



A weak comparison principle in tubular neighbourhoods of embedded manifolds

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Abstract. We study weak solutions to degenerate quasilinear elliptic equations, involving first order terms, in unbounded tubular domains. In particular we show that, under suitable hypotheses, the weak comparison principle holds if the domain is narrow enough.

1. Introduction and statement of the main results

We want to study the *weak comparison principle* for degenerate quasilinear elliptic equations with first order terms. More precisely, we consider $u, v \in C^1(\Omega) \cap C^0(\bar{\Omega}) \cap L^\infty(\Omega)$, weak solutions to the problem

$$(1.1) \quad \begin{cases} -\operatorname{div}(a(z)\nabla u) + \Lambda |\nabla u|^q \leq f(z, u) & \text{in } \Omega, \\ -\operatorname{div}(a(z)\nabla v) + \Lambda |\nabla v|^q \geq f(z, v) & \text{in } \Omega, \end{cases}$$

where $q \geq 1$, $\Omega \subseteq \mathbb{R}^n$ and denoting with z a point in \mathbb{R}^n . Equivalently for smooth solutions, by using the divergence theorem, (1.1) can be rephrased as

$$(1.2) \quad \int_{\Omega} a(z)\nabla u \cdot \nabla \psi + \Lambda |\nabla u|^q \psi \, dz \leq \int_{\Omega} f(z, u) \psi \, dz,$$

$$(1.3) \quad \int_{\Omega} a(z)\nabla v \cdot \nabla \psi + \Lambda |\nabla v|^q \psi \, dz \geq \int_{\Omega} f(z, v) \psi \, dz,$$

for every test-function $\psi \in C_c^\infty(\Omega)$ with $\psi \geq 0$. In all the paper, $a(z)$ is a nonnegative weight satisfying Assumption A1. In particular, since $a(z)$ may vanish, the operator could be degenerate at some points, causing that solutions could be not of class C^2 . In such a case, (1.2) and (1.3) represent the right way of understanding our problem and in fact all our proofs are carried out exploiting only the weak formulation.

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As customary in the literature, we say that the *weak comparison principle* for problem (1.1) holds in Ω if, imposing that $u \leq v$ on $\partial\Omega$, then it follows that $u \leq v$ in the whole of Ω . A leading example is provided by the *weak maximum principle*, namely by the case when $v = 0$. It is well known that the maximum principle does not hold in general domains, but it holds if the domain is sufficiently small or after imposing suitable a-priori estimates on the solutions. For weak solutions this can be seen e.g. as a consequence of the Poincaré inequality; on the other hand, if one considers smooth solutions then direct methods are also available, see [13].

In the present work we deal with the case of unbounded domains. Here the situation is quite well understood for domains with flat boundary and in particular for narrow strips. In fact, maximum and comparison principles in strips have many applications to the study of symmetry and monotonicity properties of the solutions, see for instance [2], [4], [3], [1], and especially [6]. The case of domains with a possible different geometry has been considered in [8], where deep results have been obtained by means of Alexandroff–Bakelmann–Pucci-type estimates.

In our analysis, we suppose that the (possibly unbounded and) smooth domain $\Omega \subset \mathbb{R}^n$ is contained in a normal tube $\mathcal{B}(M, \varepsilon)$, where $M \subset \mathbb{R}^n$ is a smoothly embedded submanifold. Our crucial idea is to manipulate the inequalities (1.2) and (1.3) by using the so-called *Fermi coordinates* associated with the normal tube $\mathcal{B}(M, \varepsilon)$, see Chapter 2 in [14], Section 3 in [20]. The weak comparison principle then follows by showing that $\tilde{\mathcal{L}}_R = 0$ for all $R > 0$, where

$$\tilde{\mathcal{L}}_R := \int_{\Omega_{2R}} a(z) (u - v)^+ |\nabla(u - v)|^2 \, \text{dvol}_g,$$

see Section 3 for notation and details. The desired vanishing is established by proving that $\tilde{\mathcal{L}}_R$ can be estimated in terms of $\tilde{\mathcal{L}}_{2R}$, see equation (3.15), allowing us to exploit the simple but useful Lemma 2.1.

This kind of technique seems to be very flexible. In particular, having in mind the fact that the technique is mainly based on the weighted Sobolev inequality, we believe that it could be also used when dealing with p -Laplace inequalities as well as with fully nonlinear problems.

Let us now introduce some notation and terminology that will be used in the sequel. Let M be a smooth submanifold of \mathbb{R}^n of dimension $n - k$, and let $i: M \hookrightarrow \mathbb{R}^n$ be the embedding map. By a *normal tube* of constant radius $\varepsilon > 0$ centred at M we mean a tubular neighbourhood of M given by a *disjoint* union

$$\mathcal{B}(M, \varepsilon) = \bigsqcup_{p \in M} B(p, \varepsilon),$$

where $B(p, \varepsilon)$ is a k -dimensional ball of radius ε centred at $p \in M$ and contained in the normal subspace $N_p M \subset \mathbb{R}^n$. Moreover, given any subset $A \subseteq M$ we define

$$(1.4) \quad \mathcal{B}(A, \varepsilon) = \bigsqcup_{p \in A} B(p, \varepsilon) \subseteq \mathcal{B}(M, \varepsilon).$$

We are now ready to make our basic hypothesis on the domain Ω .

Main Assumption. *The submanifold M admits a normal tube $\mathcal{B}(M, \varepsilon)$ for some $\varepsilon > 0$, and we have $\Omega \subseteq \mathcal{B}(M, \varepsilon)$.*

Remark 1.1. Let us consider the *normal exponential map*

$$\exp^\perp: NM \longrightarrow \mathbb{R}^n,$$

sending the pair (p, v) , with $p \in M$ and $v \in N_p M$, to $p + v$ (here we are identifying $T_p \mathbb{R}^n$ with \mathbb{R}^n in the standard way). We can define the *normal injectivity radius* of M as the strictly positive function $\rho: M \rightarrow \mathbb{R}$ that associates to $p \in M$ the supremum $\rho(p)$ of all $\delta \leq 1$ such that the restriction of \exp^\perp to the neighbourhood of the zero section

$$V_\delta(p) := \{(p', v') \in NM : |p - p'| < \delta, |v'| < \delta\}$$

is a diffeomorphism onto its image in \mathbb{R}^n . Therefore M admits a normal tube $\mathcal{B}(M, \varepsilon)$ of radius ε if and only if $\rho(p) \geq \varepsilon$ for all $p \in M$. In fact, in this case we can identify $\mathcal{B}(M, \varepsilon)$ with the open neighbourhood of M in \mathbb{R}^n defined by

$$\{x \in \mathbb{R}^n : d(x, M) < \varepsilon\},$$

see Proposition 7.26 in [19] and Theorem 10.89 in [17] for more details.

In particular, if M is compact and ε is sufficiently small then $\mathcal{B}(M, \varepsilon)$ always exists, cf. Proposition 3.7.18 in [9] and Section 3 in [20]. On the other hand, there are also plenty of non-compact submanifolds of \mathbb{R}^n admitting normal tubes of constant radius, for instance all the linear subspaces have this property.

It is important to observe that it is not possible to give sufficient conditions for the existence of $\mathcal{B}(M, \varepsilon)$ in terms of curvature bounds for M . To see this, consider the smooth curve $M \subset \mathbb{R}^3$ parametrized by $x \mapsto (\sin x, \cos x, \arctan x)$, namely a spiral with the distance between the coils tending to 0. A straightforward computation shows that its curvature $\kappa(x)$ satisfies $1/2 \leq \kappa(x) < 1$, hence it is globally bounded, but M admits no tubular neighbourhood of constant radius.

In addition to our Main Assumption, will also need Assumptions [A1](#), [A2](#), [A3](#), and [A4](#) stated below.

Assumption A1. *For all $p \in M$, write*

$$\Omega_p := \Omega \cap B(p, \varepsilon).$$

We require that the weight a in (1.1) is a non-negative function such that $a \in C^0(\overline{\Omega}) \cap L^\infty(\Omega)$ and we suppose that there exists a constant C_a such that

$$\int_{\Omega_p} (a|_{\Omega_p})^{-t} dy^1 \cdots dy^k \leq C_a \quad \text{for some } t > k, \text{ uniformly with respect to } p.$$

Here $a|_{\Omega_p}$ denotes the restriction of a to Ω_p and y^1, \dots, y^k are the Euclidean coordinates in the k -dimensional linear subspace $N_p M$ of \mathbb{R}^n containing Ω_p . We observe that this kind of condition is quite well known in the context of weighted Sobolev spaces since of the works [18] and [23].

