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Pyramids and 2-representations

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Abstract. We describe a diagrammatic procedure which lifts strict monoidal actions from additive categories to categories of complexes avoiding any use of direct sums. As an application, we prove that every simple transitive 2-representation of the 2-category of projective bimodules over a finite dimensional algebra is equivalent to a cell 2-representation.

1. Introduction and description of the results

One of the most fundamental results in the classical representation theory is the fact that the algebra $\mathrm{Mat}_{n\times n}(\Bbbk)$ of $n \times n$ matrices over an algebraically closed field $\Bbbk$ has only one isomorphism class of simple modules. The main result of the present paper is an analogue of this latter fact in 2-representation theory.

Modern 2-representation theory originates from [1], [3], [4], [21] which emphasize the use of 2-categories and 2-representations for solving various problems in algebra and topology. The series [12]-[17] of papers started a systematic study of 2-representations of so-called *finitary* 2-categories which are natural 2-analogues of finite dimensional algebras. The weak Jordan–Hölder theory developed in [16] motivates the study of so-called *simple transitive* 2-representations which are suitable 2-analogues of simple modules. It turns out that, in many cases, simple transitive 2-representations can be explicitly classified, see [16], [17], [7], [6], [8], [18], [9], [19], [20], [23], [24], [25] for the results and [11] for a detailed survey on the subject. In many, but not all, cases, simple transitive 2-representations are exhausted by so-called *cell* 2-representations defined already in [12]. Cell 2-representations are precisely the weak Jordan–Hölder subquotients of regular (a.k.a. principal) 2-representations.

For the moment, a classification of *simple* finitary 2-categories is only available under substantial additional assumptions (in particular, that of existence of a weak anti-equivalence and adjunction morphisms), see [14]. Roughly speaking, in that

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special case simple 2-categories are classified by 2-categories of projective bimodules over finite dimensional self-injective algebras. Here self-injectivity is crucial as it is equivalent to existence of a weak anti-equivalence and adjunction morphisms. For a finite dimensional algebra A, the corresponding 2-category of projective bimodules is usually denoted by $\mathscr{C}_A$. With this result in hand, it is very natural to consider 2-categories of projective bimodules for arbitrary finite dimensional algebras as a basic family of simple 2-categories. In analogy to finite-dimensional algebras, where any simple representation factors through a simple algebra, for a finitary 2-category, any simple transitive exact abelian 2-representation factors through $\mathscr{C}_A$, for some algebra A. Outside the self-injective case, a number of examples were studied in [19], [20], [18], [25] and in all cases, using rather different arguments, it was shown that simple transitive 2-representations are exhausted by cell 2-representations. Moreover, in all these examples, all cell 2-representations with a fixed apex are equivalent.

The main result of the present paper, Theorem 5.2, together with Proposition 5.1, asserts that the latter is the case for any finite dimensional algebra over an algebraically closed field. This answers (positively) [11], Question 15. Compared to all previous studies, our approach uses two crucial new ideas.

The first idea is related to creating some adjunction morphisms. In case A has a non-zero projective injective module, some of the non-identity 1-morphisms in $\mathscr{C}_A$ form adjoint pairs of functors. This was exploited in [19], [20], [25]. In the present paper we suggest to enlarge $\mathscr{C}_A$ to a 2-category $\mathscr{D}_A$ by adding right adjoints to all 1-morphisms in $\mathscr{C}_A$. The 2-category $\mathscr{D}_A$ can be realized using A-A-bimodules by adding the A-A-bimodule $A^* \otimes_{\Bbbk} A$ as the right adjoint of the A-A-bimodule $A \otimes_{\Bbbk} A$. The 2-category $\mathscr{D}_A$ loses some of the symmetries of $\mathscr{C}_A$ but compensates for this loss by possessing many pairs of adjoint 1-morphisms. With the machinery developed in [5], [6], [19], [18] and some tricks using matrix computations, we prove in Theorem 4.3 that simple transitive 2-representations of $\mathscr{D}_A$ are exhausted by cell 2-representations.

The second idea is related to the necessity of some kind of "induction" allowing us to connect 2-representations of $\mathscr{C}_A$ with 2-representations of $\mathscr{D}_A$. In classical representation theory, induction is done using tensor products. Unfortunately, this technology is not yet available in 2-representation theory, which creates major obstacles. In the particular case of the 2-categories $\mathscr{C}_A$ and $\mathscr{D}_A$, we propose a way around the problem. We observe that 1-morphisms in $\mathscr{D}_A$ can be identified with (homotopy classes of) *complexes* of 1-morphisms in $\mathscr{C}_A$. This raises the problem of lifting the strict 2-structure from $\mathscr{C}_A$ to the corresponding homotopy category of complexes. The main obstacle is the incompatibility of strictness and additivity of 1-morphisms. To resolve this, we substitute the category of complexes by a new category, which we call the category of *pyramids*, see Section 2. This category is equivalent to the category of complexes, but its tensor structure can be defined avoiding direct sums (that is, avoiding the construction of taking the total complex). We use pyramids to lift 2-representations of $\mathscr{C}_A$ to the homotopy category of pyramids over $\mathscr{C}_A$ which can then be restricted to $\mathscr{D}_A$ as the latter lives inside pyramids over $\mathscr{C}_A$. This gives a well-defined "induction" from $\mathscr{C}_A$ to $\mathscr{D}_A$ which allows us to prove Theorem 5.2 using Theorem 4.3.

The paper is organized as follows: in Section 2 we collect all the results related to the definition and properties of pyramids. Section 3 collects preliminaries on 2-categories and 2-representations. In Section 4 we study 2-representations of $\mathscr{D}_A$. The main result of this section is Theorem 4.3. In Section 5 we formulate and prove Theorem 5.2 and also give a characterization of 2-categories of the form $\mathscr{C}_A$ inside the class of finitary 2-categories, see Theorem 5.4.

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2. Pyramids

2.1. Indices

We denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{Z}_{\geq 0}$ the set of non-negative integers. Further, we denote by $\mathbb{I}$ the set of all vectors $\mathbf{a} = (a_i)_{i\in\mathbb{N}}$, written $\mathbf{a} = (a_1, a_2, a_3, \dots)$, where $a_i \in \mathbb{Z}$ and $a_i = 0$ for $i \gg 0$. Note that $\mathbb{I}$ is an abelian group with respect to component-wise addition. The zero element in $\mathbb{I}$ is $\mathbf{0} := (0, 0, \dots)$.

For $i \in \mathbb{N}$, we denote by ε_i the element $(a_i)_{j\in\mathbb{N}} \in \mathbb{I}$ such that $a_j = \delta_{i,j}$ for all $j \in \mathbb{N}$ (here $\delta_{i,j}$ is the Kronecker symbol). Then $\mathbb{I}$ is a free abelian group with basis $\mathbf{B} := \{\varepsilon_i : i \in \mathbb{N}\}$, in particular, each element of $\mathbb{I}$ can be written uniquely as a linear combination (over $\mathbb{Z}$) of elements in $\mathbf{B}$.

For $\mathbf{a} \in \mathbb{I}$, the *height* of $\mathbf{a}$ is defined to be $\mathrm{ht}(\mathbf{a}) = \sum_i a_i \in \mathbb{Z}$. Note that the latter is well-defined as only finitely many components of $\mathbf{a}$ are non-zero. For $k \in \mathbb{Z}$, we denote by $\mathbb{I}_k$ the set of all $\mathbf{a} \in \mathbb{I}$ of height k.

Denote by $\pi_0 : \mathbb{I} \to \mathbb{I}$ the map which maps all $\mathbf{a}$ to $\mathbf{0}$. Let $\sigma_0 : \mathbb{I} \to \mathbb{I}$ be the identity map. For $k \in \mathbb{N}$, define $\pi_k : \mathbb{I} \to \mathbb{I}$ as the map sending $\mathbf{a} \in \mathbb{I}$ to $(a_1, a_2, \dots, a_k, 0, 0, \dots)$. Define also $\sigma_k : \mathbb{I} \to \mathbb{I}$ as the map sending $\mathbf{a} \in \mathbb{I}$ to $(a_{k+1}, a_{k+2}, \dots)$.

