



Heisenberg quasiregular ellipticity

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Abstract. Following the Euclidean results of Varopoulos and Pankka–Rajala, we provide a necessary topological condition for a sub-Riemannian 3-manifold M to admit a nonconstant quasiregular mapping from the sub-Riemannian Heisenberg group \mathbb{H} . As an application, we show that a link complement $\mathbb{S}^3 \setminus L$ has a sub-Riemannian metric admitting such a mapping only if L is empty, an unknot or Hopf link. In the converse direction, if L is empty, a specific unknot or Hopf link, we construct a quasiregular mapping from \mathbb{H} to $\mathbb{S}^3 \setminus L$.

The main result is obtained by translating a growth condition on $\pi_1(M)$ into the existence of a supersolution to the 4-harmonic equation, and relies on recent advances in the study of analysis and potential theory on metric spaces.

1. Introduction

Given two topological manifolds X and Y , it is often quite difficult to decide whether there exists a covering map from $f: X \rightarrow Y$. The only obvious obstruction is that the universal covers of X and Y should be homeomorphic. Furthermore, two manifolds with the same universal cover may have substantially different geometries. For example, \mathbb{R}^2 covers both the torus with abelian fundamental group \mathbb{Z}^2 , and the punctured torus with fundamental group the free group \mathbb{F}_2 .

The covering problem becomes more tractable if we impose geometric restrictions on the allowed covering maps. We will ask for f to be *quasiregular*, imposing a quasiconformal-type restriction on metric distortion, but also allowing the mapping to be a branched covering map onto its image; or (for compactness purposes) a constant mapping. See Section 3 for a precise definition of quasiregularity. Examples of quasiregular mappings include isometric embeddings, conformal and quasiconformal homeomorphisms, and branched holomorphic mappings.

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In this paper, we ask which sub-Riemannian 3-manifolds admit quasiregular mappings from the Heisenberg group, that is, which such manifolds are *Heisenberg quasiregularly elliptic*. We begin by reviewing the history of the quasiregular ellipticity question.

1.1. Euclidean quasiregular ellipticity

A Riemannian n -manifold M is said to be (*Euclidean*) *quasiregularly elliptic* if there exists a nonconstant quasiregular mapping from \mathbb{R}^n to M (see §2.3 for a more nuanced discussion of the definition). The classification of quasiregularly elliptic manifolds goes back to the Picard theorem in complex analysis, and has had a profound impact on the development of geometric mapping theory. In particular, Gromov devotes a chapter of his celebrated ‘Green Book’, Chapter 6 in [32], to the interplay between isoperimetric inequalities, quasiregular ellipticity and the geometry of groups.

The classical Picard theorem states that every nonconstant entire function misses at most one point. By Stoilow factorization [71], [52], this is equivalent to saying that $\mathbb{R}^2 \setminus P$ is quasiregularly elliptic if and only if P contains at most one point. While the holomorphic interpretation does not persist in higher dimensions, Rickman showed in [66] for all $n \geq 3$ that if $\mathbb{R}^n \setminus P$ is quasiregularly elliptic, then P contains at most finitely many points. Rickman also provided a converse in three dimensions [67]. It was only in a remarkable recent paper of Drasin and Pankka [21] that Rickman’s construction was extended to all dimensions. Alternate PDE proofs of the Rickman–Picard theorem were provided by Lewis and Eremenko–Lewis [53], [23], [47].

Varopoulos ([75], pp. 146–147) proved that if a closed Riemannian n -manifold M is quasiregularly elliptic, then the fundamental group $\pi_1(M)$ is virtually nilpotent and indeed has growth rate at most n . Pankka–Rajala [62] extended Varopoulos’ theorem to open manifolds, and provided a result in the spirit of Picard’s theorem: a link complement $\mathbb{S}^3 \setminus L$ is *quasiregularly elliptic for some choice of Riemannian metric* if and only if L is empty, the unknot or the Hopf link (cf. Gromov [32], Examples 6.12). The present paper generalizes the results of Pankka–Rajala and thus contributes to the study of quasiregular ellipticity in a sub-Riemannian setting.

In the Riemannian setting, the theory is naturally more advanced. A full classification of closed quasiregularly elliptic 3-manifolds was provided by Jormakka in [48], modulo the now-resolved geometrization conjecture. Holopainen and Rickman [44] extended the Rickman–Picard theorem to more general Riemannian targets, and they studied quasiregular mappings between Riemannian manifolds in [45]. Bonk and Heinonen provided an obstruction to quasiregular ellipticity for closed manifolds in terms of cohomological dimension in [8], which allows to prove nonellipticity in some cases where the fundamental group is too small to apply Varopoulos’ theorem.

1.2. Heisenberg quasiregular ellipticity

In this paper, we leave the Riemannian framework and study a quasiregular ellipticity in the sub-Riemannian setting (§2.1). The simplest homogeneous space admit-

ting a non-Euclidean sub-Riemannian metric is the Heisenberg group \mathbb{H} , a nilpotent group of step 2 and topological dimension 3. The Heisenberg group shares much of the structure of Euclidean space, including a one-parameter family of metric dilations, and therefore serves as a natural model space for sub-Riemannian geometry and source space for quasiregular mappings. We say that a sub-Riemannian manifold is *Heisenberg quasiregularly elliptic* if it admits a non-constant quasiregular map from \mathbb{H} .

The study of Heisenberg quasiregular ellipticity is in the early stages of development. A Rickman–Picard theorem for the Heisenberg group (and more generally for H-type Carnot groups) was provided by Heinonen–Holopainen in [39] (see also [55]): if $\mathbb{H} \setminus P$ is Heisenberg-quasiregularly elliptic, then P contains at most finitely many points. However, it remains an open problem to show that a quasiregular map from \mathbb{H} to itself can miss even a single point.

The study of quasiregular mappings to more general sub-Riemannian targets was recently initiated in [26], [34], [35], [37]. Following the above results of Varopoulos and Pankka–Rajala, we prove:

Theorem 1.2.1. *Let $L \subset \mathbb{S}^3$ be a link in the three-sphere and let \mathbb{H} denote the first Heisenberg group equipped with its standard sub-Riemannian structure.*

- *If there exists an equiregular sub-Riemannian metric g on $\mathbb{S}^3 \setminus L$ admitting a nonconstant quasiregular mapping $f: \mathbb{H} \rightarrow (\mathbb{S}^3 \setminus L, g)$ then L is empty, an unknot, or a Hopf link.*
- *Conversely, there exist a smooth unknot S and a smooth Hopf link H , and for $L \in \{\emptyset, S, H\}$, there exist equiregular sub-Riemannian metrics g_\emptyset , g_S , and g_H in $\mathbb{S}^3 \setminus L$ and nonconstant quasiregular maps $f: \mathbb{H} \rightarrow (\mathbb{S}^3 \setminus L, g_L)$.*

For the second part, we provide in Section 2.2 new explicit examples of mappings from the Heisenberg group onto the 3-sphere and (specific) unknot and Hopf link complements. The first statement of the theorem is a consequence of the following more general Varopoulos-type result, which we prove in Section 4.

Theorem 1.2.2. *Let M be an equiregular sub-Riemannian 3-manifold. If there exists a nonconstant quasiregular mapping $f: \mathbb{H} \rightarrow M$, then the growth rate of $\pi_1(M)$ is at most 4.*

Here and throughout this paper, we say that a group G has *growth rate at least d* if there exists a finite set S in G and a constant $c > 0$ so that the cardinality of any ball $B(R)$ is at least cR^d for all positive integers R , where $B(R)$ denotes the ball of radius R about the identity element in the word metric on the subgroup $\langle S \rangle$ generated by S . Equivalently, G has *growth rate at most d* if for every finitely generated subgroup Γ of G and for every finite set S with $\Gamma = \langle S \rangle$, there exists a constant $C > 0$ so that $B(R)$ has cardinality at most CR^d for all positive integers R . See [62] or [32], §5B, for more details.

Remark 1.2.3. We expect that a statement as in Theorem 1.2.2 holds true also in higher dimensions, that is, for quasiregular maps from \mathbb{H}^n to an equiregular

sub-Riemannian $(2n + 1)$ -manifold M , with an analogous proof. For simplicity, we restrict our discussion to 3-manifolds.

Assuming Theorem 1.2.2 we now indicate how to derive Theorem 1.2.1.

Proof of Theorem 1.2.1. If L is not one of the links listed above, then $\pi_1(M)$ contains a free group of rank at least 2, and thus it has exponential growth, that is, in particular it has growth rate larger than 4. See the references in [62]. By Theorem 1.2.2, manifolds with this property cannot admit nonconstant quasiregular mappings from the Heisenberg group.

The examples in Section 2.2 establish the positive implication in all the remaining cases, that is, if L is empty, a specific unknot, or a specific Hopf link. \square

1.3. Outline of the proof of Theorem 1.2.2

The proof of Theorem 1.2.2 will be given in Section 4. Here we provide a brief outline, following the corresponding subsections of Section 4.

- 4.1. Starting with the assumption that M has a fundamental group with a finitely generated subgroup Γ with growth rate larger than 4, we define a “(relatively) compact core” $M' \subset M$ and a lift \widetilde{M}'' of M' to \widetilde{M} , satisfying $\widetilde{M}''/\Gamma = M'$. We define a distance on the closure of M' , lift it to \widetilde{M}'' and show that the resulting space is quasi-isometric to Γ .
- 4.2. Using the fact that \widetilde{M}'' is quasi-isometric to Γ and that Γ has growth rate larger than 4, we show that \widetilde{M}'' (or rather, a net Y on \widetilde{M}'') satisfies a ‘rough’ d -dimensional isoperimetric inequality for some $d > 4$.
- 4.3. We use the local geometry of \widetilde{M}'' to prove a weak $(4/3, 1)$ -Poincaré (or Sobolev–Poincaré) inequality and a weak relative 4-dimensional isoperimetric inequality (for balls of fixed size centred in Y). This requires a careful study of \widetilde{M}'' at and near its boundary.
- 4.4. Combining the rough and the relative isoperimetric inequality, we deduce that \widetilde{M}'' also fulfills a ‘smooth’ d -dimensional isoperimetric inequality. We formulate this implication in an axiomatic way, so that it applies also in a more abstract setting.
- 4.5 We next study the 4-capacity in \widetilde{M} of a ball in \widetilde{M}'' . Fixing an admissible function u for the capacity, we restrict u to \widetilde{M}'' and use a sub-Riemannian coarea formula to relate the horizontal gradient of $u|_{\widetilde{M}''}$ to the perimeter of its level sets. Coupled with the isoperimetric inequality established above, we obtain a uniform positive lower bound for the L^4 -norm of the horizontal gradient of u . That is, we show that \widetilde{M} is 4-hyperbolic. (In fact, we show a stronger version of 4-hyperbolicity of \widetilde{M}'' and combine this with the fact that the inclusion $\widetilde{M}'' \hookrightarrow \widetilde{M}$ is bi-Lipschitz to obtain hyperbolicity of \widetilde{M} .)
- 4.6. We conclude from the 4-hyperbolicity of \widetilde{M} the existence of a positive non-constant supersolution to the 4-harmonic equation, and the existence of a Green’s function for the sub-elliptic 4-Laplacian at every point of \widetilde{M} .

- 4.7. We show that quasiregular mappings have a morphism property: the pullback of a supersolution to the 4-harmonic equation is a supersolution to a nonlinear operator of type 4.
- 4.8. Lastly, we suppose that $f: \mathbb{H} \rightarrow M$ is a nonconstant quasiregular map. We lift it to a nonconstant quasiregular map $\tilde{f}: \mathbb{H} \rightarrow \tilde{M}$ and pull back a nonconstant supersolution to the 4-harmonic equation, contradicting the 4-parabolicity of the Heisenberg group.

While the preceding outline of the proof is largely the same as in the Riemannian case [62], the sub-Riemannian geometry enters the picture in a non-trivial way in most of the steps described above. For instance, the contact structure prevents us from constructing a double of $\overline{M'}$ as in [62]. We address this issue by carefully analyzing intrinsic balls in M' and we state properties of the intrinsic distance, which we believe to be of independent interest. Furthermore, unlike in [62], we cannot apply directly the work by Kanai [49], which has been formulated for Riemannian manifolds with Ricci curvature bounds. We take the opportunity to translate his argument into a more general axiomatic framework, which applies to our setting. The proof works in metric measure spaces with a mild condition on the volumes of balls. Throughout the individual steps of the proof, we also combine results that have been recently developed in various areas of sub-Riemannian geometry, such as classifications of uniform and Sobolev–Poincaré domains, notions and properties of horizontal perimeter, and others.

Remark 1.3.1. One could bypass the discussion of the morphism property by proving a capacity inequality for arbitrary condensers in \tilde{M} . This is the approach employed by Varopoulos, see [75], Chapter X. We expect that a similar argument works in the present setting. However, since the notion of \mathcal{A} -harmonic functions has classically strong connections with questions of quasiregular ellipticity and is of independent interest for further developments, we decided to follow a different route in the present paper.

Structure of the paper

This paper is organized as follows. In Section 2 we exhibit examples of nonconstant quasiregular mappings from the Heisenberg group onto the 3-sphere and onto the complement of the unknot and Hopf link. Section 3 contains background information about quasiregular mappings of sub-Riemannian contact manifolds. Section 4 is the heart of the paper. Here we prove Theorem 1.2.2 following the outline previously indicated. We have relegated to an appendix (Appendix A) several basic properties of the calculus of horizontal derivatives.

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2. Examples of Heisenberg quasiregularly elliptic spaces

In this section we describe the Heisenberg group and some spaces that admit quasiregular mappings from it.

2.1. Sub-Riemannian manifolds

Recall that a *sub-Riemannian manifold* is a triple (M, HM, g_M) , where M is a connected smooth manifold, $HM \subset TM$ is a smooth bracket-generating distribution, and g_M is a metric on HM . An absolutely continuous curve γ in M is *horizontal* if $\dot{\gamma}$ is almost always in the horizontal distribution HM . By Chow's theorem, any two points of M are connected by a horizontal curve, and one defines the Carnot–Carathéodory distance between two points $p, q \in M$ as the infimum of g_M -lengths of the horizontal curves joining p to q . A sub-Riemannian manifold is furthermore *equiregular* if the distribution HM and its iterated brackets are, in fact, subbundles of TM of constant dimension.

In this paper, we restrict our attention to equiregular sub-Riemannian manifolds M of dimension 3, and assume that $HM \neq TM$. It is easy to see that the bracket-generating condition is then equivalent to HM being a *contact distribution*. That is, locally HM is the kernel of a smooth *contact form* α satisfying $\alpha \wedge d\alpha \neq 0$. In particular, the Darboux theorem states that locally (M, HM) is contactomorphic to the Heisenberg group with its standard contact structure. Note that this contactomorphism need not send the metric g_M to the Heisenberg metric $g_{\mathbb{H}}$.

Example 2.1.1 (Heisenberg group). In *exponential coordinates of the first kind*, the Heisenberg group \mathbb{H} is given by \mathbb{R}^3 with group structure

$$(x, y, t) * (x', y', t') = (x + x', y + y', t + t' - 2xy' + 2yx').$$

The standard contact form on \mathbb{H} is given by

$$\alpha_{\mathbb{H}} = dt + 2(x dy - y dx).$$

Notice that $\alpha_{\mathbb{H}}$ is invariant under left translations. The horizontal distribution $H\mathbb{H}$ on \mathbb{H} is given by $\ker \alpha_{\mathbb{H}}$, and is spanned by the left-invariant vector fields

$$(2.1) \quad X = \partial_x + 2y \partial_t \quad \text{and} \quad Y = \partial_y - 2x \partial_t.$$

The sub-Riemannian path metric $d_{\mathbb{H}}$ on \mathbb{H} is induced by the inner product $g_{\mathbb{H}}$ defined by the line element $ds_{\mathbb{H}}^2 = dx^2 + dy^2$ on $H\mathbb{H}$.

2.2. Heisenberg quasiregularly elliptic spaces

Our main theorem shows that not every equiregular sub-Riemannian 3-manifold is Heisenberg quasiregularly elliptic. We now describe several Heisenberg quasiregularly elliptic spaces. The first of these is well known, while we believe that the remaining constructions are new. We will leave the formal definition of quasiregularity for the next section (Definition 3.2.6), as all the mappings we mention

(apart from Example 2.2.5) are covering mappings that are either locally isometric or conformal. For the moment, it is sufficient to think of a quasiregular map $f: (\mathbb{H}, H\mathbb{H}, g_{\mathbb{H}}) \rightarrow (M, HM, g_M)$ as a continuous branched cover with the property that $f_*(H\mathbb{H}) \subset HM$ and so that there exists a constant K such that for almost every $p \in \mathbb{H}$, one has

$$\sup g_M(f_*v, f_*v) \leq K^2 \inf g_M(f_*v, f_*v),$$

where the supremum and the infimum are taken over all horizontal vectors $v \in H_p\mathbb{H}$ with $g_{\mathbb{H}}(v, v) = 1$.

Example 2.2.1 (Sub-Riemannian 3-sphere). Consider the 3-sphere \mathbb{S}^3 , viewed as the unit sphere in \mathbb{C}^2 . The tangent space to \mathbb{S}^3 at a point p is the real orthogonal complement of the normal vector $\vec{n}(p) = p$. This tangent space is not invariant under multiplication by the imaginary unit i . The subbundle of the tangent bundle which is invariant under multiplication by i coincides with the kernel of a contact form and defines a sub-Riemannian structure. Explicitly, let $\alpha_{\mathbb{S}^3}$ be the contact form given by

$$\alpha_{\mathbb{S}^3} = \bar{w}_1 dw_1 - w_1 \overline{dw_1} + \bar{w}_2 dw_2 - w_2 \overline{dw_2}$$

where $w = (w_1, w_2)$ denote coordinates in \mathbb{C}^2 . The standard sub-Riemannian metric on \mathbb{S}^3 is given by the restriction of the Euclidean inner product to $\ker \alpha_{\mathbb{S}^3}$.

The inverse stereographic projection

$$\iota(x, y, t) = \left(\frac{2y - 2xi}{1 + x^2 + y^2 - it}, \frac{1 - x^2 - y^2 + it}{1 + x^2 + y^2 - it} \right)$$

provides a bijection between \mathbb{H} and $\mathbb{S}^3 \setminus \{(0, -1)\}$, and is furthermore well known to be both contactomorphic and conformal, see for instance [50], p. 315, or [9], Section 3.3. It follows that the sub-Riemannian \mathbb{S}^3 and the punctured sub-Riemannian \mathbb{S}^3 are Heisenberg quasiregularly elliptic.