Assumption A2. $f(z, \cdot)$ is a continuous function, uniformly Lipschitz with respect to z . In other words, for every $m > 0$ there is a positive constant $L_f = L_f(m)$ such that for every $z \in \Omega$ and every $u, v \in [-m, m]$ we have

$$\|f(z, u) - f(z, v)\| \leq L_f |u - v|.$$

As a leading example we may consider the case

$$f(z, u) = h(z)g(u),$$

where $h \in L^\infty(\Omega)$ and g is a locally Lipschitz, continuous function.

Assumption A3. We consider an oriented atlas of M ,

$$(1.5) \quad \{(U_\alpha, \phi_\alpha)\}, \quad M = \bigcup_{\alpha} U_\alpha, \quad \phi_\alpha: U_\alpha \xrightarrow{\cong} V_\alpha \subset \mathbb{R}^{n-k}$$

such that the functions $i_\alpha \circ \phi_\alpha^{-1}: V_\alpha \rightarrow \mathbb{R}^n$ (where $i_\alpha: U_\alpha \hookrightarrow \mathbb{R}^n$ denotes the restriction of $i: M \hookrightarrow \mathbb{R}^n$ to U_α), together with their first and second derivatives, are uniformly bounded with respect to α .

Remark 1.2. An atlas as above exists for every smooth manifold M . In fact, let us start with an atlas $\{(U_\alpha, \phi'_\alpha)\}$ given by a countable basis of precompact smooth coordinate balls (it exists by Lemma 1.11 in [17]). On each chart U_α , the functions $i_\alpha \circ (\phi'_\alpha)^{-1}$ and their derivatives are bounded. We now set $\phi_\alpha := A_\alpha \phi'_\alpha$, where A_α is a positive constant; then, provided that we choose the A_α large enough, we can make the bounds above uniform on α .

Assumption A4. There exist a point $\bar{p} \in M$ and constants $C_1, \gamma, R_0 > 0$ such that

$$(1.6) \quad \text{vol}(B_M(\bar{p}, R)) \leq C_1 R^\gamma$$

for all $R > R_0$. Here we define

$$B_M(\bar{p}, R) := \{p \in M \mid d_M(\bar{p}, p) < R\},$$

where d_M denotes the distance function induced on M by the pull-back of the Euclidean metric of \mathbb{R}^n , and $\text{vol}(B_M(\bar{p}, R))$ stands for the measure of $B_M(\bar{p}, R)$, computed again with respect to the Euclidean metric on M .

Remark 1.3. It is known that estimate (1.6) holds (at every point $\bar{p} \in M$) if M is a complete Riemannian manifold with non-negative Ricci curvature (since the volume is an intrinsic geometric invariant, the fact that M is a submanifold of \mathbb{R}^n is irrelevant here). Indeed, in this case, the Bishop–Gromov inequality gives

$$\text{vol}(B_M(\bar{p}, R)) \leq \Gamma_{\dim(M)},$$

where $\Gamma_{\dim(M)}$ stands for the volume of the Euclidean ball of $\mathbb{R}^{\dim(M)}$ (see [21], Lemma 1.5, p. 247).

On the other hand, without the non-negativity assumption on the Ricci curvature, (1.6) in general does not hold. For instance, the volume of a ball of radius R in a symmetric space M of non-compact type grows exponentially with R , cf. [16], p. 209. In order to have an embedded example of the last situation in any dimension m , recall that the hyperbolic space \mathbb{H}^m admits a smooth, isometric embedding into \mathbb{R}^n with $n = 6m - 5$, see [5].

We are now ready to state our main result.

Theorem 1.4. *Let $\Omega \subseteq \mathcal{B}(M, \varepsilon) \subseteq \mathbb{R}^n$ be a smooth domain and let $u, v \in C^1(\Omega) \cap C^0(\overline{\Omega}) \cap W^{1,\infty}(\Omega)$ be weak solutions to (1.1), with*

$$u \leq v \quad \text{on } \partial\Omega.$$

Assume moreover that Assumptions A1, A2, A3 and A4 are satisfied. Then there exists a positive constant

$$\varepsilon_0 := \varepsilon_0(\|u\|_{W^{1,\infty}}, \|v\|_{W^{1,\infty}}, t, a, q, n, k, f, \Lambda, \gamma)$$

such that, if $0 < \varepsilon < \varepsilon_0$, then

$$u \leq v \quad \text{in } \Omega.$$

Apart from this Introduction, the paper contains two more sections. More precisely, in Section 2 we collect some preliminary results that are needed in what follows, whereas Section 3 is devoted to the proof of Theorem 1.4.

2. Preliminaries

2.1. A useful lemma

We start by recalling a useful lemma already contained and exploited in [11]. For the reader's convenience, we also provide the proof.

Lemma 2.1. *Let $\theta > 0$ and $\gamma > 0$ be such that $\theta < 2^{-\gamma}$. Moreover, let $R_0 > 0$, $C > 0$, and let*

$$\mathcal{L}: (R_0, +\infty) \rightarrow \mathbb{R}$$

be a non-negative and non-decreasing function such that

$$(2.1) \quad \begin{cases} \mathcal{L}(R) \leq \theta \mathcal{L}(2R) + g(R) & \text{for all } R > R_0, \\ \mathcal{L}(R) \leq C R^\gamma & \text{for all } R > R_0, \end{cases}$$

where $g: (R_0, +\infty) \rightarrow \mathbb{R}^+$ is such that

$$\lim_{R \rightarrow +\infty} g(R) = 0.$$

Then

$$\mathcal{L}(R) = 0 \quad \text{for all } R > R_0.$$

Proof. Choose θ_1 such that $\theta < \theta_1 < 2^{-\gamma}$. Then there exists $R_1 = R_1(\theta_1) \geq R_0$ such that

$$\theta \mathcal{L}(2R) + g(R) \leq \theta_1 \mathcal{L}(2R) \quad \text{for all } R \geq R_1,$$

and so, using the first inequality in (2.1), we get

$$(2.2) \quad \mathcal{L}(R) \leq \theta_1 \mathcal{L}(2R) \quad \text{for all } R \geq R_1.$$

Iterating (2.2) and using the second inequality in (2.1) we obtain, for all $m \in \mathbb{N}^+$ and $R \geq R_1$,

$$(2.3) \quad 0 \leq \mathcal{L}(R) \leq \theta_1^m \mathcal{L}(2^m R) \leq C \theta_1^m (2^m R)^\gamma = C (2^\gamma \theta_1)^m R^\gamma.$$

Since $0 < 2^\gamma \theta_1 < 1$, by (2.3) we deduce

$$0 \leq \mathcal{L}(R) \leq \lim_{m \rightarrow +\infty} C (2^\gamma \theta_1)^m R^\gamma = 0 \quad \text{for all } R \geq R_1,$$

that implies $\mathcal{L}(R) = 0$ for all $R \geq R_1$. By Assumption \mathcal{L} is non-decreasing, so the claim follows. \square

2.2. Weighted Sobolev spaces

The *weighted Sobolev space* with weight ρ is defined as the space $W^{1,2}(\Omega, \rho)$ of those functions in $L^2(\Omega)$ having distributional derivative in Ω for which the norm

$$(2.4) \quad \|v\|_{W^{1,2}(\Omega, \rho)} = \|v\|_{L^2(\Omega)} + \|\nabla v\|_{L^2(\Omega, \rho)} = \left(\int_{\Omega} v^2 \right)^{1/2} + \left(\int_{\Omega} |\nabla v|^2 \rho \right)^{1/2}$$

is bounded. The case $\rho = 1$ corresponds to the classical Sobolev space $W^{1,2}(\Omega)$. We also define the space $H^{1,2}(\Omega, \rho)$ as the closure of $C^\infty(\Omega) \cap W^{1,2}(\Omega, \rho)$ and the space $H_0^{1,2}(\Omega, \rho)$ as the closure of $C_0^\infty(\Omega)$, with respect to the norm (2.4). Let us now recall from [18], [23] the following.

