we conclude that the residual image is large.

Having checked what was required on residual images and since all two-dimensional representations involved are potentially diagonalizable, we see that ALT 4.2.1 in [2] applies to this congruence.

Now we can move to characteristic p and apply Lemma 4.1 to the second component: since $p \geq 7$, the p -adic representation in the second component is Fontaine–Laffaille, so we can make a congruence with a potentially diagonalizable crystalline modular representation of arbitrarily large weight k'_2 . Thus the output weights are (in increasing order) $(2, k_3, \dots, k_m, k'_2)$. It is important to stress that the newform of weight 2 in the first component is exactly the one we started with at the beginning of the algorithm, because the only change we have performed to the first component was soon undone. Thus, it belongs to the skeleton chain, and we know (this is part of the structure of the skeleton chain) that since the next move in the skeleton chain is in characteristic p , the mod p residual image of this representation is large (there is an exception to this, at the last step of the skeleton chain where the residual image is dihedral, but we are going to dedicate Section 9 to explain what are the differences in that case). As for the residual images of the other components, we know since $p < C$ that they are large. Let us specify the lower bound for k'_2 : we take it exactly as in the other case, i.e., we declare that $k'_2 > 2C + k_3 + \dots + k_m$. With this, it is easy to see that the extra-regularity condition (4) in Theorem 2.2 holds, namely, that the $(m + 1)$ -tuple of weights

$$(2, 6, k_3, \dots, k_m, k'_2)$$

is spread (note the similarity with the previous case).

Conditions (1) to (3) of that ALT are easily seen to hold, since we have already explained that residual images of all components are known to be large (and due to their differences in ramification, these residual representations are not isomorphic to each other, not even up to twist) thus we conclude that it applies and ensures that automorphy propagates well through this congruence.

The rest is just an iteration of the above procedure and it works well for the same standard arguments. To make the moves in the i -th component, for any $i > 2$, assuming we have already done so for the previous components, one repeats similar moves with large primes u_i, v_i, w_i , the last two of them congruent to squares mod q_i and satisfying properties similar to those required to u_2, v_2, w_2 , then modulo u_i the first move is undone, and one culminates with an application of Lemma 4.1 in the i -th component to change from a weight 6 p -adic modular representation to a potentially diagonalizable and crystalline one of weight k'_i chosen as in the previous case, see formula (7.1).

As happened in the second component, it is easy to check that in all these moves some ALT applies, thus ensuring that automorphy propagates well. In particular,

the extra-regularity condition (4) in Theorem 2.2, a theorem that we apply in the last move in the i -th component, holds as in the previous case as follows from formula (7.1).

After applying these moves to the last component, we end up with a set of weights: $(2, k'_2, \dots, k'_m)$ which is C -very spread, as follows from formula (7.1) as in the previous case.

Also, all components from the second to the last are potentially diagonalizable at p , and residual images modulo p of all components are large. Thus, we can now go through the mod p congruence in the first component, either if it is of type (B) or of type (C), by applying a suitable ALT, either Theorem 2.1 or the ALT in [11].

We have thus been able, through some manipulations, to reduce to a situation where even if the congruence in the skeleton chain is of type (B) or (C), a suitable ALT can be applied. The modifications we have performed in the components from the second to the last are harmless: on one hand, weights have changed, but they still are C -very spread, thus regularity in the following steps is ensured; on the other hand, some extra ramification has been introduced into them, but in such a way that the corresponding good-dihedral prime q_i ($i > 1$) still plays the same role, thus in particular ensuring that when working in any characteristic $p < C$ the corresponding residual image will be large.

Since we have explained in the previous section that in the type (A) case a suitable ALT applies, this means that with these manipulations we have been able to go through all congruences in the skeleton chain in such a way that a suitable ALT applies at each move, thus reducing the proof of the main theorem (which is Theorem 1.1) to a case where one component is a CM form. But since automorphy is preserved by tensoring by a CM form (this is one of the key observations that makes Harris' trick work) this means that we have reduced the proof to a case of a regular tensor product of $m - 1$ modular Galois representations. In section 10 we will explain that despite the fact that the newforms that we found in this $(m - 1)$ -fold tensor product are no longer of level 1, still we can conclude the proof of the main theorem by induction on m . Before that, we dedicate two sections to clarify a couple of technical points, for the reader's convenience. But it is fair to say that we have already done most of the job, what the three following sections contain are just minor issues.

8. Remarks on Galois conjugation, and on twists

In the skeleton chain, there are two minor points that should be clarified:

Twists: sometimes congruences are up to twist by a finite order character, i.e., we are changing the output representation by a twist of it in order, for example, to minimize the residual Serre's weight. When we put both representations inside m -fold tensor products, the fact that we are working up to twist is harmless because automorphy is preserved by twists and all conditions in all the ALT are also preserved by twisting (in particular, potential diagonalizability).