Example 2.2.2 (Lens spaces). Let \mathbb{L} be a lens space equipped with its standard contact structure and sub-Riemannian metric arising from its representation as a quotient of \mathbb{S}^3 . (See Section 3.1 in [26] for details on the sub-Riemannian structure on lens spaces). We assume that $\mathbb{L} \neq \mathbb{S}^3$. Composing the embedding $\iota: \mathbb{H} \hookrightarrow \mathbb{S}^3$ with the quotient projection $\pi: \mathbb{S}^3 \rightarrow \mathbb{L}$ gives a conformal map from \mathbb{H} onto \mathbb{L} .

More generally, if Γ is any group of isometries of the sub-Riemannian \mathbb{S}^3 such that \mathbb{S}^3/Γ is a smooth manifold and $\pi: \mathbb{S}^3 \rightarrow \mathbb{S}^3/\Gamma$ denotes the quotient map, then the composition $\pi \circ \iota$ is a quasiregular mapping from \mathbb{H} to \mathbb{S}^3/Γ with its standard contact structure and sub-Riemannian metric. See [19] and [33] for examples of finite isometry groups of \mathbb{S}^3 arising in the study of proper holomorphic mappings between balls and CR representation theory.

Example 2.2.3 (Unknot complement). Let $L_1 \subset \mathbb{S}^3$ be an unknot, whose complement $M_1 := \mathbb{S}^3 \setminus L_1$ is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}^2$. We claim that M_1 has a sub-Riemannian metric admitting a surjective quasiregular map from \mathbb{H} .

Consider first the quotient $M'_1 := \mathbb{H}/\langle(0, 0, 1)\rangle$ of the Heisenberg group by integer translations along the t -axis. Since vertical translations are isometries

of \mathbb{H} , the sub-Riemannian metric $g_{\mathbb{H}}$ projects to a well-defined sub-Riemannian metric on M'_1 , with the projection map $\pi: \mathbb{H} \rightarrow M'_1$ a surjective local isometry.

Note now that M_1 and M'_1 are both diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}^2$. Let $f: M'_1 \rightarrow M_1$ be a diffeomorphism. Give M_1 the contact structure and sub-Riemannian metric induced by this diffeomorphism. Then the map $f \circ \pi: \mathbb{H} \rightarrow M_1$ is a surjective local isometry, as desired.

For an explicit example, let L' denote the t -axis in \mathbb{H} , and consider the mapping $h(x, y, t) = (\cos(2\pi t)e^x, \sin(2\pi t)e^x, y)$ from \mathbb{H} to itself. This mapping commutes with integer translations along the t -axis, and so induces a sub-Riemannian metric g on $\mathbb{H} \setminus L'$. Then $h: (\mathbb{H}, g_{\mathbb{H}}) \rightarrow (\mathbb{H} \setminus L', g)$ is a quasiregular surjection.

Example 2.2.4 (Hopf link complement). Let $L_2 \subset \mathbb{S}^3$ be the Hopf link, with $M_2 := \mathbb{S}^3 \setminus L_2$ diffeomorphic to $\mathbb{R} \times \mathbb{S}^1 \times \mathbb{S}^1$. We claim that M_2 has a sub-Riemannian metric admitting a surjective quasiregular map from \mathbb{H} .

Note first that the integer group \mathbb{Z}^2 acts on \mathbb{H} by the isometries

$$(0, b, c) * (x, y, t) = (x, y + b, t + c + 2bx).$$

The quotient space $M'_2 = \mathbb{H}/\mathbb{Z}^2$ then inherits a sub-Riemannian structure from \mathbb{H} , and the projection map $\pi: \mathbb{H} \rightarrow M'_2$ is a surjective local isometry.

Note that M_2 and M'_2 are both diffeomorphic to $\mathbb{R} \times \mathbb{S}^1 \times \mathbb{S}^1$. Let $f: M'_2 \rightarrow M_2$ be a diffeomorphism. Give M_2 the contact structure and sub-Riemannian metric induced by this diffeomorphism. Then the map $f \circ \pi: \mathbb{H} \rightarrow M_2$ is a surjective local isometry, as desired.

An explicit example is easy to construct as in Example 2.2.3, taking L to be the union of the equators in $\mathbb{S}^3 \subset \mathbb{C}^2$.

Example 2.2.5 (Surjection to the sub-Riemannian 3-sphere). We conclude this section by providing a *surjective* quasiregular mapping from the Heisenberg group to the sub-Riemannian 3-sphere.

Fix an integer $a > 1$ and let $f'_a: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ be the continuous extension of the map

$$f'_a: (r_1 e^{i\theta_1}, r_2 e^{i\theta_2}) \mapsto (r_1 e^{ai\theta_1}, r_2 e^{ai\theta_2}),$$

defined on $\Omega_0 := \{(r_1 e^{i\theta_1}, r_2 e^{i\theta_2}) : r_1 \neq 0, r_2 \neq 0\}$.

The *multi-twist map* $f_a: \mathbb{S}^3 \rightarrow \mathbb{S}^3$ is given by restricting f'_a to the sphere. It is clear that f_a is a branched covering map, with branching along the standard Hopf link, and it was shown in [26] that f_a is quasiregular according to the so-called metric definition. The equivalence of various definitions of quasiregularity (metric, geometric, and analytic) on equiregular sub-Riemannian manifolds has been established in [34], [35]. To keep our discussion self-contained, we verify here directly that the map appearing in the following lemma is quasiregular in the sense of Definition 3.2.6.

Lemma 2.2.6. *Let ι be the conformal embedding of \mathbb{H} into \mathbb{S}^3 as in Example 2.2.1. Then the map $f := f_a \circ \iota: \mathbb{H} \rightarrow \mathbb{S}^3$ is a surjective quasiregular map. Here, \mathbb{H} and \mathbb{S}^3 are endowed with their standard contact structures and sub-Riemannian metrics.*

Proof. Since we have chosen a to be an integer larger than 1, the map f is surjective from \mathbb{H} to \mathbb{S}^3 . Recall that ι is a diffeomorphism and f_a is smooth on $\Omega_0 = \{(r_1 e^{i\theta_1}, r_2 e^{i\theta_2}) : r_1 \neq 0, r_2 \neq 0\}$, so f is smooth on $\iota^{-1}(\Omega_0)$. By a short computation as in Section 3.2 of [26], one sees that

$$g_{\mathbb{S}^3}(v, v) \leq g_{\mathbb{S}^3}((f_a)_*v, (f_a)_*v) \leq a^2 g_{\mathbb{S}^3}(v, v), \quad v \in H_p \mathbb{S}^3, p \in \Omega_0.$$

Combined with the fact that ι is conformal, it follows that $f = f_a \circ \iota$ fulfills the distortion estimate required for quasiregularity on the set $\iota^{-1}(\Omega_0)$. As $\mathbb{H} \setminus \iota^{-1}(\Omega_0)$ is the union of the t -axis with a unit circle in the xy -plane, it is negligible and we know that the distortion estimate holds almost everywhere on \mathbb{H} as required.

The remaining property in Definition 3.2.6 to verify is the existence of weak horizontal derivatives of f in L^4_{loc} . Since f is smooth outside the t -axis and a planar circle in \mathbb{H} , it follows that it is absolutely continuous along almost every fiber in a fibration given by a horizontal left invariant vector field. The corresponding directional derivatives exist pointwise almost everywhere and are weak derivatives. (See, for instance, Theorem 2.2 in [73] and Remark 3.2.3.) It remains to establish their local integrability. For this purpose, it is useful to introduce the complex operators

$$Z = \frac{1}{2}(X - iY) \quad \text{and} \quad \bar{Z} = \frac{1}{2}(X + iY)$$

on \mathbb{H} , and to use coordinates $w = (w_1, w_2)$ on \mathbb{S}^3 . A direct computation gives

$$(2.2) \quad \iota_* Z = -i \frac{(1 + w_2)^2}{1 + \bar{w}_2} W \quad \text{and} \quad \iota_* \bar{Z} = i \frac{(1 + \bar{w}_2)^2}{1 + w_2} \bar{W},$$

where

$$W = \bar{w}_2 \partial_{w_1} - \bar{w}_1 \partial_{w_2} \quad \text{and} \quad \bar{W} = w_2 \partial_{\bar{w}_1} - w_1 \partial_{\bar{w}_2};$$

cf. [50], p. 320. Writing $w_j = r_j e^{i\theta_j}$ for $j \in \{1, 2\}$ and $w' = (w'_1, w'_2) = f_a(w)$, we find

$$\partial_{w_1} w'_1 = e^{i(a-1)\theta_1} \left(\frac{1+a}{2} \right) \quad \text{and} \quad \partial_{\bar{w}_1} w'_1 = e^{i(a+1)\theta_1} \left(\frac{1-a}{2} \right)$$

on Ω_0 . Analogous formulae hold for w'_2 . It follows that $Z(h \circ f_a \circ \iota)$ and $\bar{Z}(h \circ f_a \circ \iota)$ are in L^∞_{loc} for an arbitrary smooth function $h: \mathbb{S}^3 \rightarrow \mathbb{R}$. This shows that f has weak horizontal derivatives in L^4_{loc} and concludes the proof. \square

2.3. Notions of quasiregular ellipticity

In this section we discuss some subtleties in the definition of quasiregular ellipticity, and provide a few more examples.

We start with Euclidean quasiregular ellipticity. A Riemannian n -manifold (M, g_M) is *quasiregularly elliptic* if there exists a non-constant quasiregular mapping $f: \mathbb{R}^n \rightarrow M$. A differentiable n -manifold M (without specifying a Riemannian metric) is *quasiregularly elliptic as a manifold* if it supports some Riemannian metric g' such that (M, g') is quasiregularly elliptic.

Example 2.3.1. The Rickman–Picard theorem states that any quasiregular map $f: \mathbb{S}^3 \rightarrow \mathbb{S}^3$ of the Euclidean sphere misses at most finitely many points. Thus, if L is a non-empty link, $\mathbb{S}^3 \setminus L$ is not quasiregularly elliptic with the induced standard metric from \mathbb{S}^3 . However, if L is either a smooth unknot or a smooth Hopf link, then $\mathbb{S}^3 \setminus L$ is quasiregularly elliptic as a manifold [62].

Analogous definitions apply for sub-Riemannian 3-manifolds with \mathbb{H} as the source space: one can consider Heisenberg quasiregular ellipticity of the sub-Riemannian manifold (M, HM, g_M) , of the contact manifold (M, HM) with some choice of Riemannian tensor on HM , or of the manifold M with some choice of bundle and Riemannian tensor. In the examples above, we have shown that the sub-Riemannian 3-sphere and lens spaces are quasiregularly elliptic with their standard sub-Riemannian metric, while the unknot complement and Hopf link complement are quasiregularly elliptic as manifolds.

Question 2.3.2. Is there an unknot complement or a Hopf link complement that is Heisenberg quasiregularly elliptic when endowed with the contact structure induced by the standard contact form $\alpha_{\mathbb{S}^3}$?

We conclude with four more examples of quasiregularly elliptic manifolds, leaving the constructions of quasiregular mappings to the reader.

Example 2.3.3. The half-space $\mathbb{H}_{x^+} = \{(x, y, t) \in \mathbb{H} : x > 0\}$ is Heisenberg quasiregularly elliptic as a contact manifold. One can construct an explicit mapping using polarized coordinates. (For the definition of polarized coordinates, see for instance §2.1 in [9]).

Example 2.3.4. Any domain $\Omega \subset \mathbb{H}$ diffeomorphic to \mathbb{H} is Heisenberg quasiregularly elliptic as a contact manifold. This follows from the uniqueness of tight contact structures on \mathbb{R}^3 , see Eliashberg [22].

In the next two examples, our source space is \mathbb{H}_{x^+} .

Example 2.3.5. Equip $M = \mathbb{H} \setminus \{(0, 0, t) : t \in \mathbb{R}\}$ with the standard contact structure. Then there is a Riemannian tensor on HM for which there exists a quasiregular map $f: \mathbb{H}_{x^+} \rightarrow \mathbb{H} \setminus \{(0, 0, t) : t \in \mathbb{R}\}$. One can construct an explicit map using polarized coordinates. The map f is not defined on all of \mathbb{H} and the contactomorphism from Example 2.3.3 distorts the standard metric. Hence the existence of f does not imply that the unknot complement in \mathbb{S}^3 with the induced standard contact structure is Heisenberg quasiregularly elliptic as a contact manifold.

Example 2.3.6. Let $M = \mathbb{S}^3 \setminus L$, where L is the union of equators $w_1 = 0$ and $w_2 = 0$ in \mathbb{S}^3 . Give M the standard contact structure HM . Then there is a Riemannian tensor on HM for which there exists a quasiregular map $f: \mathbb{H}_{x^+} \rightarrow M$. One can construct an explicit map using a contactomorphism from \mathbb{H} to the roto-translation group (as given for instance in [25]). Note that this does not imply that the Hopf link complement in \mathbb{S}^3 is Heisenberg quasiregularly elliptic as a contact manifold with the standard contact structure, as f is not defined on all of \mathbb{H} .

3. Quasiregular mappings: preliminaries

Quasiregular mappings have first been studied in Euclidean space as a generalization of complex analytic functions. They also arise naturally as non-injective counterparts for quasiconformal maps. We refer to [65], [68] for in-depth introductions to the subject. The definitions generalize to Riemannian manifolds, see for instance [43]. In the sub-Riemannian setting, quasiregular mappings were first investigated in the Heisenberg group and other Carnot groups [39], [18]. Many properties of quasiregular mappings in Euclidean spaces carry over to the Carnot group setting; for an example we mention the results in [55]. The theory for more general equiregular sub-Riemannian manifolds has been initiated in [26] and further developed in a recent series of papers [34], [35], [37], [36]. Versions of quasiregularity on metric spaces of locally bounded geometry were discussed in [15].

Even though quasiregular mappings have been studied already in greater generality, we decided to include in this section a self-contained discussion, which focuses on the specific setting of this paper. Restricting our attention to contact 3-manifolds allows us to exploit properties of the Heisenberg group using contact geometry. Moreover, oriented sub-Riemannian contact 3-manifolds can be endowed with a CR structure [24]. While we do not make use of this structure in our proofs, the reader interested in CR geometry may read our results as a continuation of the research on quasiconformal maps in CR 3-manifolds [51], [73], [72], [57].

Sections 3.1 and 3.2 contain definitions related to sub-Riemannian contact manifolds and quasiregular mappings, respectively. Section 3.3 is devoted to the interplay between contact geometry and quasiregular mappings, and we provide auxiliary results that allow us to make use of the rich theory in the Heisenberg group.

3.1. Definition of contact forms and measures adapted to a metric

As discussed in Section 2.1, a 3-manifold M with a subbundle $HM \subsetneq TM$ is an equiregular sub-Riemannian manifold precisely if it is a contact manifold. We use this fact throughout the paper, and we alternate our perspective between sub-Riemannian and contact geometry. We do not assume that the horizontal distribution of the manifolds under consideration is the kernel of an a priori given contact form. Instead, we will now describe how to choose a specific such form α_M canonically associated with the sub-Riemannian metric g_M on (M, HM) . The related volume form $\alpha_M \wedge d\alpha_M$ is useful since a meaningful geometric study of quasiregular mappings requires a canonical choice of measure in order to define notions such as Jacobian and distributional Laplacians. In Riemannian manifolds, the choice is the Riemannian volume form. A natural generalization to the sub-Riemannian setting is provided by the *Popp volume*: a smooth volume form canonically associated to an equiregular sub-Riemannian manifold M and the metric g_M thereon.

Definition 3.1.1. Let (M, HM, g_M) be an equiregular 3-manifold with a co-orientable horizontal distribution. Let α_M be the contact form uniquely determined by the conditions that $\ker \alpha_M = HM$ and that $d\alpha_M|_{HM}$ coincides with

the volume form induced by g_M on HM . The *Popp volume* vol_M is given by $\alpha_M \wedge d\alpha_M$. The associated measure μ_M is called the *Popp measure*.

Equivalently, one can choose a local orthonormal frame $\{e_1, e_2\}$ on (HM, g_M) and let e_3 be the Reeb vector field determined by α_M . If $\{\nu_1, \nu_2, \nu_3\}$ denotes the dual orthonormal basis to $\{e_1, e_2, e_3\}$, then $\nu_1 \wedge \nu_2 \wedge \nu_3$ agrees with vol_M , independently of the choice we made for the orthonormal frame. One can also see vol_M as the Riemannian volume associated to the extended Riemannian metric obtained from g_M by declaring $\{e_1, e_2, e_3\}$ orthonormal. For a more thorough discussion of Popp measures on contact manifolds, the reader may consult for instance [3], [5].

Remark 3.1.2. In order to keep the presentation in the Definition 3.1.1 simple, we have assumed that the subbundle HM is co-orientable, that is, given by the kernel of a globally defined contact form. If this is not the case, we cannot choose a global orientation for HM and the form α_M is defined only up to a sign. However, while we cannot globally promote g_M to a Riemannian metric, the Popp volume vol_M is still well-defined. See also Remark 9 in [2]. In fact, Popp measures can be introduced much more generally on arbitrary equiregular sub-Riemannian manifolds (see [58], 10.6).

For an equiregular sub-Riemannian 3-manifold the Popp measure equals a constant multiple of spherical 4-dimensional Hausdorff measure with respect to the sub-Riemannian distance; see Theorem 4 in [1]. In the case of \mathbb{H} with the standard sub-Riemannian metric $g_{\mathbb{H}}$, the measure $\mu_{\mathbb{H}}$ coincides with the Haar measure on \mathbb{H} , which is the 3-dimensional Lebesgue measure (all up to a possible multiplicative factor).

3.2. Definition of horizontal derivatives and quasiregularity

We begin our formal discussion of quasiregularity by introducing certain classes of functions. Let U be an open set in an equiregular sub-Riemannian 3-manifold N with an orthonormal frame $\{e_1, e_2\}$ of the subbundle HN . The *horizontal Sobolev space* $HW^{1,4}(U)$ is defined as the space of functions $u \in L^4(U)$ whose distributional derivatives in direction e_1 and e_2 exist and belong to $L^4(U)$, and the *local horizontal Sobolev space* $HW_{\text{loc}}^{1,4}(U)$ is defined accordingly. See for instance Section 2.2 in [10] for the precise definitions.

We also consider the regularity of mappings that take values in a manifold. However, we do not need the full structure of a Sobolev space in this setting, so we confine ourselves to the following definition. It is stated in terms of the (divergence-free) vector fields X and Y given in (2.1).

Definition 3.2.1. We say that a continuous map $f: \mathbb{H} \rightarrow M$ has L_{loc}^4 *weak horizontal derivatives* if for any smooth function $h: M \rightarrow \mathbb{R}$ and any open set $U \Subset \mathbb{H}$, there is a function $g \in L_{\text{loc}}^4(\mathbb{H})$ so that

$$\int_U X\varphi \cdot (h \circ f) \, d\mu_{\mathbb{H}} = - \int_U \varphi \cdot g \, d\mu_{\mathbb{H}}, \quad \text{for all } \varphi \in \mathcal{C}_0^\infty(U),$$

and analogously for the vector field Y . We write $g = X(h \circ f)$.