Theorem 2.2. *Let $\mathcal{D} \subset \mathbb{R}^k$ be a smooth, bounded domain, $\rho \in L^1(\mathcal{D})$ and C_ρ a constant such that*

$$\int_{\mathcal{D}} \frac{1}{\rho^t} \leq C_\rho \quad \text{for some } t > \frac{k}{2}.$$

Then, for $1 + 1/t > 2/k$, setting

$$\frac{1}{2^*(t)} := \frac{1}{2} - \frac{1}{k} + \frac{1}{2t},$$

it results that $2 < 2^(t) < +\infty$, and it follows that the space $H_0^{1,2}(\mathcal{D}, \rho)$ is continuously embedded in $L^q(\mathcal{D})$ for $1 \leq q \leq 2^*(t)$. More precisely, there exists a constant $C_{t,k}$ such that*

$$(2.5) \quad \|w\|_{L^{2^*(t)}(\mathcal{D})}^2 \leq C_{t,k} C_\rho^{-t} \|\nabla w\|_{L^2(\mathcal{D}, \rho)}^2 = C_S \int_{\mathcal{D}} |\nabla w|^2 \rho,$$

for any $w \in H_0^{1,2}(\mathcal{D}, \rho)$, where we set $C_S = C_S(t, k, \rho) := C_{t,k} C_\rho^{-t}$.

2.3. Fermi coordinates for Ω

A key ingredient for the proof of Theorem 1.4 is the choice of suitable local coordinates in order to parametrize the normal tube $\mathcal{B}(M, \varepsilon)$; such coordinates are sometimes called *Fermi coordinates*, cf. Chapter 2 in [14] and Section 3 in [20].

Up to refinements of the atlas (1.5), we can assume that on each coordinate neighbourhood $U_\alpha \subset \mathbb{R}^n$ is defined a moving orthonormal frame $e_\alpha^1, \dots, e_\alpha^k$ for the normal bundle $NM|_{U_\alpha}$. This means that, for all $p \in U_\alpha$, the k vectors

$$e_\alpha^1(p), \dots, e_\alpha^k(p) \in \mathbb{R}^n$$

are an orthonormal basis for the normal space $N_p M$ and that, for each i , the assignation $p \mapsto e_\alpha^i(p)$ defines a smooth vector field. We take $\mathcal{B}(U_\alpha, \varepsilon)$ as in (1.4), and we set

$$\varphi_\alpha = \phi_\alpha^{-1}: V_\alpha \longrightarrow U_\alpha, \quad E_\alpha^i = e_\alpha^i \circ \varphi_\alpha.$$

Thus we have a diffeomorphism

$$(2.6) \quad \Phi_\alpha: V_\alpha \times B_{\mathbb{R}^k}(0, \varepsilon) \longrightarrow \mathcal{B}(U_\alpha, \varepsilon)$$

$$\left((x_\alpha^1, \dots, x_\alpha^{n-k}), (y^1, \dots, y^k) \right) \mapsto \varphi_\alpha(x_\alpha^1, \dots, x_\alpha^{n-k}) + \sum_{i=1}^k y^i E_\alpha^i(x_\alpha^1, \dots, x_\alpha^{n-k}),$$

where we denoted by $x_\alpha^1, \dots, x_\alpha^{n-k}$ the Euclidean coordinates in the open set $V_\alpha \subseteq \mathbb{R}^{n-k}$ and by y^1, \dots, y^k the Euclidean coordinates in the open ball $B_{\mathbb{R}^k}(0, \varepsilon)$. This shows that

$$(2.7) \quad \left\{ (\mathcal{B}(U_\alpha, \varepsilon), \Phi_\alpha^{-1}) \right\}_\alpha, \quad \Phi_\alpha^{-1}: \mathcal{B}(U_\alpha, \varepsilon) \xrightarrow{\simeq} V_\alpha \times B_{\mathbb{R}^k}(0, \varepsilon) \subset \mathbb{R}^{n-k} \times \mathbb{R}^k$$

is an oriented atlas for $\mathcal{B}(M, \varepsilon)$.

Denoting by z^1, \dots, z^n the Euclidean coordinates in \mathbb{R}^n , in the local chart $\mathcal{B}(U_\alpha, \varepsilon)$ we have the Euclidean metric

$$g = (dz^1)^2 + \dots + (dz^n)^2.$$

On the other hand, on V_α we have the induced metric

$$h'_\alpha = (i_\alpha \circ \varphi_\alpha)^* g,$$

where $i_\alpha: U_\alpha \hookrightarrow \mathbb{R}^n$ denotes the embedding map as in Assumption A3, whereas on $B_{\mathbb{R}^k}(0, \varepsilon)$ we have the metric h'' induced by the Euclidean metric on \mathbb{R}^k , namely

$$h'' = (dy^1)^2 + \dots + (dy^k)^2.$$

The next result allows us to compare the two metrics $\Phi_\alpha^* g$ and $h_\alpha = h'_\alpha + h''$ on $V_\alpha \times B_{\mathbb{R}^k}(0, \varepsilon)$, cf. Lemma 3.1 in [20].

Lemma 2.3. *With the notation introduced above, we have*

$$(2.8) \quad \Phi_\alpha^* g = h_\alpha + \sum_{i=1}^k y^i (r_\alpha^i + 2t_\alpha^i) + \sum_{i,j=1}^k y^i y^j s_\alpha^{ij},$$

where r_α^i , t_α^i and s_α^{ij} are 2-tensors on $V_\alpha \times B_{\mathbb{R}^k}(0, \varepsilon)$ whose coefficients are smooth functions on V_α .

Proof. For simplicity of notation, let us temporarily drop the subscript α . Given any pair X, Y of vector fields in $V \times B_{\mathbb{R}^k}(0, \varepsilon)$ we have

$$(\Phi^*g)(X, Y) = g(d\Phi(X), d\Phi(Y)) = d\Phi(X) \cdot d\Phi(Y),$$

where the dot stands for the Euclidean scalar product. Then, in order to determine the coefficients of Φ^*g with respect to the frame

$$\partial_{x^1}, \dots, \partial_{x^{n-k}}, \partial_{y^1}, \dots, \partial_{y^k}$$

of $TV \times TB_{\mathbb{R}^k}(0, \varepsilon)$, we must compute the quantities

$$\begin{aligned} (\Phi^*g)(\partial_{y^i}, \partial_{y^j}) &= d\Phi(\partial_{y^i}) \cdot d\Phi(\partial_{y^j}) = \partial_{y^i}\Phi \cdot \partial_{y^j}\Phi, \\ (\Phi^*g)(\partial_{x^a}, \partial_{y^j}) &= d\Phi(\partial_{x^a}) \cdot d\Phi(\partial_{y^j}) = \partial_{x^a}\Phi \cdot \partial_{y^j}\Phi, \\ (\Phi^*g)(\partial_{x^a}, \partial_{x^b}) &= d\Phi(\partial_{x^a}) \cdot d\Phi(\partial_{x^b}) = \partial_{x^a}\Phi \cdot \partial_{x^b}\Phi. \end{aligned}$$

Observing that $\partial_{x^a}\varphi$ is a tangent vector field on V , whereas E^i is a normal vector field, we obtain $\partial_{x^a}\varphi \cdot E^i = 0$. Moreover, the fact that e^1, \dots, e^k is an orthonormal frame implies $E^i \cdot E^j = \delta_{ij}$. Therefore we get

$$\begin{aligned} \partial_{y^i}\Phi \cdot \partial_{y^j}\Phi &= E^i \cdot E^j = \delta_{ij} = h''_{ij}, \\ \partial_{x^a}\Phi \cdot \partial_{y^j}\Phi &= \sum_{i=1}^k y^i \partial_{x^a} E^i \cdot E^j \end{aligned}$$

and

$$\begin{aligned} &\partial_{x^a}\Phi \cdot \partial_{x^b}\Phi \\ &= \partial_{x^a}\varphi \cdot \partial_{x^b}\varphi + \sum_{i=1}^k y^i (\partial_{x^a}\varphi \cdot \partial_{x^b} E^i + \partial_{x^b}\varphi \cdot \partial_{x^a} E^i) + \sum_{i,j=1}^k y^i y^j \partial_{x^a} E^i \cdot \partial_{x^b} E^j \\ &= h'_{ab} + \sum_{i=1}^k y^i (\partial_{x^a}\varphi \cdot \partial_{x^b} E^i + \partial_{x^b}\varphi \cdot \partial_{x^a} E^i) + \sum_{i,j=1}^k y^i y^j \partial_{x^a} E^i \cdot \partial_{x^b} E^j. \end{aligned}$$

Now, setting

$$\begin{aligned} (2.9) \quad r^i &= \sum_{a,b=1}^{n-k} (\partial_{x^a}\varphi \cdot \partial_{x^b} E^i + \partial_{x^b}\varphi \cdot \partial_{x^a} E^i) dx^a dx^b, \\ t^i &= \sum_{a=1}^{n-k} \sum_{j=1}^k (\partial_{x^a} E^i \cdot E^j) dx^a dy^j, \\ s^{ij} &= \sum_{a,b=1}^{n-k} (\partial_{x^a} E^i \cdot \partial_{x^b} E^j) dx^a dx^b, \end{aligned}$$

we obtain (2.8). □

Example 2.4. Let us explicitly compute the expression (2.8) in three simple cases, namely when M is either a plane curve, or a space curve or a surface in \mathbb{R}^3 .