2.2. Pyramids over an additive category

Let $\mathcal{A}$ be an additive category. A *pyramid* $(X_\bullet, d_\bullet, n)$ over $\mathcal{A}$ is a tuple

$$(X_\bullet := \{X_\mathbf{a}; \mathbf{a} \in \mathbb{I}\}, d_\bullet := \{d_{\mathbf{a},i} : \mathbf{a} \in \mathbb{I}, i \in \mathbb{N}\}, n),$$

where

- $n \in \mathbb{Z}_{\geq 0}$,
- all $X_\mathbf{a}$ are objects in $\mathcal{A}$,
- each $d_{\mathbf{a},i}$ is a morphism in $\mathcal{A}$ from $X_\mathbf{a}$ to $X_{\mathbf{a}+\varepsilon_i}$,

satisfying the following axioms:

(I) we have $X_\mathbf{a} = 0$ unless $a_i = 0$, for all $i > n$,

(II) there is $m \in \mathbb{Z}$ such that $X_\mathbf{a} = 0$, unless all $a_i < m$,

(III) we have $d_{\mathbf{a}+\varepsilon_i,i} \circ d_{\mathbf{a},i} = 0$, for all $\mathbf{a}$ and i,

(IV) we have $d_{\mathbf{a}+\varepsilon_i,j} \circ d_{\mathbf{a},i} = -d_{\mathbf{a}+\varepsilon_j,i} \circ d_{\mathbf{a},j}$, for all $\mathbf{a}$, i and j.

For a pyramid $(X_\bullet, d_\bullet, n)$ over $\mathcal{A}$ and $k \in \mathbb{Z}$, we define $d^{(k)}$ as the matrix $(d^{(k)}_{\mathbf{a},\mathbf{b}})^{\mathbf{b}\in\mathbb{I}_k}_{\mathbf{a}\in\mathbb{I}_{k+1}}$, where

$$d^{(k)}_{\mathbf{a},\mathbf{b}} = \begin{cases} d_{\mathbf{b},i}, & \text{if } \mathbf{a} = \mathbf{b} + \varepsilon_i; \\ 0, & \text{otherwise.} \end{cases}$$

Let $(X_\bullet, d_\bullet, n)$ and $(Y_\bullet, \partial_\bullet, m)$ be two pyramids over $\mathcal{A}$. A *morphism*

$$\alpha : (X_\bullet, d_\bullet, n) \to (Y_\bullet, \partial_\bullet, m)$$

of pyramids is defined as $\alpha = \{\alpha^{(k)} : k \in \mathbb{Z}\}$, where each $\alpha^{(k)}$ is a matrix $(\alpha^{(k)}_{\mathbf{a},\mathbf{b}})^{\mathbf{b}\in\mathbb{I}_k}_{\mathbf{a}\in\mathbb{I}_k}$ with $\alpha^{(k)}_{\mathbf{a},\mathbf{b}} \colon X_\mathbf{b} \to Y_\mathbf{a}$, such that the following condition is satisfied, for every k:

$$\alpha^{(k+1)} \cdot d^{(k)} = \partial^{(k)} \cdot \alpha^{(k)}. \tag{2.1}$$

Here both sides of the equality should be understood as products of the corresponding matrices. This is well-defined as, for each k, the matrix $\alpha^{(k)}$ contains only finitely many non-zero components.

Let $(X_\bullet, d_\bullet, n)$, $(Y_\bullet, \partial_\bullet, m)$ and $(Z_\bullet, \daleth_\bullet, l)$ be three pyramids over $\mathcal{A}$. Let further

$$\alpha : (X_\bullet, d_\bullet, n) \to (Y_\bullet, \partial_\bullet . m) \quad \text{and} \quad \beta : (Y_\bullet, \partial_\bullet, m) \to (Z_\bullet, \daleth_\bullet, l)$$

be morphisms of pyramids. Then their *composition* $\beta \circ \alpha \colon (X_\bullet, d_\bullet, n) \to (Z_\bullet, \daleth_\bullet, l)$ is defined as the morphism γ of pyramids such that $\gamma^{(k)} = \beta^{(k)} \cdot \alpha^{(k)}$, for each k. Again, the right hand side should be understood as the usual product of matrices. Thanks to the finiteness properties mentioned in the previous paragraph, the product is well-defined. Condition (2.1) is satisfied because of the computation

$$\beta^{(k+1)} \cdot \alpha^{(k+1)} \cdot d^{(k)} = \beta^{(k+1)} \cdot \partial^{(k)} \cdot \alpha^{(k)} = \daleth^{(k)} \cdot \beta^{(k)} \cdot \alpha^{(k)},$$

where the first equality is justified by the fact that α is a morphism of pyramids and the second equality is justified by the fact that β is a morphism of pyramids.

For a pyramid $(X_\bullet, d_\bullet, n)$, the corresponding identity morphism ω is defined by declaring each $\omega^{(k)}$ to be the matrix such that

$$\omega^{(k)}_{\mathbf{a},\mathbf{b}} = \begin{cases} \mathrm{id}_{X_\mathbf{a}}, & \mathbf{a} = \mathbf{b}, \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 2.1. *Let $\mathcal{A}$ be an additive category. The construct consisting of all pyramids over $\mathcal{A}$, morphisms of pyramids, composition of morphisms and identity morphisms forms a category, denoted $\mathcal{P}(\mathcal{A})$.*

Proof. This follows directly from the definitions using the interpretation via matrix multiplication. □

The category $\mathcal{P}(\mathcal{A})$ inherits from $\mathcal{A}$ the obvious preadditive structure given by component-wise addition of morphisms. Furthermore, $\mathcal{P}(\mathcal{A})$ also inherits from $\mathcal{A}$ the additive structure by taking component-wise direct sums.

We have the canonical embedding of $\mathcal{A}$ into $\mathcal{P}(\mathcal{A})$ which sends an object $X \in \mathcal{A}$ to a pyramid concentrated at position $\mathbf{0}$ with the obvious assignment on morphisms.

2.3. Homotopy category of pyramids

Let $(X_\bullet, d_\bullet, n)$ and $(Y_\bullet, \partial_\bullet, m)$ be two pyramids over $\mathcal{A}$. A *homotopy*

$$\chi : (X_\bullet, d_\bullet, n) \to (Y_\bullet, \partial_\bullet, m)$$

of pyramids is defined as $\chi = \{\chi^{(k)} : k \in \mathbb{Z}\}$, where each $\chi^{(k)}$ is a matrix $(\chi^{(k)}_{\mathbf{a},\mathbf{b}})^{\mathbf{b}\in\mathbb{I}_k}_{\mathbf{a}\in\mathbb{I}_{k-1}}$, with $\chi^{(k)}_{\mathbf{a},\mathbf{b}} \colon X_{\mathbf{b}} \to Y_{\mathbf{a}}$.

If $\alpha\colon (X_\bullet, d_\bullet, n) \to (Y_\bullet, \partial_\bullet, m)$ is a morphism of pyramids, we will say that the morphism α is *homotopic to zero*, denoted $\alpha \sim 0$, provided that there exists a homotopy $\chi\colon (X_\bullet, d_\bullet, n) \to (Y_\bullet, \partial_\bullet, m)$ such that

$$\alpha^{(k)} = \chi^{(k+1)} \circ d^{(k)} + \partial^{(k-1)} \cdot \chi^{(k)}.$$

As usual, null homotopic morphisms form an ideal of $\mathcal{P}(\mathcal{A})$, denoted $\mathcal{I}$, and hence we may form the quotient $\mathcal{H}(\mathcal{A}) := \mathcal{P}(\mathcal{A})/\mathcal{I}$, which we will call the *homotopy category of pyramids*.

2.4. Pyramids versus complexes

Let $\mathcal{A}$ be an additive category. We denote by $\mathrm{Com}^-(\mathcal{A})$ the category of *bounded from the right* complexes over $\mathcal{A}$ and by $\mathcal{K}^-(\mathcal{A})$ the corresponding homotopy category.

The category $\mathrm{Com}^-(\mathcal{A})$ can be regarded as a subcategory of $\mathcal{P}(\mathcal{A})$ in the obvious way, that is, by identifying the complex

$$\cdots \longrightarrow M_{k-1} \xrightarrow{f_{k-1}} M_k \xrightarrow{f_k} M_{k+1} \longrightarrow \ldots \tag{2.2}$$

with the pyramid $(X_\bullet, d_\bullet, 1)$, where

$$X_{\mathbf{a}} = \begin{cases} M_k, & \mathbf{a} = k\varepsilon_1, \\ 0, & \text{otherwise;} \end{cases} \qquad \text{and} \qquad d_{\mathbf{a},i} = \begin{cases} f_k, & \mathbf{a} = k\varepsilon_1 \text{ and } i = 1, \\ 0, & \text{otherwise.} \end{cases}$$

We can also define a functor $\mathcal{F}\colon \mathcal{P}(\mathcal{A}) \to \mathrm{Com}^-(\mathcal{A})$ by sending a pyramid $(X_\bullet, d_\bullet, n)$ to the complex of the form (2.2) where

$$M_k := \bigoplus_{\mathrm{ht}(\mathbf{a})=k} X_{\mathbf{a}} \quad \text{and} \quad f_k = d^{(k)},$$

with the action of $d^{(k)}$ on M_k being the obvious one. Conditions (I) and (II) guarantee that M_k is well-defined while conditions (III) and (IV) imply that (2.2) is indeed a complex. On morphisms in $\mathcal{P}(\mathcal{A})$ the functor $\mathcal{F}$ is defined in the obvious way (using the natural action of a matrix on a direct sum whose components index the columns of the matrix).

