On the factorization of iterated polynomials

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Abstract. Let \( \mathbb{F}_q \) be the finite field with \( q \) elements and \( f, g \in \mathbb{F}_q[x] \) be polynomials of degree at least one. This paper deals with the asymptotic growth of certain arithmetic functions associated to the factorization of the iterated polynomials \( f(g^{(n)}(x)) \) over \( \mathbb{F}_q \), such as the largest degree of an irreducible factor and the number of irreducible factors. In particular, we provide significant improvements on the results of D. Gómez-Pérez, A. Ostafe and I. Shparlinski (2014).

1. Introduction

For a field \( K \) and a polynomial \( f \in K[x] \), let \( f^{(0)}(x) = x \) and \( f^{(n)}(x) = f(f^{(n-1)}(x)) \) for \( n \geq 1 \). The polynomial \( f^{(n)}(x) \) is called the \( n \)-th iterate of \( f \). Algebraic aspects of \( f^{(n)}(x) \) have been extensively considered by many authors in the past few years [1], [2], [7], [8], [10]; in most of the cases, the main object of study is the class of polynomials \( f \in K[x] \) for which its iterates \( f^{(n)}(x) \) are all irreducible over \( K \). Such polynomials are called stable. In the case \( K \) a finite field, a recent work [6] extends the notion of stability to finite sets of polynomials \( \{f_1, \ldots, f_r\} \). More specifically, the authors explore the sets of quadratic polynomials \( C = \{f_1, \ldots, f_r\} \) over \( \mathbb{F}_q \) such that any polynomial obtained by compositions of elements in \( C \) is irreducible.

In [5], the authors explore further arithmetic properties of the factorization of \( f^{(n)}(x) \) when \( K = \mathbb{F}_q \) is the finite field with \( q \) elements. They obtain lower bounds on the values of some arithmetic functions related to the factorization of \( f^{(n)}(x) \) such as the number of irreducible factors and the degree of its squarefree part.

Theorem 1.1. For any fixed \( \varepsilon > 0 \) and a positive integer \( d \) such that \( \gcd(d, q) = 1 \), all but \( o(q^{d+1}) \) polynomials \( f \in \mathbb{F}_q[x] \) of degree \( d \) satisfy the following:

Mathematics Subject Classification (2010): Primary 12E05; Secondary 37P05.
Keywords: Dynamics over finite fields, factorization, irreducible polynomials.
(i) If \( \Delta_n(f) \) is the degree of the squarefree part of \( f^{(n)}(x) \), then \( \Delta_n(f) \gg n^{1-\varepsilon} \). Moreover, the implicit constant in the previous inequality can be taken uniformly on \( q \).

(ii) If \( r_n(f) \) is the number of distinct irreducible factors of \( f^{(n)}(x) \), we have that \( r_n(f) \geq (0.5 + o(1)) \cdot n \) when \( n \to \infty \) provided that

\[
\frac{1}{2 \log d} - \varepsilon \right) \log q.
\]

Additionally, if \( f \in \mathbb{F}_q[x] \) has degree \( d \) with \( \gcd(d, q) = 1 \) and \( f \) is not of the form \( ax^d \), then the largest degree \( D_n(f) \) of an irreducible factor of \( f^{(n)}(x) \) satisfies

\[
D_n(f) > \frac{(n-1) \log d - \log 2}{\log q} \approx n.
\]

In particular, the functions \( \Delta_n(f) \), \( r_n(f) \) and \( D_n(f) \) grow (roughly) at least linearly with respect to \( n \) under not too restricted conditions. It is worth mentioning that the exclusion of \( o(q^{d+1}) \) polynomials in Theorem 1.1 is due to the character sums techniques employed in its proof.

In this paper, we explore similar questions to the ones discussed in [5] over a more general class of iterated polynomials. For polynomials \( f, g \in \mathbb{F}_q[x] \), we define the following sequence of polynomials over \( \mathbb{F}_q \):

\[
P_0[f, g](x) = f(x), \quad P_n[f, g](x) = f(g^{(n)}(x)), \quad n \geq 1.
\]

We call \( P_n[f, g](x) \) the \( n \)-th \( g \)-iterate of \( f \). Motivated by Question 18.9 in [3], we introduce the following arithmetic functions associated to the factorization of \( P_n[f, g](x) \) over \( \mathbb{F}_q \).

**Definition 1.2.** For \( f, g \in \mathbb{F}_q[x] \), let

\[
P_n[f, g](x) = p_1, n(x)^{e_1, n} \cdots p_{N_n, n}(x)^{e_{N_n, n}}, \quad n \geq 0,
\]

be the factorization of \( P_n[f, g](x) \) into irreducible polynomials over \( \mathbb{F}_q \). For each \( n \geq 0 \), we consider the following arithmetic functions:

(a) \( E_{f, g}(n) := \max_{1 \leq i \leq N_n} e_{i, n} \) is the largest multiplicity of a root of \( P_n[f, g](x) \) (recall that finite fields are perfect fields);

(b) \( e_{f, g}(n) := \min_{1 \leq i \leq N_n} e_{i, n} \) is the smallest multiplicity of a root of \( P_n[f, g](x) \);

(c) \( \Delta_{f, g}(n) := \deg(p_{1, n}(x) \cdots p_{N_n, n}(x)) \) is the degree of the squarefree part of \( P_n[f, g](x) \);

(d) \( M_{f, g}(n) := \max_{1 \leq i \leq N_n} \deg(p_{i, n}(x)) \) is the largest degree of an irreducible factor of \( P_n[f, g](x) \) over \( \mathbb{F}_q \);

(e) \( m_{f, g}(n) := \min_{1 \leq i \leq N_n} \deg(p_{i, n}(x)) \) is the smallest degree of an irreducible factor of \( P_n[f, g](x) \) over \( \mathbb{F}_q \);

(f) \( N_{f, g}(n) := N_n \) is the number of distinct irreducible factors of \( P_n[f, g](x) \) over \( \mathbb{F}_q \);

(g) \( A_{f, g}(n) := \Delta_{f, g}(n)/N_{f, g}(n) \) is the average degree of the distinct irreducible factors of \( P_n[f, g](x) \) over \( \mathbb{F}_q \).
If \( f \) is a constant or \( g \) has degree at most one, the arithmetic functions above are all constant. For this reason, we may restrict ourselves to the case \( \deg(f) \geq 1 \) and \( \deg(g) \geq 2 \). Our results have implications in the study of the asymptotic behaviour of the functions in Definition 1.2 and, in particular, they extend some results in Theorem 1.1. Most notably, in contrast to the lower bound \( \Delta_n(f) \gg n^{1-\varepsilon} \) given in Theorem 1.1 we show that, up to genuine exceptions, \( \Delta_n(f) \) has exponential growth. Furthermore, the condition \( \gcd(\deg(f), q) = 1 \) is replaced by a less restrictive one. For more details, see Corollary 2.5.

The structure of the paper is given as follows. In Section 2, we present the main results and provide useful remarks. Section 3 is devoted to provide some definitions and background material that is used along the way. In Sections 4 and 5 we prove our main results. Finally, in Section 6, we present conclusions and some open problems.

2. Main results

In this section, we provide the main results of this paper. We start with the following definition.