A few remarks concerning this definition are in order.

Remark 3.2.2. Our definition of weak horizontal derivatives essentially agrees with the one given by Tang in [73], §2, for maps between smooth strongly pseudoconvex CR 3-manifolds, and in case the target manifold is the Heisenberg group, it matches the standard definition employed in connect with the horizontal Sobolev space as for instance used in [18].

Remark 3.2.3. According to Theorem 2.2 in [73], a continuous map $f: \mathbb{H} \rightarrow M$ has L_{loc}^4 weak horizontal derivatives if and only if it is ACL, i.e., f is absolutely continuous along almost every fiber in the fibration given by X and Y , and, moreover, it has L_{loc}^4 horizontal derivatives in these directions. In this case, the weak and pointwise horizontal derivatives coincide almost everywhere. Analogous definitions and statements apply for other integrability exponents $1 \leq p < \infty$.

We employ the horizontal derivatives to introduce a formal horizontal differential. Given local coordinates (x_1, x_2, x_3) on M , we can define for every continuous map $f: \mathbb{H} \rightarrow M$ with L_{loc}^4 weak horizontal derivatives the following notions:

$$(3.1) \quad Xf = \sum_{i=1}^3 X(x_i \circ f) \partial_{x_i} \quad \text{and} \quad Yf = \sum_{i=1}^3 Y(x_i \circ f) \partial_{x_i}$$

The vector fields Xf and Yf are well defined almost everywhere. Indeed, given charts Φ and Ψ , we write

$$X(\Phi_i \circ f) = X((\Phi_i \circ \Psi^{-1}) \circ (\Psi \circ f))$$

and apply the chain rule from Proposition A.0.1 in the Appendix with $h = \Phi_i$. This yields

$$\begin{aligned} Xf &= \sum_{i=1}^3 X(\Phi_i \circ f) \partial_{\Phi_i} = \sum_{i=1}^3 \left(\sum_{j=1}^3 \frac{\partial(\Phi_i \circ \Psi^{-1})}{\partial \Psi_j} (\Psi(f(\cdot))) X(\Psi_j \circ f) \right) \partial_{\Phi_i} \\ &= \sum_{j=1}^3 X(\Psi_j \circ f) \sum_{i=1}^3 \frac{\partial(\Phi_i \circ \Psi^{-1})}{\partial \Psi_j} \partial_{\Phi_i} = \sum_{j=1}^3 X(\Psi_j \circ f) \partial_{\Psi_j}, \end{aligned}$$

and analogously for X replaced by Y . Thus we can interpret $Xf(p)$ and $Yf(p)$ as elements in $T_p M$ for almost every $p \in M$ and formulate the following definition, which generalizes Definition 1.1 in [18].

Definition 3.2.4. We say that a continuous map $f: \mathbb{H} \rightarrow M$ with L_{loc}^4 weak horizontal derivatives is *weakly contact* if Xf and Yf lie in HM almost everywhere.

Definition 3.2.5. Assume that $f: \mathbb{H} \rightarrow M$ is weakly contact. At almost every $p \in \mathbb{H}$, the *formal horizontal differential* of f ,

$$D_H f(p) : H_p \mathbb{H} \rightarrow H_p M,$$

is defined by

$$(3.2) \quad D_H f(p)(X_p) = Xf(p) \quad \text{and} \quad D_H f(p)(Y_p) = Yf(p),$$

extended to the entire horizontal plane $H_p\mathbb{H}$ by linearity.

To formulate distortion conditions for a weakly contact map $f: \mathbb{H} \rightarrow M$, it is convenient to work with the quantities $\|D_H f(p)\|$ and $\ell[D_H f(p)]$. These are standard notations in quasiconformal analysis. For a linear map $A: (H_p\mathbb{H}, g_{\mathbb{H}}) \rightarrow (H_q M, g_M)$, we set

$$\|A\| := \sup_{g_{\mathbb{H}}(v,v)=1} \sqrt{g_M(Av, Av)} \quad \text{and} \quad \ell[A] := \inf_{g_{\mathbb{H}}(v,v)=1} \sqrt{g_M(Av, Av)}.$$

We are now prepared to state an analytic definition for quasiregularity.

Definition 3.2.6. Let M be a smooth orientable 3-manifold endowed with an equiregular distribution HM and a sub-Riemannian metric g_M . We call a map $f: \mathbb{H} \rightarrow M$ *quasiregular* if

- f is continuous,
- f has L_{loc}^4 weak horizontal derivatives,
- f is weakly contact,
- f satisfies the distortion estimate, that is, there exists a positive and finite constant K such that

$$(3.3) \quad \frac{\|D_H f(p)\|}{\ell[D_H f(p)]} \leq K, \quad \text{for almost every } p \in \mathbb{H}.$$

The quotient on the left-hand side of (3.3) is by convention set equal to 1 if $\|D_H f(p)\| = \ell[D_H f(p)] = 0$.

If (3.3) holds for some K we also say that f is K -*quasiregular*.

Example 3.2.7. Let us spell out explicitly Definition 3.2.6 for the case of the standard sub-Riemannian Heisenberg group, $M = \mathbb{H}$ and $g_M = g_{\mathbb{H}}$.

Assume that $f: \mathbb{H} \rightarrow \mathbb{H}$ is a nonconstant quasiregular according to Definition 3.2.6. By setting h (in the definition of L_{loc}^4 weak horizontal derivatives) equal to a projection on one of the coordinates, one sees that the components of f lie in $HW_{\text{loc}}^{1,4}(\mathbb{H})$. Moreover, in this case, $\alpha_{\mathbb{H}}(Xf) = \alpha_{\mathbb{H}}(Yf) = 0$ and

$$D_H f = \begin{pmatrix} Xf_1 & Yf_1 \\ Xf_2 & Yf_2 \end{pmatrix}$$

with respect to the basis $\{X, Y\}$. It is well known that (3.3) is equivalent to

$$\|D_H f\|^4 \leq K'(\det D_H f)^2, \quad \text{a.e.}$$

for $K' = K^2$, see also Proposition 3.3.7.

This discussion shows that f is quasiregular according to the definition commonly used for mappings in the Heisenberg group [18] (called “quasiregular in the sense of Dairbekov” in [35]). The two definitions are in fact equivalent in this setting. In [18], the regularity condition requires the components of the mapping to belong to the horizontal Sobolev space $HW_{\text{loc}}^{1,4}$, but this implies that f has L_{loc}^4 weak horizontal derivatives. To see this, we have to verify that $h \circ f$ belongs to $HW_{\text{loc}}^{1,4}$ for all smooth functions $h: \mathbb{H} \rightarrow \mathbb{R}$, yet this follows from Proposition A.0.1 applied to $M = \mathbb{H}$ and Ψ_i , $i \in \{1, 2, 3\}$, the i -th coordinate function in our model.

3.3. Equivalent characterizations of quasiregularity

In this section, we give equivalent formulations of Definition 3.2.6. The first one allows to interpret quasiregularity in charts. The second one is essentially a reformulation of the distortion condition (3.3) in terms of the Jacobian.

3.3.1. Contactomorphic coordinates.

Definition 3.3.1. Let $\Psi = (x, y, t)$ be a system of smooth \mathbb{H} -valued coordinate charts on (M, HM, g_M) with the property that for every point $p \in M$, there exists a neighborhood U and a coordinate function $\Psi: U \rightarrow \Psi(U) \subseteq \mathbb{H}$ for which

$$\Psi_{*,q}(HM) = H_{\Psi(q)}\mathbb{H}, \quad q \in U.$$

We call such coordinates *contactomorphic*.

An example of contactomorphic coordinates is provided by *Darboux’s theorem*, which allows us to arrange locally $\Psi^*\alpha_{\mathbb{H}} = \alpha_M$ for a contact form α_M on M ; see for instance Theorem 3.1 in [7].

In Proposition 3.3.3 below we will show how the quasiregularity condition can be expressed in contactomorphic coordinate charts. In the proof, the following auxiliary result is used.

Lemma 3.3.2. *Assume that U is a domain in \mathbb{H} . Let $f: U \rightarrow M$ be continuous with weak horizontal derivatives in L_{loc}^4 . Assume further that f is weakly contact and let $\Psi: f(U) \rightarrow \Psi(f(U)) \subset \mathbb{H}$ be a Darboux chart so that $\Psi \circ f$ is weakly contact. Then, for almost every $q \in U$, one has*

$$(3.4) \quad D_H f(q) = \begin{pmatrix} X(\Psi_1 \circ f)(q) & Y(\Psi_1 \circ f)(q) \\ X(\Psi_2 \circ f)(q) & Y(\Psi_2 \circ f)(q) \end{pmatrix},$$

where the matrix on the right is computed with respect to the bases

$$(3.5) \quad \{X_q, Y_q\}$$

for $H_q\mathbb{H}$, and

$$(3.6) \quad \{X_{\Psi(f(q))} = \partial_{\Psi_1} + 2\Psi_2(f(q))\partial_{\Psi_3}, Y_{\Psi(f(q))} = \partial_{\Psi_2} - 2\Psi_1(f(q))\partial_{\Psi_3}\}$$

for $H_{f(q)}M$.

Proof. By the definition of the formal horizontal differential, we have

$$D_H f(q)(aX_q + bY_q) = aXf(q) + bYf(q)$$

pointwise almost everywhere or in the sense of distributions. As the definition of Xf and Yf in (3.1) is independent of the choice of coordinates, we may in particular work in the coordinates given by Ψ . Thus,

$$(3.7) \quad aXf(q) + bYf(q) = a \sum_{i=1}^3 X(\Psi_i \circ f)(q) \partial_{\Psi_i}|_{f(q)} + b \sum_{i=1}^3 Y(\Psi_i \circ f)(q) \partial_{\Psi_i}|_{f(q)}$$

which in turn equals

$$\begin{pmatrix} X(\Psi_1 \circ f)(q) & Y(\Psi_1 \circ f)(q) \\ X(\Psi_2 \circ f)(q) & Y(\Psi_2 \circ f)(q) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

when expressed with respect to the basis $\{X_q, Y_q\}$ in the source space and the basis (3.6) in the target. Here we have used the fact that $\Psi \circ f$ is a weakly contact map between domains in the Heisenberg group, so that the right hand side of (3.7) can be rewritten using the contact equations in the Heisenberg group ([50], §B) and the local frames. \square

Proposition 3.3.3. *Let \mathbb{H} be the standard sub-Riemannian Heisenberg group and let M be a contact sub-Riemannian 3-manifold as above. A map $f: \mathbb{H} \rightarrow M$ is quasiregular in the sense of Definition 3.2.6 if and only if*

- (i) *at every point there exists a contactomorphic coordinate chart Ψ such that $\Psi \circ f$ is quasiregular between domains in \mathbb{H} with its standard sub-Riemannian structure, and*
- (ii) *the coordinate-free distortion estimate (3.3) holds for a fixed constant K .*

If the first condition holds for one contactomorphic chart Ψ at p , it holds in fact for all such charts Φ . Moreover, assuming condition (i) of the proposition, we have that $\Psi_i \circ f$ has weak horizontal derivatives in L^4_{loc} for $i \in \{1, 2, 3\}$. Then the same holds true for $h \circ f$ where h is an arbitrary smooth function by Proposition A.0.1. Thus Xf and Yf can be defined, and condition (ii) makes sense in this situation.

Proof. First, assume that f satisfies conditions (i) and (ii) in Proposition 3.3.3. Then it is continuous, has weak horizontal derivatives in L^4_{loc} and is weakly contact. The distortion estimate (3.3) holds by assumption. This proves one implication.

Second, suppose that f is quasiregular in the sense of Definition 3.2.6. Then f already satisfies condition (ii) in Proposition 3.3.3 and it suffices to check condition (i). In a neighborhood V of every point $p \in M$, we can consider a contactomorphic chart $\Psi: V \rightarrow \mathbb{H}$ given by Darboux's theorem. The map $\Psi \circ f: U \rightarrow \mathbb{H}$, for $U \subseteq \mathbb{H}$ small enough so that $f(U) \subseteq V$, is continuous. By definition, $\Psi_i \circ f \in HW^{1,4}_{\text{loc}}(U)$ for $i \in \{1, 2, 3\}$. Moreover, the weak contact condition of f implies that $\Psi \circ f$ is weakly contact with respect to the standard structure in source and target. \square

Having established the expression of the formal horizontal differential in Darboux coordinates in Lemma 3.3.2, we proceed with the proof. We may choose U small enough such that there exist constants $c, C > 0$ so that

$$(3.8) \quad c g_{\mathbb{H}, q'}(v, v) \leq (\Psi_* g_M)_{q'}(v, v) \leq C g_{\mathbb{H}, q'}(v, v)$$

for all $q' \in \Psi(f(U))$ and $v \in H_{q'}\mathbb{H}$.

By assumption, the distortion estimate

$$\frac{\|Df(q)\|}{\ell[Df(q)]} \leq K$$

holds for a.e. $q \in U$, where $\|\cdot\|$ and $\ell[\cdot]$ are computed with respect to the metric g_M in the target (which corresponds to $(\Psi_*)g_M$ if $D_H f(q)$ is expressed in coordinates as in (3.4)). The inequalities in (3.8) allow us to switch to the norm $g_{\mathbb{H}}$ in the target. We conclude that $\Psi \circ f$ satisfies the distortion estimate (3.3) in a neighborhood of p for some constant K' (depending on c and C). \square

Remark 3.3.4. Several deep properties of nonconstant quasiregular mappings (such as discreteness, openness, Lusin property and vanishing measure of the branch set) follow trivially from Proposition 3.3.3 by expressing the mappings in charts and by relying on the rich theory in the Heisenberg group as developed in [17] and [39]. It also follows that quasiregular maps are differentiable almost everywhere in the sense of [34], [35] with the formal horizontal differential almost everywhere equal to the restriction of the Margulis–Mostow derivative; cf. the discussion in §3.2 of [10].

3.3.2. Distortion estimate in terms of the Jacobian. The distortion estimate (3.3) in Definition 3.2.6 can be reformulated in terms of a Jacobian determinant of f .

Definition 3.3.5. For almost every $p \in \mathbb{H}$, the *formal (horizontal) Jacobian determinant* $\det D_H f(p)$ of a map $f: \mathbb{H} \rightarrow M$ with weak horizontal derivatives is defined as the determinant of the matrix representation of $D_H f(p)$ with respect to the orthonormal bases $\{X_p, Y_p\}$ and $\{e_1, e_2\}$ for $(H_p\mathbb{H}, g_{\mathbb{H}})$ and $(H_{f(p)}M, g_M)$.

Note that the sign of $\det D_H f(p)$ depends on the orientation of $\{e_1, e_2\}$, but this is irrelevant for the following proposition.

Proposition 3.3.6. *Let $f: \mathbb{H} \rightarrow (M, HM, g_M)$ be a weakly contact map and, for $p \in \mathbb{H}$, let $\{e_1, e_2\}$ be a local orthonormal frame on HM around $f(p) \in M$. Then, with respect to the bases $\{X_p, Y_p\}$ and $\{e_{1,f(p)}, e_{2,f(p)}\}$,*

$$(3.9) \quad D_H f(p) = \begin{pmatrix} g_M(Xf, e_1) & g_M(Yf, e_1) \\ g_M(Xf, e_2) & g_M(Yf, e_2) \end{pmatrix}$$

and

$$(3.10) \quad \|D_H f(p)\| \ell[D_H f(p)] = |\det D_H f(p)|.$$

Proof. The expression (3.9) is immediate if one expands the vectors Xf and Yf in $\{e_1, e_2\}$ as

$$Xf = g_M(Xf, e_1)e_1 + g_M(Xf, e_2)e_2 \quad \text{and} \quad Yf = g_M(Yf, e_1)e_1 + g_M(Yf, e_2)e_2.$$

Once $D_H f$ is expressed as a matrix with respect to the basis $\{X, Y\}$ in the source and the basis $\{e_1, e_2\}$ in the target, the identity (3.10) becomes a standard fact from linear algebra; one simply has to observe that the eigenvalues of the symmetric matrix $(D_H f)^T(D_H f)$ are $\|D_H f\|^2$ and $\ell[D_H f]^2$. \square

Proposition 3.3.6 yields the following characterization.

Proposition 3.3.7. *Let M be a smooth 3-manifold endowed with an equiregular distribution HM and a sub-Riemannian metric g_M . A map $f: \mathbb{H} \rightarrow M$ is K -quasi-regular if and only if*

- f is continuous,
- f has L^4_{loc} weak horizontal derivatives,
- f is weakly contact,
- the estimate

$$(3.11) \quad \|D_H f\|^4 \leq K^2 (\det D_H f)^2$$

holds almost everywhere.

The formal horizontal Jacobian is related to the usual Jacobian (with respect to Popp measure) if the map is smooth. For a diffeomorphism ϕ between domains in \mathbb{H} and a contact 3-manifold N , endowed with the contact forms $\alpha_{\mathbb{H}}$ and α_N respectively, the Jacobian J_ϕ is given by the equation

$$\phi^*(\alpha_N \wedge d\alpha_N) = J_\phi \alpha_{\mathbb{H}} \wedge d\alpha_{\mathbb{H}}.$$

Proposition 3.3.8. *Let V be a domain in \mathbb{H} and V' a domain in N . Assume that $\phi: V \rightarrow V'$ is a smooth contact transformation. Then $J_\phi = (\det D_H \phi)^2$.*

Proof. Since ϕ is smooth we may apply the usual calculus for differential forms to obtain

$$J_\phi \alpha_{\mathbb{H}} \wedge d\alpha_{\mathbb{H}} = \phi^*(\alpha_N \wedge d\alpha_N) = \lambda^2 \alpha_{\mathbb{H}} \wedge d\alpha_{\mathbb{H}},$$

where $\phi^* \alpha_N = \lambda \alpha_{\mathbb{H}}$. Hence $J_\phi = \lambda^2$. On the other hand, if $\{e_1, e_2\}$ is a local orthonormal frame on V' , and $\{\nu_1, \nu_2\}$ is the dual frame of 1-forms, then $X\phi = \nu_1(X\phi)e_1 + \nu_2(X\phi)e_2$ and $Y\phi = \nu_1(Y\phi)e_1 + \nu_2(Y\phi)e_2$, whence

$$D_H \phi = \begin{pmatrix} \nu_1(X\phi) & \nu_1(Y\phi) \\ \nu_2(X\phi) & \nu_2(Y\phi) \end{pmatrix}$$

with respect to the bases $\{X, Y\}$ and $\{e_1, e_2\}$. Since $X\phi = \phi_* X$ and $Y\phi = \phi_* Y$, it follows that

$$\begin{aligned} \det D_H \phi &= \nu_1(\phi_* X) \nu_2(\phi_* Y) - \nu_1(\phi_* Y) \nu_2(\phi_* X) \\ &= d\alpha_N(\phi_* X, \phi_* Y) = \lambda d\alpha_{\mathbb{H}}(X, Y) = \lambda. \end{aligned}$$

The claim follows. \square

4. Proof of the main theorem

We now prove Theorem 1.2.2, following the steps outlined in the introduction. We fix an equiregular sub-Riemannian 3-manifold M whose fundamental group has growth rate larger than 4.