(i) Let $M \subset \mathbb{R}^2$ be a plane curve. Then $n = 2$, $k = 1$ and the function Φ can be chosen of the form

$$\Phi(x, y) = \varphi(x) + yN(x),$$

where φ is the arc-length parametrization and N is the principal normal vector, see Chapter 1 in [10]. Then

$$\partial_x \Phi = \partial_x \varphi + y \partial_x E = (1 - \kappa y)T, \quad \partial_y \Phi = N,$$

where $T = T(x)$ is the unit tangent vector and $\kappa = \kappa(x)$ is the curvature. From this we get

$$\begin{aligned} \Phi^* g &= (dx)^2 + (dy)^2 - 2y\kappa dx^2 + y^2 \kappa^2 (dx)^2 \\ &= h - 2y\kappa(dx)^2 + y^2 \kappa^2 (dx)^2. \end{aligned}$$

(ii) Let $M \subset \mathbb{R}^3$ be a space curve. Then $n = 3$, $k = 2$ and the function Φ can be chosen of the form

$$\Phi(x, y^1, y^2) = \varphi(x) + y^1 E^1(x) + y^2 E^2(x),$$

where φ is the arc-length parametrization, E^1 is the principal normal vector, and E^2 is the binormal vector. Thus $\partial_x \varphi = T$, where $T = T(x)$ is the unit tangent vector, and by the Frénet formulas ([10], p. 19), we obtain

$$\partial_x E^1 = -\kappa T - \tau E^2, \quad \partial_x E^2 = \tau E^1,$$

where $\kappa = \kappa(x)$ is the curvature and $\tau = \tau(x)$ is the torsion. Now, using (2.9), we easily find

$$\begin{aligned} r^1 &= -2\kappa(dx)^2, & t^1 &= -\tau dx dy^2, & r^2 &= 0, & t^2 &= \tau dx dy^1, \\ s^{11} &= (\kappa^2 + \tau^2)(dx)^2, & s^{12} &= 0, & s^{22} &= \tau^2 (dx)^2, \end{aligned}$$

so we get

$$\begin{aligned} \Phi^* g &= h - 2y^1(\kappa(dx)^2 + \tau dx dy^2) + 2y^2(\tau dx dy^1) \\ &\quad + (y^1)^2((\kappa^2 + \tau^2)(dx)^2) + (y^2)^2(\tau^2(dx)^2). \end{aligned}$$

(iii) Let $M \subset \mathbb{R}^3$ be a surface. Then $n = 3$, $k = 1$ and the function Φ can be chosen of the form

$$\Phi(x^1, x^2, y) = \varphi(x^1, x^2) + yE(x^1, x^2),$$

where we take $E(x^1, x^2) = \partial_{x^1} \varphi \wedge \partial_{x^2} \varphi / \|\partial_{x^1} \varphi \wedge \partial_{x^2} \varphi\|$. Denoting as customary by

$$E = \partial_{x^1} \varphi \cdot \partial_{x^1} \varphi, \quad F = \partial_{x^1} \varphi \cdot \partial_{x^2} \varphi, \quad G = \partial_{x^2} \varphi \cdot \partial_{x^2} \varphi$$

the coefficients of the first fundamental form I_M of M , and by

$$e = -\partial_{x^1} E \cdot \partial_{x^1} \varphi, \quad f = -\partial_{x^2} E \cdot \partial_{x^1} \varphi, \quad g = -\partial_{x^2} E \cdot \partial_{x^2} \varphi$$

the coefficients of the second fundamental form Π_M , from the formulas in p. 92 and p. 154 of [10] we deduce

$$\begin{aligned} \partial_{x^1} \Phi \cdot \partial_{x^1} \Phi &= E - 2y e + y^2 \psi_{11}, \\ \partial_{x^1} \Phi \cdot \partial_{x^2} \Phi &= F - 2y f + y^2 \psi_{12}, \\ \partial_{x^2} \Phi \cdot \partial_{x^2} \Phi &= G - 2y g + y^2 \psi_{22}, \end{aligned}$$

and

$$\partial_{x^1} \Phi \cdot \partial_y \Phi = 0, \quad \partial_{x^2} \Phi \cdot \partial_y \Phi = 0, \quad \partial_y \Phi \cdot \partial_y \Phi = 1,$$

where

$$\psi_{11} = \frac{e^2 G - 2efF + f^2 E}{EG - F^2}, \quad \psi_{12} = \frac{-f^2 F + efG - egF + gfE}{EG - F^2}, \quad \psi_{22} = \frac{f^2 G - 2fgF + g^2 E}{EG - F^2}.$$

Then, using the expressions of the mean curvature H_M and the Gaussian curvature K_M of M in terms of the coefficients of the first two fundamental forms (cf. [10], pp. 155–156), we can rewrite (2.8) as

$$\Phi^* g = I_M + (dy)^2 - 2y \Pi_M + y^2 \text{III}_M,$$

where $I_M + (dy)^2 = h$ and $\text{III}_M = -K_M I_M + 2H_M \Pi_M$ is the so-called *third fundamental form* of M , see [22], p. 98.

As a consequence of Lemma 2.3 we have the result below, comparing the volume forms of the two metrics $\Phi_\alpha^* g$ and h_α on $V_\alpha \times B_{\mathbb{R}^k}(0, \varepsilon)$.

Corollary 2.5. *With the notation introduced above, we have*

$$(2.10) \quad \text{dvol}_{\Phi_\alpha^* g} = \lambda_\alpha \text{dvol}_{h_\alpha}$$

for a smooth function λ on $\mathcal{B}(M, \varepsilon)$ such that $\lambda \equiv 1$ on M . If moreover $\varepsilon < 1$, for every

$$(x_\alpha, y) = ((x_\alpha^1, \dots, x_\alpha^{n-k}), (y^1, \dots, y^k)) \in V_\alpha \times B_{\mathbb{R}^k}(0, \varepsilon)$$

we have

$$|\lambda_\alpha(x_\alpha, y) - 1| \leq K_1 \|y\|,$$

where K_1 is a positive constant, independent on α .

It follows that there exists $\varepsilon_1 = \min\{1, 1/(2K_1)\}$, independent on α , such that if $\varepsilon < \varepsilon_1$ then

$$(2.11) \quad \frac{1}{2} \leq \lambda_\alpha(x_\alpha, y) \leq \frac{3}{2} \quad \text{for all } (x_\alpha, y) \in V_\alpha \times B_{\mathbb{R}^k}(0, \varepsilon).$$

Proof. In order to simplify the notation, we drop the subscript α . Since $\det h \neq 0$ everywhere, by Lemma 2.3 we can write the following finite expansion of $\det \Phi^*g$ in terms of the normal coordinates y^i :

$$(2.12) \quad \det \Phi^*g = \det h + \sum_{i=1}^k y^i A^i + \sum_{i,j=1}^k y^i y^j A^{ij} + \sum_{i,j,l=1}^k y^i y^j y^l A^{ijl} + \dots,$$

where all the coefficients are smooth functions depending on the first and second derivatives of Φ . Now $\varepsilon < 1$ implies $\|y\| < 1$ and so $|y^i| < 1$ for all i , thus from (2.12) we obtain

$$(2.13) \quad \det \Phi^*g = \det h + \sum_{i=1}^k \tilde{K}_1^i(x, y) y^i$$

where $|\tilde{K}_1^i(x, y)| \leq \tilde{K}_1$ and \tilde{K}_1 is a positive constant, which is moreover independent on the chart because of Assumption A3. Setting $\lambda(x, y) := \det \Phi^*g \cdot \det h^{-1}$, from (2.13) we get¹

$$\begin{aligned} |\lambda(x, y) - 1| &= \left| \sqrt{\det \Phi^*g \cdot \det h^{-1}} - 1 \right| = \left| \sqrt{1 + \sum_{i=1}^k \frac{\tilde{K}_1^i(x, y)}{\det h} y^i} - 1 \right| \\ &\leq \left| \sum_{i=1}^k \frac{\tilde{K}_1^i(x, y)}{\det h} y^i \right| \leq \sum_{i=1}^k \left| \frac{\tilde{K}_1^i(x, y)}{\det h} \right| |y^i| \leq \sum_{i=1}^k \frac{\tilde{K}_1}{\det h} |y^i| \leq k \frac{\tilde{K}_1}{\det h} \|y\| = K_1 \|y\|, \end{aligned}$$

where we put $K_1 := k\tilde{K}_1/\det h$. Again by Assumption A3, the function $\det h$ is bounded, uniformly with respect to α , so the positive constant K_1 is also independent on the chart. The proof of the last statement is now straightforward. \square

Remark 2.6. Using the same arguments as in the proof of Corollary 2.5, we can show that on V_α we have

$$(2.14) \quad d\text{vol}_{h'_\alpha} = \mu_\alpha(x_\alpha) dx_\alpha$$

for a smooth positive function $\mu_\alpha(x_\alpha)$, uniformly bounded with respect to α .