**Definition 2.1.** Let \( p \) be the characteristic of \( \mathbb{F}_q \) and consider \( f, g \in \mathbb{F}_q[x] \) such that \( \deg(f) = k \geq 1 \) and \( \deg(g) = D \geq 2 \).

- (a) The \( p \)-reduction of \( g \) is the unique polynomial \( G \in \mathbb{F}_q[x] \) such that \( g(x) = G(x)^p^h \) for some \( h \geq 0 \) and the formal derivative \( G'(x) \) of \( G(x) \) is not the zero polynomial.

- (b) The pair \( (f, g) \) is \( p \)-critical if the \( p \)-reduction of \( g(x) \) is a linear polynomial, i.e., \( g(x) = ax^p^h + b \) for some \( h \geq 0 \) and \( a, b \in \mathbb{F}_q \).

- (c) The pair \( (f, g) \) is critical if there exist elements \( \alpha, \beta \) and \( \gamma \) in \( \mathbb{F}_q \) such that \( f(x) = \beta(x - \alpha)^k \) and \( g(x) = \gamma(x - \alpha)^D + \alpha \).

A trivial upper bound for all the arithmetic functions in Definition 1.2 is \( kD^m \), where \( k = \deg(f) \) and \( D = \deg(g) \). It is a routine exercise to check that, if \( (f, g) \) is critical or \( p \)-critical, \( E_{f,g}(n) = E_{f,g}(0) \cdot D^n \), \( e_{f,g}(n) = e_{f,g}(0) \cdot D^n \) and all the other arithmetic functions are constant, equal to their values at \( n = 0 \). So, from now and on, we take the following assumption:

The pair \( (f, g) \) is neither critical nor \( p \)-critical.

The main results of this paper provide finer information on the growth the arithmetic functions related to \( P_n[f, g](x) \) and can be stated as follows.

**Theorem 2.2.** Let \( f, g \in \mathbb{F}_q[x] \) be polynomials of degrees \( k \geq 1 \) and \( D \geq 2 \), respectively, such that the pair \( (f, g) \) is neither critical nor \( p \)-critical. Let \( G \in \mathbb{F}_q[x] \) be the \( p \)-reduction of \( g \), write \( g(x) = G(x)^p^h \) and set \( d = \deg(G) > 1 \). Then the following hold:

- (i) \( e_{f,g}(n) \geq p^{nh} \) for \( n \geq 0 \).
Effectively, such that the pair following hold.

**Theorem 2.4.** Let $A$ or $L$. Reis

result. “roughly sharp” in the sense that the bounds on the “growth-type” of the functions

Moreover, the following are equivalent

In particular, there exists $\Delta_{f,g} > 0$ such that $\Delta_{f,g}(n) \geq C_{f,g} \cdot d^n$ for $n \geq 0$.

In particular, the pair $(f,g)$ is neither critical nor $p$-critical. Then $M_{f,g}(n) \gg n$.

Our second result concerns the degree of the irreducible factors of $P_n[f,g](x)$.

**Theorem 2.4.** Let $f, g \in \mathbb{F}_q[x]$ be polynomials of degrees $k \geq 1$ and $D \geq 2$, respectively, such that the pair $(f,g)$ is neither critical nor $p$-critical. Then $M_{f,g}(n) \gg n$.

Moreover, the following are equivalent:

(i) there exists a constant $c > 0$ such that $m_{f,g}(n) \leq c$ for any $n \geq 0$;

(ii) the sequence $m_{f,g}(n)$ is eventually constant;

(iii) $f$ has a root $\beta \in \mathbb{F}_q$ such that $g^{(i)}(\beta) = \beta$ for some $i \geq 1$.

Following the notation of Theorem 1.1, we observe that, for a generic polynomial $f \in \mathbb{F}_q[x]$ of degree at least one, $A_{f,f}(n) \leq M_{f,f}(n) = D_{n+1}(f)$, $N_{f,f}(n) = r_{n+1}(f)$ and $\Delta_{f,f}(n) = \Delta_{n+1}(f)$ for any $n \geq 0$. In addition, the pair $(f,f)$ is critical or $p$-critical according to $f(x)$ is of the form $ax^k$ or of the form $ax^k + b$, respectively. Therefore, if $\deg(f) > 1$ is relatively prime with $q$, the pair $(f,f)$ cannot be $p$-critical and it is critical if and only if $f$ is of the form $ax^k$. In particular, Theorems 2.2 and 2.4 and Remark 2.3 provide extensions and significant improvements on the results in Theorem 1.1.

**Corollary 2.5.** Suppose that $f \in \mathbb{F}_q[x]$ has degree $D > 1$ and is not of the form $ax^k$ or of the form $ax^k + b$. Let $d > 1$ be the degree of the $p$-reduction of $f$. Then the following hold:

(i) $\Delta_n(f) \approx d^n$ and so $\Delta_n(f)$ grows exponentially;

(ii) $D_n(f) \gg n$;

(iii) there exists $C_f > 0$ such that, for any $n \geq 0$, either $r_n(f) \geq C_f \cdot d^{n/2}$ or $D_n(f) \geq C_f \cdot d^{n/2}$.

We emphasize that the results of Theorems 2.2 and 2.4 and Remark 2.3 are “roughly sharp” in the sense that the bounds on the “growth-type” of the functions provided in such theorem can be reached. More specifically, we have the following result.

**Theorem 2.6.** Let $f \in \mathbb{F}_q[x]$ be any polynomial of degree at least one. Then the following hold.

(i) If $f$ is not of the form $ax^k$, then for any nonnegative integer $t$, there exist infinitely many polynomials $g$ such that $N_{f,g}(n) \approx n^t$ and $M_{f,g}(n) \approx \deg(g)^n$. 
Let $3.1$. Notations and basic results

Throughout this paper, we fix $\mathbb{F}_q$ a finite field of characteristic $p$. In this section, we introduce some useful definitions and provide background material that is used along the way. We further show that Theorems 2.2 and 2.4 need to be proved only for a restricted class of pairs $(f,g)$.

3. Preliminaries

Throughout this paper, we fix $\mathbb{F}_q$ a finite field of characteristic $p$. In this section, we introduce some useful definitions and provide background material that is used along the way. We further show that Theorems 2.2 and 2.4 need to be proved only for a restricted class of pairs $(f,g)$.

3.1. Notations and basic results

Let $\overline{\mathbb{F}}_q$ be the algebraic closure of $\mathbb{F}_q$. We usually denote by $Q$ a power of $q$ so $\mathbb{F}_Q$ is an extension of $\mathbb{F}_q$.

Definition 3.1. If $F \in \mathbb{F}_Q[x]$, we consider the following functions:

(a) $\nu(F)$ is the largest multiplicity of a root of $F$ over $\overline{\mathbb{F}}_q$;
(b) $\nu^*(F)$ is the smallest multiplicity of a root of $F$ over $\overline{\mathbb{F}}_q$;
(c) $M_Q(F)$ is the largest degree of an irreducible factor of $F$ over $\mathbb{F}_Q$;
(d) $m_Q(F)$ is the smallest degree of an irreducible factor of $F$ over $\mathbb{F}_Q$;
(e) $\Delta(F)$ is the degree of the squarefree part of $F$;
(f) $N_Q(F)$ is the number of irreducible factors of $F$ over $\mathbb{F}_Q$.