4.1. Topology and covering theory

By definition, there exists a finitely generated subgroup Γ of $\pi_1(M)$ so that Γ has growth rate larger than d , for some number $d > 4$. We will associate to Γ a relatively compact “core” $M' \subset M$. The goal of this section is to prove that a specific lift \widetilde{M}'' of M' inside the universal cover \widetilde{M} of M is quasi-isometric to Γ .

Fix a basepoint x_0 in M and smooth closed curves $\gamma_1, \dots, \gamma_s$ for some $s \geq 2$ such that Γ is generated by $[\gamma_1], \dots, [\gamma_s] \in \pi_1(M)$. We may assume, without loss of generality, that the curves are simple, intersect only at the basepoint, and intersect transversally at the basepoint. Let $\overline{M'}$ be a closed, connected manifold with \mathcal{C}^∞ boundary such that $M' := \text{int}(\overline{M'})$ satisfies:

1. $\text{int}(M') = M' \subset \overline{M'} \Subset M$,
2. $\gamma_i \subset M'$ for all $i \in \{1, \dots, s\}$,
3. $\pi_1(M')$ is the free group generated by $[\gamma_1], \dots, [\gamma_s]$.

Let \widetilde{M} and \widetilde{M}' denote the universal covers of M and M' respectively. While \widetilde{M}' is (generically) not a subset of \widetilde{M} , there is some intermediate cover of M' inside \widetilde{M} . Indeed, the inclusion of M' in M induces a map $\widetilde{M}' \rightarrow \widetilde{M}$ whose image we denote by \widetilde{M}'' .

Lemma 4.1.1. *The following properties hold:*

- (1) \widetilde{M}'' is a cover of M' under the standard projection $\pi: \widetilde{M} \rightarrow M$,
- (2) the action of Γ on \widetilde{M} leaves \widetilde{M}'' invariant,
- (3) $\widetilde{M}''/\Gamma = M'$.

All these properties also hold if M' is replaced by its closure $\overline{M'}$. That is, one can define in the same way a cover space of $\overline{M'}$ so that Γ acts on this cover, and the quotient of the action can be identified with $\overline{M'}$.

The universal cover \widetilde{M} can be endowed with a contact sub-Riemannian structure by lifting the contact form and metric from M . Let us denote by d_M , respectively $d_{\widetilde{M}}$, the sub-Riemannian metric on M , respectively \widetilde{M} . The embeddings $M' \hookrightarrow M$ and $\widetilde{M}'' \hookrightarrow \widetilde{M}$ equip M' and \widetilde{M}'' with sub-Riemannian metrics (denoted $\delta_{M'}$ and $\delta_{\widetilde{M}''}$) such that the covering map $\pi: \widetilde{M}'' \rightarrow M'$ becomes a local isometry. For example, the distance between two points of M' is the infimal g_M -length of horizontal curves contained in M' joining these two points. We call these quantities the *intrinsic distance* on M' and \widetilde{M}'' .

Understanding the topological and metric properties of a submanifold endowed with an *intrinsic* distance is more challenging in the present sub-Riemannian setting than it would be in the Riemannian case. The difficulties arise already in

case M is the Heisenberg group \mathbb{H} itself. For instance, Monti and Rickly showed in [60] that the only geodetically convex subsets of \mathbb{H} are the empty set, points, geodesic arcs, and the whole space. Moreover, there exist domains in \mathbb{H} , even \mathcal{C}^1 -smooth ones, for which some points on the boundary cannot be joined from inside the domain by rectifiable curves [4]. These complications hint at the subtleties involved in analyzing the intrinsic distance on submanifolds with boundary in a sub-Riemannian manifold. In the present section, we discuss properties of the intrinsic distances on M' and \widetilde{M}'' .

First note that although $\overline{M'}$ is compact in the topology of M , the intrinsic distance $\delta_{M'}$ might induce a different topology on M' and we do not a priori know whether M' is bounded with respect to this distance. Our first goal is to show that $\delta_{M'}$ is bi-Lipschitz equivalent to the *extrinsic* distance $d_M|_{M'}$, in a way that extends to the boundary of M' .

We define the function

$$(4.1) \quad d_{\text{intr}} : \overline{M'} \times \overline{M'} \rightarrow [0, +\infty], \quad d_{\text{intr}}(p, q) = \inf \text{length}(\gamma),$$

where the infimum is taken over curves $\gamma: [0, 1] \rightarrow \overline{M'}$ such that $\gamma(0) = p$, $\gamma(1) = q$, and most importantly, $\gamma(x) \in M'$ for $x \in (0, 1)$. The length in (4.1) is computed with respect to the sub-Riemannian metric tensor g_M , so it agrees with the length in the extrinsic metric $d_M|_{M'}$. In particular, non-horizontal curves have infinite length. It is clear that the restriction of d_{intr} to M' agrees with $\delta_{M'}$.

A priori, the value of $d_{\text{intr}}(p, q)$ could be infinite. The following proposition shows that this is not the case.

Proposition 4.1.2. *The function d_{intr} defines a metric on $\overline{M'}$ that is bi-Lipschitz equivalent to $d_M|_{\overline{M'}}$.*

Proof. It is immediate that

$$d_{\text{intr}}(p, q) \geq d_M(p, q), \quad \text{for all } p, q \in \overline{M'}.$$

Since d_M is a metric, this shows that d_{intr} is non-degenerate, and it suffices to prove that also the reverse inequality holds – up to a multiplicative constant.

We first show that the two distances are locally bi-Lipschitz equivalent. This implies that d_{intr} induces the original topology on $\overline{M'}$. We conclude that $(\overline{M'}, d_{\text{intr}})$ is compact and use this information to show that the two metrics are globally bi-Lipschitz equivalent. A similar argument can be found in §3 of [20].

We fix a point $p \in \overline{M'}$ and a Darboux chart $\Phi: U \rightarrow V$ mapping a neighborhood U of 0 in \mathbb{H} to a neighborhood V of $p = \Phi(0)$ in M . This is possible since M is a contact manifold. The chart map Φ pulls the metric tensor g_M back to a sub-Riemannian metric g_U on U , which, on compact sets, is comparable to the standard sub-Riemannian metric $g_{\mathbb{H}}$.

Furthermore, there exist relatively compact neighborhoods U_0 of 0 in U , and V_0 of p in V , with $\Phi(U_0) = V_0$, such that the $d_{\mathbb{H}}$ -distance between points of U_0 is realized by curves contained in U , while the d_M -distance between points in V_0 is realized by curves contained in V . Thus, in order to prove that d_{intr} and $d_M|_{\overline{M'}}$

are bi-Lipschitz equivalent on $V_0 \cap \overline{M'}$, it suffices to prove that there exists a finite constant c_p such that

$$(4.2) \quad \inf_{\gamma((0,1)) \subset \Phi^{-1}(V \cap M')} \text{length}_{g_{\mathbb{H}}}(\gamma) \leq c_p \inf_{\gamma} \text{length}_{g_{\mathbb{H}}}(\gamma)$$

for all $u, u' \in \Phi^{-1}(V_0 \cap \overline{M'})$ and for $\gamma: [0, 1] \rightarrow U$, with $\gamma(0) = u$ and $\gamma(1) = u'$.

First, suppose that $p \in M'$. In this case, by choosing U smaller if necessary, we may assume that $\Phi(U) \cap \partial M' = \emptyset$, whence $\Phi^{-1}(V \cap M') = U$ and (4.2) clearly holds.

Now suppose that $p \in \partial \overline{M'}$. By making U smaller if necessary, we may assume that $\Phi^{-1}(V \cap \partial M')$ is a smoothly embedded disk which separates U in two domains. Denote by U' the domain $\Phi^{-1}(V \cap M')$ in U . We may further assume that the subdomain U_0 is chosen so that $\Omega := \Phi^{-1}(V_0 \cap M') \subset U_0$ is a domain with $C^{1,1}$ boundary with $0 \in \partial \Omega$ and

$$\partial \Omega = (\Phi^{-1}(V_0 \cap \partial M')) \dot{\cup} (\partial \Omega \cap U').$$

By Theorem 1.3 in [59], Ω is an NTA domain in $(\mathbb{H}, d_{\mathbb{H}})$ and hence also a uniform domain (see, for instance, Proposition 4.2 in [11]). By a limiting argument (see the remark on p. 270 of [11]), it follows that points in $\overline{\Omega}$ can be joined by uniform curves. In particular, $\overline{\Omega}$ is quasiconvex: there exists $C > 0$ so that for any pair of points $u, u' \in \overline{\Omega}$ there exists a rectifiable curve $\gamma: [0, 1] \rightarrow \overline{\Omega}$ such that $\gamma((0, 1)) \subset \Omega$, $\gamma(0) = u$, $\gamma(1) = u'$, and $\text{length}_{g_{\mathbb{H}}}(\gamma) \leq C d_{\mathbb{H}}(u, u')$. Since u and u' lie in U_0 , to compute $d_{\mathbb{H}}(u, u')$ it suffices to consider curves contained in U . Thus we have established (4.2) when $p \in \partial M'$. We conclude that for every point $p \in \overline{M'}$ there exists an open neighborhood V_p of p in M and a finite constant $C_p > 0$ such that

$$d_M(q, q') \leq d_{\text{intr}}(q, q') \leq C_p d_M(q, q'), \quad \text{for all } q, q' \in V_p \cap \overline{M'}.$$

In particular, the intrinsic and extrinsic topologies on $\overline{M'}$ agree. Hence $\overline{M'}$ is compact and therefore bounded with respect to the metric d_{intr} .

Since $\overline{M'}$ is compact, we can cover $\overline{M'}$ by finitely many open sets V_{p_1}, \dots, V_{p_N} of the above form. By the Lebesgue number lemma, there exists $r_0 > 0$ such that for all $0 < r < r_0$ and $p \in \overline{M'}$, we have

$$(4.3) \quad \{q \in \overline{M'} : d_M(p, q) < r\} \subseteq V_{p_i} \quad \text{for some } i \in \{1, \dots, N\}.$$

In order to prove that d_{intr} and d_M are bi-Lipschitz equivalent on $\overline{M'}$, we have to show that

$$\sup_{p \neq q} \frac{d_{\text{intr}}(p, q)}{d_M(p, q)}$$

is uniformly bounded. Considering the cases $d_M(p, q) < r$ and $d_M(p, q) \geq r$ in turn, we see that

$$\frac{d_{\text{intr}}(p, q)}{d_M(p, q)} \leq \max \left\{ C_{p_1}, \dots, C_{p_N}, \frac{\text{diam}_{d_{\text{intr}}}(\overline{M'})}{r} \right\}.$$

for all $p \neq q$. This completes the proof. \square

The restriction of the metric d_{intr} to M' is by definition a length metric on M' and agrees with the sub-Riemannian distance $\delta_{M'}$ induced by $g_{M'}$. Using the covering map π , this distance can be lifted to a length metric $d_{\widetilde{M}''}$ on \widetilde{M}'' which agrees with the sub-Riemannian distance $\delta_{\widetilde{M}''}$ induced on \widetilde{M}'' by the pull-back of g_M under π .

Lemma 4.1.3. *The metric space $(\widetilde{M}'', d_{\widetilde{M}''})$ is quasi-isometric to Γ endowed with a word metric.*

Recall that a map $f: X \rightarrow Y$ between metric spaces is (A, B) -quasi-isometric for $A \geq 1$ and $B > 0$ if $A^{-1}d_X(x_1, x_2) - B \leq d_Y(f(x_1), f(x_2)) \leq Ad_X(x_1, x_2) + B$ for all $x_1, x_2 \in X$ and if $f(X)$ is B -coarsely dense in Y , i.e., every point of Y is within distance B of $f(X)$.

Proof. The set $\overline{M'}$ can be endowed with a length metric $d_{\overline{M'}}$, where the distance between two points p, q is defined by minimizing the g_M -length of curves in $\overline{M'}$ connecting p and q . This distance is bounded from below by the restriction of d_M to M' , and bounded from above by d_{intr} . By Proposition 4.1.2, $d_{\overline{M'}}$ is comparable to both $d_M|_{\overline{M'}}$ and d_{intr} . The metric $d_{\overline{M'}}$ lifts to a length metric d_N on the component N of $\pi^{-1}(\overline{M'})$ that contains \widetilde{M}'' .

Since (N, d_N) and $(\widetilde{M}'', d_{\widetilde{M}''})$ are quasi-isometric, it suffices to prove that (N, d_N) is quasi-isometric to Γ . This follows from the Milnor–Švarc lemma, upon observing that (N, d_N) is a length space on which Γ acts properly discontinuously and cocompactly by isometries. Here we have used Lemma 4.1.1 and the fact that $\overline{M'}$ is compact in the metric $d_{\overline{M'}}$, which follows from Proposition 4.1.2. \square

4.2. Proof of the rough isoperimetric inequality

We will show in Section 4.4 that \widetilde{M} satisfies a d -dimensional isoperimetric inequality for some $d > 4$. To this end, we first establish in Proposition 4.2.4 a rough, or coarse, d -dimensional isoperimetric inequality for a net on \widetilde{M}'' .

Following the terminology used by Kanai in [49], we call a countable set Y a *net* if there exists a set-valued function

$$N : Y \rightarrow \{A : A \subseteq Y\}, \quad y \mapsto N(y)$$

with the property that

1. $N(y) \subseteq Y$ is finite,
2. $x \in N(y)$ if and only if $y \in N(x)$.

The points in $N(y)$ are called the *neighbors* of y , and Y is said to be *connected* if for any two points y and x in Y there exists a chain of finitely many points connecting y and x so that any two consecutive points are neighbors. The *combinatorial distance* $\delta(y, x)$ is the minimal length (number of elements in the chain) a path must have to connect y and x .

The nets considered in this paper are all connected, and one might think of them as graphs. We encounter two types of nets:

1. finitely generated groups, where the combinatorial metric agrees with the word metric with respect to the system of generators,
2. point sets Y on a metric space (X, d) that are ε -separated ($d(y, x) \geq \varepsilon$ for all $y, x \in Y$) and *maximal* (with respect to order of inclusion), and where $N(y) = \{x \in P : 0 < d(y, x) \leq 2\varepsilon\}$.

We recall that in every metric space there exist maximal ε -separated nets for every $\varepsilon > 0$. Moreover, a totally bounded metric space contains a *finite* maximal ε -separated net for every $\varepsilon > 0$. In particular, every compact set in a metric space contains a finite maximal ε -separated net for every $\varepsilon > 0$.

A net N is *uniform* if

$$\sup_{y \in Y} \#N(y) < \infty.$$

Nets derived from finitely generated groups in the above way are always uniform. The metric space $(\widetilde{M}'', d_{\widetilde{M}''})$ defined at the beginning of this section admits a uniform net, which is moreover maximally ε -separated for some $\varepsilon > 0$.

Lemma 4.2.1. *There exists $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$, the manifold \widetilde{M}'' contains a maximal ε -separated net Y such that*

$$(4.4) \quad \sup_{y \in Y} \#(B_{\widetilde{M}''}(y, r) \cap Y) < \infty, \quad \text{for all } 0 < r < \infty.$$

In particular, Y is uniform.

Proof. Since $\pi|_{\widetilde{M}''} : \widetilde{M}'' \rightarrow M'$ is a locally isometric cover, and since $\overline{M'}$ is compact, there exists a constant $\varepsilon_0 > 0$ such that π is an isometry when restricted to ε_0 -balls in \widetilde{M}'' . We let ε be a positive number less than ε_0 and pick a maximal ε -separated net $Y_{M'}$ in M' . Notice that this net is finite since $\overline{M'}$ is compact.

We denote the lift of $Y_{M'}$ to \widetilde{M}'' by Y , that is,

$$Y = \{\gamma.x : \gamma \in \Gamma, x \in (\pi|_{\widetilde{M}''})^{-1}(y), y \in Y_{M'}\}.$$

Clearly, Y is ε -separated. It is also maximal, for if we could add a point $p \in \widetilde{M}'' \setminus Y$ with $d_{\widetilde{M}''}(p, y) \geq \varepsilon$ for all $y \in Y$, then $\pi(p)$ would be at distance at least ε from every point in $Y_{M'}$, which contradicts the maximality of $Y_{M'}$. We observe further that, by construction, Y is Γ -invariant.

Denote by n the cardinality of $Y_{M'}$. We choose points y_1, \dots, y_n in Y such that

$$\pi\{y_1, \dots, y_n\} = Y_{M'}.$$

Each $y \in Y$ can be written as $y = \gamma.y_i$ for a unique $i \in \{1, \dots, n\}$ and a unique $\gamma \in \Gamma$. Thus

$$d_{\widetilde{M}''}(y, \gamma.y_1) \leq \max_{1 \leq i \leq n} d_{\widetilde{M}''}(y_i, y_1).$$

This shows that there exists a n -to-1 quasi-isometry from $(Y, d_{\widetilde{M}''})$ to

$$Y_0 = \{\gamma.y_1 : \gamma \in \Gamma\} \quad \text{with} \quad d_{\widetilde{M}''}$$

given by $y = \gamma.y_i \mapsto \gamma.y_1$. On the other hand, we know from the proof of the Milnor–Švarc lemma, see for instance Theorem 1.18 in [70], that $(Y_0, d_{\widetilde{M}''})$ is quasi-isometric to Γ with the word metric via the map $\gamma.y_1 \mapsto \gamma$. It follows that there is

an n -to-1 quasi-isometry $\varphi: Y \rightarrow \Gamma$. Hence, for every $r > 0$, there exists r' , depending on r and the quasi-isometry constants of φ , such that

$$\varphi(B_{\widetilde{M}''}(y, r) \cap Y) \subset B_{\Gamma}(\varphi(y), r'), \quad \text{for all } y \in Y.$$

The ball on the right hand side contains only a finite set of elements in Γ , whose cardinality can be bounded depending on r' , but independently of y . It follows that $B_{\widetilde{M}''}(y, r) \cap Y$ contains at most this number times n net points. This proves (4.4). \square

Remark 4.2.2. Condition (4.4) in Lemma 4.2.1 can be extended to all points in \widetilde{M}'' . Indeed, if x is a point in $\widetilde{M}'' \setminus Y$, then by maximality of Y , there exists $y \in Y$ with $d_{\widetilde{M}''}(x, y) < \varepsilon$. Thus, for all $r > 0$, we have that

$$B_{\widetilde{M}''}(x, r) \subset B_{\widetilde{M}''}(y, r + \varepsilon).$$

It follows by (4.4) that for all $x \in \widetilde{M}''$, we have

$$(4.5) \quad \#(B_{\widetilde{M}''}(x, r) \cap Y) \leq \sup_{y \in Y} \#(B_{\widetilde{M}''}(y, r + \varepsilon) \cap Y) =: \nu(r, \varepsilon) < \infty.$$

The net Y from Lemma 4.2.1 is quasi-isometric to $(\widetilde{M}'', d_{\widetilde{M}''})$ if it is seen as a subset of \widetilde{M}'' and endowed with $d_{\widetilde{M}''}$. The same holds true, but is less immediate, if Y is equipped with the *combinatorial distance* δ . Clearly

$$d_{\widetilde{M}''}(y, y') \leq 2\varepsilon\delta(y, y')$$

for all $y, y' \in Y$. One can use property (4.5) to prove that δ is controlled also from above in terms of $d_{\widetilde{M}''}$ (up to multiplicative and additive constants). An analogous statement is known for complete Riemannian manifolds with lower Ricci curvature bound, and an inspection of Lemma 2.5 in [49] shows that it carries over to length spaces satisfying condition (4.5).