3. The weak comparison principle: Proof of Theorem 1.4

3.1. Test functions

For the sake of brevity, we write B_R instead of $B_M(\bar{p}, R)$, see Assumption A4. Let $\varphi_R \in C_c^\infty(M)$ be such that

$$(3.1) \quad \begin{cases} 0 \leq \varphi_R \leq 1 & \text{in } M, \\ \varphi_R \equiv 1 & \text{in } B_R, \\ \varphi_R \equiv 0 & \text{in } M \setminus B_{2R}, \\ |\nabla \varphi_R| \leq 2/R & \text{in } B_{2R} \setminus B_R. \end{cases}$$

¹Here we use the inequality $|\sqrt{1+x} - 1| \leq |x|$ for $x \geq -1$.

For instance, we could take φ_R of the form

$$\varphi_R(p) = \begin{cases} 1 & \text{in } B_R, \\ \left(\frac{2R-d_M(\bar{p}, p)}{R}\right)^2 & \text{in } B_{2R} \setminus B_R, \\ 0 & \text{in } M \setminus B_{2R}; \end{cases}$$

in fact, the distance function $d_M(\bar{p}, p)$ has distributional gradient whose L^∞ -norm is at most 1, see Theorem 11.3 on p. 296 of [15]. Since $\mathcal{B}(M, \varepsilon)$ is locally trivial, in view of diffeomorphisms in (2.6), we can extend φ_R to the whole tubular neighbourhood and we continue to denote this extension by φ_R . In what follows we will consider the bounded domain

$$\Omega_R := \Omega \cap \mathcal{B}(B_R, \varepsilon),$$

see Figure 1 below.

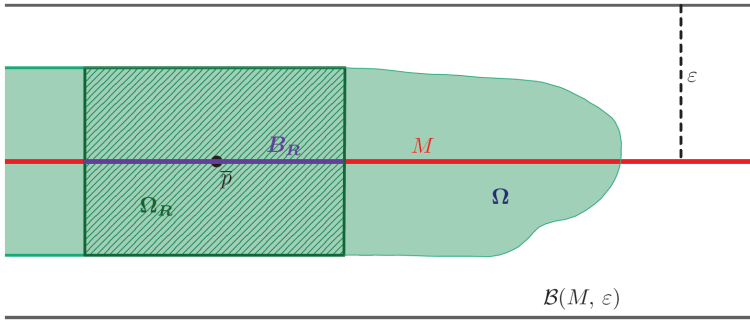


FIGURE 1. The bounded domain Ω_R .

Moreover, for a fixed real number $\beta \geq 1$, we will define

$$\psi_R := [(u - v)^+]^\beta \varphi_R^2.$$

With this notation, we can state the following crucial result, whose proof relies on our assumption that $u \leq v$ on $\partial\Omega$.

Lemma 3.1. *The function ψ_R belongs to $H_0^1(\Omega_{2R})$, and it can be plugged into (1.2) and (1.3).*

Proof. We have $\partial\Omega_{2R} \subseteq \partial\Omega \cup \partial\mathcal{B}(B_{2R}, \varepsilon)$; furthermore, by assumption, $(u - v)^+ = 0$ on $\partial\Omega$, whereas $\varphi_R = 0$ on $\partial\mathcal{B}(B_{2R}, \varepsilon)$. This means that $\psi_R = 0$ on $\partial\Omega_{2R}$, so the first claim follows from [13], Chapter 7; cf. also Theorem 8.12 in [7]. The second claim is an immediate consequence of the density of $H_0^1(\Omega_{2R})$ in $C_c^\infty(\Omega_{2R})$. \square

Plugging ψ_R as a test function into (1.2) and (1.3) and subtracting we get

$$\begin{aligned}
& \underbrace{\beta \int_{\Omega_{2R}} a(z) [(u-v)^+]^{\beta-1} |\nabla(u-v)|^2 \varphi_R^2 \, d\text{vol}_g}_{\mathcal{L}_{2R}} \\
& \leq \int_{\Omega_{2R}} |\Lambda| [(u-v)^+]^\beta \left| |\nabla u|^q - |\nabla v|^q \right| \varphi_R^2 \, d\text{vol}_g \\
& \quad + \int_{\Omega_{2R}} |f(z, u) - f(z, v)| [(u-v)^+]^\beta \varphi_R^2 \, d\text{vol}_g \\
& \quad + 2 \int_{\Omega_{2R}} a(z) [(u-v)^+]^\beta |\nabla(u-v)| |\nabla \varphi_R| \varphi_R \, d\text{vol}_g \\
(3.2) \quad & \leq |\Lambda| q \left(\|\nabla u\|_\infty + \|\nabla v\|_\infty \right)^{q-1} \underbrace{\int_{\Omega_{2R}} [(u-v)^+]^\beta |\nabla(u-v)| \varphi_R^2 \, d\text{vol}_g}_{\mathcal{A}_{2R}} \\
& \quad + L_f \underbrace{\int_{\Omega_{2R}} [(u-v)^+]^{\beta+1} \varphi_R^2 \, d\text{vol}_g}_{\mathcal{B}_{2R}} \\
& \quad + 2 \underbrace{\int_{\Omega_{2R}} a(z) [(u-v)^+]^\beta |\nabla(u-v)| |\nabla \varphi_R| \varphi_R \, d\text{vol}_g}_{\mathcal{C}_{2R}},
\end{aligned}$$

where we estimated $\left| |\nabla u|^q - |\nabla v|^q \right|$ via the Lagrange theorem applied to the function t^q , and we exploited the reverse triangular inequality. Moreover, the constant $L_f = L_f(\|u\|_\infty + \|v\|_\infty)$ is the one contained in Assumption A2.

In the sequel, we will estimate the terms \mathcal{A}_{2R} , \mathcal{B}_{2R} and \mathcal{C}_{2R} in (3.2), making repeated use of the Young inequality

$$ab \leq \tau a^2 + \frac{1}{4\tau} b^2, \quad a, b \in \mathbb{R}, \quad \tau > 0.$$

3.2. Estimate for \mathcal{A}_{2R}

Dealing first with the term \mathcal{A}_{2R} , we obtain

$$\begin{aligned}
\mathcal{A}_{2R} &= \int_{\Omega_{2R}} [(u-v)^+]^\beta |\nabla(u-v)| \varphi_R^2 \, d\text{vol}_g \\
&= \int_{\Omega_{2R}} a(z)^{1/2} [(u-v)^+]^{\frac{\beta-1}{2}} |\nabla(u-v)| \varphi_R a(z)^{-1/2} [(u-v)^+]^{\frac{\beta+1}{2}} \varphi_R \, d\text{vol}_g \\
&\leq \tau \int_{\Omega_{2R}} a(z) [(u-v)^+]^{\beta-1} |\nabla(u-v)|^2 \varphi_R^2 \, d\text{vol}_g \\
&\quad + \frac{1}{4\tau} \int_{\Omega_{2R}} a(z)^{-1} [(u-v)^+(\beta+1)/2]^2 \varphi_R^2 \, d\text{vol}_g \\
(3.3) \quad &= \tau \mathcal{L}_{2R} + \frac{1}{4\tau} \int_{\Omega_{2R}} a(z)^{-1} [(u-v)^+(\beta+1)/2]^2 \varphi_R^2 \, d\text{vol}_g.
\end{aligned}$$

In order to estimate the integral on the right-hand side of the above equality, we will consider the oriented atlas of $\mathcal{B}(M, \varepsilon)$ given by

$$\{(\mathcal{B}(U_\alpha, \varepsilon), \Phi_\alpha^{-1})\},$$

cf. (2.7). We choose a partition of unity $\{\bar{\rho}_\alpha\}_\alpha$, subordinate to the open cover $\{\mathcal{B}(U_\alpha, \varepsilon)\}_\alpha$ of $\mathcal{B}(M, \varepsilon)$ and obtained by pulling back a partition of unity subordinate to the open cover $\{U_\alpha\}_\alpha$ of M via the projection map

$$\mathcal{B}(M, \varepsilon) \longrightarrow M, \quad x \in B(p, \varepsilon) \mapsto p.$$