For any integer $i$, let $\sigma_i : \overline{\mathbb{F}}_q \to \overline{\mathbb{F}}_q$ be the $i$-th power of the Frobenius automorphism, i.e., $\sigma_i(\alpha) = \alpha^q$. For simplicity, $\sigma_i : \mathbb{F}_q[x] \to \mathbb{F}_q[x]$ also denotes the natural extension of $\sigma_i$ to the polynomial ring $\mathbb{F}_q[x]$, i.e., for $f \in \mathbb{F}_q[x]$ with $f(x) = \sum_{j=0}^ka_jx^j$, we have $\sigma_i(f) = \sum_{j=0}^k a_j^q x^j$. The following result is classical.

Lemma 3.2. A polynomial $f \in \mathbb{F}_q[x]$ is irreducible if and only if $\sigma_i(f) \in \mathbb{F}_q[x]$ is irreducible for any $i \in \mathbb{Z}$.

Definition 3.3. For an element $\alpha \in \overline{\mathbb{F}}_q$, $\deg_Q(\alpha)$ denotes the degree of the minimal polynomial $m_\alpha, Q$ of $\alpha$ over $\mathbb{F}_Q$. Equivalently, $\deg_Q(\alpha)$ is the least positive integer $s$ such that $\alpha \in \mathbb{F}_Q^s$. 

(ii) There exist infinitely many polynomials $g$ such that $M_{f,g}(n) \approx n$.

(iii) If $q > 2$ is a prime power and $f$ writes as $f(x) = x^k \cdot F(x)$, where $F(x) = x^n + \sum_{i=0}^{m-1} a_i x^i$ with $m \geq 1$ and $a_0 \neq 0, 1$, then there exists a polynomial $g \in \mathbb{F}_q[x]$ such that $M_{f,g}(n) \geq 2^n$ and $N_{f,g}(n) \gg 2^n$.

We comment that the proof of Theorem 2.6 is constructive and, in particular, items (i), (ii) and (iii) are proved considering $g$ a monomial, a $q$-linearized polynomial (i.e., a polynomial of the form $\sum_{i=0}^{m} a_i x^i$) and a mix of them, respectively. Moreover, for $g \in \mathbb{F}_q[x]$ a monomial or $q$-linearized and generic irreducible $f \in \mathbb{F}_q[x]$, we obtain exact implicit formulas for all the functions given in Definition 1.2. This is done using the main results of [4] (the monomial case) and [11] (the linearized case). For more details, see Section 5.
Remark 3.4. If \( f \in \mathbb{F}_Q[x] \) is irreducible and \( \alpha \in \mathbb{F}_Q = \mathbb{F}_q \) is such that \( f(\alpha) = 0 \), then \( \deg_Q(\alpha) = \deg(f) \) and \( f(x) = a \cdot m_{\alpha,Q}(x) \) for some \( a \in \mathbb{F}_Q \).

3.2. Factorization of composed polynomials

Here we provide a finite field generalization of the well-known Capelli’s lemma, that gives a criterion on the irreducibility of composed polynomials \( f(g(x)) \) with \( f \) irreducible. This is done via the theory of spins of polynomials, introduced in [9].

We just state the results without proof since they are quite simple and are proved in Section 4 of [9]. If \( Q = p^m \), set \( \tau_{Q,j} = \sigma_{mj} \) for \( j \geq 0 \), i.e., \( \tau_{Q,j}(\alpha) = \alpha^{Q^j} \) for any \( \alpha \in \mathbb{F}_q \). Of course, \( \tau_j \) naturally extends to the polynomial ring \( \mathbb{F}_q[x] \). We start with the following definition.

Definition 3.5. For a polynomial \( f \in \mathbb{F}_Q[x] \), let \( s = s_Q(f) \) be the least positive integer such that all the coefficients of \( f \) lie in \( \mathbb{F}_Q^s \). We define the spin of \( f \) over \( \mathbb{F}_Q \) as

\[
S_Q(f) = \prod_{j=0}^{s_Q(f)-1} \tau_{Q,j}(f).
\]

The following result provides a way of obtaining the factorization of composed polynomials via spins.

Lemma 3.6 (See Lemmas 11 and 13 of [9]). Let \( g \in \mathbb{F}_Q[x] \) be a polynomial of degree at least one. For an element \( \lambda \in \mathbb{F}_Q \) with \( \deg_Q(\lambda) = s \), we have the following equality:

\[
S_Q(g(x) - \lambda) = \prod_{j=0}^{s-1} (g(x) - \lambda^{Q^j}).
\]

In addition, if \( f \in \mathbb{F}_Q[x] \) is irreducible of degree \( k \) and \( \alpha \in \mathbb{F}_Q \) is any of its roots, then the factorization of \( f(g(x)) \) over \( \mathbb{F}_Q \) is given by

\[
f(g(x)) = \lambda_f \cdot \prod_{R} S_Q(R(x)),
\]

where \( R \) runs over all the irreducible factors of \( g(x) - \alpha \) over \( \mathbb{F}_{Q^k} \) and \( \lambda_f \) is the leading coefficient of \( f \).

From the previous lemma, we obtain the following corollary.

Corollary 3.7. Let \( \alpha \in \mathbb{F}_Q \) be an element with \( \deg_Q(\alpha) = k \), and let \( g \in \mathbb{F}_Q[x] \) be a polynomial of degree at least one. If

\[
g(x) - \alpha = p_1(x)^{e_1} \cdots p_r(x)^{e_r},
\]

is the factorization of \( g(x) - \alpha \) over \( \mathbb{F}_{Q^k} \) then, for any \( 1 \leq i \leq r \), we have that \( \deg(S_Q(p_i(x))) = k \cdot \deg(p_i) \).
Proof. Let $f$ be the minimal polynomial of $\alpha$ over $\mathbb{F}_q$, hence $f$ is irreducible and $\deg(f) = k$. From Lemma 3.6, we have that

$$f(g(x)) = \prod_{i=0}^{r} S_Q(p_i(x))^{e_i}. \quad (3.1)$$

Since $p_i \in \mathbb{F}_q[x]$, it follows that $s_Q(p_i) \leq k$. In particular, from the definition, we have that

$$\deg(S_Q(p_i(x))) = s_Q(p_i(x)) \cdot \deg(p_i) \leq k \cdot \deg(p_i).$$

Taking degrees on equation (3.1) we obtain

$$k \cdot \deg(g) = \sum_{i=1}^{r} e_i \cdot \deg(S_Q(p_i(x))) \leq k \cdot \sum_{i=1}^{r} e_i \cdot \deg(p_i) = k \cdot \deg(g).$$

Therefore, we necessarily have the equality $\deg(S_Q(p_i(x))) = k \cdot \deg(p_i)$. \qed

Lemma 3.6 and Corollary 3.7 immediately give the following results.

**Lemma 3.8.** Let $G, F_1, F_2 \in \mathbb{F}_q[x]$ be polynomials of degree at least one. If $F_1$ and $F_2$ are relatively prime, then $F_1(G(x))$ and $F_2(G(x))$ are relatively prime.