Let Y be a net endowed with the combinatorial distance δ . We define the *boundary* of a set $S \subseteq Y$ as

$$\partial S = \{y \in Y : \delta(y, S) = 1\}.$$

Definition 4.2.3. We say that Y satisfies a *rough d -dimensional isoperimetric inequality* if there exists a constant $0 < C < \infty$ such that

$$(\#S)^{(d-1)/d} \leq C \#\partial S$$

for all nonempty finite subsets S of Y .

Proposition 4.2.4. *There exists $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$, the manifold \widetilde{M}'' contains a maximal ε -separated net Y which satisfies a rough d -dimensional isoperimetric inequality for some $d > 4$.*

Proof. The first part of the proof is not specific to the sub-Riemannian setting, but instead proceeds exactly the same way as in the proof of Theorem 1.3 in [62].

Namely one uses the fact that Γ has a growth rate $d > 4$ in order to deduce by Théorème 1 in [14] that it satisfies a rough d -dimensional isoperimetric inequality.

Then we choose ε_0 and Y as in Lemma 4.2.1. By the comment after (4.5), we know that (Y, δ) is quasi-isometric to $(\widetilde{M}'', d_{\widetilde{M}''})$, and thus, by Lemma 4.1.3, also quasi-isometric to Γ endowed with the word metric.

Kanai has shown in [49], Lemma 4.2, that the validity of a rough d -dimensional isoperimetric inequality is a quasi-invariant for uniform nets. Recall that Γ is uniform (since the group is finitely generated), and that Y is uniform by Lemma 4.2.1. It follows that Y satisfies a rough d -dimensional isoperimetric inequality. \square

We will later apply Proposition 4.2.4 to prove a (smooth) isoperimetric inequality on \widetilde{M}'' . A close inspection of Kanai's proof in the Riemannian setting reveals that the full strength of volume comparison geometry is not needed. For our purposes, a much weaker estimate suffices.

Lemma 4.2.5. *There exists $\varepsilon_1 > 0$ such that for every ε -net Y as above with $\varepsilon < \varepsilon_1$, there are constants $0 < c_- \leq c_+ < \infty$ such that*

$$c_- \leq \mu_{\widetilde{M}}(B_{\widetilde{M}''}(y, \varepsilon)) \leq c_+, \quad \text{for all } y \in Y.$$

The constants c_- and c_+ may depend on the data of the manifold and on ε , but not on y .

Proof. Recall that $Y = \{\gamma.y_i : \gamma \in \Gamma, i = 1, \dots, n\}$ for a finite set of points $\{y_1, \dots, y_n\} \in \widetilde{M}''$. Since γ acts by isometries and $\mu_{\widetilde{M}}$ agrees up to a multiplicative constant with the 4-dimensional Hausdorff measure with respect to $d_{\widetilde{M}''}$, it suffices to consider the mass of the balls $B_{\widetilde{M}''}(y_i, \varepsilon)$ for $i = 1, \dots, n$, more precisely, to prove that this volume is positive and finite. If we choose ε_1 no larger than the constant r_0 from the proof of Proposition 4.1.2 and the constant ε_0 from the proof of Lemma 4.2.1, then every such ball is isometric to a ball in M' which is contained in one of finitely many sets that cover M' and that can be mapped bi-Lipschitzly onto a domain in the Heisenberg group by a Darboux chart. It is well known that $\mu_{\mathbb{H}}(B_{\mathbb{H}}(p, r)) = cr^4$ for all $r > 0$, $p \in \mathbb{H}$, and a positive and finite constant c . The claim follows. \square

This concludes our discussion of the coarse geometry of \widetilde{M}'' . In the next section we will focus on the local geometry.

4.3. Proof of the relative isoperimetric inequality

In order to derive from a rough isoperimetric inequality a smooth, global one, we need local information on the geometry of the manifold \widetilde{M}'' . In this section, we prove a weak Sobolev–Poincaré inequality and a weak relative 4-dimensional isoperimetric inequality for balls of a fixed radius centered in the net Y . In the continuous version of the isoperimetric inequality, the cardinality of a finite set S and of its boundary are replaced by the volume of a domain Ω and the perimeter of its boundary.

We now explain the notion of perimeter in our setting. We follow the presentation in [29], which is specific to sub-Riemannian contact 3-manifolds. In [29], the manifolds are assumed to be endowed with a *global* contact form, but in light of Remark 3.1.2, this assumption is not necessary for our application.

Definition 4.3.1. The *divergence* of a smooth horizontal vector field V on a manifold (N, HN, g_N) is the real-valued function $\operatorname{div}_N V$ characterized by the identity

$$(4.6) \quad \mathcal{L}_V \operatorname{vol}_N = (\operatorname{div}_N V) \operatorname{vol}_N,$$

where \mathcal{L}_V denotes the Lie derivative along V .

Definition 4.3.2. Let N be a contact sub-Riemannian 3-manifold. The relative *perimeter* of a measurable set $E \subset N$ in an open set $\Omega \subseteq N$ is defined to be

$$\mathcal{P}(E, \Omega) := \sup \left\{ \int_{E \cap \Omega} \operatorname{div}_N V \, d\mu_N : V \text{ horizontal, } \|V\|_\infty \leq 1 \right\},$$

where the supremum is taken over C^1 vector fields V with compact support in Ω .

The perimeter $\mathcal{P}(E, \Omega)$ can be seen as a measure for the area of the boundary of E in Ω , see for instance [29], 2.3, and [27], (2.7).

Our first goal is to prove a local relative isoperimetric inequality on M' . We follow the argument of Galli and Ritoré in [28], where such an inequality is proved for balls centred in a compact subset of a contact sub-Riemannian manifold. For our argument we only need a weak form of this relative isoperimetric inequality, namely a statement about ε -balls centred in the points of a given ε -net in M' . The reason why we cannot directly apply Lemma 3.7 in [28], is that this statement holds only for small enough balls, where the smallness condition depends on the compact set. In other words, if we consider balls centered in the ε -net $Y_{M'}$, we might only get an estimate for balls at a scale much smaller than ε . Since M' is an open subset of M , this issue cannot be fixed by a simple compactness argument. We therefore give a direct proof for the result we are going to apply. This includes analyzing intrinsic ε -balls in M' whose closure with respect to d_M might intersect the boundary of M' .

Proposition 4.3.3 (Weak local relative isoperimetric inequality). *There exists $\delta_1 > 0$ such that for every ε -net Y , $\varepsilon < \delta_1$, on \widetilde{M}'' , there exist constants $C_I > 0$ and $1 < c < \infty$, depending only on Y and the data of the manifold, such that for any bounded set $E \subset \widetilde{M}''$ with finite perimeter, one has, for all $y \in Y$,*

$$C_I (\min\{\mu_{\widetilde{M}}(E \cap B_{\widetilde{M}''}(y, \varepsilon)), \mu_{\widetilde{M}}((\widetilde{M}'' \setminus E) \cap B_{\widetilde{M}''}(y, \varepsilon))\})^{\frac{d-1}{d}} \leq \mathcal{P}(E, B_{\widetilde{M}''}(y, c\varepsilon)).$$

The result follows from a suitable weak Sobolev–Poincaré inequality for the balls $B_{\widetilde{M}''}(y, \varepsilon)$, $y \in Y$. While we may assume that M and thus also \widetilde{M} are complete (by a conformal change of metric which does not affect quasiregularity), the same is not true for M' and \widetilde{M}'' and we must prove by hand that the considered balls are John domains.

We denote by

$$u_E = \frac{1}{\mu(E)} \int_E u \, d\mu = \mathop{\int}\limits_E u \, d\mu$$

the mean value of a function $u: X \rightarrow \mathbb{R}$ over a measurable set E with positive mass in a metric measure space (X, d, μ) . To establish the desired Poincaré inequality for ε -balls centred in the points of a ε -net Y_M on M' , we will use Darboux charts to transfer the problem to the Heisenberg group. Balls with respect to the intrinsic distance on a domain U in \mathbb{H} need not be John domains even if the boundary of the ball is smooth, but they can be compared to subsets of U that are John domains. In this context the following lemma is useful. The proof is a standard argument which works much more generally and which we reproduce here for completeness.

We will use the following definition of horizontal gradient.

Definition 4.3.4. Let (N, HN, g_N) be a sub-Riemannian contact 3-manifold. The *horizontal gradient* $\nabla_H u$ of a C^1 function $u: N \rightarrow \mathbb{R}$ is the unique horizontal vector field on N with the property that

$$g_N(\nabla_H u, V) = du(V), \quad \text{for all } V \in HN.$$

See Section 2.2 in [2] for an expression of the gradient in a local orthonormal frame.

Lemma 4.3.5. *Let Ω be a domain in \mathbb{H} and $B \subset \Omega$ a measurable subset with the property that*

$$\left(\int_B |u - u_B|^{4/3} \, d\mu_{\mathbb{H}} \right)^{3/4} \leq C \operatorname{diam}(B) \left(\int_B |\nabla_H u| \, d\mu_{\mathbb{H}} \right)$$

for some real-valued function u defined in a neighborhood of Ω . If B_1 and B_2 are measurable sets of positive mass such that $B_1 \subset B \subset B_2 \subset \Omega$, then

$$\left(\int_{B_1} |u - u_{B_1}|^{4/3} \, d\mu_{\mathbb{H}} \right)^{3/4} \leq C' \operatorname{diam}(B_2) \left(\int_{B_2} |\nabla_H u| \, d\mu_{\mathbb{H}} \right),$$

where C' depends only on C and on the ratios $\mu_{\mathbb{H}}(B)/\mu_{\mathbb{H}}(B_1)$ and $\mu_{\mathbb{H}}(B_2)/\mu_{\mathbb{H}}(B)$.

Proof. For simplicity we write $\mu = \mu_{\mathbb{H}}$. First we observe that

$$\begin{aligned} \left(\int_{B_1} |u - u_{B_1}|^{4/3} \, d\mu \right)^{3/4} &\leq \left(\int_{B_1} |u - u_B|^{4/3} \, d\mu \right)^{3/4} + \left(\int_{B_1} |u_{B_1} - u_B|^{4/3} \, d\mu \right)^{3/4} \\ &= \left(\int_{B_1} |u - u_B|^{4/3} \, d\mu \right)^{3/4} + \mu(B_1)^{3/4} \left| \int_{B_1} (u - u_B) \, d\mu \right| \\ &\leq \left(\int_{B_1} |u - u_B|^{4/3} \, d\mu \right)^{3/4} + \mu(B_1)^{3/4} \int_{B_1} |u - u_B| \, d\mu \\ &\leq 2 \left(\int_{B_1} |u - u_B|^{4/3} \, d\mu \right)^{3/4}, \end{aligned}$$

where we have used Hölder's inequality in the last step.

With this estimate and the assumed inequality for B in hand, it now follows that

$$\begin{aligned} \left(\int_{B_1} |u - u_{B_1}|^{4/3} d\mu \right)^{3/4} &\leq 2 \left(\int_{B_1} |u - u_B|^{4/3} d\mu \right)^{3/4} \\ &\leq C \left(\frac{\mu(B)}{\mu(B_1)} \right)^{3/4} \text{diam}(B) \left(\int_B |\nabla_H u| d\mu \right) \\ &\leq C \left(\frac{\mu(B)}{\mu(B_1)} \right)^{3/4} \left(\frac{\mu(B_2)}{\mu(B)} \right) \text{diam}(B_2) \int_{B_2} |\nabla_H u| d\mu. \end{aligned}$$

The proof is complete. \square

Proposition 4.3.6 (Weak Sobolev–Poincaré inequality). *There exists $\delta_0 > 0$ such that for every ε -net $Y_{M'}$ on M' as above with $\varepsilon < \delta_0$, there are constants $0 < C < \infty$ and $1 \leq c < \infty$, depending only on M' , such that*

$$\left(\int_{B(x,\varepsilon)} |u - u_{B(x,\varepsilon)}|^{4/3} d\mu_M \right)^{3/4} \leq C\varepsilon \int_{B(x,c\varepsilon)} |\nabla_H u| d\mu_M,$$

for all $x \in Y_{M'}$ and $u \in C^\infty(M')$.

Proof. For convenience, we endow \mathbb{H} with the left-invariant distance $d_\infty : (x, x') \mapsto \|x^{-1} * x'\|$, induced by the gauge function $\|(z, t)\| := \max\{|z|, \sqrt{|t|}\}$, $(z, t) \in \mathbb{C} \times \mathbb{R}$, which is bi-Lipschitz equivalent to the standard sub-Riemannian distance $d_{\mathbb{H}}$. This gives explicit information on the shape of balls.

We fix a finite collection of sets $\{V_1, \dots, V_n\}$ in M which cover M' so that each V_i can be mapped L -bi-Lipschitzly to a domain in (\mathbb{H}, d_∞) and $V_i \cap M'$ has positive mass. Let r_0 be as in the proof of Proposition 4.1.2, and fix $\delta_0 \leq r_0$ small enough such that every $2L^2\varepsilon$ -ball in M intersected with M' lies inside one of the sets V_1, \dots, V_n provided that $\varepsilon < \delta_0$.

The Darboux chart map pushes forward the Popp measure on M' (induced by g_M) to the standard volume on \mathbb{H} . Let $Y_{M'}$ be a ε -net on M' as before and let x be a point in $Y_{M'}$. As previously explained there exists a neighborhood V_i of x which under a Darboux chart map Ψ is mapped L -bi-Lipschitzly to a domain U_i in \mathbb{H} . An argument similar to the one in the proof of Lemma 4.3.5 shows that it suffices to verify the inequality

$$(4.7) \quad \left(\int_A |u - u_A|^{4/3} d\mu_{\mathbb{H}} \right)^{3/4} \leq C\varepsilon \int_{2A} |\nabla_H u| d\mu_{\mathbb{H}}$$

for all $u \in C^\infty(U_i)$, where

$$A = B_{d_\infty}(\Psi(x), L\varepsilon) \cap \Psi(M' \cap V_i), \quad 2A = B_{d_\infty}(\Psi(x), 2L\varepsilon) \cap \Psi(M' \cap V_i),$$

and C is a finite constant depending on A . Here we have used the fact that d_∞ -balls at $\Psi(x)$ have positive and finite $\mu_{\mathbb{H}}$ -measure when intersected with $\Psi(M' \cap V_i)$.

It is not clear whether the domain A is a John domain, but we will prove (4.7) by applying Lemma 4.3.5 with $B_1 := A$ and $B_2 := 2A$. Fix a $C^{1,1}$ domain B with

$B_1 \subset B \subset B_2$. By Theorem 1.3 in [59], B is a uniform domain, and in particular a John domain in $(\mathbb{H}, d_{\mathbb{H}})$. By Theorem 1.5 and Section 6 in [30] applied to \mathbb{H} , it follows that the desired strong $(4/3, 1)$ -Poincaré inequality holds for B . Then (4.7) is a consequence of Lemma 4.3.5 and the proof is complete if we observe that there are only finitely many points in $Y_{M'}$. \square

We are now ready to prove the local relative isoperimetric inequality for \widetilde{M}'' .

Proof of Proposition 4.3.3. We let $\delta_1 \leq \delta_0$, where δ_0 is the parameter from Proposition 4.3.6, which ensures that ε -balls centred at the points of an ε -net $Y_{M'}$ on M' for $\varepsilon < \delta_0$ satisfy a weak $(4/3, 1)$ -Poincaré inequality.

We further require that δ_1 is small enough such that $\pi|_{\widetilde{M}''}$ is an isometry on $c\varepsilon$ -balls for $\varepsilon < \delta_1$ and c as in Proposition 4.3.6. Thus the same weak $(4/3, 1)$ -Poincaré inequality holds for points in the lifted net Y on \widetilde{M}'' . It then follows that there exist constants $C_I > 0$ and $1 < c < \infty$, depending only on Y and the data of the manifold, such that for any set $E \subset \widetilde{M}''$ with locally finite perimeter, one has

$$(4.8) \quad \begin{aligned} C_I (\min\{\mu_{\widetilde{M}}(E \cap B_{\widetilde{M}''}(y, \varepsilon)), \mu_{\widetilde{M}}((\widetilde{M}'' \setminus E) \cap B_{\widetilde{M}''}(y, \varepsilon))\})^{3/4} \\ \leq \mathcal{P}(E, B_{\widetilde{M}''}(y, c\varepsilon)), \end{aligned}$$

for all $y \in Y$. The proof in the sub-Riemannian setting follows the same methods as in the Euclidean case, see [28] and for instance Corollary 1.29 in [31]: inequality (4.8) follows by applying the Poincaré inequality to the function $u = \chi_E$. Since χ_E is not smooth but merely of bounded variation, this requires an approximation result of BV functions by smooth functions. This is given by Proposition 2.4 in [28] in the setting of sub-Riemannian contact manifolds. Finally we notice that $\mathcal{P}(E, \Omega)$ is the total variation of the characteristic function χ_E . The fact that we only have a *weak* Poincaré inequality accounts for the enlarged ball on the right-hand side of the above isoperimetric inequality.