Theorem 2.2. *The functor $\mathcal{F}$ and the inclusion of $\mathrm{Com}^-(\mathcal{A})$ into $\mathcal{P}(\mathcal{A})$ form a pair of mutually quasi-inverse equivalences of categories.*

Proof. Including and then applying $\mathcal{F}$ does nothing and hence is obviously isomorphic to the identity functor. On the other hand, given a pyramid $(X_\bullet, d_\bullet, n)$, we can define a morphism from $(X_\bullet, d_\bullet, n)$ to $\mathcal{F}(X_\bullet, d_\bullet, n)$ using the obvious inclusion of each $X_{\mathbf{a}}$ into the corresponding M_k. This gives a natural transformation from the identity functor to $\mathcal{F}$ followed by the inclusion of $\mathrm{Com}^-(\mathcal{A})$ into $\mathcal{P}(\mathcal{A})$. Projecting M_k onto every $X_{\mathbf{a}}$ defines an inverse natural transformation. Therefore applying $\mathcal{F}$ and then including is also isomorphic to the identity functor. The claim follows. □

The following is now clear by comparing the definitions.

Corollary 2.3. *The mutually inverse equivalences in Theorem 2.2 induce mutually inverse equivalences between $\mathcal{K}^-(\mathcal{A})$ and $\mathcal{H}(\mathcal{A})$.*

2.5. Tensoring pyramids

This subsection will hopefully clarify why we need pyramids. Let $\mathcal{A}$ be an additive strict monoidal category. We denote the tensor product in $\mathcal{A}$ by $\circ$ and the identity object in $\mathcal{A}$ by $\mathbb{1}$. We assume that $\circ$ is biadditive. We think of $\mathcal{A}$ as a 2-category with one object and denote by $\circ_0$ the tensor product of morphisms and by $\circ_1$ the usual composition of morphisms in $\mathcal{A}$. We would like to extend the monoidal structure on $\mathcal{A}$ to $\mathrm{Com}^-(\mathcal{A})$ and to $\mathcal{K}^-(\mathcal{A})$. However, we do not know how to do that. The problem is that to make this work one has to use the construction of taking the total complex, which involves taking direct sums. However, there is usually no *strict* distributivity in $\mathcal{A}$ and hence it is not possible to ensure strict associativity of the product of complexes. Our idea is to substitute the category of complexes by the category of pyramids where the tensor structure can be extended without taking any direct sums. Here it will also become clear how the last component of the pyramid tuple is used. The following construction is inspired by and generalizes Section 3 in [8].

For two pyramids $(X_\bullet, d_\bullet, n)$ and $(Y_\bullet, \partial_\bullet, m)$, we define their tensor product

$$(X_\bullet, d_\bullet, n) \circ (Y_\bullet, \partial_\bullet, m)$$

as the pyramid $(Z_\bullet, \urcorner_\bullet, n+m)$, where, for $\mathbf{a} \in \mathbb{I}$, we have

$$Z_{\mathbf{a}} := X_{\pi_n(\mathbf{a})} \circ Y_{\sigma_n(\mathbf{a})},$$

and, for $\mathbf{a} \in \mathbb{I}$ and $i \in \mathbb{N}$, we define

$$\urcorner_{\mathbf{a},i} := \begin{cases} d_{\pi_n(\mathbf{a}),i} \circ_0 \mathrm{id}, & \text{if } i \leq n, \\ (-1)^{\mathrm{ht}(\pi_n(\mathbf{a}))} \; \mathrm{id} \circ_0 \partial_{\sigma_n(\mathbf{a}),i} & \text{otherwise.} \end{cases}$$

Let $\alpha\colon (X_\bullet, d_\bullet, n) \to (\tilde{X}_\bullet, \tilde{d}_\bullet, \tilde{n})$ and $\beta\colon : (Y_\bullet, \partial_\bullet, m) \to (\tilde{Y}_\bullet, \tilde{\partial}_\bullet, \tilde{m})$ be morphisms of pyramids. Their tensor product

$$\alpha \circ_0 \beta : (X_\bullet, d_\bullet, n) \circ (Y_\bullet, \partial_\bullet, m) \to (\tilde{X}_\bullet, \tilde{d}_\bullet, \tilde{n}) \circ (\tilde{Y}_\bullet, \tilde{\partial}_\bullet, \tilde{m})$$

is defined by

$$(\alpha \circ_0 \beta)^{(k)}_{\mathbf{a},\mathbf{b}} := \begin{cases} \alpha^{(l)}_{\pi_n(\mathbf{a}),\pi_{\tilde{n}}(\mathbf{b})} \circ_0 \beta^{(k-l)}_{\sigma_n(\mathbf{a}),\sigma_{\tilde{n}}(\mathbf{b})}, & \text{if } l = \mathrm{ht}(\pi_n(\mathbf{a})) = \mathrm{ht}(\pi_{\tilde{n}}(\mathbf{b})), \\ 0, & \text{otherwise}, \end{cases}$$

for any $k \in \mathbb{Z}$ and any $\mathbf{a}, \mathbf{b} \in \mathbb{I}_k$. Note that, under the assumption $\mathbf{a}, \mathbf{b} \in \mathbb{I}_k$, the conditions $l = \mathrm{ht}(\pi_n(\mathbf{a})) = \mathrm{ht}(\pi_{\tilde{n}}(\mathbf{b}))$ and $k - l = \mathrm{ht}(\sigma_n(\mathbf{a})) = \mathrm{ht}(\sigma_{\tilde{n}}(\mathbf{b}))$ are equivalent.

Proposition 2.4. *The above endows $\mathcal{P}(\mathcal{A})$ with the structure of a strict monoidal category.*

Proof. We start by checking that $(Z_\bullet, \daleth_\bullet, n + m)$ is indeed a pyramid. It follows directly from the definitions that (I), (II) and (III) are satisfied. So, we only need to check (IV). Let $i, j \in \mathbb{N}$ be different. If both $i, j \leq n$, then the corresponding part of (IV) for $(Z_\bullet, \daleth_\bullet, n + m)$ follows directly from the definitions and (IV) for $(X_\bullet, d_\bullet, n)$.

Assume that both $i, j > n$. Then the anti-commutative square

$$\begin{array}{ccc} Y_{\mathbf{c}+\varepsilon_i} & \xrightarrow{\partial_{\mathbf{c}+\varepsilon_i,j}} & Y_{\mathbf{c}+\varepsilon_j+\varepsilon_i} \\ {\scriptstyle \partial_{\mathbf{c},i}}\uparrow & & \uparrow{\scriptstyle \partial_{\mathbf{c}+\varepsilon_j,i}} \\ Y_{\mathbf{c}} & \xrightarrow[\partial_{\mathbf{c},j}]{} & Y_{\mathbf{c}+\varepsilon_j} \end{array}$$

given by (IV) for $(Y_\bullet, \partial_\bullet, m)$ induces one of the following squares:

$$\begin{array}{ccc} X_{\mathbf{b}} \otimes Y_{\mathbf{c}+\varepsilon_i} & \xrightarrow{-\mathrm{id}\otimes\partial_{\mathbf{c}+\varepsilon_i,j}} & X_{\mathbf{b}} \otimes Y_{\mathbf{c}+\varepsilon_j+\varepsilon_i} \\ {\scriptstyle -\mathrm{id}\otimes\partial_{\mathbf{c},i}}\uparrow & & \uparrow{\scriptstyle -\mathrm{id}\otimes\partial_{\mathbf{c}+\varepsilon_j,i}} \\ X_{\mathbf{b}} \otimes Y_{\mathbf{c}} & \xrightarrow[-\mathrm{id}\otimes\partial_{\mathbf{c},j}]{} & X_{\mathbf{b}} \otimes Y_{\mathbf{c}+\varepsilon_j} \end{array}$$

or

$$\begin{array}{ccc} X_{\mathbf{b}} \otimes Y_{\mathbf{c}+\varepsilon_i} & \xrightarrow{\mathrm{id}\otimes\partial_{\mathbf{c}+\varepsilon_i,j}} & X_{\mathbf{b}} \otimes Y_{\mathbf{c}+\varepsilon_j+\varepsilon_i} \\ {\scriptstyle \mathrm{id}\otimes d_{\mathbf{c},i}}\uparrow & & \uparrow{\scriptstyle \mathrm{id}\otimes d_{\mathbf{c}+\varepsilon_j,i}} \\ X_{\mathbf{b}} \otimes Y_{\mathbf{c}} & \xrightarrow[\mathrm{id}\otimes\partial_{\mathbf{c},j}]{} & X_{\mathbf{b}} \otimes Y_{\mathbf{c}+\varepsilon_j} \end{array}$$

(depending on the parity of $\mathrm{ht}(\mathbf{b})$). Clearly, both of them give the corresponding part of (IV) for $(Z_\bullet, \daleth_\bullet, n + m)$.