**Proposition 3.9.** Let $f \in \mathbb{F}_q[x]$ be an irreducible polynomial of degree $k$ and let $\alpha \in \mathbb{F}_q$ be any of its roots, where $Q = q^k$. For any polynomial $g \in \mathbb{F}_q[x]$ of degree at least one and any $n \geq 0$, the following hold:

(i) $E_{f,g}(n) = \nu(g^{(n)}(x) - \alpha)$;
(ii) $e_{f,g}(n) = \nu^*(g^{(n)}(x) - \alpha)$;
(iii) $M_{f,g}(n) = k \cdot M_Q(g^{(n)}(x) - \alpha)$;
(iv) $m_{f,g}(n) = k \cdot m_Q(g^{(n)}(x) - \alpha)$;
(v) $\Delta_{f,g}(n) = k \cdot \Delta(g^{(n)}(x) - \alpha)$;
(vi) $N_{f,g}(n) = N_Q(g^{(n)}(x) - \alpha)$.

**Corollary 3.10.** For any polynomial $g \in \mathbb{F}_q[x]$ of degree at least two and any $\alpha$ in an extension $\mathbb{F}_k$ of $\mathbb{F}_q$, the function $m_{Q_k}(g^{(n)}(x) - \alpha)$ is nondecreasing.

The following lemma provides bounds for the values of $\nu$ and $\nu^*$ at iterated polynomials $g^{(n)}(x) - \alpha$.

**Lemma 3.11.** Let $\alpha$ be an element in $\mathbb{F}_q$, $g \in \mathbb{F}_q[x]$ be a polynomial of degree at least one and $n \geq 1$. The following hold:

\begin{align*}
\nu(g^{(n)}(x) - \alpha) &\leq \max_{\Gamma \in C_n(\alpha)} \prod_{\lambda \in \Gamma} \nu(g(x) - \lambda), \\
\nu^*(g^{(n)}(x) - \alpha) &\geq \min_{\Gamma \in C_n(\alpha)} \prod_{\lambda \in \Gamma} \nu^*(g(x) - \lambda),
\end{align*}

where $C_n(\alpha)$ comprises the n-tuples $\Gamma = (\gamma_1, \ldots, \gamma_n) \in (\mathbb{F}_q)^n$ such that $\gamma_1 = \alpha$ and, if $n \geq 2$, $g(\gamma_i) = \gamma_{i-1}$ for $2 \leq i \leq n$. 


Proof. We only consider the inequality (3.2) since the proof of inequality (3.3) is entirely similar. We proceed by induction on \( n \). The case \( n = 1 \) is trivial. Suppose that the result holds for an integer \( N \geq 1 \) and let \( n = N + 1 \). If \( g(x) - \alpha = \prod_{i=1}^{s} (x - \delta_i)^{n_i} \), where the \( \delta_i \) are pairwise distinct, we have that
\[
\nu(g^{(n)}(x) - \alpha) = \max_{1 \leq i \leq s} e_i \cdot \nu(g^{(N)}(x) - \delta_i).
\]
From the induction hypothesis and the trivial inequality \( e_i \leq \nu(g(x) - \alpha) \), we have that
\[
\nu(g^{(n)}(x) - \alpha) \leq \nu(g(x) - \alpha) \cdot \max_{\Gamma \in C_N(\delta_i), 1 \leq i \leq s} \prod_{\lambda \in \Gamma} \nu(g(x) - \lambda).
\]
It follows by the definition that the elements of \( C_n(\alpha) \) are exactly the sets of the form \( \{\alpha, \gamma_2, \ldots, \gamma_{n-1}\} \), where \( \{\gamma_2, \ldots, \gamma_{n-1}\} \in C_N(\delta_i) \) for some \( 1 \leq i \leq s \). In particular,
\[
\nu(g(x) - \alpha) \cdot \max_{\Gamma \in C_N(\delta_i), 1 \leq i \leq s} \prod_{\lambda \in \Gamma} \nu(g(x) - \lambda) = \max_{\Gamma \in C_n(\alpha)} \prod_{\lambda \in \Gamma} \nu(g(x) - \lambda),
\]
and the result follows. \( \square \)

3.3. A reduction of Theorems 2.2 and 2.4

Here we show that Theorems 2.2 and 2.4 need only to be proved for pairs \((f, g)\) such that \( f \) is irreducible.

Lemma 3.12. Let \( f, g \in \mathbb{F}_q[x] \) be polynomials such that \( \deg(f) \geq 1 \) and \( \deg(g) \geq 2 \). Suppose that the irreducible factorization of \( f \) over \( \mathbb{F}_q \) is
\[
f(x) = f_1(x)^{e_1} \cdots f_s(x)^{e_s}.
\]
Then the pair \((f, g)\) is critical (resp. \( p\)-critical) if and only if \((f_1, g)\) is critical (resp. \( p\)-critical) for any \( 1 \leq i \leq s \). In addition, for any \( n \geq 0 \), the following hold:
(a) \( E_{f,g}(n) = \max_{1 \leq i \leq s} \{e_i \cdot E_{f_i,g}(n)\} \) if \( n \geq 1 \);
(b) \( e_{f,g}(n) = \min_{1 \leq i \leq s} \{e_i \cdot e_{f_i,g}(n)\} \) if \( n \geq 1 \);
(c) \( M_{f,g}(n) = \max_{1 \leq i \leq s} \{M_{f_i,g}(n)\} \);
(d) \( m_{f,g}(n) = \min_{1 \leq i \leq s} \{m_{f_i,g}(n)\} \);
(e) \( \Delta_{f,g}(n) = \sum_{1 \leq i \leq s} \Delta_{f_i,g}(n) \);
(f) \( N_{f,g}(n) = \sum_{1 \leq i \leq s} N_{f_i,g}(n) \).

In particular, if Theorems 2.2 and 2.4 hold for the pairs \((f, g)\) with \( f \) irreducible, then they hold for any pair \((f, g)\).

Proof. The first statement follows directly from the definition of critical and \( p\)-critical pairs. Also, items (a–f) follow from Lemma 3.8 and the fact that
\[
P_n[f, g](x) = \prod_{i=1}^{s} P_n[f_i, g](x)^{e_i}, \quad n \geq 0.
\]
For the last statement we observe that, for a fixed \( f \), the number \( s \) is bounded by \( \deg(f) \), that does not depend on \( n \). In particular, from the bounds in the items (a–f) above, Theorems 2.2 and 2.4 hold for the pair \((f,g)\) whenever they hold for each pair \((f_i,g)\) with \(1 \leq i \leq s\).

\[ \square \]

4. Proof of Theorems 2.2 and 2.4

In this section we provide the proof of Theorems 2.2 and 2.4, that is divided in many parts. We observe that, from Lemma 3.12, we only need to consider pairs \((f,g)\) such that \( f \) is irreducible. So we take the following assumption:

The pair \((f,g)\) is neither critical nor \(p\)-critical and \( f \) is irreducible.

Before we proceed to the proof of Theorems 2.2 and 2.4, we comment on the main ideas that are employed. If \( f \in \mathbb{F}_q[x] \) has degree \( k \) and is irreducible, Proposition 3.9 entails that the arithmetic functions associated to \( f(g^{(i)}(x)) \) depend on the factorization of the polynomial \( g^{(i)}(x) - \alpha \) over \( \mathbb{F}_q \), where \( \alpha \) is any root of \( f \). So we focus on the factorization of polynomials of the form \( g^{(n)}(x) - \alpha \) in our statements and explain how they imply the main results. When studying the polynomials of the form \( g^{(n)}(x) - \alpha \), we will frequently look at the reversed \( g \)-orbit of \( \alpha \), that is, the set \( \{ \beta \in \mathbb{F}_q \mid g^{(i)}(\beta) = \alpha \text{ for some } i \geq 0 \} \).