Galois conjugation: finally, in the skeleton chain we also sometimes replace a modular form by a Galois conjugate of it, in particular, this is why the fact that the “bottom space” contains a single orbit is good enough to use it as a bridge to propagate automorphy. But when we put the modular form (as first component) inside an m -fold tensor product, automorphy will in principle not necessarily be preserved by Galois conjugation of the first component. To remedy this problem, we do the following: whenever in the first component one applies Galois conjugation, at the same time one conjugates all other components by the same Galois element (i.e., by a compatible choice of Galois elements). This way, we are inducing a conjugation on the m -fold tensor product, which is a valid move since it preserves automorphy. The alteration infringed in the components from the second to the last is harmless: after conjugating, they preserve their weights and their levels and their good-dihedral primes q_i , so this is an optimal way to deal with the issue of Galois conjugation.

9. In which we explain why when one component has a dihedral but not bad-dihedral residual image ALT still work well

At several points, we have used the argument that the residual image of the m -fold tensor product satisfies the adequacy condition required to apply certain ALT because the residual images of all components were large and not isomorphic to each other, not even up to twist.

This is true at all congruences between m -fold tensor products considered in this paper, except for the case where we get to the end of the skeleton chain and there we have a mod 13 congruence between two representations, in the first component, with dihedral residual image. Moreover, since the congruence is of type (C), we have to go through the “genetic manipulation” here, and this involves working in characteristic 13 several times during the process of changing the weights of the components from the second to the last one, and this is done with a first component having dihedral residual image.

Thus, we need to make sure that at all these moves in characteristic 13 the condition on the size of the residual image required to apply an ALT is satisfied, despite the fact that residual images of the components are not all large.

For the components from the second to the last, we know that due to their good-dihedral primes the mod 13 residual images are large, and we can argue in the usual way to see that for the $(m - 1)$ -fold tensor product having them as components the residual image is adequate, even after restriction to the cyclotomic field of 13-th roots of unity. Then, we can apply Lemma A.3.1 in [1] to check that tensoring with the first component gives a residual image that also satisfies this property. The only thing that has to be checked is that the dihedral representation that we have residually in the first component is irreducible when restricted to the compositum M of the cyclotomic field of 13-th roots of unity and the fixed field of the kernel of the $(m - 1)$ -fold tensor product of the other residual components

(observe also that the residual representation in the first component has image of order prime to 13 as required in Lemma A.3.1 of op. cit. because it is dihedral). But this follows from the combination of two facts: first, this dihedral representation is induced from a character of $\mathbb{Q}(\sqrt{-3})$ (recall that the congruence modulo 13 in the first component is with a CM elliptic curve of conductor 27) and since the size of the projectivized residual image is larger than 4 (this can be checked easily by looking at the inertia group at 13) it will become reducible after restriction to a Galois number field only if such field contains $\mathbb{Q}(\sqrt{-3})$, and second, the only component that ramifies at 3 is the first one, thus M does not contain $\mathbb{Q}(\sqrt{-3})$.

Thus we conclude that even if in this case the residual image of the first component is not large, the adequacy condition on the residual image of the m -fold tensor product is satisfied.

10. Completing the induction process: rewinding

At the end of our chain of congruences, we have reduced the proof of automorphy of the given m -fold tensor product of level 1 cuspform to that of an $(m - 1)$ -fold regular tensor product, but this time the components are newforms of level greater than 1. In order to have a proof by induction on m , we should now “safely connect” this tensor product with another one whose components are level 1 newforms.

But what may seem complicated is rather simple: we are looking for a chain of congruences propagating automorphy ending in an $(m - 1)$ -fold tensor product of level 1 newform, but we have already constructed such a chain! Simply consider all the moves we have done with the m -fold tensor products, starting in a “level 1 case”, through our proof, delete from all this the first component and forget about the congruences involving exclusively the first component, and reverse the resulting chain of congruences of $(m - 1)$ -fold tensor products. This is clearly a chain of $(m - 1)$ -fold tensor products starting where we wanted to start and ending, as we wanted, in a level 1 case. Is it safe, i.e., do ALT apply through this chain (as usual, we need to apply them the other way round, i.e., starting in the level 1 tensor product)? The answer is yes, and this is due to the following facts:

- (1) The chain of m -fold tensor products that we have built in previous sections is safe.
- (2) Moreover, at each step we have applied ALT that work in both directions, so that they can be used to propagate automorphy not only from right to left, but also from left to right (this allows us to reverse now).
- (3) For all the ALT that we have applied to m -fold tensor products, we have checked the required local conditions on each component (because these are conditions that behave well under tensor products) and we have also checked that the residual image of the tensor product was sufficiently large by checking that the images of the components were so (and that there were no unwanted isomorphisms between them). As for regularity, it is clear that if it holds for the case of m components it will also hold after dropping the first

component. The same applies to the extra-regularity condition (4) in Theorem 2.2. Thus we conclude that by dropping the first component we are not altering the fact that the chain is safe.

This concludes the reduction of the proof to the case of an $(m - 1)$ -fold regular tensor product of level 1 newforms. Thus, since the claim is trivial for $m = 1$, the proof of the main theorem follows by induction on m .

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