If $i \leq n$ and $j > n$, then we obtain one of the following two situations:

$$\begin{array}{ccc}
X_{\mathbf{b}+\varepsilon_i} \otimes Y_{\mathbf{c}} & \xrightarrow{-\mathrm{id}\otimes\partial_{\mathbf{c},j}} & X_{\mathbf{b}+\varepsilon_i} \otimes Y_{\mathbf{c}+\varepsilon_j} \\
{\scriptstyle d_{\mathbf{b},i}\otimes\mathrm{id}}\uparrow & & \uparrow{\scriptstyle d_{\mathbf{b},i}\otimes\mathrm{id}} \\
X_{\mathbf{b}} \otimes Y_{\mathbf{c}} & \xrightarrow[\mathrm{id}\otimes\partial_{\mathbf{c},j}]{} & X_{\mathbf{b}} \otimes Y_{\mathbf{c}+\varepsilon_j}
\end{array}$$

or

$$\begin{array}{ccc}
X_{\mathbf{b}+\varepsilon_i} \otimes Y_{\mathbf{c}} & \xrightarrow{\mathrm{id}\otimes\partial_{\mathbf{c},j}} & X_{\mathbf{b}+\varepsilon_i} \otimes Y_{\mathbf{c}+\varepsilon_j} \\
{\scriptstyle d_{\mathbf{b},i}\otimes\mathrm{id}}\uparrow & & \uparrow{\scriptstyle d_{\mathbf{b},i}\otimes\mathrm{id}} \\
X_{\mathbf{b}} \otimes Y_{\mathbf{c}} & \xrightarrow[-\mathrm{id}\otimes\partial_{\mathbf{c},j}]{} & X_{\mathbf{b}} \otimes Y_{\mathbf{c}+\varepsilon_j}
\end{array}$$

(again, depending on the parity of $\mathrm{ht}(\mathbf{b})$). Both of them give the corresponding part of (IV) for $(Z_\bullet, \urcorner_\bullet, n+m)$.

This, together with the observation that our tensor product of morphisms produces the usual tensor product of morphisms of complexes after applying $\mathcal{F}$, implies that our tensor product is well-defined. All axioms of strict monoidal category now follow directly from our construction as soon as we observe that the unit in $\mathcal{P}(\mathcal{A})$ is the pyramid $(X_\bullet, d_\bullet, 0)$, where $X_{\mathbf{0}} = \mathbb{1}$, all other $X_{\mathbf{c}} = 0$ and all $d_{\mathbf{c},i} = 0$. □

Corollary 2.5. *The equivalences of Theorem 2.2 and Corollary 2.3 are compatible with the monoidal structure and are hence biequivalences.*

Proof. This follows directly from the definitions. □

2.6. Pyramids and strict monoidal actions

Let $\mathcal{A}$ be as in the previous subsection and $\mathcal{C}$ an additive category equipped with a strict monoidal action $\Diamond\colon \mathcal{A} \times \mathcal{C} \to \mathcal{C}$ by additive functors.

For a pyramid $(X_\bullet, d_\bullet, n) \in \mathcal{P}(\mathcal{A})$ and a pyramid $(Y_\bullet, \partial_\bullet, m) \in \mathcal{P}(\mathcal{C})$, we define

$$(X_\bullet, d_\bullet, n) \blacklozenge (Y_\bullet, \partial_\bullet, m)$$

as the pyramid $(Z_\bullet, \urcorner_\bullet, n+m) \in \mathcal{P}(\mathcal{C})$, where, for $\mathbf{a} \in \mathbb{I}$, we have

$$Z_{\mathbf{a}} := X_{\pi_n(\mathbf{a})} \Diamond Y_{\sigma_n(\mathbf{a})},$$

and, for $\mathbf{a} \in \mathbb{I}$ and $i \in \mathbb{N}$, we define

$$\urcorner_{\mathbf{a},i} := \begin{cases} d_{\pi_n(\mathbf{a}),i} \Diamond \mathrm{id}, & \text{if } i \leq n, \\ (-1)^{\mathrm{ht}(\pi_n(\mathbf{a}))} \quad \mathrm{id} \Diamond \partial_{\sigma_n(\mathbf{a}),i} & \text{otherwise.} \end{cases}$$

Let $\beta\colon (Y_\bullet, \partial_\bullet, m) \to (\tilde{Y}_\bullet, \tilde{\partial}_\bullet, \tilde{m})$ be a morphism of pyramids. We define the morphism $(X_\bullet, d_\bullet, n) \blacklozenge \beta$ as

$$\gamma : (X_\bullet, d_\bullet, n) \blacklozenge (Y_\bullet, \partial_\bullet, m) \to (X_\bullet, d_\bullet, n) \blacklozenge (\tilde{Y}_\bullet, \tilde{\partial}_\bullet, \tilde{m})$$

such that

$$(\gamma)^{(k)}_{\mathbf{a},\mathbf{b}} := \omega^{(\mathrm{ht}(\pi_n(\mathbf{a})))}_{\pi_n(\mathbf{a}),\pi_{\tilde{n}}(\mathbf{b})} \diamondsuit \beta^{(k-\mathrm{ht}(\pi_n(\mathbf{a}))}_{\sigma_n(\mathbf{a}),\sigma_{\tilde{n}}(\mathbf{b})},$$

for any $k \in \mathbb{Z}$ and any $\mathbf{a}, \mathbf{b} \in \mathbb{I}_k$ (recall the definition of $\omega^{(l)}$ from Section 2.2). This turns $(X_\bullet, d_\bullet, n)\blacklozenge\_$ into an additive endofunctor of $\mathcal{P}(\mathcal{C})$.

Finally, let $\alpha\colon (X_\bullet, d_\bullet, n) \to (\tilde{X}_\bullet, \tilde{d}_\bullet, \tilde{n})$ be a morphism of pyramids. We define

$$\alpha\blacklozenge(Y_\bullet, \partial_\bullet, m) : (X_\bullet, d_\bullet, n)\blacklozenge(Y_\bullet, \partial_\bullet, m) \to (\tilde{X}_\bullet, \tilde{d}_\bullet, \tilde{n})\blacklozenge(Y_\bullet, \partial_\bullet, m)$$

as the morphism η given by

$$(\eta)^{(k)}_{\mathbf{a},\mathbf{b}} := \alpha^{(\mathrm{ht}(\pi_n(\mathbf{a})))}_{\pi_n(\mathbf{a}),\pi_{\tilde{n}}(\mathbf{b})} \diamondsuit \omega^{(k-\mathrm{ht}(\pi_n(\mathbf{a})))}_{\sigma_n(\mathbf{a}),\sigma_{\tilde{n}}(\mathbf{b})},$$

for any $k \in \mathbb{Z}$ and any $\mathbf{a}, \mathbf{b} \in \mathbb{I}_k$.

Proposition 2.6. *The construct $\blacklozenge\colon \mathcal{P}(\mathcal{A}) \times \mathcal{P}(\mathcal{C}) \to \mathcal{P}(\mathcal{C})$ is a strict monoidal action by additive functors. This action descends to a strict monoidal action*

$$\blacklozenge : \mathcal{H}(\mathcal{A}) \times \mathcal{H}(\mathcal{C}) \to \mathcal{H}(\mathcal{C}).$$

Proof. Mutatis mutandis the proof of Proposition 2.4. □

3. Finitary 2-categories and their 2-representations

In this section we recall basic facts from the classical 2-representation theory of finitary (strict) 2-categories developed in [12]-[17], see also [11] for a survey and [10] for more details.

3.1. Finitary 2-categories

We say that a $\Bbbk$-linear category is finitary if it is a small category equivalent to the category of projective modules for some finite dimensional associative $\Bbbk$-algebra.

Following [12], a *finitary* 2-category $\mathscr{C}$ over $\Bbbk$ is a 2-category with finitely many objects in which each $\mathscr{C}(\mathtt{i}, \mathtt{j})$ is a $\Bbbk$-linear finitary category and where all compositions are (bi)additive and $\Bbbk$-linear and all identity 1-morphisms are indecomposable.

In what follows, $\mathscr{C}$ is always assumed to be a finitary 2-category over $\Bbbk$. All functors are assumed to be additive and $\Bbbk$-linear.

3.2. 2-representations

A 2-representation of $\mathscr{C}$ is a 2-functor to some fixed *target* 2-category. All 2-representations of $\mathscr{C}$ form a 2-category where 1-morphisms are strong natural transformations and 2-morphisms are modifications, see Subsection 2.3 in [14]. All 2-representations will be denoted by bold capital roman letters $\mathbf{M}$, $\mathbf{N}$, etc.