We start with the following definition.

Definition 4.1. For a polynomial \( g \in \mathbb{F}_q[x] \), an element \( \lambda \in \mathbb{F}_q \) is said to be \( g \)-periodic if there exists a positive integer \( i \) such that \( g^{(i)}(\lambda) = \lambda \).

We have the following proposition.

Proposition 4.2. Let \( g \in \mathbb{F}_q[x] \) be a polynomial of degree \( D \geq 2 \) and \( \alpha \in \mathbb{F}_q \). Suppose that \( G \in \mathbb{F}_q[x] \) is the \( p \)-reduction of \( g \) with \( g(x) = G(x)^p \), where \( h \geq 0 \). If \( g(x) \) is not of the form \( a(x - \alpha)^D + \alpha \) or of the form \( ax^h + b \) then, for any \( n \geq 0 \), the following holds:

\[ p^n h \leq \nu^*(g^{(n)}(x) - \alpha) \leq \nu(g^{(n)}(x) - \alpha) \leq \sqrt{\frac{D}{D - 1}} \cdot \kappa_g^n \leq \sqrt{2} \kappa_g^n, \]

where \( \kappa_g = \sqrt{D(D - 1)} \).

Proof. The result is trivial for \( n = 0 \) and so we suppose that \( n \geq 1 \). We observe that, for any \( \lambda \in \mathbb{F}_q \), \( \nu(g(x) - \lambda) = p^h \cdot \nu(G(x) - \sigma_{-h}(\lambda)) \) and \( \nu^*(g(x) - \lambda) = p^h \cdot \nu^*(G(x) - \sigma_{-h}(\lambda)) \). In particular, the inequality

\[ p^n h \leq \nu^*(g^{(n)}(x) - \alpha) \leq \nu(g^{(n)}(x) - \alpha), \]

follows trivially from inequality (3.3). Following the notation of Lemma 3.11, \( C_n(\alpha) \) comprises the \( n \)-tuples

\[ \Gamma = (\gamma_1, \ldots, \gamma_n) \in (\mathbb{F}_q)^n, \]
such that \( \gamma_1 = \alpha \) and, if \( n \geq 2 \), \( g(\gamma_i) = \gamma_{i-1} \) for \( 2 \leq i \leq n \). From inequality (3.2), we have that

\[
\nu(g^{(n)}(x) - \alpha) \leq \max_{\gamma \in C_n(\alpha)} \left\{ \prod_{\lambda \in \Gamma} \nu(g(x) - \lambda) \right\}.
\]

(4.1)

We observe that \( \nu(g(x) - \lambda) \leq D = \deg(g) \), for any \( \lambda \in \mathbb{F}_q \).

**Claim 1.** The equality \( \nu(g(x) - \gamma) = D \) holds for at most one element \( \gamma \in \mathbb{F}_q \).

**Proof of Claim 1.** Since \( g(x) \) is not of the form \( ax^p + b \) and \( G \in \mathbb{F}_q[x] \) is the \( p \)-reduction of \( g \), we have that \( G \) has degree \( d := D/p^h \geq 2 \) and \( G'(x) \) is not the zero polynomial. Since \( g(x) = G(x)^{p^h} \), it suffices to prove that \( \nu(G(x) - \delta) = d \) for at most one element \( \delta \in \mathbb{F}_q \). However, if \( \nu(G(x) - \delta) = \nu(G(x) - \delta_0) = d \) with \( \delta \neq \delta_0 \), it follows that the polynomial \( G'(x) \) vanishes at two distinct elements of \( \mathbb{F}_q \) with multiplicity at least \( d - 1 \). Since \( G'(x) \) is not the zero polynomial, it follows that \( \deg(G'(x)) \geq 2(d - 1) \). However, \( \deg(G') \geq \deg(G) - 1 = d - 1 \) and so \( 2(d - 1) \leq d - 1 \), a contradiction since \( d \geq 2 \).

If it does not exist \( \gamma \in \mathbb{F}_q \) such that \( \nu(g(x) - \gamma) = D \), inequality (4.1) yields

\[
\nu(g^{(n)}(x) - \alpha) \leq (D - 1)^n < \sqrt{\frac{D}{D - 1} \cdot k_q^n}.
\]

On the contrary, let \( B \in \mathbb{F}_q \) be such that \( \nu(g(x) - B) = D \). From Claim 1, \( \nu(g(x) - \gamma) \leq D - 1 \) whenever \( \gamma \neq B \). For any \( \Gamma = (\gamma_1, \ldots, \gamma_n) \in C_n(\alpha) \) and \( \gamma \in \mathbb{F}_q \), let \( e(\gamma, \Gamma) \) be the number of indexes \( 1 \leq i \leq n \) for which \( \gamma = \gamma_i \). In particular, inequality (4.1) yields

\[
\nu(g^{(n)}(x) - \alpha) \leq \max_{\Gamma \in C_n(\alpha)} \left\{ D^{e(B, \Gamma)} \cdot (D - 1)^{n - e(B, \Gamma)} \right\}.
\]

(4.2)

Therefore, if \( e(B, \Gamma) \leq 1 \) for any \( n \geq 1 \) and any \( \Gamma \in C_n(\alpha) \), inequality (4.2) yields

\[
\nu(g^{(n)}(x) - \alpha) \leq D(D - 1)^{n - 1} < \sqrt{\frac{D}{D - 1} \cdot k_q^n},
\]

as desired. Otherwise, there exist \( N \geq 2 \) and \( \Gamma_0 \in C_N(\alpha) \) such that \( B \in \Gamma_0 \) and \( e(B, \Gamma_0) \geq 2 \). If \( B = \gamma_i = \gamma_j \in \Gamma_0 \) with \( 1 \leq i < j \leq N \), then \( g^{(j-i)}(B) = B \) and \( g^{(i)}(B) = \alpha \) and so \( B, \alpha \) are \( g \)-periodic and lie in the same orbit. Additionally, since \( \nu(g(x) - B) = D \), there exist \( a, b \in \mathbb{F}_q \) such that \( g(x) = a(x-b)^D + B \). Let \( M \) be the least positive integer such that \( g^{(M)}(B) = B \).

**Claim 2.** For any \( n \geq 1 \) and any \( \Gamma \in C_n(\alpha) \), we have that

\[
e(B, \Gamma) \leq \frac{(n - 1)}{M} + 1.
\]

**Proof of Claim 2.** Set \( s = e(B, \Gamma) \); if \( s = 0, 1 \) the result is trivial. Otherwise, \( s \geq 2 \) and following the proof that \( B \) is periodic, we see that there exist integers
On the factorization of iterated polynomials

\[0 \leq i_1 < \cdots < i_s \leq n - 1\] such that \(g^{(i_l - i_{l-1})}(B) = B\) for any \(2 \leq l \leq s\). In particular, \(i_l - i_{l-1} \geq M\) for any \(2 \leq l \leq s\) and so
\[n - 1 \geq i_s - i_1 = \sum_{2 \leq l \leq s} (i_l - i_{l-1}) \geq (s - 1)M,\]
proving the claim.