To conclude, we observe that for all $d > 4$, we have $3/4 < (d-1)/d$. Since the considered volumes are finite, we see that the desired weak local relative d -dimensional isoperimetric inequality follows from the above version. \square

4.4. Proof of the global isoperimetric inequality

Kanai ([49], Lemma 4.5) established the transition from rough and local isoperimetric inequalities to global isoperimetric inequalities for Riemannian manifolds with lower bound on the Ricci curvature. In this section, we present a similar transition in the abstract setting of metric measure spaces. Our results apply in particular to the case of the sub-Riemannian manifold \widetilde{M}'' .

Throughout this section we let (X, μ) be a metric measure space, which we further assume to be equipped with a *perimeter measure* \mathcal{P} . The perimeter measure \mathcal{P} should act on pairs E and Ω , where E is measurable and Ω is open. Further, $\mathcal{P}(\cdot, \Omega)$ should be a Borel measure for each Ω , and $\Omega \mapsto \mathcal{P}(E, \Omega)$ should be monotonic with respect to set inclusion. A set E is said to be of *finite perimeter* if $\mathcal{P}(E, X) < \infty$.

Definition 4.4.1. Let (X, μ) be a metric measure space equipped with a notion of perimeter \mathcal{P} as above. We say that X satisfies a *weak relative d -dimensional isoperimetric inequality at scale $\varepsilon > 0$ in $Y \subset X$* if there exist finite constants $c, C \geq 1$ such that

$$(\min\{\mu(E \cap B(x, \varepsilon)), \mu((X \setminus E) \cap B(x, \varepsilon))\})^{(d-1)/d} \leq C\mathcal{P}(E, B(x, c\varepsilon))$$

for all $x \in Y$ and for all non-empty relative compact domains $E \subset X$ of finite perimeter. We say that X satisfies a *d -dimensional isoperimetric inequality* if there exists a constant $0 < C < \infty$, such that

$$\mu(E)^{(d-1)/d} \leq C\mathcal{P}(E, X)$$

for all non-empty relative compact domains $E \subset X$ of finite perimeter.

Our goal in this section is to prove the following theorem.

Theorem 4.4.2. *Let (X, μ) be a metric measure space equipped with a perimeter function \mathcal{P} . Suppose that there exists $d \geq 1$ and $\varepsilon > 0$ such that*

- (1) *X contains a maximal ε -separated net Y with a d -dimensional isoperimetric inequality,*
- (2) *X satisfies a weak relative d -dimensional isoperimetric inequality at scales ε and 3ε in Y ,*
- (3) *$\sup_{y \in Y} \sharp(B(y, r) \cap Y) < \infty$ for all $r > 0$,*
- (4) *there are constants $0 < c_-, c_+ < \infty$ such that $c_- \leq \mu(B(y, \varepsilon)) \leq c_+$ for all $y \in Y$.*

Then X satisfies a d -dimensional isoperimetric inequality.

The proof below reveals that it suffices in fact to assume that the estimate in (3) holds for $r \simeq \varepsilon$, where the exact value of r depends on the constant c from the weak relative isoperimetric inequality. Adapting the argument given in Lemma 2.3 of [49], one can see that such a weakened version of (3) holds true as soon as X satisfies a weak condition on the volume of balls as in (4) – at suitable scales depending on ε .

Proof of Theorem 4.4.2. Let E be an arbitrary non-empty relatively compact domain in X with finite perimeter. We wish to show that

$$\mu(E)^{(d-1)/d} \leq C\mathcal{P}(E, X)$$

for a universal constant C that does not depend on E . The strategy is to use the large-scale information provided by the rough isoperimetric inequality for the net Y combined with the local information provided by the weak local relative isoperimetric inequality. Without loss of generality, we may assume that the relative isoperimetric inequality holds with the same constants at scale ε and at scale 3ε .

We observe further that it suffices to consider points of Y that lie in the open ε -neighborhood $N_\varepsilon(E)$ of E . We divide these points into two categories by setting

$$S := \{y \in Y : \mu(E \cap B(y, \varepsilon)) > \frac{1}{2} \mu(B(y, \varepsilon))\}$$

and

$$P_0 := \{y \in Y \cap N_\varepsilon(E) : \mu(E \cap B(y, \varepsilon)) \leq \frac{1}{2} \mu(B(y, \varepsilon))\}.$$

We will apply the weak local relative isoperimetric inequality to points in P_0 and in the combinatorial boundary ∂S . For the set S we will also apply the rough isoperimetric inequality. Note that relative compactness of E ensures that the cardinality of S is finite.

By maximality of the net, the ε -balls centered at points in $S \cup P_0$ cover the set E and thus

$$(4.9) \quad \mu(E) \leq \sum_{y \in P_0} \mu(B(y, \varepsilon) \cap E) + \sum_{y \in S} \mu(B(y, \varepsilon) \cap E).$$

If y belongs to P_0 , then by definition and by the weak local relative isoperimetric inequality at scale ε ,

$$\begin{aligned} \mu(B(y, \varepsilon) \cap E)^{(d-1)/d} &\leq \min\{\mu(B(y, \varepsilon) \cap E), \mu(B(y, \varepsilon) \cap (X \setminus E))\}^{(d-1)/d} \\ &\leq C \mathcal{P}(E, B(y, c\varepsilon)). \end{aligned}$$

Summing over all such points y , we find that

$$\begin{aligned} \sum_{y \in P_0} \mu(E \cap B(y, \varepsilon)) &\leq \left(\sum_{y \in P_0} (\mu(E \cap B(y, \varepsilon)))^{(d-1)/d} \right)^{d/(d-1)} \\ &\leq \left(C \sum_{y \in P_0} \mathcal{P}(E, B(y, c\varepsilon)) \right)^{d/(d-1)} \leq (\nu C \mathcal{P}(E, X))^{d/(d-1)}, \end{aligned}$$

where the constant ν is derived from assumption (3), which controls the overlap of $c\varepsilon$ -balls centred in points of the net; see the argument in Remark 4.2.2. If S is empty, then this estimate combined with (4.9) gives the desired bound. Otherwise we estimate the sum over the points in S as follows:

$$(4.10) \quad \sum_{y \in S} \mu(B(y, \varepsilon) \cap E) \leq c_+ \cdot \#S \leq (c_+^{(d-1)/d} C \# \partial S)^{d/(d-1)}.$$

If y is a point in ∂S , then by definition, y does not belong to S , but there is a point $s \in S$ such that $d(y, s) \leq 2\varepsilon$. Since the ε -ball centred at s intersects E significantly, but the corresponding ball centred at y does not, we can show that the slightly enlarged ball $B(y, 3\varepsilon)$ intersects both E and its complement in sets of large mass. Indeed, for y and s as above, one has

$$B(s, \varepsilon) \cap E \subseteq B(y, 3\varepsilon) \cap E$$

and thus

$$(4.11) \quad \mu(B(y, 3\varepsilon) \cap E) \geq \mu(B(s, \varepsilon) \cap E) > \frac{1}{2} \mu(B(s, \varepsilon)) \geq \frac{c_-}{2}.$$

On the other hand, to estimate $\mu(B(y, 3\varepsilon) \cap (X \setminus E))$ from below, we observe that since $y \notin S$, the point y either lies outside a ε -neighborhood of E , in which case $B(y, \varepsilon)$ is entirely contained in $X \setminus E$ and we have

$$(4.12) \quad \mu(B(y, 3\varepsilon) \cap (X \setminus E)) \geq \mu(B(y, \varepsilon) \cap (X \setminus E)) \geq \mu(B(y, \varepsilon)) \geq c_-,$$

or \bar{y} belongs to P_0 . If the latter happens, then

$$(4.13) \quad \begin{aligned} \mu(B(y, 3\varepsilon) \cap (X \setminus E)) &\geq \mu(B(y, \varepsilon) \cap (X \setminus E)) \\ &= \mu(B(y, \varepsilon)) - \mu(B(y, \varepsilon) \cap E) \geq \frac{1}{2} \mu(B(y, \varepsilon)) \geq \frac{c_-}{2}. \end{aligned}$$

Combining (4.11), (4.12) and (4.13) with the weak local relative isoperimetric inequality at scale 3ε , we find that

$$\left(\frac{c_-}{2}\right)^{\frac{d-1}{d}} \leq \min\{\mu(B(y, 3\varepsilon) \cap E), \mu(B(y, 3\varepsilon) \cap (X \setminus E))\}^{\frac{d-1}{d}} \leq C\mathcal{P}(E, B(y, 3c\varepsilon)).$$

Since this estimate holds uniformly for all $y \in \partial S$, we conclude that

$$\#\partial S \leq \left(\frac{2}{c_-}\right)^{(d-1)/d} C \sum_{y \in \partial S} \mathcal{P}(E, B(y, 3c\varepsilon)) \leq \left(\frac{2}{c_-}\right)^{(d-1)/d} C\nu\mathcal{P}(E, X),$$

where ν is again a finite constant which controls the overlap of balls, guaranteed by assumption (3).

We insert this estimate in (4.10), and return to the volume estimate (4.9) at the beginning of the proof, which now reads

$$\mu(E) \leq C\mathcal{P}(E, X)^{d/(d-1)}$$

for a suitable universal constant C . This concludes the proof. \square

As an application of Theorem 4.4.2, we obtain the following statement relevant for the proof of our main result.

Corollary 4.4.3 (Global isoperimetric inequality). *The manifold \widetilde{M}'' satisfies a d -dimensional isoperimetric inequality for $d > 4$, that is, there exists a constant $C > 0$ such that*

$$(\mu_{\widetilde{M}}(E))^{(d-1)/d} \leq C\mathcal{P}(E, \widetilde{M}'')$$

for all non-empty relatively compact domains $E \subset \widetilde{M}''$ with piecewise \mathcal{C}^1 -boundary.

Here $\mathcal{P}(E, \widetilde{M}'')$ denotes the (sub-Riemannian) perimeter of E in \widetilde{M}'' .

Proof. We verify that $X = \widetilde{M}''$ endowed with the metric $d_{\widetilde{M}''}$ and the measure $\mu_{\widetilde{M}}$ fulfills the assumptions of Theorem 4.4.2.

We choose $\varepsilon < \min\{\varepsilon_0, \varepsilon_1, \delta_1/3\}$, where the bound for ε is given by constants that have appeared earlier. The fact that $\varepsilon < \varepsilon_0$ allows us by Lemma 4.2.1 and Proposition 4.2.4 to choose a maximal ε -separated net Y on \widetilde{M}'' which satisfies condition (3) in Theorem 4.4.2 and fulfills a rough d -dimensional isoperimetric inequality for $d > 4$. Since ε has also been chosen smaller than ε_1 , Lemma 4.2.5 yields assumption (4) in Theorem 4.4.2. Finally, the fact that 3ε is smaller than δ_1 ensures that \widetilde{M}'' satisfies a weak relative d -dimensional isoperimetric inequality at scales ε and 3ε by Proposition 4.3.3. The claim follows. \square

4.5. Computation of the capacity at infinity

The goal of this section is to prove that the 4-capacity in \widetilde{M} of a closed ball (or a more general compact set) in \widetilde{M}'' is positive. We will follow a standard proof relying on the isoperimetric and coarea inequalities. We first formulate a suitable horizontal coarea formula on sub-Riemannian contact manifolds.

Proposition 4.5.1 (Coarea formula). *Let N be a contact sub-Riemannian 3-manifold. Then, for all $u \in \mathcal{C}^3(N)$, one has that*

$$(4.14) \quad \int_N |\nabla_H u| \, d\mu_N = \int_{-\infty}^{\infty} \mathcal{P}_N(\{x \in N : u(x) > t\}, N) \, dt,$$

where $|\nabla_H u| = \sqrt{g_N(\nabla_H u, \nabla_H u)}$.

Here, as before, μ_N denotes the Popp volume measure on N induced by the metric g_N , while \mathcal{P}_N denotes the horizontal perimeter. Equation (4.14) has been established in Theorem 4.2 of [61] for Lipschitz functions in Carnot–Carathéodory spaces, that is, for the case when the manifold is \mathbb{R}^n with a sub-Riemannian metric. While we have to consider other manifolds as well, the coarea formula for \mathcal{C}^3 functions suffices for our purposes. This case is considerably easier to prove as it follows directly from the Riemannian coarea formula. The idea of using a Riemannian coarea formula to derive a statement in the sub-Riemannian setting is not new; see Theorem 7.2.2 in [54] (for sub-Riemannian groups).

Proof of Proposition 4.5.1. We assume first that the contact structure of N is co-orientable. This allows to promote the sub-Riemannian metric g_N to a Riemannian metric on N by declaring the Reeb vector field orthonormal to the distribution HN . We continue to write this metric as g_N . Recall that the Riemannian volume associated to g_N agrees with the Popp volume. Given a $\mathcal{C}^3(N)$ function u , we denote by ∇u the usual, Riemannian, gradient of u . Let us further abbreviate

$$E_t := \{x \in N : u(x) > t\} \quad \text{and} \quad \Sigma_t := \{x \in N : u(x) = t\}, \quad t \in \mathbb{R}.$$

As u is \mathcal{C}^3 , it follows by Sard’s theorem and the discussion in §2.3 of [29] and (2.9) in [28] that for almost every $t \in \mathbb{R}$ (for the regular values of u), one has

$$(4.15) \quad \mathcal{P}_N(E_t, N) = \int_{\Sigma_t} |\nu| \, d\sigma^2,$$

where σ^2 is the Riemannian measure on Σ_t and ν the orthogonal projection to HN of the unit vector field n that is normal to Σ_t . Let us fix such a regular value t . Since Σ_t is a level set of u and t is regular, a unit vector field normal to Σ_t is given by $n := \nabla u / |\nabla u|$. Here $|\cdot|$ is computed with respect to g_N .

Next, the Riemannian coarea formula, as stated for instance in Corollary I.3.1 of [12], says that

$$\int_N |\nabla u(x)| \phi(x) \, d\mu_N = \int_{-\infty}^{\infty} \int_{\Sigma_t} \phi(y) \, d\sigma^2(y) \, dt,$$

for all nonnegative measurable functions ϕ on N . We apply this to the function $\phi = h|\nabla_H u|/|\nabla u|$ for some nonnegative measurable function h on N . It follows that

$$(4.16) \quad \int_N h|\nabla_H u| \, d\mu_N = \int_{-\infty}^{\infty} \int_{\Sigma_t} h \left| \frac{\nabla_H u}{|\nabla u|} \right| \, d\sigma^2 \, dt.$$

We observe that $\nabla_H u/|\nabla u|$ agrees with the orthogonal projection ν of the unit normal n to HN . The desired coarea formula (in the case of a co-orientable contact structure) then follows from (4.15) and (4.16) with $h \equiv 1$.

Since a general contact 3-manifold can be covered by open subsets restricted to which the horizontal distribution is orientable, the general case can be proved by a partition of unity argument. \square

Definition 4.5.2. Let N be a contact sub-Riemannian 3-manifold. The p -capacity, $1 < p < \infty$, of a compact set $C \subseteq N$ is defined as

$$\text{cap}_p(N, C) = \inf \int_N |\nabla_H u|^p \, d\mu_N,$$

where the infimum is taken over $u \in \mathcal{C}_0^\infty(N)$ with $u|_C \geq 1$, and the norm $|\cdot|$ is defined using the sub-Riemannian metric g_N . The pair (N, C) is called a *condenser*.

In the current section, we apply the above definition with $N = \widetilde{M}$. In Section 4.7, we will apply it in the case where N is an open subset of \mathbb{H} .

Proposition 4.5.3. Let $K \subset \widetilde{M}''$ be a compact set of positive $\mu_{\widetilde{M}}$ measure. Then

$$(4.17) \quad \text{cap}_4(\widetilde{M}, K) > 0.$$

We proceed as in Theorem 1.3 of [62], yet we work on \widetilde{M} at first, and only deal with \widetilde{M}'' when the isoperimetric inequality comes into play, rather than deriving for instance a Sobolev inequality on \widetilde{M}'' in greatest possible generality. However, the reader will surely recognize in what follows arguments similar to those used in the proof of Sobolev inequalities; see for instance [64].

Proof. We fix $u \in \mathcal{C}_0^\infty(\widetilde{M})$ such that $u|_K \geq 1$. We have to find a uniform positive lower bound for $\int_{\widetilde{M}} |\nabla_H u|^4 \, d\mu_{\widetilde{M}}$. The coarea formula will naturally lead to an integral of $|\nabla_H u|$, rather than $|\nabla_H u|^4$, but this issue can be solved by applying Hölder's inequality with a suitable exponent. For any $\gamma > 1$, it holds that

$$(4.18) \quad (\mu_{\widetilde{M}}(K))^{d/(d-1)-4/3} \leq \left(\int_K |u|^{\gamma d/(d-1)} \, d\mu_{\widetilde{M}} \right)^{d/(d-1)-4/3}.$$

We will later choose the exponent γ appropriately depending on d . The classical real-variables inequality $(\delta^{-1} \int_0^\infty s^{1/\delta-1} F(s) \, ds)^\delta \leq \int_0^\infty F(s)^\delta \, ds$, valid for decreasing functions F and $0 < \delta \leq 1$ (see, for instance, (3.34) in [38] or (II.2.2) in [12]) yields

$$\left(\int_{\widetilde{M}''} (|u|^\gamma)^{d/(d-1)} \, d\mu_{\widetilde{M}} \right)^{(d-1)/d} \leq \int_0^\infty (\mu_{\widetilde{M}}\{x \in \widetilde{M}'' : |u(x)|^\gamma > s\})^{(d-1)/d} \, ds$$

when applied to $F(s) = \mu_{\widetilde{M}}(\{x \in \widetilde{M}'' : |u(x)|^\gamma > s\})$ and $\delta = (d-1)/d$. Using the isoperimetric inequality, this can be further estimated from above by

$$C \int_0^\infty \mathcal{P}(\{x \in \widetilde{M}'' : |u(x)|^\gamma > s\}, \widetilde{M}'') ds.$$

If $\gamma > 3$, then $|u|^\gamma \in \mathcal{C}^3$ since $u \in \mathcal{C}_0^\infty(\widetilde{M})$. Thus we can apply the coarea formula to $N = \widetilde{M}''$ and $|u|^\gamma$. We obtain

$$(4.19) \quad \left(\int_{\widetilde{M}''} (|u|^\gamma)^{d/(d-1)} d\mu_{\widetilde{M}''} \right)^{(d-1)/d} \leq C \int_{\widetilde{M}''} |\nabla_H(|u|^\gamma)| d\mu_{\widetilde{M}''}.$$

For the rest of the computation, we fix

$$\gamma := \frac{4(d-1)}{d-4}.$$

Since $d > 4$, it holds that $\gamma > 3$ as required. Moreover, γ is chosen so that

$$\frac{\gamma}{\gamma-1} \frac{d}{d-1} = \frac{4}{3},$$

Then Hölder's inequality and (4.19) yield

$$\left(\int_{\widetilde{M}''} (|u|^\gamma)^{d/(d-1)} d\mu_{\widetilde{M}''} \right)^{(d-1)/d-3/4} \leq C \gamma \left(\int_{\widetilde{M}''} |\nabla_H u|^4 d\mu_{\widetilde{M}''} \right)^4.$$

Returning to (4.18), we have found that

$$(\mu_{\widetilde{M}}(K))^{(d-1)/d-3/4} \leq C \gamma \left(\int_{\widetilde{M}''} |\nabla_H u|^4 d\mu_{\widetilde{M}''} \right)^4 \leq C \gamma \left(\int_{\widetilde{M}} |\nabla_H u|^4 d\mu_{\widetilde{M}} \right)^4.$$

Taking the infimum over all such u completes the proof. \square

The following notions are standard in the Riemannian setting.