Taking, as the target 2-category, the 2-category of finitary additive $\Bbbk$-linear categories, we obtain the 2-category $\mathscr{C}$-afmod of *finitary* 2-representations of $\mathscr{C}$. Taking, as the target 2-category, the 2-category of finitary abelian $\Bbbk$-linear categories, we obtain the 2-category $\mathscr{C}$-mod of *abelian* 2-representations of $\mathscr{C}$.

There is a diagrammatic *abelianization* 2-functor $\overline{\cdot}$: $\mathscr{C}$-afmod $\to$ $\mathscr{C}$-mod, see Subsection 4.2 in [13].

For each $\mathtt{i} \in \mathscr{C}$, we have the *principal* 2-representation $\mathbf{P}_\mathtt{i} := \mathscr{C}(\mathtt{i}, {}_-)$, for which we have the usual Yoneda lemma, see Lemma 3 in [14].

3.3. Simple transitive 2-representations

A finitary 2-representation $\mathbf{M}$ of $\mathscr{C}$ is called *transitive* provided that, for any $\mathtt{i}$ and $\mathtt{j}$ and any indecomposable objects $X \in \mathbf{M}(\mathtt{i})$ and $Y \in \mathbf{M}(\mathtt{j})$, there is a 1-morphism F of $\mathscr{C}$ such that Y is isomorphic to a direct summand of $\mathbf{M}(\mathrm{F})\, X$.

A finitary 2-representation $\mathbf{M}$ of $\mathscr{C}$ is called *simple* provided that $\coprod_{\mathtt{i}\in\mathscr{C}} \mathbf{M}(\mathtt{i})$ does not have any non-zero proper $\mathscr{C}$-invariant ideals (cf. Subsection 2.6 in [16]). We note that simplicity implies transitivity, however, we will always speak about *simple transitive* 2-representations. There is a weak Jordan–Hölder theory for finitary 2-representations of $\mathscr{C}$ developed in [16].

3.4. Cells and cell 2-representations

For indecomposable 1-morphisms F and G in $\mathscr{C}$, we write $\mathrm{F} \geq_L \mathrm{G}$ provided that F is isomorphic to a direct summand of $\mathrm{H} \circ \mathrm{G}$, for some 1-morphism H. This defines the *left* preorder $\geq_L$, equivalence classes of which are called *left cells*. Similarly one defines the *right* preorder $\geq_R$ and *right cells*, and also the *two-sided* preorder $\geq_J$ and *two-sided cells*.

A two-sided cell $\mathcal{J}$ is called *strongly regular* provided that no two of its left (or right) cells are comparable with respect to the left (respectively right) order and the intersection of any left and any right cell in $\mathcal{J}$ contains precisely one element.

Given a left cell $\mathcal{L}$, there is a unique $\mathtt{i}$ such that all 1-morphisms in $\mathcal{L}$ start at $\mathtt{i}$. The corresponding *cell 2-representation* $\mathbf{C}_\mathcal{L}$ is defined as the subquotient of $\mathbf{P}_\mathtt{i}$ obtained by taking the unique simple transitive quotient of the subrepresentation of $\mathbf{P}_\mathtt{i}$ given by the additive closure of all 1-morphisms F such that $\mathrm{F} \geq_L \mathcal{L}$. The 2-representation $\mathbf{C}_\mathcal{L}$ is simple transitive. We refer to Subsection 6.5 in [13] for details.

If $\mathbf{M}$ is a simple transitive 2-representation of $\mathscr{C}$, then the set of two-sided cells whose elements do not annihilate $\mathbf{M}$ contains a unique maximal element called the *apex* of $\mathbf{M}$, see Subsection 3.2 in [2].

3.5. Bookkeeping tools

Let $\mathbf{M}$ be a finitary 2-representation of $\mathscr{C}$. Then, to each 1-morphism F, we can associate a matrix $[\mathrm{F}]$ with non-negative integer coefficients, whose rows and columns are indexed by isomorphism classes of indecomposable objects in

$$\mathcal{M} := \coprod_{\mathtt{i}} \mathbf{M}(\mathtt{i}),$$

and the $X \times Y$-entry gives the multiplicity of X as a direct summand of $\mathbf{M}(\mathrm{F})\, Y$.

If we additionally know that $\overline{\mathbf{M}}(\mathrm{F})$ is exact, we also have the matrix $[\![\mathrm{F}]\!]$ with non-negative integer coefficients, whose rows and columns are indexed by isomorphism classes of simple objects in $\overline{\mathcal{M}}$ and the $X \times Y$-entry gives the composition multiplicity of X in $\overline{\mathbf{M}}(\mathrm{F})\,Y$.

If (F,G) is an adjoint pair of 1-morphisms, then $\overline{\mathbf{M}}(\mathrm{G})$ is exact and $[\mathrm{F}]^t = [\![\mathrm{G}]\!]$, see Lemma 10 in [16].

4. The 2-category $\mathscr{D}_A$ and its 2-representations

4.1. Definition of $\mathscr{D}_A$

Let $\Bbbk$ be an algebraically closed field and A a connected, basic, finite dimensional associative (unital) $\Bbbk$-algebra. Let $\mathcal{C}$ be a small category equivalent to A-mod. As usual, we denote by $*$ the $\Bbbk$-duality $\mathrm{Hom}_\Bbbk({}_-,\Bbbk)$. We write $\mathrm{add}(X)$ for the closure under direct sums and direct summands of an object X. We define the 2-category $\mathscr{D}_A = \mathscr{D}_{A,\mathcal{C}}$ to have

- one object $\mathtt{i}$ (which we identify with $\mathcal{C}$);
- as 1-morphisms all endofunctors of $\mathcal{C}$ isomorphic to tensoring with A-A-bimodules in $\mathrm{add}\big(A \oplus (A \otimes_\Bbbk A) \oplus (A^* \otimes_\Bbbk A)\big)$;
- as 2-morphisms all natural transformations of functors.

We denote by F and G the functors given by tensoring with $A \otimes_\Bbbk A$ and $A^* \otimes_\Bbbk A$, respectively. We have the multiplication table for these functors given by

(4.1)

$X \setminus Y$	F	G
F	$\mathrm{F}^{\oplus \dim(A)}$	$\mathrm{F}^{\oplus \dim(A)}$
G	$\mathrm{G}^{\oplus \dim(A)}$	$\mathrm{G}^{\oplus \dim(A)}$

and an adjoint pair (F,G) of 1-morphisms in $\mathscr{D}_A$ (see e.g. [12], Section 7.3).

Since our main theorem (Theorem 5.2) is trivial if $A \cong \Bbbk$, and furthermore, covered by the main theorem of [16] (as $\Bbbk$ is a symmetric algebra), we will from now on assume that $\dim_\Bbbk(A) > 1$.

Let $\epsilon_1, \epsilon_2, \dots, \epsilon_n$ be a complete list of pairwise orthogonal primitive idempotents in A. The 2-category $\mathscr{D}_A$ has two two-sided cells, one which only consists of the identity 1-morphism, and the unique two-sided cell $\mathcal{J}$ that consists of the indecomposable constituents of F and G. The latter correspond to $A\epsilon_i \otimes_\Bbbk \epsilon_j A$ and $A^*\epsilon_i \otimes_\Bbbk \epsilon_j A$, for $i,j = 1,2,\dots,n$, respectively. Left cells in $\mathscr{D}_A$ are indexed by $j = 1,\dots,n$, with $\mathcal{L}_j$ consisting of 1-morphisms corresponding to $A\epsilon_i \otimes_\Bbbk \epsilon_j A$ and $A^*\epsilon_i \otimes_\Bbbk \epsilon_j A$, for this fixed j. Right cells are given by $\mathcal{R}_i$, for $i = 1,\dots,n$, consisting of 1-morphisms corresponding to $A\epsilon_i \otimes_\Bbbk \epsilon_j A$, for this fixed i, and $\mathcal{R}'_i$, for $i = 1,\dots,n$, consisting of 1-morphisms corresponding to $A^*\epsilon_i \otimes_\Bbbk \epsilon_j A$, for this fixed i. Note that some $\mathcal{R}_i$ might coincide with some $\mathcal{R}'_j$ (which happens when A has projective-injective modules).

Proposition 4.1. *The 2-category $\mathscr{D}_A$ is $\mathcal{J}$-*simple *in the sense that any non-zero 2-ideal of $\mathscr{D}_A$ contains the identity 2-morphisms for all 1-morphisms in $\mathcal{J}$.*

Proof. Given a non-zero endomorphism of $\mathrm{F}\oplus\mathrm{G}$ corresponding to an A-A-bimodule homomorphism

$$\varphi : (A \oplus A^*) \otimes_{\Bbbk} A \to (A \oplus A^*) \otimes_{\Bbbk} A,$$

we can pre- and post-compose it with the identity on $A \otimes_{\Bbbk} A$ to obtain a non-zero non-radical endomorphism of a direct sum of copies of F. Indeed, $\mathrm{id}_{A\otimes_{\Bbbk} A} \circ_0 \varphi \circ_0 \mathrm{id}_{A\otimes_{\Bbbk} A}$ is an endomorphism of $A \otimes_{\Bbbk} (A \oplus A^*) \otimes_{\Bbbk} A \otimes_{\Bbbk} A$ acting only on the middle two tensor factors, which serve as multiplicity spaces. The claim follows. □

4.2. Cell 2-representations of $\mathscr{D}_A$

Let $\mathbf{N}$ denote the 2-representation of $\mathscr{D}_A$ given by the natural action of $\mathscr{D}_A$ on the additive category generated by all projective and all injective objects in $\mathcal{C}$.