From \(e(B, \Gamma) \leq (n - 1)/M + 1\) and inequality (4.2) we have that, for any \(n \geq 1\), the following holds:
\[\nu(g^n(x) - \alpha) \leq D \cdot D(n-1)/M(D-1)^{(n-1)(M-1)/M} \leq \sqrt{D \over D - 1} \cdot \kappa^n,\]
provided that \(M \geq 2\). In particular, we only need to consider the case \(M = 1\). However, in this case, \(B = g(B) = a(B - b)D + B\) and so \(b = B\) and \(g(x) = a(x - B)D + B\). Additionally, since \(\alpha\) is \(g\)-periodic and lies in the same orbit of \(B\), we have that \(\alpha = B\). In other words, \(g(x) = a(x - \alpha)D + \alpha\), contradicting our hypothesis. \(\square\)

We observe that, from Proposition 4.2, items (i) and (ii) of Theorem 2.2 are easily deduced. In fact, from Lemma 3.12, we only need prove them for \(f\) an irreducible polynomial. If \(f\) has degree \(k\) and \(\alpha \in \mathbb{F}_q^*\) is any of its roots, Proposition 3.9 entails that \(e_{f,g}(n) = \nu^*(g^n(x) - \alpha)\) and \(E_{f,g}(n) = \nu(g^n(x) - \alpha)\). Moreover, the assumption in Theorem 2.2 that the pair \((f, g)\) is neither critical nor \(p\)-critical directly implies that \(g\) and \(\alpha\) satisfy the conditions in Proposition 4.2.

We proceed to the proof of item (iii) of Theorem 2.2 and the comment thereafter. The key idea is to prove that, if \(g\) is not of the form \(ax^b + \beta(x - \alpha)^D + \alpha\), there exists an element \(\gamma\) in the reversed \(g\)-orbit
\[\{\beta \in \mathbb{F}_q \mid g^{(i)}(\beta) = \alpha \text{ for some } i \geq 0\}\]
of \(\alpha\) such that \(\Delta(g^n(x) - \gamma) \geq \deg(G)^n\) for any \(n \geq 0\), where \(G\) is the \(p\)-reduction of \(g\). If \(g^{(i)}(\gamma) = \alpha\), the polynomial \(g^n(x) - \gamma\) divides \(g^{(n+i)}(x) - \alpha\) for any \(n \geq 0\). The latter implies that \(\Delta(g^n(x) - \alpha) \gg \deg(G)^n\). These observations are compiled in the following lemma.

**Lemma 4.3.** Let \(\alpha \in \mathbb{F}_q\) and let \(g \in \mathbb{F}_q[x]\) be a polynomial of degree \(D \geq 2\) such that \(g\) is not of the form \(ax^b + \beta\) or \(a(x - \alpha)^D + \alpha\). Let \(G \in \mathbb{F}_q[x]\) be the \(p\)-reduction of \(g\), \(d = \deg(G) > 1\) and write \(D = d \cdot p^h\) with \(h \geq 0\). Then there exists a positive integer \(i\) and an element \(\gamma \in \mathbb{F}_q\) such that \(g^{(i)}(\gamma) = \alpha\) and \(\nu(g^n(x) - \gamma) = p^n\)
for any \(n \geq 0\). In particular, there exists a constant \(C_{\alpha, g} > 0\) such that
\[d^n,\]
for any \(n \geq 0\) and so \(\Delta(g^n(x) - \alpha) \approx d^n\).

**Proof.** First, we claim that there exists a positive integer \(j = j(\alpha)\) and an element \(\lambda \in \mathbb{F}_q\) such that \(\lambda\) is not \(g\)-periodic and \(g^{(j)}(\lambda) = \alpha\). Proceeding by contradiction, suppose that all the roots of \(g(x) - \alpha\) are \(g\)-periodic and, if \(\eta\) is any root of \(g(x) - \alpha\), all the roots of \(g(x) - \eta\) are also \(g\)-periodic. We observe that, for any \(\beta \in \mathbb{F}_q\), at
most one root of the polynomial $g(x) - \beta$ is $g$-periodic. In particular, the latter implies that $g(x) - \alpha = a_1(x - \eta)^D$ and $g(x) - \eta = a_2(x - \eta_0)^D$ for some $\eta, \eta_0 \in \mathbb{F}_q$. In other words, $\nu(g(x) - \alpha) = \nu(g(x) - \eta) = D = \deg(g)$. From Claim 1 in Proposition 4.2, we conclude that $\alpha = \eta$ and so $g(x)$ is of the form $a(x - \alpha)^D + \alpha$, contradicting our hypothesis.

Let $\lambda \in \mathbb{F}_q$ and $j > 0$ be such that $g^{(j)}(\lambda) = \alpha$ and $\lambda$ is not $g$-periodic. Since $G$ is such that $G'(x)$ is not the zero polynomial, it follows that the set of roots of $G'(x)$ is finite. It is direct to verify that, for any $m \geq 0$, no root of $g^{(m)}(x) - \lambda$ is $g$-periodic. In particular, the polynomials $g^{(m)}(x) - \lambda$ with $m \geq 0$ are pairwise relatively prime. Therefore, there exists a positive integer $M = M(\lambda)$ such that the polynomial $g^{(k)}(x) - \lambda$ is relatively prime with $G'(x)$ for any $k \geq M$. Let $\gamma$ be any root of $g^{(M)}(x) - \lambda$.

Claim 1. For any $n \geq 0$, $\nu(g^{(n)}(x) - \gamma) = p^{nh}$.

Proof of Claim 1. From Proposition 4.2 and inequality (3.2), we have that

$$p^{nh} \leq \nu(g^{(n)}(x) - \gamma) \leq \max_{\Gamma \in C_n(\gamma)} \left\{ \prod_{\beta \in \Gamma} \nu(g(x) - \beta) \right\} = p^{nh} \cdot \max_{\Gamma \in C_n(\gamma)} \left\{ \prod_{\beta \in \Gamma} \nu(G(x) - \sigma_{-h}(\beta)) \right\},$$

where $C_n(\gamma)$ is as in Lemma 3.11. We observe that, if there exists $n \geq 0$, $\Gamma \in C_n(\gamma)$ and $\beta \in \Gamma$ such that $\nu(G(x) - \sigma_{-h}(\beta)) > 1$, then there exists an element $\beta_0 \in \mathbb{F}_q$ such that $G(\beta_0) = \sigma_{-h}(\beta)$ and $G'(\beta_0) = 0$. However, $g(\beta_0) = \beta \in C_n(\gamma)$ and so $\beta_0$ is a root of $g^{(i)}(x) - \gamma$ for some $0 \leq t \leq n + 1$. The latter implies that $\beta_0$ is a root of $g^{(t+M)}(x) - \lambda$, which is relatively prime with $G'(x)$ since $t + M \geq M$. But this is a contradiction with the equality $G'(\beta_0) = 0$. In particular, for any $n \geq 0$, we have that

$$\max_{\Gamma \in C_n(\gamma)} \left\{ \prod_{\beta \in \Gamma} \nu(G(x) - \sigma_{-h}(\beta)) \right\} = 1,$$

proving the claim.