This implies that the functions $\rho_\alpha := \Phi_\alpha^*(\bar{\rho}_\alpha)$, defined on $V_\alpha \times B_{\mathbb{R}^k}(0, \varepsilon)$, only depend on the first factor, namely $\rho_\alpha = \rho_\alpha(x_\alpha)$. Therefore for $\varepsilon < \varepsilon_1$ we can write, by using (2.10), (2.11), (2.14) and the Fubini–Tonelli theorem,

$$\begin{aligned} & \int_{\Omega_{2R}} a(z)^{-1} [((u-v)^+)^{(\beta+1)/2}]^2 \varphi_R^2 \, \text{dvol}_g \\ &= \sum_\alpha \int_{V_\alpha \times B_{\mathbb{R}^k}(0, \varepsilon)} \rho_\alpha(x) a(x, y)^{-1} [((u-v)^+)^{(\beta+1)/2}]^2 \varphi_R^2(x) \lambda(x, y) \, \text{dvol}_{h_\alpha} \\ &\leq \frac{3}{2} \sum_\alpha \int_{V_\alpha} \rho_\alpha(x) \varphi_R^2(x) \left(\int_{B_{\mathbb{R}^k}(0, \varepsilon)} a(x, y)^{-1} [((u-v)^+)^{(\beta+1)/2}]^2 \, dy \right) \mu(x) \, dx. \end{aligned}$$

Remark 3.2. In order to simplify the notation, in the above equation and in the following computations we drop the subscript α whenever it is not strictly necessary. Moreover, with a slight abuse of notation, we write $a(x, y)$, $\varphi_R^2(x)$ and $(u-v)^+$ instead of $\Phi_\alpha^*(a(z))$, $\Phi_\alpha^*(\varphi_R^2(x))$ and $\Phi_\alpha^*((u-v)^+)$, respectively, highlighting for clarity the dependence on the variables x and y if necessary.

Now we estimate the integral on $B_{\mathbb{R}^k}(0, \varepsilon)$ by exploiting Hölder’s inequality, obtaining

$$\begin{aligned} (3.4) \quad & \int_{B_{\mathbb{R}^k}(0, \varepsilon)} a(x, y)^{-1} [((u-v)^+)^{\frac{\beta+1}{2}}]^2 \, dy \\ &\leq \left(\int_{B_{\mathbb{R}^k}(0, \varepsilon)} \left(a(x, y)^{-1} \right)^{\frac{kt}{2t-k}} \, dy \right)^{\frac{2t-k}{kt}} \cdot \left(\int_{B_{\mathbb{R}^k}(0, \varepsilon)} [((u-v)^+)^{\frac{\beta+1}{2}}]^{2^*(t)} \, dy \right)^{2/2^*(t)}. \end{aligned}$$

By exploiting again Hölder’s inequality and making use of Assumption A1, we have

$$\begin{aligned} (3.5) \quad & \int_{B_{\mathbb{R}^k}(0, \varepsilon)} \left(a(x, y)^{-1} \right)^{\frac{kt}{2t-k}} \, dy \\ &\leq \left(\int_{B_{\mathbb{R}^k}(0, \varepsilon)} a(x, y)^{-t} \, dy \right)^{\frac{k}{2t-k}} \cdot \left(\int_{B_{\mathbb{R}^k}(0, \varepsilon)} 1 \, dy \right)^{\frac{2t-2k}{2t-k}} \\ &\leq C_a^{\frac{k}{2t-k}} (\Gamma_k \varepsilon^k)^{\frac{2t-2k}{2t-k}} = C_a^{\frac{k}{2t-k}} \Gamma_k^{\frac{2t-2k}{2t-k}} \varepsilon^{\frac{k(2t-2k)}{2t-k}}, \end{aligned}$$

were Γ_k is the volume of the unit ball in \mathbb{R}^k . Moreover we have, by using (2.5),

$$\begin{aligned}
 & \left(\int_{B_{\mathbb{R}^k}(0, \varepsilon)} [((u-v)^+)^{(\beta+1)/2}]^{2^*(t)} dy \right)^{2/2^*(t)} \\
 (3.6) \quad & = \|((u-v)^+)^{\frac{\beta+1}{2}}\|_{L^{2^*(t)}(B_{\mathbb{R}^k}(0, \varepsilon))}^2 \leq C_S^2 \|\nabla_{\mathbb{R}^k}((u-v)^+)^{\frac{\beta+1}{2}}\|_{L^2(B_{\mathbb{R}^k}(0, \varepsilon), a)}^2 \\
 & = C_S^2 \int_{B_{\mathbb{R}^k}(0, \varepsilon)} |\nabla_{\mathbb{R}^k}((u-v)^+)^{(\beta+1)/2}|^2 a(x, y) dy.
 \end{aligned}$$

Remark 3.3. By Theorem 2.2, see also the proof of Theorem 5.1 in [12], it follows that the constant C_S in (3.6) does not depend on α because of Assumption A1.

Plugging (3.5) and (3.6) into (3.4), we infer

$$\begin{aligned}
 & \int_{B_{\mathbb{R}^k}(0, \varepsilon)} a(x, y)^{-1} [((u-v)^+)^{(\beta+1)/2}]^2 dy \\
 & \leq C_a^{-t} C_S^2 \Gamma_k^{\frac{2t-2k}{kt}} \varepsilon^{\frac{2t-2k}{t}} \int_{B_{\mathbb{R}^k}(0, \varepsilon)} a(x, y) |\nabla_{\mathbb{R}^k}((u-v)^+)^{(\beta+1)/2}|^2 dy \\
 & = C_a^{-t} C_S^2 \Gamma_k^{\frac{2t-2k}{kt}} \varepsilon^{\frac{2t-2k}{t}} \left(\frac{\beta+1}{2}\right)^2 \\
 & \quad \cdot \int_{B_{\mathbb{R}^k}(0, \varepsilon)} a(x, y) [((u-v)^+)^{(\beta-1)/2} |\nabla_{\mathbb{R}^k}(u-v)|]^2 dy \\
 & \leq \frac{C_a^{-t} C_S^2 \Gamma_k^{\frac{2t-2k}{kt}} \varepsilon^{\frac{2t-2k}{t}} (\beta+1)^2}{2} \\
 & \quad \cdot \int_{B_{\mathbb{R}^k}(0, \varepsilon)} a(x, y) [(u-v)^+]^{\beta-1} |\nabla_{\mathbb{R}^k}(u-v)|^2 \lambda(x, y) dy,
 \end{aligned}$$

where the last inequality follows because we have $1 \leq 2\lambda(x, y)$ for $\varepsilon < \varepsilon_1$, see (2.11).

Remark 3.4. By using Lemma 2.3, we can write

$$|\nabla_{\Phi_\alpha^* g}(u-v)|^2 = |\nabla_{\mathbb{R}^k}(u-v)|^2 + |\nabla_{h'_\alpha}(u-v)|^2 + o(\varepsilon)$$

in $V_\alpha \times B_{\mathbb{R}^k}(0, \varepsilon)$, uniformly on α . So, making ε_1 smaller if necessary, we can assume that the inequality

$$|\nabla_{\mathbb{R}^k}(u-v)|^2 \leq |\nabla_{\Phi_\alpha^* g}(u-v)|^2$$

holds in every coordinate chart $V_\alpha \times B_{\mathbb{R}^k}(0, \varepsilon)$, as soon as $\varepsilon < \varepsilon_1$.