Proposition 4.2. *Let $\mathcal{L}$ be a left cell in $\mathcal{J}$. Then $\mathbf{C}_{\mathcal{L}}$ is equivalent to $\mathbf{N}$.*

Proof. Without loss of generality, we may assume that $\mathcal{L} = \mathcal{L}_1$ (as described in the previous subsection), i.e. 1-morphisms in $\mathcal{L}$ correspond to the bimodules in the additive closure of $(A \oplus A^*) \otimes_{\Bbbk} \epsilon_1 A$.

We denote by $\mathbf{M}$ the defining 2-representation of $\mathscr{D}_A$ on $\mathcal{C}$.

Let L_1 be a simple object in $\mathcal{C}$ corresponding to ϵ_1. Then we have a unique morphism $\Phi\colon \mathbf{P}_{\mathtt{i}} \to \mathbf{M}$ sending $\mathbb{1}_{\mathtt{i}}$ to L_1. For $\mathrm{H} \in \mathcal{L}$, we have $\mathrm{H}\, L_1 \in \mathbf{N}(\mathtt{i})$, so Φ restricts to a morphism of 2-representations from the 2-representation $\mathbf{R}$ on the additive closure of 1-morphisms in $\mathcal{L}$ to $\mathbf{N}$. By the usual argument, see e.g. [13], Proposition 22, Φ induces an equivalence from $\mathbf{C}_{\mathcal{L}}$ to $\mathbf{N}$. Explicitly, the sum of indecomposable 1-morphisms in $\mathcal{L}$ acts via $\mathbf{N}$ by tensoring with $(A \oplus A^*) \otimes_{\Bbbk} \epsilon_1 A$, and when evaluated on L_1, just produces $A \oplus A^*$, an additive generator of $\mathbf{N}$. In terms of 2-morphisms, the kernel of the surjective evaluation map is precisely given by $\mathrm{End}_A(A \oplus A^*) \otimes \mathrm{rad}(\epsilon_1 A \epsilon_1)$, which is the unique maximal left ideal factored out when constructing the $\mathbf{C}_{\mathcal{L}}$ as a quotient of $\mathbf{R}$. □

4.3. Simple transitive 2-representations of $\mathscr{D}_A$

Here we formulate our main result about the 2-category $\mathscr{D}_A$.

Theorem 4.3. *Each simple transitive 2-representation of $\mathscr{D}_A$ is equivalent to a cell 2-representation.*

Before proving this theorem, we need some preparation.

4.4. Some quasi-idempotent bimodules

For a positive integer k, we denote by $\underline{k}$ the set $\{1, 2, \ldots, k\}$. Let B be a finite dimensional associative (unital) $\Bbbk$-algebra. Let $M_1, M_2, \ldots, M_k$ be a list of pairwise non-isomorphic indecomposable left B-modules. Let ${}_1N, {}_2N, \ldots, {}_lN$ be a list of pairwise non-isomorphic indecomposable right B-modules. Let H be a B-B-bimodule of the form

$$H = \bigoplus_{i=1}^{k} \bigoplus_{j=1}^{l} (M_i \otimes_{\Bbbk} {}_jN)^{\oplus h_{i,j}}, \tag{4.2}$$

where all $h_{i,j} \in \mathbb{Z}_{\geq 0}$.

Proposition 4.4. *Assume that the following conditions are satisfied.*

(a) $H \otimes_B H \cong H^{\oplus d}$*, for some* $d \in \mathbb{N}$.

(b) *For each* $i, j \in \{1, 2, \ldots, k\}$*, the module* M_i *is isomorphic to a direct summand of* $H \otimes_B M_j$.

(c) *There is a decomposition* $H \cong H_1 \oplus H_2$ *of* B-B*-bimodules such that we have* $H_1 \otimes_B H \cong H_1^{\oplus d'}$*, for some* $d' \in \mathbb{N}$.

Then $H_1 \cong M \otimes_{\Bbbk} N$*, for some left* B*-module* M *and some right* B*-module* N.

Proof. Define the $k \times l$-matrix $\mathtt{H} = (h_{i,j})_{i\in\underline{k}}^{j\in\underline{l}}$ describing the multiplicities in (4.2). Define a $l \times k$-matrix $\mathtt{C} = (c_{j,i})_{j\in\underline{l}}^{i\in\underline{k}}$ via $c_{j,i} := \dim({}_jN \otimes_B M_i)$. From (a), we deduce that

$$\mathtt{HCH} = d\mathtt{H} \tag{4.3}$$

and hence also $\mathtt{HCHC} = d\mathtt{HC}$.

The matrix $\mathtt{HC}$ describes multiplicities of M_i in $H \otimes_B M_j$, for $i, j \in \underline{k}$, and hence is positive by (b). From $(\mathtt{HC})^2 = d\mathtt{HC}$ and Proposition 4.1 in [22] we see that $\operatorname{rank}(\mathtt{HC}) = 1$. Now, (4.3) implies that $\operatorname{rank}(\mathtt{H}) = 1$.

Define the $k \times l$-matrix $\mathtt{H}_1 = (h^{(1)}_{i,j})_{i\in\underline{k}}^{j\in\underline{l}}$ describing the multiplicities of $M_i \otimes_{\Bbbk} {}_jN$ in H_1. By (c), we also have $\mathtt{H}_1\mathtt{CH} = d'\mathtt{H}_1$ and thus $\operatorname{rank}(\mathtt{H}_1) = 1$. Write $\mathtt{H}_1 = vw^t$, for some $v \in \mathbb{Z}^k_{\geq 0}$ and $w \in \mathbb{Z}^l_{\geq 0}$. Then, for

$$M := \bigoplus_{i=1}^{k} M_i^{\oplus v_i} \quad \text{and} \quad N := \bigoplus_{j=1}^{l} {}_jN^{\oplus w_j},$$

we obtain $H_1 \cong M \otimes_{\Bbbk} N$. □

4.5. Proof of Theorem 4.3

Proof. Let $\mathbf{M}$ be a simple transitive 2-representation of $\mathscr{D}_A$. If all 1-morphisms in $\mathcal{J}$ annihilate $\mathbf{M}$, then $\mathbf{M}$ is a cell 2-representation by Theorem 18 in [16].

Assume now that $\mathbf{M}$ has apex $\mathcal{J}$. We denote by B the basic algebra such that $\mathbf{M}(\mathtt{i})$ is equivalent to B-proj. Let $e_1, e_2, \ldots, e_m$ be a complete set of pairwise orthogonal primitive idempotents in B. Then the Cartan matrix of B is

$$\mathtt{C} := \left(\dim(e_iBe_j)\right)_{i\in\underline{m}}^{j\in\underline{m}}.$$

Let X and Y be the B-B-bimodules corresponding to the actions of $\mathbf{M}(\mathrm{F})$ and $\mathbf{M}(\mathrm{G})$, respectively. By Theorem 11 (i) in [6] (which does not make use of the flatness assumption), both X and Y have the property that the B-modules $X \otimes_B L$ and $Y \otimes_B L$ are projective, for any B-module L. Therefore, by Theorem 1 in [18], both X and Y are of the form

$$\bigoplus_{i=1}^{m} Be_i \otimes_{\Bbbk} {}_iN,$$

for some right B-modules ${}_iN$. By Lemma 4 in [19] for $\mathrm{K} = \mathbf{M}(\mathrm{F})$ and $\mathrm{K}' = \mathbf{M}(\mathrm{G})$, the latter is exact and hence all of the right tensor factors in Y are projective. By Lemma 8 in [19] for $\mathrm{K} = \mathbf{M}(\mathrm{F})$ and $\mathrm{H} = \mathbf{M}(\mathrm{G})$, the functor $\mathbf{M}(\mathrm{FG})$ is a projective functor, and hence so is $\mathbf{M}(\mathrm{F})$ as a direct summand in it. Therefore, all ${}_iN$ are right projective.