Therefore, for $i = j + M$, $g^{(i)}(\gamma) = \alpha$ and $\nu(g^{(n)}(x) - \gamma) = p^{nh}$ for any $n \geq 0$. We observe that, since $g^{(i)}(\gamma) = \alpha$, $g^{(n)}(x) - \gamma$ divides $g^{(n+1)}(x) - \alpha$ and so we have that

$$\Delta(g^{(n)}(x) - \alpha) \geq \Delta(g^{(\chi(n))}(x) - \gamma),$$

where $\chi(n) = \max\{n - i, 0\}$. In addition, we have the trivial inequality

$$\Delta(g^{(m)}(x) - \gamma) \geq \frac{\deg(g^{(n)}(x) - \gamma)}{\nu(g^{(m)}(x) - \gamma)} = d_m,$$

for any $m \geq 0$ and so inequality (4.3) holds with $C_{\alpha, q} = 1/d^i$. To finish the proof we observe that, from Proposition 4.2, $\nu^* (g^{(n)}(x) - \alpha) \geq p^{nh}$ and so

$$\Delta(g^{(n)}(x) - \alpha) \leq \frac{\deg(g^{(n)}(x) - \alpha)}{\nu^* (g^{(n)}(x) - \alpha)} \leq d^n.$$

In particular, $\Delta(g^{(n)}(x) - \alpha) \approx d^n$. \qed
We observe that Lemma 4.3 immediately implies item (iii) of Theorem 2.2 and the comment thereafter. In fact, from Lemma 3.12, we only need to consider the pairs \((f, g)\) as in Theorem 2.2 with \(f \in \mathbb{F}_q[x]\) irreducible. Moreover, if \(f\) is irreducible of degree \(k\) and \(\alpha \in \mathbb{F}_q^*/\) is any of its roots, Proposition 3.9 entails that 
\[
\Delta_{f,g}(n) = k \cdot \Delta(g^{(k)}(x) - \alpha).
\]
Since the pair \((f, g)\) is neither critical nor \(p\)-critical, we have that \(\alpha\) and \(g\) satisfy the conditions of Lemma 4.3 and so \(\Delta(g^{(k)}(x) - \alpha) \approx d^n\), where \(d\) is the degree of the \(p\)-reduction \(G \in \mathbb{F}_q[x]\) of \(g\).

### 4.1. On the degree and number of irreducible factors of \(P_n[f, g](x)\)

Here we provide the proof of Theorem 2.4. We start with the bound on \(M_{f,g}(n)\).

**Lemma 4.4.** Let \(f, g \in \mathbb{F}_q[x]\) be polynomials of degrees \(k \geq 1\) and \(d \geq 2\), respectively, such that \(f\) is irreducible and the pair \((f, g)\) is neither critical nor \(p\)-critical. Then \(M_{f,g}(n) \gg n\).

**Proof.** Let \(\alpha \in \mathbb{F}_q\) be any root of \(f\). From Proposition 3.9, we only need to prove that \(M_Q(g^{(n)}(x) - \alpha) \gg n\) for \(Q = q^k\). If we set \(M(n) = M_Q(g^{(n)}(x) - \alpha)\), it follows that the roots of \(g^{(n)}(x) - \alpha\) lie in \(\mathcal{C}_{M(n)} = \bigcup_{1 \leq t \leq M(n)} P_{Q^t}\). Since the pair \((f, g)\) is neither critical or \(p\)-critical, \(g\) is not of the form \(\beta(x - \alpha)^p + \alpha\) or of the form \(ax^d + b\). Therefore, from Lemma 4.3, it follows that \(\Delta(g^{(n)}(x) - \alpha) \geq c \cdot d^n\) for some \(c > 0\) (that does not depend on \(n\)), where \(d > 1\) is the degree of the \(p\)-reduction of \(g\). In particular, we have that

\[
c \cdot d^n \leq \Delta(g^{(n)}(x) - \alpha) \leq |\mathcal{C}_{M(n)}| \leq Q^{M(n)} + \cdots + Q < 2Q^{M(n)}.
\]

Thus, \(M(n) \geq \frac{n \log(d) + \log c - \log 2}{\log Q} \gg n\), since \(Q, d > 1\). \(\square\)

The following result proves the remaining statement in Theorem 2.4, concerning the arithmetic function \(m_{f,g}(n)\).

**Lemma 4.5.** Let \(f \in \mathbb{F}_q[x]\) be an irreducible polynomial of degree \(k \geq 1\) and let \(\alpha \in \mathbb{F}_q^*/\) be any of its roots. If \(g\) is any polynomial of degree \(D \geq 1\), then the following are equivalent:

(i) there exists a constant \(c > 0\) such that \(m_{f,g}(n) \leq c\) for any \(n \geq 0\);

(ii) the sequence \(\{m_{f,g}(n)\}_{n \geq 0}\) is eventually constant;

(iii) \(\alpha\) is \(g\)-periodic;

(iv) \(m_{f,g}(n) = k\) for any \(n \geq 0\).

**Proof.** From Proposition 3.9 and Corollary 3.10, \(m_{f,g}(n)\) is nondecreasing and so (i) implies (ii). To see that (ii) implies (iii), suppose that \(m_{f,g}(n) = R\) for some positive integer \(R\) and any \(n \geq n_0\), where \(n_0 > 0\). Since there exist a finite number of irreducible polynomials of degree \(R\) over \(\mathbb{F}_q\), there exist positive integers \(M > N\) and an irreducible polynomial \(h \in \mathbb{F}_q[x]\) of degree \(R\) such that \(h(x)\) divides \(P_N[f, g](x)\) and \(P_M[f, g](x)\). The latter implies that there exists a root \(\gamma \in \mathbb{F}_q\) of \(h\) that is a root of \(g^{(N)}(x) - \alpha^t\) and \(g^{(M)}(x) - \alpha^t\) for some \(1 \leq s, t \leq k\). Therefore,
that
g
Proof.
m
polynomial, i.e., a polynomial of the form
σ
factorization of polynomials
be the factorization of
arithmetic functions associated to the
5. The cases g monomial and q-linearized
In this section we explicitly compute the arithmetic functions associated to the
factorization of polynomials $P_n[f, g](x)$ when $g$ is a monomial or a $q$-linearized
polynomial, i.e., a polynomial of the form $\sum_{i=0}^m a_i x^{q^i}$. In particular, we provide a constructive proof of Theorem 2.6. We are mainly interested in the case in
which $f$ is irreducible, since the general case can be obtained from the identities
of Lemma 3.12. In the following lemma, we provide a reduction to the case in
which $g'(x)$ is not the zero polynomial.

Lemma 5.1. Suppose that $g(x) = x^m$ or $g(x) = \sum_{i=0}^m a_i x^{q^i} \in \mathbb{F}_q[x]$, let $G \in \mathbb{F}_q[x]$ be the $p$-reduction of $g$ and write $g(x) = G(x)^{p^h}$. Then, for any $n \geq 0$, $e_{f, g}(n) = p^{nh} \cdot e_{f, G}(n)$, $E_{f, g}(n) = p^{nh} \cdot E_{f, G}(n)$ and all the other arithmetic functions in
Definition 1.2 coincide at every $n \geq 0$ when evaluated for the pairs $(f, g)$ and $(f, G)$.