Moreover, since the gradient ∇ is the vector field representing the differential map with respect to the metric (see [21], p. 20), it is straightforward to check that

$$\Phi_\alpha^* \nabla g = \nabla_{\Phi_\alpha^* g} \Phi_\alpha^*.$$

Using Remark 3.4, and noticing that all the constants involved in the compu-

tations are independent on the local chart U_α , we obtain

$$\begin{aligned}
& \int_{\Omega_{2R}} a(z)^{-1} [(u-v)^+]^{(\beta+1)/2}]^2 \varphi_R^2 \, d\text{vol}_g \\
& \leq \frac{3C_a^{-t} C_S^2 \Gamma_k^{\frac{2t-2k}{kt}} \varepsilon^{\frac{2t-2k}{t}} (\beta+1)^2}{4} \sum_\alpha \int_{V_\alpha} \rho_\alpha(x) \\
& \quad \cdot \left(\int_{B_{\mathbb{R}^k}(0, \varepsilon)} a(x, y) [(u-v)^+]^{\beta-1} |\nabla_{\mathbb{R}^k}(u-v)|^2 \varphi_R^2(x) \lambda(x, y) dy \right) \mu(x) \, dx \\
& \leq \frac{3C_a^{-t} C_S^2 \Gamma_k^{\frac{2t-2k}{kt}} \varepsilon^{\frac{2t-2k}{t}} (\beta+1)^2}{4} \sum_\alpha \int_{V_\alpha} \rho_\alpha(x) \\
(3.7) \quad & \cdot \left(\int_{B_{\mathbb{R}^k}(0, \varepsilon)} a(x, y) [(u-v)^+]^{\beta-1} |\nabla_{\Phi_\alpha^* g}(u-v)|^2 \varphi_R^2(x) \lambda(x, y) dy \right) \mu(x) \, dx \\
& = \frac{3C_a^{-t} C_S^2 \Gamma_k^{\frac{2t-2k}{kt}} \varepsilon^{\frac{2t-2k}{t}} (\beta+1)^2}{4} \sum_\alpha \int_{V_\alpha \times B_{\mathbb{R}^k}(0, \varepsilon)} \rho_\alpha(x) \\
& \quad \cdot (a(x, y) [(u-v)^+]^{\beta-1} |\nabla_{\Phi_\alpha^* g}(u-v)|^2 \varphi_R^2(x)) \, d\text{vol}_{\Phi_\alpha^* g} \\
& = \frac{3C_a^{-t} C_S^2 \Gamma_k^{\frac{2t-2k}{kt}} \varepsilon^{\frac{2t-2k}{t}} (\beta+1)^2}{4} \\
& \quad \cdot \int_{\Omega_{2R}} a(z) [(u-v)^+]^{\beta-1} |\nabla(u-v)|^2 \varphi_R^2 \, d\text{vol}_g \\
& = \frac{3C_a^{-t} C_S^2 \Gamma_k^{\frac{2t-2k}{kt}} \varepsilon^{\frac{2t-2k}{t}} (\beta+1)^2}{4} \mathcal{L}_{2R}.
\end{aligned}$$

Summing up, if $\varepsilon \leq \varepsilon_1$, from (3.3) and (3.7) we finally get

$$(3.8) \quad \mathcal{A}_{2R} \leq \left(\tau + \frac{3C_a^{-t} C_S^2 \Gamma_k^{\frac{2t-2k}{kt}} \varepsilon^{\frac{2t-2k}{t}} (\beta+1)^2}{16\tau} \right) \mathcal{L}_{2R}.$$

3.3. Estimate for \mathcal{B}_{2R}

Let us now estimate the term \mathcal{B}_{2R} . As before, for $\varepsilon \leq \varepsilon_1$ we have

$$\begin{aligned}
\mathcal{B}_{2R} &= \int_{\Omega_{2R}} [(u-v)^+]^{\beta+1} \varphi_R^2 \, d\text{vol}_g \\
&= \sum_\alpha \int_{V_\alpha \times B_{\mathbb{R}^k}(0, \varepsilon)} \rho_\alpha(x) [(u-v)^+]^{\beta+1} \varphi_R^2(x) \lambda(x, y) \, d\text{vol}_{h_\alpha} \\
&\leq \frac{3}{2} \sum_\alpha \int_{V_\alpha} \rho_\alpha(x) \varphi_R^2(x) \left(\int_{B_{\mathbb{R}^k}(0, \varepsilon)} [(u-v)^+]^{\beta+1} dy \right) \mu(x) \, dx.
\end{aligned}$$

Now we estimate the integral on $B_{\mathbb{R}^k}(0, \varepsilon)$ by using Hölder's inequality and (3.6), obtaining

$$\begin{aligned}
& \int_{B_{\mathbb{R}^k}(0, \varepsilon)} [(u-v)^+]^{\beta+1} dy \\
& \leq \left(\int_{B_{\mathbb{R}^k}(0, \varepsilon)} 1 dy \right)^{\frac{2t-k}{kt}} \cdot \left(\int_{B_{\mathbb{R}^k}(0, \varepsilon)} [((u-v)^+)^{\frac{\beta+1}{2}}]^{2^*(t)} dy \right)^{2/2^*(t)} \\
& \leq (\Gamma_k \varepsilon^k)^{\frac{2t-k}{kt}} C_S^2 \int_{B_{\mathbb{R}^k}(0, \varepsilon)} |\nabla_{\mathbb{R}^k} ((u-v)^+)^{(\beta+1)/2}|^2 a(x, y) dy \\
(3.9) \quad & = \Gamma_k^{\frac{2t-k}{kt}} \varepsilon^{\frac{2t-k}{t}} C_S^2 \left(\frac{\beta+1}{2} \right)^2 \\
& \quad \cdot \int_{B_{\mathbb{R}^k}(0, \varepsilon)} ([(u-v)^+]^{(\beta-1)/2} |\nabla_{\mathbb{R}^k} (u-v)|)^2 a(x, y) dy \\
& = \frac{\Gamma_k^{\frac{2t-k}{kt}} \varepsilon^{\frac{2t-k}{t}} C_S^2 (\beta+1)^2}{4} \int_{B_{\mathbb{R}^k}(0, \varepsilon)} [(u-v)^+]^{\beta-1} |\nabla_{\mathbb{R}^k} (u-v)|^2 a(x, y) dy \\
& \leq \frac{\Gamma_k^{\frac{2t-k}{kt}} \varepsilon^{\frac{2t-k}{t}} C_S^2 (\beta+1)^2}{2} \\
& \quad \cdot \int_{B_{\mathbb{R}^k}(0, \varepsilon)} [(u-v)^+]^{\beta-1} |\nabla_{\mathbb{R}^k} (u-v)|^2 a(x, y) \lambda(x, y) dy.
\end{aligned}$$

Arguing as in (3.7) and using Remark 3.4, we deduce

$$\begin{aligned}
\mathcal{B}_{2R} & \leq \frac{3\Gamma_k^{\frac{2t-k}{kt}} \varepsilon^{\frac{2t-k}{t}} C_S^2 (\beta+1)^2}{4} \sum_{\alpha} \int_{V_{\alpha}} \rho_{\alpha}(x) \\
& \quad \cdot \left(\int_{B_{\mathbb{R}^k}(0, \varepsilon)} a(x, y) [(u-v)^+]^{\beta-1} |\nabla_{\mathbb{R}^k} (u-v)|^2 \lambda(x, y) \varphi_R^2(x) dy \right) \mu(x) dx \\
& \leq \frac{3\Gamma_k^{\frac{2t-k}{kt}} \varepsilon^{\frac{2t-k}{t}} C_S^2 (\beta+1)^2}{4} \sum_{\alpha} \int_{V_{\alpha} \times B_{\mathbb{R}^k}(0, \varepsilon)} \rho_{\alpha}(x) \\
& \quad \cdot (a(x, y) [(u-v)^+]^{\beta-1} |\nabla_{\Phi_{\alpha}^* g} (u-v)|^2 \varphi_R^2(x)) \operatorname{dvol}_{\Phi_{\alpha}^* g} \\
& = \frac{3\Gamma_k^{\frac{2t-k}{kt}} \varepsilon^{\frac{2t-k}{t}} C_S^2 (\beta+1)^2}{4} \int_{\Omega_{2R}} a(z) [(u-v)^+]^{\beta-1} |\nabla (u-v)|^2 \varphi_R^2 \operatorname{dvol}_g.
\end{aligned}$$

Summing up, for $\varepsilon < \varepsilon_1$ we get

$$(3.10) \quad \mathcal{B}_{2R} \leq \frac{3 C_S^2 \Gamma_k^{\frac{2t-k}{kt}} \varepsilon^{\frac{2t-k}{t}} (\beta+1)^2}{4} \mathcal{L}_{2R}.$$

3.4. Estimate for \mathcal{C}_{2R}

Finally, let us estimate the term \mathcal{C}_{2R} . Exploiting Young's inequality and the last inequality in (3.1), we get

$$\begin{aligned}
\mathcal{C}_{2R} &= \int_{\Omega_{2R}} a(z) [(u-v)^+]^\beta |\nabla(u-v)| |\nabla\varphi_R| \varphi_R \, d\text{vol}_g \\
&= \int_{\Omega_{2R}} a(z)^{1/2} [(u-v)^+]^{\frac{\beta-1}{2}} |\nabla(u-v)| \varphi_R a(z)^{1/2} [(u-v)^+]^{\frac{\beta+1}{2}} |\nabla\varphi_R| \, d\text{vol}_g \\
&\leq \tau' \int_{\Omega_{2R}} a(z) [(u-v)^+]^{\beta-1} |\nabla(u-v)|^2 \varphi_R^2 \, d\text{vol}_g \\
&\quad + \frac{1}{4\tau'} \int_{\Omega_{2R}} a(z) [(u-v)^+]^{\beta+1} |\nabla\varphi_R|^2 \, d\text{vol}_g \\
&\leq \tau' \mathcal{L}_{2R} + \frac{\|a\|_\infty}{\tau' R^2} \int_{\Omega_{2R}} [(u-v)^+]^{\beta+1} \, d\text{vol}_g.
\end{aligned}$$