By transitivity of $\mathbf{M}$ and (4.1), we can apply Proposition 4.4 both to the pair $H = X \oplus Y$ and $H_1 = X$ and to the pair $H = X \oplus Y$ and $H_1 = Y$. By Proposition 4.4, we can write $X \cong M \otimes_{\Bbbk} N$, where M is left B-projective and N is right B-projective. Similarly, we can write $Y \cong M' \otimes_{\Bbbk} N'$, where M' is left B-projective and N' is right B-projective. Define $\mathbf{a}, \mathbf{b}, \mathbf{a}', \mathbf{b}' \in \mathbb{Z}^m_{\geq 0}$ by

$$M \cong \bigoplus_{i=1}^{m} Be_i^{\oplus a_i}, \quad M' \cong \bigoplus_{i=1}^{m} Be_i^{\oplus a'_i}, \quad N \cong \bigoplus_{i=1}^{m} e_iB^{\oplus b_i}, \quad N' \cong \bigoplus_{i=1}^{m} e_iB^{\oplus b'_i}. \tag{4.4}$$

Set $d := \dim(A)$. On the one hand, $X \otimes_B Y \cong X^{\oplus d}$ and, on the other hand,

$$X \otimes_B Y \cong M \otimes_{\Bbbk} N \otimes_B M' \otimes_{\Bbbk} N' \cong \big(M \otimes_{\Bbbk} N'\big)^{\oplus \mathbf{b}^t \mathtt{C} \mathbf{a}'}.$$

Consequently

$$d\mathbf{b} = \mathbf{b}^t \mathtt{C} \mathbf{a}' \mathbf{b}'. \tag{4.5}$$

Similarly, on the one hand, $Y \otimes_B X \cong Y^{\oplus d}$ and, on the other hand,

$$Y \otimes_B X \cong M' \otimes_{\Bbbk} N' \otimes_B M \otimes_{\Bbbk} N \cong \big(M' \otimes_{\Bbbk} N\big)^{\oplus (\mathbf{b}')^t \mathtt{C} \mathbf{a}}.$$

Therefore

$$d\mathbf{b}' = (\mathbf{b}')^t \mathtt{C} \mathbf{a} \mathbf{b}. \tag{4.6}$$

Due to the adjunction (F, G), we have $[\mathbf{M}(\mathrm{F})]^t = [\![\mathbf{M}(\mathrm{G})]\!]$. Using (4.4), directly from the definitions we deduce that the i,j-th component of the matrix $[\mathbf{M}(\mathrm{F})]$ is

$$\sum_{r=1}^{m} a_i \dim(e_r B e_j)\, b_r = \sum_{r=1}^{m} a_i\, c_{r,j}\, b_r$$

and therefore $[\mathbf{M}(\mathrm{F})] = \mathbf{a}\mathbf{b}^t \mathtt{C}$ which yields $[\mathbf{M}(\mathrm{F})]^t = \mathtt{C}^t \mathbf{b} \mathbf{a}^t$. Similarly, we have $[\![\mathbf{M}(\mathrm{G})]\!] = \mathtt{C}\mathbf{a}'(\mathbf{b}')^t$. This implies

$$\mathtt{C}^t \mathbf{b} \mathbf{a}^t = \mathtt{C} \mathbf{a}' (\mathbf{b}')^t. \tag{4.7}$$

By adjunction, we have

$$\mathrm{End}_{B\text{-}B}(M \otimes_{\Bbbk} N) \cong \mathrm{End}_{B\text{-}}(M) \otimes_{\Bbbk} \mathrm{End}_{\text{-}B}(N),$$

and hence

$$\dim(\mathrm{End}_{B\text{-}B}(M \otimes_{\Bbbk} N)) = \dim(\mathrm{End}_{B\text{-}}(M)) \cdot \dim(\mathrm{End}_{\text{-}B}(N)).$$

From (4.4) we obtain

$$\dim(\mathrm{End}_{B\text{-}}(M)) = \mathbf{a}^t\mathtt{C}\mathbf{a} \quad \text{and} \quad \dim(\mathrm{End}_{\text{-}B}(N)) = \mathbf{b}^t\mathtt{C}\mathbf{b}.$$

This allows us to compute

$$\begin{aligned}\dim(\mathrm{End}_{B\text{-}B}(M \otimes_{\Bbbk} N)) &= (\mathbf{a}^t\mathtt{C}\mathbf{a})(\mathbf{b}^t\mathtt{C}\mathbf{b}) = (\mathbf{b}^t\mathtt{C}\mathbf{b})(\mathbf{a}^t\mathtt{C}\mathbf{a}) = (\mathbf{b}^t\mathtt{C}\mathbf{b})^t(\mathbf{a}^t\mathtt{C}\mathbf{a})\\ &= (\mathbf{b}^t\mathtt{C}^t\mathbf{b})(\mathbf{a}^t\mathtt{C}\mathbf{a}) = \mathbf{b}^t\mathtt{C}\mathbf{a}'(\mathbf{b}')^t\mathtt{C}\mathbf{a},\end{aligned}$$

where in the third equality we used that the transpose of a number is the same number, and in the last one we used (4.7). We have no representation theoretic interpretation for this crucial computation. Then we have

$$(\mathbf{b}^t\mathtt{C}\mathbf{a}')((\mathbf{b}')^t\mathtt{C}\mathbf{a})\mathbf{b} \overset{(4.6)}{=} d(\mathbf{b}^t\mathtt{C}\mathbf{a}')\mathbf{b}' \overset{(4.5)}{=} d^2\,\mathbf{b}.$$

As $\mathbf{b} \neq 0$, it follows that $(\mathbf{b}^t\mathtt{C}\mathbf{a}')((\mathbf{b}')^t\mathtt{C}\mathbf{a}) = d^2$ and hence

$$\dim(\mathrm{End}_{B\text{-}B}(M \otimes_{\Bbbk} N)) = \dim(A^{\mathrm{op}} \otimes A).$$

Due to Proposition 4.1, the 2-functor $\mathbf{M}$ induces an embedding of $A^{\mathrm{op}} \otimes A$, which is the endomorphism algebra of F, into $\mathrm{End}_{B\text{-}B}(X)$. This embedding must be an isomorphism by the above dimension count. As A is basic, the algebra $A^{\mathrm{op}} \otimes A$ is also basic and hence

$$\mathrm{End}_{B\text{-}B}(M \otimes_{\Bbbk} N) \cong \mathrm{End}_{B\text{-}}(M) \otimes_{\Bbbk} \mathrm{End}_{\text{-}B}(N)$$

is basic as well. This means that both M and N are basic. Moreover, since primitive idempotents in $A \otimes A^{\mathrm{op}}$ and $\mathrm{End}_{B\text{-}B}(X)$ correspond, each indecomposable direct summand of F is represented by an indecomposable projective B-B-bimodule. As (F, G) is an adjoint pair, the same is true for G by taking component-wise adjunctions. Therefore, all indecomposable 1-morphisms in $\mathcal{J}$ correspond to indecomposable projective B-B-bimodules.

Let $\mathcal{L}$ be a left cell in $\mathcal{J}$ and L a simple object in $\overline{\mathbf{M}}(\mathtt{i})$ which is not annihilated by 1-morphisms in $\mathcal{L}$. Such L exists since otherwise all 1-morphisms in $\mathcal{J}$ would act as zero. Let $\mathrm{K}_1, \mathrm{K}_2, \dots, \mathrm{K}_s$ be a complete list of pairwise non-isomorphic 1-morphisms in $\mathcal{L}$ and

$$\mathrm{K} := \mathrm{K}_1 \oplus \mathrm{K}_2 \oplus \cdots \oplus \mathrm{K}_s.$$

Then $\mathbf{M}(\mathrm{K})\,L$ is a basic projective generator of $\overline{\mathbf{M}}(\mathtt{i})$.

We have the evaluation morphism

$$\Phi : \mathrm{End}_{\mathscr{D}_A}(\mathrm{K}) \to \mathrm{End}_{\overline{\mathbf{M}}(\mathtt{i})}(\mathbf{M}(\mathrm{K})\,L).$$

Similarly, consider the simple transitive 2-representation $\mathbf{C}_{\mathcal{L}}$, and a simple object L' in its abelianization corresponding to the indecomposable 1-morphism $A\epsilon_1 \otimes_{\Bbbk} \epsilon_1 A$. Let Q be the underlying algebra of $\mathbf{C}_{\mathcal{L}}$. Then, by construction of cell 2-representations, the kernel of the corresponding evaluation morphism

$$\Psi : \mathrm{End}_{\mathscr{D}_A}(\mathrm{K}) \to \mathrm{End}_{\overline{\mathbf{C}_{\mathcal{L}}}(\mathtt{i})}(\mathbf{C}_{\mathcal{L}}(\mathrm{K})\, L'),$$

for the cell 2-representation $\mathbf{C}_{\mathcal{L}}$, which corresponds to the natural quotient morphism $Q \otimes \epsilon_1 A\epsilon_1 \twoheadrightarrow Q \otimes \epsilon_1 A\epsilon_1/\mathrm{rad}(\epsilon_1 A\epsilon_1)$, is the unique maximal left 2-ideal. Therefore the kernel of Φ is contained in the kernel of Ψ. Hence, we have a surjective map from the image of Φ, which is a subalgebra of B, to Q. At the same time, the above computation shows that the Cartan matrix of Q and that of B coincide (as both encode the structure constants of multiplication of 1-morphisms in $\mathcal{J}$). Consequently, the kernel of Φ must coincide with the kernel of Ψ and $Q \cong B$.