Definition 4.5.4. We say that a contact sub-Riemannian 3-manifold N is p -parabolic, $1 < p < \infty$, if

$$\text{cap}_p(N, C) = 0$$

for all compact sets $C \subseteq N$. A manifold that is not p -parabolic is called p -hyperbolic.

In this language, Proposition 4.5.3 states that \widetilde{M} is 4-hyperbolic. By way of contrast, it is well known ([39], p. 130) that the sub-Riemannian Heisenberg group \mathbb{H} is 4-parabolic.

Next, we will introduce some machinery of nonlinear potential theory which, in coordination with the above hyperbolicity and parabolicity results, will complete the proof of Theorem 1.2.2.

4.6. Nonlinear potential theory

In this section and the next, we give a brief digression into some aspects of nonlinear potential theory on sub-Riemannian manifolds. The main goal of this section is to

conclude from the 4-hyperbolicity of \widetilde{M} the existence of a positive nonconstant supersolution to the 4-harmonic equation, and the existence of a Green's function for the 4-Laplacian at every point of \widetilde{M} .

For an introduction to the classical Euclidean nonlinear potential theory, we refer the reader to [40]. For a discussion of \mathcal{A} -harmonic functions in the *Riemannian* setting, see [42]. Nonlinear potential theory on Carnot groups has been initiated in [39]. An in-depth study of Q -harmonic functions on sub-Riemannian manifolds is part of [10]. Nonlinear potential theory in metric measure spaces of bounded geometry has been discussed in [6]. Here we will merely provide the results that are needed to prove our main theorem in the setting of manifolds modelled on the Heisenberg group.

Let (N, HN, g_N) be a sub-Riemannian contact 3-manifold. The definition of harmonic functions $u: N \rightarrow \mathbb{R}$ requires the notion of a horizontal gradient, divergence, and a Laplacian on N . The divergence of a smooth vector field has been defined in Definition 4.3.1, the horizontal gradient in Definition 4.3.4. We now extend these notions to the nonsmooth case.

We say that a horizontal vector field $\nabla_H u \in L^1_{\text{loc}}(N)$ is a *weak horizontal gradient* of $u \in L^1_{\text{loc}}(N)$ if

$$\int_N g_N(\nabla_H u, \Phi) \, d\mu_N = - \int_N u \operatorname{div}_N \Phi \, d\mu_N$$

for all smooth compactly supported horizontal vector fields Φ on N .

Remark 4.6.1. If a continuous map $f: \mathbb{H} \rightarrow N$ has weak horizontal derivatives in L^4_{loc} , then $u \circ f \in HW^{1,4}_{\text{loc}}(\mathbb{H})$ for every smooth function $u: N \rightarrow \mathbb{R}$, and the weak horizontal gradient of $u \circ f$ exists and agrees with $X(u \circ f)X + Y(u \circ f)Y$.

We say that $\operatorname{div}_N V \in L^1_{\text{loc}}(N)$ is a *weak divergence* of a locally integrable horizontal vector field V on N if

$$\int_N g_N(\nabla_H \varphi, V) \, d\mu_N = - \int_N \varphi \operatorname{div}_N V \, d\mu_N$$

for all $\varphi \in C^\infty_0(N)$.

Definition 4.6.2. A continuous $HW^{1,4}_{\text{loc}}$ -function $u: N \rightarrow \mathbb{R}$ is said to be *4-harmonic* if the equation

$$\operatorname{div}_N (g_N(\nabla_H u, \nabla_H u) \nabla_H u) = 0$$

holds in a weak sense.

In the following, we will also use a generalization of this concept. We consider operators $\mathcal{A}: HN \rightarrow HN$ for which there exist constants $0 < \alpha \leq \beta < \infty$ such that

1. $\mathcal{A}_x: H_x N \rightarrow H_x N$ is continuous for almost every x ,
2. $x \mapsto \mathcal{A}_x(V)$ is measurable for all horizontal measurable vector fields V ,
3. for almost every $x \in N$ and all $h \in H_x N$:

- (a) $g_N(\mathcal{A}_x(h), h) \geq \alpha g_N(h, h)^2$,
- (b) $g_N(\mathcal{A}_x(h), \mathcal{A}_x(h))^{1/2} \leq \beta g_N(h, h)^{3/2}$,
- (c) $g_N(\mathcal{A}_x(h_1) - \mathcal{A}_x(h_2), h_1 - h_2) > 0$ for $h_1 \neq h_2$,
- (d) $\mathcal{A}_x(\lambda h) = |\lambda|^2 \lambda \mathcal{A}_x(h)$ for all $\lambda \in \mathbb{R} \setminus \{0\}$.

Here and in what follows we have written $\mathcal{A}_x(h) := \mathcal{A}(x, h)$ for $h \in H_x N$. We will call such \mathcal{A} operators of type 4 on N .

Definition 4.6.3. A $HW_{\text{loc}}^{1,4}$ -function $u: N \rightarrow \mathbb{R}$ is called *solution of the \mathcal{A} -harmonic equation*, or for short an *\mathcal{A} -solution*, if

$$-\text{div}_N(\mathcal{A}(\nabla_H u)) = 0$$

holds in the weak sense for a suitable $\mathcal{A}: HN \rightarrow HN$ as above. *\mathcal{A} -sub-* and *super-*solutions can be defined accordingly using the signs \leq and \geq , and nonnegative test functions. If the solution u is continuous, it is called *\mathcal{A} -harmonic*.

The *standard operator of type 4* is $\mathcal{A}_x(h) = g_N(h, h)h$; continuous solutions to the \mathcal{A} -harmonic equation for this \mathcal{A} are precisely the 4-harmonic functions of Definition 4.6.2.

We will also encounter solutions with a singularity. Let Ω be a relatively compact domain in N and y a point in N . We say that a positive function

$$G = G(\cdot, y) \in \mathcal{C}(\Omega \setminus \{y\}) \cap HW_{\text{loc}}^{1,4}(\Omega \setminus \{y\})$$

is a *Green's function in Ω with pole y for the \mathcal{A} -harmonic equation* if

1. $\lim_{x \rightarrow z} G(x) = 0$ for all $z \in \partial\Omega$,
2. $\int_{\Omega} g_N(\nabla_H \varphi, \mathcal{A}(\nabla_H G)) \, d\mu_N = \varphi(y)$ for all $\varphi \in \mathcal{C}_0^\infty(\Omega)$.

Note that such $G(\cdot, y)$ is \mathcal{A} -harmonic in $\Omega \setminus \{y\}$.

We say that a function $G = G(\cdot, y)$ is a *Green's function in N with pole y for the \mathcal{A} -harmonic equation* if there exists an exhaustion of N by relatively compact domains $\Omega_i \subset \Omega_{i+1}$, $i \in \mathbb{N}$, and associated Green's functions G_i with pole at y , such that G equals $\lim_{i \rightarrow \infty} G_i$ and is not identically equal to infinity.

The 4-parabolicity of the Heisenberg group implies a Liouville-type consequence for \mathcal{A} -supersolutions. The following theorem is stated in the context of general Carnot groups in [39], p. 131; see also §3.2 of [13] and Theorem 4 in [77].

Theorem 4.6.4. *Let \mathcal{A} be an operator of type 4 on \mathbb{H} . Then every nonnegative \mathcal{A} -supersolution u with*

$$(4.20) \quad u(x) = \text{ess} \liminf_{y \rightarrow x} u(y), \quad x \in \mathbb{H},$$

is constant.

The statements in [39] and [77] are formulated for so called “superharmonic functions” rather than for “supersolutions”. Yet it is not difficult to see that a nonnegative supersolution with the property (4.20) is lower semicontinuous and fulfills the comparison principle required in the usual definition of superharmonic functions. The proof in the Euclidean setting (Theorem 7.16 in [40]) can be easily

adapted to the Heisenberg group; the only point worth observing is that a horizontal Sobolev function with almost everywhere vanishing horizontal gradient has to be constant almost everywhere. (See also Proposition 9.4 in [6] for superharmonic functions in an abstract metric measure space setting.)

Complementing Theorem 4.6.4, we have the following result.

Theorem 4.6.5. *If a contact sub-Riemannian 3-manifold N is 4-hyperbolic, then it admits a nonconstant nonnegative supersolution of the standard operator \mathcal{A} of type 4, with the property that*

$$(4.21) \quad u_N(x) = \operatorname{ess\,lim\,inf}_{y \rightarrow x} u_N(y), \quad x \in N.$$

Moreover, N supports a positive Green's function $G(\cdot, y)$ for \mathcal{A} at any $y \in N$.

We sketch an argument for Theorem 4.6.5. The approach is standard, and we refer to Theorem 9.22 in [40] (for Euclidean spaces), Theorem 3.27 in [41] (Riemannian setting), Section 3.2 of [13] and Theorems 3 and 4 in [77] (sub-Riemannian setting), and Theorem 3.14 in [46] (for abstract metric measure spaces).

Proof. Since N is 4-hyperbolic, it contains a compact set K whose 4-capacity at infinity is positive and bounded. That is, there is a $C > 0$ such that for any open set Ω with $K \subset \Omega \Subset N$, one has

$$(4.22) \quad \inf_{\Omega} |\nabla_H u|^4 \, d\mu_N \geq C,$$

where the infimum is taken over $u \in C_0^\infty(\Omega)$ satisfying $u|_K \geq 1$. Without loss of generality, we may restrict to *nonnegative* functions u .

For a fixed Ω , consider a minimizing sequence u_j for (4.22). Each u_j is in $C_0^\infty(\Omega)$, and therefore in $HW_0^{1,4}(\Omega)$. One shows that the sequence u_j is a Cauchy sequence in this Sobolev space and that the limit *potential* function $u_\Omega := \lim_{j \rightarrow \infty} u_j \in HW_0^{1,4}$ is nonnegative and 4-harmonic outside of K . Furthermore, one shows that

$$(4.23) \quad \inf_{\Omega} |\nabla_H u|^4 \, d\mu_N = \int_{\Omega} |\nabla_H u_\Omega|^4 \, d\mu_N.$$

Consider now an exhaustion of N by domains $\Omega_j \Subset N$, with associated potential functions u_{Ω_j} . Again, one shows that these converge in the Sobolev space to a potential u_N , now defined on all of N . The limiting function is nonnegative, and satisfies

$$(4.24) \quad \inf_N |\nabla_H u|^4 \, d\mu_N = \int_N |\nabla_H u_N|^4 \, d\mu_N$$

In particular, u_N is nonconstant. By Theorem 8.22 in [6], we may choose an $HW_{\text{loc}}^{1,4}$ representative of u_N (which we continue to denote by the same letter) for which (4.21) holds. To show that u_N is a supersolution, one considers the variational kernel

$$F(x, \xi) := |\xi|^4 = g_N(\xi, \xi)^2, \quad \xi \in H_x N$$

and the associated variational integral

$$I_F(u) := \int_N F(x, \nabla_H u) \, d\mu_N.$$

By construction, $u = u_N$ is a superminimizer for I_F in the sense of Definition 7.7 in [6] and thus one shows analogously as in the Euclidean case (Theorem 5.13 in [40]) that

$$\int_N g_N(\nabla_H u_N, \nabla_H u_N) \nabla_H u_N, \nabla_H v - \nabla_H u_N \, d\mu_N \geq 0$$

for all admissible $v = u_N + \varepsilon\varphi$. This shows that u_N is a supersolution of the 4-Laplacian on all of N .

To construct a Green's function, one takes a sequence of balls $K_j = B(y, r_j)$ with $r_j \rightarrow 0$ and shows that the global potential functions associated to K_j converge, up to renormalization, to a Green's function. \square

4.7. Morphism property

In this section, we show that if $u: N \rightarrow \mathbb{R}$ is a 4-harmonic function and $f: \mathbb{H} \rightarrow N$ is a quasiregular mapping, then the composition $f \circ u$ is \mathcal{A} -harmonic for a suitable operator \mathcal{A} of type 4 on \mathbb{H} .

This so-called *morphism property* has been proved in Theorem 3.14 of [39] (under an additional smoothness assumption on the mapping) and in [16] (without such assumption) for arbitrary quasiregular maps between domains in the sub-Riemannian Heisenberg group. A morphism property for 1-*quasiconformal* maps between equiregular sub-Riemannian manifolds has recently been proved in [10]. None of these results covers exactly the case we are interested in, on the other hand, unlike in the setting of the mentioned results, we can rely on an already well established theory of quasiregular mappings in the Heisenberg group.

The *pullback* of an operator \mathcal{A} of type 4 under a quasiregular mapping f is the operator $f^\# \mathcal{A}$ whose value at a point x on a horizontal tangent vector h is

$$f^\# \mathcal{A}_x(h) := \det(D_H f(x))^2 D_H f(x)^{-1} \mathcal{A}_{f(x)}((D_H f(x)^{-1})^T h)$$

if $\det D_H f(x) \neq 0$, and $f^\# \mathcal{A}_x$ equal to the standard operator of type 4 at x otherwise. See Section 3.3.2 for the definition of the formal Jacobian $\det D_H f$.

Lemma 4.7.1. *Let $f: V \rightarrow V'$ be a quasiregular mapping, for $V \subset \mathbb{H}$ and $V' \subset N$ domains and N a sub-Riemannian 3-manifold. Let \mathcal{A} be the standard operator of type 4 in V' . Then $f^\# \mathcal{A}$ is an operator of type 4 in V .*

Proof. The proof goes analogously to the Euclidean case (see Lemma 14.38 in [40]), using the characterization of quasiregularity provided in Proposition 3.3.7 and the fact that for a nonconstant quasiregular map f , a set A has measure zero if and only if $f(A)$ has measure zero (see Remark 3.3.4). \square

Lemma 4.7.2. *Assume that V and V' are domains in \mathbb{H} , and V'' is a domain in an arbitrary sub-Riemannian contact 3-manifold. For quasiregular mappings $h: V \rightarrow V'$ and $f: V' \rightarrow V''$ and an operator \mathcal{A} of type 4 in V'' we have*

$$h^\# f^\# \mathcal{A} = (f \circ h)^\# \mathcal{A}.$$

Proof. The statement is a simple computation based on the chain rule

$$D_H(f \circ h)(p) = D_H f(h(p)) \circ D_H h(p), \quad \text{for almost every } p \in V.$$

The latter follows from Proposition A.0.2 applied to the map g and the components $u = f_i$, $i \in \{1, 2\}$, of f in coordinates. \square

Proposition 4.7.3. *Let N be a smooth sub-Riemannian contact 3-manifold, $V \subset \mathbb{H}$ and $V' \subset N$ be domains, and $\phi: V \rightarrow V'$ a quasiconformal diffeomorphism. If u is a 4-harmonic function, then $v := u \circ \phi$ is $\phi^\# \mathcal{A}$ -harmonic for the standard operator \mathcal{A} of type 4. An analogous statement holds for supersolutions.*

Proof. By Lemma 4.7.1, $\phi^\# \mathcal{A}$ is an operator of type 4. We would like to show that v is a weak solution for the $\phi^\# \mathcal{A}$ -Laplacian. That is, for any $\Psi \in C_0^\infty(V)$,

$$\int_V g_{\mathbb{H}}(\phi^\# \mathcal{A}_x(\nabla_H v), \nabla_H \Psi) \, d\mu_{\mathbb{H}} = 0.$$

We push Ψ forward via ϕ , obtaining $\phi^\# \Psi = \Psi \circ \phi^{-1}: V' \rightarrow \mathbb{R}$, and compute with the help of Proposition 3.3.8 that

$$\begin{aligned} & \int_V g_{\mathbb{H}}(\phi^\# \mathcal{A}_x(\nabla_H v), \nabla_H \Psi) \, d\mu_{\mathbb{H}} \\ &= \int_V g_{\mathbb{H}}(\det(D_H \phi(x))^2 D_H \phi(x)^{-1} \mathcal{A}_{\phi(x)}((D_H \phi(x)^{-1})^T \nabla_H v), \nabla_H \Psi) \, d\mu_{\mathbb{H}} \\ &= \int_V \det(D_H \phi(x))^2 g_{\mathbb{H}}(\mathcal{A}_{\phi(x)}((D_H \phi(x)^{-1})^T \nabla_H v), (D_H \phi(x)^{-1})^T \nabla_H \Psi) \, d\mu_{\mathbb{H}} \\ &= \int_{V'} g_N(\mathcal{A}_x(\nabla_H u), \nabla_H(\phi^\# \Psi)) \, d\mu_N, \end{aligned}$$

which is equal to zero since $\phi^\# \Psi \in C_0^\infty(V')$. \square

In the next step, we will pull \mathcal{A} back by a quasiregular function that need not be a diffeomorphism.

Proposition 4.7.4. *Let $U, V \subset \mathbb{H}$ be domains and let $h: U \rightarrow V$ be a quasiregular mapping. If $v: V \rightarrow \mathbb{R}$ is \mathcal{A} -harmonic for some operator \mathcal{A} of type 4, then $w = v \circ h$ is $h^\# \mathcal{A}$ -harmonic. An analogous statement holds for supersolutions.*

This result has also been stated in Theorem 9 of [77] for \mathcal{A} -harmonic functions. The main technical difficulty in the proof is to push forward a test function Ψ under a quasiregular mapping h . If h was a homeomorphism, such a push-forward could be simply defined as $\Psi \circ h^{-1}$. If h is not injective, it is still possible to define a function $h^\# \Psi$ which plays the role of a push-forward, but it is more difficult to verify the necessary regularity properties. To do so, we use some terminology from

topology, for which we refer to [68] or 14.9 in [40]. For the moment, let us just recall that a relatively compact domain is called a *normal domain* for a map h if $h(\partial D) = \partial h(D)$. Recall further that nonconstant quasiregular mappings on \mathbb{H} are discrete and open [18], and by the latter property we have $\partial h(D) \subset h(\partial D)$ for every domain. We employ some terminology related to path lifting. For an interval $[a, b]$ and $c \in (a, b)$, we write $I_c = [a, c]$ if $c < b$, and $I_c = [a, b]$ if $c = b$. Given a path $\beta: [a, b] \rightarrow \mathbb{H}$, we say that a path $\alpha: I_c \rightarrow \mathbb{H}$ is an *h -lifting* of β starting at a point $x \in \mathbb{H}$ if $\alpha(a) = x$ and $h \circ \alpha = \beta|_{I_c}$. We call α a *total h -lifting* if $I_c = [a, b]$.