If $\varepsilon < \varepsilon_1$, the integral at the right-hand side can be estimated by (3.9), obtaining

$$\begin{aligned}
\int_{\Omega_{2R}} [(u-v)^+]^{\beta+1} \, d\text{vol}_g &= \sum_\alpha \int_{V_\alpha \times B_{\mathbb{R}^k}(0, \varepsilon)} \rho_\alpha(x) [(u-v)^+]^{\beta+1} \lambda(x, y) \, d\text{vol}_{h_\alpha} \\
&\leq \frac{3}{2} \sum_\alpha \int_{V_\alpha} \rho_\alpha(x) \left(\int_{B_{\mathbb{R}^k}(0, \varepsilon)} [(u-v)^+]^{\beta+1} \, dy \right) \mu(x) \, dx \\
&\leq \frac{3 \Gamma_k^{\frac{2t-k}{kt}} \varepsilon^{\frac{2t-k}{t}} C_S^2 (\beta+1)^2}{4} \sum_\alpha \int_{V_\alpha} \rho_\alpha(x) \\
&\quad \cdot \left(\int_{B_{\mathbb{R}^k}(0, \varepsilon)} a(x, y) [(u-v)^+]^{\beta-1} |\nabla_{\mathbb{R}^k}(u-v)|^2 \lambda(x, y) \, dy \right) \mu(x) \, dx \\
&\leq \frac{3 \Gamma_k^{\frac{2t-k}{kt}} \varepsilon^{\frac{2t-k}{t}} C_S^2 (\beta+1)^2}{4} \sum_\alpha \int_{V_\alpha \times B_{\mathbb{R}^k}(0, \varepsilon)} \rho_\alpha(x) \\
&\quad \cdot (a(x, y) [(u-v)^+]^{\beta-1} |\nabla_{\Phi_\alpha^* g}(u-v)|^2 \varphi_R^2(x)) \, d\text{vol}_{\Phi_\alpha^* g} \\
&= \frac{3 \Gamma_k^{\frac{2t-k}{kt}} \varepsilon^{\frac{2t-k}{t}} C_S^2 (\beta+1)^2}{4} \int_{\Omega_{2R}} a(z) [(u-v)^+]^{\beta-1} |\nabla(u-v)|^2 \, d\text{vol}_g.
\end{aligned}$$

Summing up, for $\varepsilon < \varepsilon_1$ we get

$$\begin{aligned}
(3.11) \quad \mathcal{C}_{2R} &\leq \tau' \mathcal{L}_{2R} + \frac{3 C_S^2 \Gamma_k^{\frac{2t-k}{kt}} \varepsilon^{\frac{2t-k}{t}} (\beta+1)^2 \|a\|_\infty}{4\tau' R^2} \\
&\quad \cdot \int_{\Omega_{2R}} a(z) [(u-v)^+]^{\beta-1} |\nabla(u-v)|^2 \, d\text{vol}_g.
\end{aligned}$$

3.5. End of the proof

We can now specialize the computations above, using particular values for the parameters. Fixing, for instance,

$$\beta = 2 \quad \text{and} \quad \tau' = \frac{1}{4},$$

we can rewrite inequalities (3.8), (3.10) and (3.11) as follows:

$$\begin{aligned} \mathcal{A}_{2R} &\leq \left(\tau + \frac{\Theta_{\mathcal{A}}}{\tau} \right) \mathcal{L}_{2R}, \\ \mathcal{B}_{2R} &\leq \Theta_{\mathcal{B}} \mathcal{L}_{2R}, \\ \mathcal{C}_{2R} &\leq \frac{1}{4} \mathcal{L}_{2R} + \frac{\Theta_{\mathcal{C}}}{R^2} \int_{\Omega_{2R}} a(z) (u-v)^+ |\nabla(u-v)|^2 \, \text{dvol}_g, \end{aligned}$$

where

$$\begin{aligned} \Theta_{\mathcal{A}} &= \frac{27 C_a^{-t} C_S^2 \Gamma_k^{(2t-2k)/(kt)} \varepsilon^{(2t-2k)/t}}{16}, \\ \Theta_{\mathcal{B}} &= \frac{27 C_S^2 \Gamma_k^{(2t-k)/(kt)} \varepsilon^{(2t-k)/t}}{4}, \\ \Theta_{\mathcal{C}} &= 27 C_S^2 \Gamma_k^{(2t-k)/(kt)} \varepsilon^{(2t-k)/t} \|a\|_{\infty}, \end{aligned} \tag{3.12}$$

so that (3.2) becomes

$$\begin{aligned} 2\mathcal{L}_{2R} &\leq |\Lambda| q (\|\nabla u\|_{\infty} + \|\nabla v\|_{\infty})^{q-1} \left(\tau + \frac{\Theta_{\mathcal{A}}}{\tau} \right) \mathcal{L}_{2R} + L_f \Theta_{\mathcal{B}} \mathcal{L}_{2R} \\ &\quad + \frac{1}{2} \mathcal{L}_{2R} + \frac{2\Theta_{\mathcal{C}}}{R^2} \int_{\Omega_{2R}} a(z) (u-v)^+ |\nabla(u-v)|^2 \, \text{dvol}_g. \end{aligned}$$

Finally, we fix

$$\tau = \frac{1}{2|\Lambda|q(\|\nabla u\|_{\infty} + \|\nabla v\|_{\infty})^{q-1}}$$

and we resume the previous computations as

$$\mathcal{L}_{2R} \leq \Theta_1 \mathcal{L}_{2R} + \frac{\Theta_2}{R^2} \int_{\Omega_{2R}} a(z) (u-v)^+ |\nabla(u-v)|^2 \, \text{dvol}_g, \tag{3.13}$$

where

$$\begin{aligned} \Theta_1 &:= (2|\Lambda|^2 q^2 (\|\nabla u\|_{\infty} + \|\nabla v\|_{\infty})^{2q-2} \Theta_{\mathcal{A}} + L_f \Theta_{\mathcal{B}}), \\ \Theta_2 &:= 2\Theta_{\mathcal{C}}. \end{aligned} \tag{3.14}$$

Let us now set

$$\tilde{\mathcal{L}}_R := \int_{\Omega_R} a(z) (u-v)^+ |\nabla(u-v)|^2 \, \text{dvol}_g,$$

so that, from the properties of φ_R stated in (3.1), it follows that

$$\tilde{\mathcal{L}}_R \leq \mathcal{L}_{2R} \leq \tilde{\mathcal{L}}_{2R}.$$

Therefore, by (3.13) we get

$$(3.15) \quad \tilde{\mathcal{L}}_R \leq \theta \tilde{\mathcal{L}}_{2R},$$

where

$$\theta := \left(\Theta_1 + \frac{\Theta_2}{R^2} \right).$$

Looking at (3.12) and (3.14) we see that, taking $\varepsilon < \varepsilon_1$ sufficiently small and R sufficiently large, we have $\theta < 2^{-\gamma}$, where γ is as in Assumption A4. Moreover, we are supposing $a, u, v, \nabla u, \nabla v \in L^\infty(\Omega)$ so, using Assumption A4, we obtain

$$\tilde{\mathcal{L}}_R \leq CR^\gamma,$$

where we set $C := C_1 \|a(z)(u-v)^+ |\nabla(u-v)|^2\|_{L^\infty(\Omega)}$.

Summing up, there exists $0 < \varepsilon_0 < \varepsilon_1$ such that, if $\varepsilon < \varepsilon_0$, we can apply Lemma 2.1 in order to deduce $\tilde{\mathcal{L}}_R = 0$ for all $R > 0$. This in turn implies $(u-v)^+ |\nabla(u-v)^+|^2 = 0$ in the whole of Ω . Since $(u-v)^+ = 0$ on $\partial\Omega$, we get $(u-v)^+ = 0$ everywhere, that is $u \leq v$ on the whole of Ω and the weak comparison principle is proved.

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<https://mathoverflow.net/questions/283467>,

<https://mathoverflow.net/questions/284388>,

<https://mathoverflow.net/questions/285061>.

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