We have a unique homomorphism from $\mathbf{P}_{\mathtt{i}}$ to $\overline{\mathbf{M}}(\mathtt{i})$ sending $\mathbb{1}_{\mathtt{i}}$ to L. By the above, this restricts to an equivalence between $\mathbf{C}_{\mathcal{L}}$ and $\mathbf{M}$. The proof is complete. □

5. The 2-category $\mathscr{C}_A$ and its 2-representations

5.1. Definition of $\mathscr{C}_A$

Let A and $\mathcal{C}$ be as in Subsection 4.1. Define the 2-category $\mathscr{C}_A = \mathscr{C}_{A,\mathcal{C}}$ to have

- one object $\mathtt{i}$ (which we identify with $\mathcal{C}$);
- as 1-morphisms all endofunctors of $\mathcal{C}$ isomorphic to tensoring with A-A-bimodules in $\mathrm{add}\big(A \oplus (A \otimes_{\Bbbk} A)\big)$;
- as 2-morphisms all natural transformations of functors.

Note that, by definition, $\mathscr{C}_A$ is a 2-subcategory of $\mathscr{D}_A$.

We denote by F the functor corresponding to tensoring with $A \otimes_{\Bbbk} A$. We also denote by $\mathcal{J}'$ the two-sided cell for $\mathscr{C}_A$ that does not contain the identity 1-morphism.

5.2. Cell 2-representations of $\mathscr{C}_A$

Here we formulate a similar statement to Proposition 4.2. Let $\mathbf{N}$ denote the 2-representation of $\mathscr{C}_A$ given by the natural action of $\mathscr{C}_A$ on the additive category generated by all projective objects in $\mathcal{C}$.

Proposition 5.1. *Let $\mathcal{L}'$ be a left cell in $\mathcal{J}'$. Then $\mathbf{C}_{\mathcal{L}'}$ is equivalent to $\mathbf{N}$.*

Proof. Mutatis mutandis the proof of Proposition 4.2. □

5.3. Simple transitive 2-representations of $\mathscr{C}_A$

Our main result is the following statement.

Theorem 5.2. *Each simple transitive 2-representation of $\mathscr{C}_A$ is equivalent to a cell 2-representation.*

Special cases of this result were obtained in [16], Theorem 15, [17], Theorem 33, [19], Theorem 1, [18], Theorem 6, [20], Theorem 19, and [25], Theorem 3.1.

Proof. Let $\mathbf{M}$ be a simple transitive 2-representation of $\mathscr{C}_A$. If all 1-morphisms in $\mathcal{J}'$ annihilate $\mathbf{M}$, then $\mathbf{M}$ is a cell 2-representation by [16], Theorem 18.

Assume now that the apex of $\mathbf{M}$ is $\mathcal{J}'$. Let $\mathcal{L}'$ be a left cell in $\mathcal{J}'$. Consider the 2-category $\mathcal{H}(\mathscr{C}_A)$ and its action on $\mathcal{H}(\mathbf{M}(\mathtt{i}))$. Let

$$\cdots \to Q_2 \to Q_1 \to Q_0 \to 0$$

be a projective resolution of the A-A-bimodule $A^* \otimes_{\Bbbk} A$. Let Q be an object in $\mathcal{H}(\mathscr{C}_A)$ which corresponds to this resolution under the biequivalence between $\mathcal{H}(\mathscr{C}_A)$ and $\mathcal{K}^-(\mathrm{add}\big(A \oplus (A \otimes_{\Bbbk} A)\big))$ in Corollary 2.5.

Note that all 1-morphisms in $\mathscr{D}_A$ correspond to right A-projective A-A-bimodules and hence define exact endofunctors of both $\mathcal{C}$ and its derived category $\mathcal{D}^-(\mathcal{C})$. Let $\mathscr{A}$ be the 2-subcategory of $\mathcal{H}(\mathscr{C}_A)$ generated by Q and F. Then $\mathscr{A}$ acts, after applying $\mathcal{F}$ from Theorem 2.2, on $\mathcal{D}^-(\mathcal{C})$ by functors which are isomorphic to the corresponding functors in $\mathscr{D}_A$. As both actions are 2-full and 2-faithful, this induces a biequivalence between $\mathscr{D}_A$ and $\mathscr{A}$.

Denote by $\mathbf{N}(\mathtt{i})$ the additive closure in $\mathcal{H}(\mathbf{M}(\mathtt{i}))$ of $\mathbf{M}(\mathtt{i})$ and $\mathrm{Q}\,\mathbf{M}(\mathtt{i})$. By construction, this is a finitary additive 2-representation of $\mathscr{A}$. Note that the original 2-representation $\mathbf{M}$ of $\mathscr{C}_A$ is a 2-subrepresentation of the restriction of $\mathbf{N}$ to $\mathscr{C}_A$. Let $\mathbf{N}'$ be the simple transitive 2-subquotient of $\mathbf{N}$ containing this copy of $\mathbf{M}$.

By Theorem 4.3, every simple transitive 2-representation of $\mathscr{A}$ is a cell 2-representation. In particular, $\mathbf{N}'$ must be equivalent to $\mathbf{C}_{\mathcal{L}}$, where $\mathcal{L}$ is a left cell in $\mathcal{J}$. Since

$$A \otimes_{\Bbbk} A \otimes_A (A \oplus A^*) \otimes_{\Bbbk} \epsilon_i A \subset \mathrm{add}(A \otimes_{\Bbbk} \epsilon_i A),$$

the restriction of $\mathbf{C}_{\mathcal{L}}$ to $\mathscr{C}_A$ contains a unique simple transitive subquotient with apex $\mathcal{J}'$ given by the action of $\mathscr{C}_A$ on the additive closure in $\mathbf{C}_{\mathcal{L}}$ of all elements corresponding to the unique left cell $\mathcal{L}'$ of $\mathscr{C}_A$ contained in $\mathcal{L}$. By construction, the latter is equivalent to the cell 2-representation of $\mathscr{C}_A$ for $\mathcal{L}'$. The claim follows. □

Remark 5.3. Theorem 5.2 admits a straightforward generalization to the case when A is not connected. In this general case objects of $\mathscr{C}_A$ are in a one-to-one correspondence with connected components of A.

5.4. A characterization of $\mathscr{C}_A$

In this subsection we give a characterization of 2-categories of the form $\mathscr{C}_A$ inside the class of finitary 2-categories.

Theorem 5.4. *Let $\mathscr{C}$ be a finitary 2-category. Assume that the following conditions are satisfied.*

(a) *$\mathscr{C}$ has one object $\mathtt{i}$ and exactly two two-sided cells, namely, one consisting of the identity 1-morphism and one other, called $\mathcal{J}$.*

(b) *$\mathcal{J}$ is strongly regular and has the same number of left cells as of right cells.*

(c) *$\mathscr{C}$ is $\mathcal{J}$-simple.*

(d) *There is a left cell $\mathcal{L}$ in $\mathcal{J}$ such that the corresponding cell 2-representation $\mathbf{C}_{\mathcal{L}}$ is exact and 2-full. We denote by A the algebra underlying $\mathbf{C}_{\mathcal{L}}$.*

Then $\mathscr{C}$ is biequivalent to $\mathscr{C}_A$.

Proof. We consider the cell 2-representation $\mathbf{C}_{\mathcal{L}}$ of $\mathscr{C}$. It is simple transitive by construction. By Theorem 11 (i) in [6], all indecomposable 1-morphisms in $\mathcal{J}$ act on $\overline{\mathbf{C}}_{\mathcal{L}}(\mathtt{i})$ as functors which send any object to a projective object. By Theorem 1 in [18], all indecomposable 1-morphisms in $\mathcal{J}$ act on $\overline{\mathbf{C}}_{\mathcal{L}}(\mathtt{i})$ as functors isomorphic to tensoring with $\Bbbk$-split bimodules. Thanks to the exactness part in (d), all indecomposable 1-morphisms in $\mathcal{J}$ act on $\overline{\mathbf{C}}_{\mathcal{L}}(\mathtt{i})$ as projective functors, moreover, as indecomposable projective functors due to the 2-fullness part of (d).

By $\mathcal{J}$-simplicity, the 2-representation $\mathbf{C}_{\mathcal{L}}$ is 2-faithful. Condition (b) guarantees that all indecomposable projective functors on $\overline{\mathbf{C}}_{\mathcal{L}}(\mathtt{i})$ are in the essential image of $\overline{\mathbf{C}}_{\mathcal{L}}$. Now (d) implies that this 2-representation is also 2-full and hence induces a biequivalence with $\mathscr{C}_A$. □

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