Proof of Proposition 4.7.4. Without loss of generality, we may assume that h is nonconstant and that U is a normal neighborhood whose h -image is a ball V . As in Proposition 4.7.3, we need to push forward a test function $\Psi \in C_0^\infty(U)$. By a result of Dairbekov [17], quasiregular mappings between domains in the Heisenberg group are discrete and open, hence index theory is applicable and the branch set and its image both have measure zero. We can then define the push-forward of Ψ as

$$h^\# \Psi(y) = \sum_{x \in h^{-1}y} \text{index}(h, x) \Psi(x),$$

where $\text{index}(h, x)$ is the local topological index of h at x .

As in Lemma 14.30 of [40], one verifies that $h^\# \Psi \in \mathcal{C}_0(V)$ and the support of $h^\# \Psi$ is contained in $h(\text{spt} \Psi)$. While $h^\# \Psi$ is not necessarily smooth, one can show that it is in $HW_0^{1,4}(V)$. This can be done along the lines of Lemma 14.31 in [40]: one works locally and proves absolute continuity of $h^\# \Psi$ along almost every horizontal line in V . These lines are first lifted under h to curves in U , and then one shows that almost every such h -lifting is absolutely continuous. This is the content of Lemma 4.7.5 below. Once the *ACL*-property of $h^\# \Psi$ is established, one shows by the same argument as in Lemma 14.31 of [40] that

$$\int_V |\nabla_H(h^\# \Psi)|^4 \, d\mu_{\mathbb{H}} < \infty.$$

We can then carry out the calculations as in Proposition 4.7.3 and use the density of smooth functions in $HW_{\text{loc}}^{1,4}$ to conclude

$$\int_V \langle \mathcal{A}_x(\nabla_H v), \nabla_H h^\# \Psi \rangle \, d\mu_{\mathbb{H}} = 0$$

as desired. □

In the preceding proof we made use of the following result.

Lemma 4.7.5. *Let $h: U \rightarrow \mathbb{H}$ be a nonconstant quasiregular mapping in a domain $U \subseteq \mathbb{H}$ and let $\Psi \in C_0^\infty(U)$. Then $h^\# \Psi \in \text{ACL}(h(U))$.*

Lemma 4.7.5 can be found in [74], Lemma 7, in a more general setting. Here we specialize to the Heisenberg group and the class of quasiregular mappings. The proof in [74] is based on a series of results in other papers, which we will list below. The method of proof differs from the argument in Euclidean spaces, where it is

used that quasiregular mappings have bounded inverse metric dilatation. Instead, the proof in [74] makes use of a capacity estimate, which we think deserves to be better known. In Lemma 5 of [78], the following was proved (in greater generality): if $E \subset \mathbb{H}$ is connected and $G \subset \mathbb{H}$ is an open set contained in the metric $c_0 \text{diam} E$ -neighborhood of E for a given universal constant $c_0 > 0$, then

$$(4.25) \quad (\text{cap}_4(G, E))^3 \geq c \frac{(\text{diam} E)^4}{\mu_{\mathbb{H}}(G)}$$

for an absolute constant $0 < c < \infty$. (Note that the smoothness assumption on the admissible functions in our definition of capacity can be relaxed by an approximation argument, so as to make it agree with the definition given in [78].)

For the Euclidean antecedent of (4.25), see Lemma 5.9 in [56]. The estimate (4.25) is useful when coupled with a distortion inequality for quasiregular mappings and condensers. It is straightforward to verify, see for instance Proposition 2 in [74], that for every quasiregular mapping $h: U \rightarrow \mathbb{H}$, $U \subseteq \mathbb{H}$, there exists a constant $1 \leq K < \infty$, such that for every normal domain $A \subset U$ and every condenser (A, C) , one has

$$(4.26) \quad \text{cap}_4(A, C) \leq K N(h, A) \text{cap}_4(h(A), h(C)),$$

where $N(h, A) := \sup_{x \in \mathbb{H}} \#(h^{-1}(x) \cap A)$.

For the benefit of the reader we will work out in detail that part of the proof of Lemma 4.7.5 which concerns the application of (4.25). We sketch the remaining part of the argument and refer the reader to the cited references for more details.

Proof of Lemma 4.7.5. Throughout the proof we assume that \mathbb{H} is endowed with the Korányi distance

$$d(p, q) := \|p^{-1} * q\|_K, \quad \|(x, y, t)\|_K = \sqrt[4]{(x^2 + y^2)^2 + t^2},$$

which is bi-Lipschitz equivalent to the sub-Riemannian distance $d_{\mathbb{H}}$. Let $x_0 \in \text{supp}(h\#\Psi)$ and $h^{-1}(x_0) \cap \text{supp}\Psi = \{q_1, \dots, q_s\}$. Without loss of generality, we may assume that x_0 is the origin. One chooses small enough normal neighborhoods $U_k := U(q_k, h, r_1)$ around q_k with $h(U_k) = B(x_0, r_1)$ as described in [74]. We may assume that all the U_k are compact; cf. [69], I, Lemma 4.9.

Following [74], we construct a “cube” Q inside $B(x_0, r_1)$ which is fibered by segments β_z along the flow lines of a left invariant horizontal vector field V , where z ranges in a domain of a hyperplane transversal to V . The first task is to show for almost every $z \in S$ that every total h -lifting $\alpha: [a, b] \rightarrow U_k$ of the horizontal curve $\beta_z: [a, b] \rightarrow Q$ is absolutely continuous. To do so, one introduces the set function

$$\Phi(V) := \mu_{\mathbb{H}}\left(\bigcup_{k=1}^s U_k \cap h^{-1}(V \cap Q)\right), \quad V \text{ Borel.}$$

In Proposition 1 of [76], it was shown that the upper volume derivative

$$\overline{\Phi}'(z) := \limsup_{r \rightarrow 0^+} \frac{\Phi(N_r(\beta_z) \cap Q)}{r^3}$$

exists and is finite for almost all $z \in S$. Here, $N_r(\beta_z)$, $r > 0$, denotes the metric r -neighborhood of β_z . We fix a point $z \in S$ where Φ has finite upper volume derivative and argue that all the h -liftings α of $\beta_z: [a, b] \rightarrow Q$ in U_k , $1 \leq k \leq s$, are absolutely continuous.

To do this, we choose disjoint closed arcs I_1, \dots, I_l with respective lengths $\Delta_1, \dots, \Delta_l$ on β_z such that

$$\sum_{i=1}^l \Delta_i < \delta.$$

We define $[a_i, b_i] := \beta_z^{-1}(I_i) \subset [a, b]$. Then $E_i := \alpha([a_i, b_i])$ is a connected set in U_k . We will show that $\sum_{i=1}^l d(\alpha(a_i), \alpha(b_i)) \leq \sum_{i=1}^l \text{diam}(E_i)$ can be made smaller than any given constant if $\delta > 0$ is chosen small enough. To achieve this goal, we will construct small open neighborhoods G_i of E_i such that – among other assumptions – the conditions for the estimate (4.25) are satisfied for the condensers (G_i, E_i) for all $1 \leq i \leq l$.

First, by continuity of h , there exists $0 < r_2 < c_0 \min\{\text{diam}(E_i) : 1 \leq i \leq l\}$ such that $h(N_{r_2}(E_i))$ is compactly contained in $B(x_0, r_1)$. We may further assume that r_2 is small enough so that all the sets $h(N_{r_2}(E_i))$, $1 \leq i \leq l$, are disjoint.

Second, since h is an open mapping, it follows that I_i is at positive distance from the boundary of $h(N_{r_2}(E_i))$ for all $1 \leq i \leq l$, and so we can choose $0 < r_3 < \min\{\text{dist}(I_i, \partial h(N_{r_2}(E_i))) : 1 \leq i \leq l\}$ such that

$$N_{r_3}(I_i) \subset h(N_{r_2}(E_i)), \quad 1 \leq i \leq l.$$

We may assume that $r_3 < \delta$.

Third, since U_k is a normal domain, for every $r < r_3$, the components of $h^{-1}(N_r(I_i)) \cap U_k$ are mapped by h onto $N_r(I_i)$; see [69], I, Lemma 4.8. This shows that the E_i -component of $h^{-1}(N_r(I_i)) \cap U_k$ is contained entirely inside $N_{c_0 \text{diam}(E_i)}(E_i)$. For if this was not the case, then part of the boundary of $N_{r_2}(E_i)$ would have to be mapped inside $N_r(I_i)$, which is impossible by the choice of r_3 .

Finally, we define G_i to be the E_i -component of $h^{-1}(N_r(I_i)) \cap U_k$. Note that G_i is a normal domain by [69], I, Lemma 4.7, and (G_i, E_i) is a condenser which satisfies the conditions for the capacity lower bound (4.25).

The image $(h(G_i), h(E_i))$, $h(E_i) = I_i$ is again a condenser. By (4.26), it follows that

$$c^{1/3} \frac{\text{diam}(E_i)^{4/3}}{\mu_{\mathbb{H}}(G_i)^{1/3}} \leq \text{cap}_4(G_i, E_i) \leq K N(h, G_i) \text{cap}_4(h(G_i), h(E_i)).$$

This implies that

$$\begin{aligned} \text{diam}(E_i) &\leq c^{-1/4} K^{3/4} N(h, G_i)^{3/4} \cdot \left(\frac{\mu_{\mathbb{H}}(G_i)}{r^3} \right)^{1/4} \cdot (r \text{cap}_4(h(G_i), h(E_i)))^{3/4} \\ (4.27) \quad &\leq c' \cdot \left(\frac{\Phi(N_r(\beta_z) \cap Q)}{r^3} \right)^{1/4} \cdot (r \text{cap}_4(h(G_i), h(E_i)))^{3/4} \end{aligned}$$

for $c' = c^{-1/4} K^{3/4} N(h, U_k)^{3/4}$.

The proof is nearly complete if we find a constant c'' (which is allowed to depend on z , k and h) such that

$$(4.28) \quad \text{cap}_4(h(G_i), h(E_i)) \leq \frac{c'' \Delta_i}{r}, \quad 1 \leq i \leq l.$$

By construction, $h(G_i) = N_r(E_i)$. This shows that

$$w : \mathbb{H} \rightarrow \mathbb{R}, \quad w(q) := \begin{cases} \frac{\text{dist}(q, \partial N_{r/2}(E_i))}{r/2}, & w \in N_{r/2}(\beta_z), \\ 0, & w \in \mathbb{H} \setminus N_r(\beta_z), \end{cases}$$

is admissible for $\text{cap}_4(h(G_i), h(E_i))$. Indeed, by definition of the metric neighborhood, it follows that $w(q) \geq 1$ for $q \in E_i$; and w vanishes in a neighborhood of $\partial N_r(E_i)$. By the 1-Lipschitz continuity of $\text{dist}(\cdot, C)$,

$$|\nabla_H \text{dist}(\cdot, C)| \leq 1$$

for any compact set C . Thus

$$\text{cap}_4(h(G_i), h(E_i)) \leq \int_{N_r(E_i)} |\nabla_H w(q)|^4 d\mu_{\mathbb{H}}(q) \leq 2^4 \frac{\mu_{\mathbb{H}}(N_r(E_i))}{r^4} \leq c'' \frac{\Delta_i r^3}{r^4},$$

for some constant c'' which does not depend on δ and the choice of I_1, \dots, I_l . This yields (4.28). The proof of the absolute continuity of α concludes by (4.27) as in [74]. It is then straightforward to verify that $h^\# \Psi$ is in ACL , see the arguments in [40] or [74]. \square

Combining the results of this subsection with the Darboux theorem and using Lemma 4.7.2, we deduce the following theorem.

Theorem 4.7.6. *Let $f : \mathbb{H} \rightarrow N$ be a quasiregular map from the Heisenberg group to a smooth sub-Riemannian contact 3-manifold N . If u is a 4-harmonic function, then $w := u \circ f$ is $f^\# \mathcal{A}$ -harmonic, where \mathcal{A} is the standard operator of type 4. An analogous statement holds for supersolutions.*

Proof. We fix $x \in \mathbb{H}$ and a Darboux chart ϕ mapping an open set of \mathbb{H} to an open neighborhood of $f(x)$ in N . We then write locally $u \circ f = v \circ h$ where $v = u \circ \phi$ and $h = \phi^{-1} \circ f$. By Proposition 4.7.3, v is $\phi^\# \mathcal{A}$ -harmonic. By Proposition 4.7.4, $w = v \circ h$ is $(\phi^{-1} \circ f)^\# \phi^\# \mathcal{A}$ -harmonic. Finally, Lemma 4.7.2 implies that $(\phi^{-1} \circ f)^\# \phi^\# \mathcal{A} = (\phi \circ \phi^{-1} \circ f)^\# \mathcal{A} = f^\# \mathcal{A}$. The proof is complete. \square

4.8. Application to quasiregular mappings

Applying the following theorem with $N = \widetilde{M}$ concludes the proof of Theorem 1.2.2 by the argument given in the introduction.

Theorem 4.8.1. *If $f : \mathbb{H} \rightarrow N$ is quasiregular and N is 4-hyperbolic, then f is constant.*

Proof. The proof rests on the fact that, unlike N , the Heisenberg group is 4-parabolic. If f is surjective, we arrive at a contradiction by the morphism property (Theorem 4.7.6) and Theorems 4.6.4 and 4.6.5.

If f is not constant, but $f(\mathbb{H})$ is a strict subset of N , then we choose a point y on the boundary of $f(\mathbb{H})$. The 4-hyperbolicity of N allows us to select a positive Green's function $G = G(\cdot, y)$ for the 4-Laplacian on N (see Theorem 4.6.5). By Theorem 4.7.6, $G \circ f$ would be a positive nonconstant solution to an \mathcal{A} -harmonic equation for some operator \mathcal{A} of type 4 on \mathbb{H} . However, such solutions cannot exist (see Theorem 4.6.4). \square

A. Calculus for horizontal derivatives

In this section we discuss chain rules for horizontal derivatives that are used especially in connection with the morphism property.

Proposition A.0.1. *Let \mathbb{H} be the standard sub-Riemannian Heisenberg group, and suppose that (M, HM, g_M) is a contact sub-Riemannian manifold. Let U be a domain in \mathbb{H} , V a domain in M , and $f: U \rightarrow V$ a continuous function. Assume further that $u: V \rightarrow \mathbb{R}$ is smooth and $\Psi: V \rightarrow \mathbb{H}$ is a smooth chart so that $X(\Psi_i \circ f)$ and $Y(\Psi_i \circ f)$, $i \in \{1, 2, 3\}$, exist in the weak sense and belong to L^p_{loc} for some $1 \leq p < \infty$. Then the weak derivatives $X(u \circ f)$, $Y(u \circ f)$ exist, belong to L^p_{loc} and are given almost everywhere by the following formulae:*

$$X(u \circ f)(p) = \sum_{i=1}^3 \frac{\partial(u \circ \Psi^{-1})}{\partial \Psi_i}(\Psi(f(p))) X(\Psi_i \circ f)(p),$$

$$Y(u \circ f)(p) = \sum_{i=1}^3 \frac{\partial(u \circ \Psi^{-1})}{\partial \Psi_i}(\Psi(f(p))) Y(\Psi_i \circ f)(p).$$

Proof. We use charts to write locally $u \circ f = (u \circ \Psi^{-1}) \circ (\Psi \circ f)$. Note that $u \circ \Psi^{-1}$ is a smooth function on a domain in \mathbb{H} and thus the usual derivative $(u \circ \Psi^{-1})_*$ exists everywhere in the domain of $u \circ \Psi^{-1}$. Moreover, $u \circ \Psi^{-1}$ is locally Lipschitz both with respect to the Euclidean and the sub-Riemannian metric on \mathbb{H} and thus $u \circ \Psi^{-1}$ is ACL (see Remark 3.2.3).

Concerning the factor $\Psi \circ f$, we denote

$$\gamma_{\Psi \circ f, p}(s) := \Psi(f(p \exp(sX))).$$

By assumption and Remark 3.2.3, for almost every p in the plane transversal to X , the tangent vector $\dot{\gamma}_{\Psi \circ f, p}(s)$ exists for almost every s and it equals

$$X(\Psi \circ f)(p \exp(sX)) = \sum_{i=1}^3 X(\Psi_i \circ f)(p \exp(sX)) \partial \Psi_i.$$

Thus, for

$$\gamma_{u \circ f, p}(s) := u(f(p \exp(sX))) = (u \circ \Psi^{-1}) \circ (\Psi(f(p \exp(sX)))),$$

we obtain

$$X(u \circ f)(p \exp(sX)) = \dot{\gamma}_{u \circ f, p}(s) = (u \circ \Psi^{-1})_{*, \Psi(f(p \exp(sX)))} X(\Psi \circ f)(p \exp(sX)),$$

for almost every s , and analogously for X replaced by Y . This yields the formula for the chain rule. The fact that $X(u \circ f)$ and $Y(u \circ f)$ belong to L^p_{loc} is immediate

from the corresponding property of the horizontal derivatives of $\Psi_i \circ f$ and the fact that $u \circ \Psi^{-1}$ is smooth. Finally, we refer again to Remark 3.2.3 to deduce that the horizontal derivatives exist also in a weak sense. \square

Next we consider the case where the function u is not smooth but only belongs to some Sobolev space. In this case we have to impose a stronger assumption on the map f , namely we will assume that it is quasiregular. For our purposes it suffices to discuss mappings between domains in the Heisenberg group.

Proposition A.0.2. *Let $f: \Omega \rightarrow \Omega'$ be a nonconstant quasiregular map between domains in \mathbb{H} , and let $u: \Omega' \rightarrow \mathbb{R}$ be an $HW_{\text{loc}}^{1,4}$ -function. Then $u \circ f$ belongs to $HW_{\text{loc}}^{1,4}$. Moreover,*

$$\nabla_H(u \circ f)(p) = (D_H f(p))^T \nabla_H u(f(p)), \quad a.e. \ p \in \Omega.$$

Proof. The proof goes along the same lines as in the Euclidean case, ([40], Theorem 14.28), using the fact that quasiregular mappings on the Heisenberg group are weakly contact and differentiable almost everywhere in the sense of Pansu [63]. Moreover, as shown in §5 of [17], the Pansu differential agrees almost everywhere with the map that is obtained by extending $D_H f$ to a homomorphism of the Lie algebra of \mathbb{H} . \square

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