Proof. We observe that, if $g(x) = \sum_{i=0}^m a_i x^{q^i}$, then $p^h$ must be a power of $q$ and so
$G(x^{p^h}) = G(x)^{p^h}$; the last equality trivially holds if $g$ is a monomial. In particular,
$\sigma_n(G) = G$, i.e., $G \in \mathbb{F}_{p^h}[x]$. Therefore, it follows by induction on $n \geq 0$ that

$P_n[f, g](x) = f(G^{\sigma_n}(x))^{p^{nh}} = [\sigma_{-nh}(f)(G^{\sigma_n}(x))]^{p^{nh}} = \sigma_{-nh}(P_n[f, G](x))^{p^{nh}}.$

Fix $n \geq 0$ and let

$P_n[f, G](x) = h_1(x)^{c_1} \cdots h_r(x)^{c_r}$

be the factorization of $P_n[f, G]$ into irreducible polynomials over $\mathbb{F}_q$. It follows
from Lemma 3.2 that the factorization of $\sigma_{-nh}(P_n[f, G](x))^{p^{nh}}$ is $P_n[f, g](x)$ into irreducible polynomials over $\mathbb{F}_q$
is

$P_n[f, g](x) = H_1(x)^{p^{nh} c_1} \cdots H_r(x)^{p^{nh} c_r},$

where $H_i(x) = \sigma_{-nh}(h_i(x))$ for any $1 \leq i \leq r$. 

We emphasize that the previous lemma holds for any polynomial $g \in \mathbb{F}_q[x]$ with
$p$-reduction $G$ such that $g(x) = G(x)^{p^h}$ and $G \in \mathbb{F}_{p^h}[x]$. So, for the rest of this
section, we may assume that $g(x)$ is a monomial or a $q$-linearized polynomial such
that $g'(x)$ is not the zero polynomial.
5.1. The monomial case

Here we consider \( g(x) = x^D \), where \( \gcd(D, p) = 1 \) and \( f \in \mathbb{F}_q[x] \) an irreducible polynomial not of the form \( ax \). We introduce some useful definitions.

Definition 5.2. Let \( f, g \in \mathbb{F}_q[x] \) be polynomials such that \( \gcd(f, g) = 1 \), and let \( a \) and \( b \) be positive integers such that \( \gcd(a, b) = 1 \).

(a) The order \( O(g, f) \) of \( g \) modulo \( f \) is the least positive integer \( s \) such that \( f(x) \) divides \( g(x)^s - 1 \).

(b) If \( f \) is not divisible by \( x \), \( \text{ord}(f) := O(x, f) \) is the order of \( f \).

(c) The order \( \text{ord}(a, b) \) of \( a \) modulo \( b \) is the least positive integer \( i \) such that \( a^i \equiv 1 \pmod{b} \).

Butler [4] obtained the following result.

Theorem 5.3. Let \( f \in \mathbb{F}_q[x] \) be an irreducible polynomial of degree \( k \), not of the form \( ax \) such that \( e = \text{ord}(f) \). Let \( m \) be a positive integer such that \( \gcd(m, q) = 1 \) and write \( D = d_1 d_2 \), where \( \gcd(d_1, e) = 1 \) and each prime factor of \( d_2 \) divides \( e \). Then, for \( g(x) = x^D \) and any \( n \geq 0 \), the following hold:

(i) \( E_{f, g}(n) = e_{f, g}(n) = 1 \);

(ii) \( \Delta_{f, g}(n) = k \cdot D^n \);

(iii) \( M_{f, g}(n) = \text{ord}(q, D^n e) \);

(iv) \( m_{f, g}(n) = \text{ord}(q, d_2^n e) \);

(v) \( N_{f, g}(n) = \sum_{M | d_1^k} k d_2^n \varphi(M) / \text{ord}(q, Md_2^n e) \).

In order to estimate the functions appearing in Corollary 5.4, we need to find bounds for orders \( \text{ord}(a, b) \), where the prime factors of \( b \) are fixed. In this context, the prime valuation of integers is required.

Definition 5.5. For an integer \( a \neq 0 \) and \( r \) a prime number, \( \nu_r(a) \) denotes the greatest nonnegative integer \( s \) such that \( r^s \) divides \( a \).

The following lemma is a particular case of the ‘lifting the exponent lemma’ (LTE), a famous result in the Olympiad folklore. Its proof easily follows by induction and so we omit the details.
Lemma 5.6. Let \( r \) be a prime and \( a \) a positive integer such that \( r \) divides \( a - 1 \). For any positive integer \( n \), the following hold:

(i) if \( r \) is odd, \( \nu_r(a^n - 1) = \nu_r(a - 1) + \nu_r(n) \);
(ii) if \( r = 2, \nu_2(a^n - 1) = \nu_2(a^2 - 1) + \nu_2(n) - 1 \) for \( n \) even, and \( \nu_2(a^n - 1) = \nu_2(a - 1) \) for \( n \) odd. In particular, if \( a \equiv 1 \pmod{4} \), \( \nu_2(a^n - 1) = \nu_2(a - 1) + \nu_2(n) \).

From the previous lemma, we have the following result.

Proposition 5.7. Let \( a \) be a positive integer not divisible by the primes in the set \( C = \{ r_1, \ldots, r_u \} \). Then there exist constants \( L(C, a), U(C, a) > 0 \) (only depending on \( a \) and the primes \( r_i \)) such that, for any positive integer \( b \) whose set of prime factors is \( C \), the following holds:

\[
L(C, a) \leq \frac{\varphi(b)}{\text{ord}(a,b)} \leq U(C, a).
\]

Proof. Set \( R = \prod_{i=1}^u r_i, S = \text{ord}(a, R) \) and \( e_i = \nu_{r_i}(a^S - 1) \), where \( 1 \leq i \leq u \). Let \( b = r_1^{E_1} \cdots r_u^{E_u} \) be a positive integer with \( E_i \geq 1 \) for any \( 1 \leq i \leq u \). If \( b \) is odd or \( a \equiv 1 \pmod{4} \), Lemma 5.6 entails that

\[
\text{ord}(a,b) = S \cdot \prod_{1 \leq i \leq u} r_i^{m_i},
\]

where \( m_i = \max\{E_i - e_i, 0\} \). Therefore,

\[
\frac{\varphi(b)}{\text{ord}(a,b)} = \frac{\varphi(R)}{S} \cdot \prod_{1 \leq i \leq u} r_i^{m_i^* - 1},
\]

where \( m_i^* = E_i - m_i = \min\{E_i, e_i\} \). From the definition of \( m_i^* \), we have that

\[
1 \leq \prod_{1 \leq i \leq u} r_i^{m_i^* - 1} \leq \prod_{1 \leq i \leq u} r_i^{e_i - 1} \leq \frac{a^S - 1}{R}.
\]

The result follows with \( L(C, a) = \varphi(R)/S \) and \( U(C, a) = \varphi(R)(a^S - 1)/(RS) \). For \( b \) even and \( a \equiv 3 \pmod{4} \), we observe that \( A = a^2 \) satisfies \( A \equiv 1 \pmod{4} \) and

\[
\text{ord}(A, b) \leq \text{ord}(a, b) \leq 2 \cdot \text{ord}(A, b).
\]

In this case, the result follows with \( L(C, a) = L(C, A)/2 \) and \( U(C, a) = U(C, A) \). \( \square \)

The following result provides sharp estimates on the growth on the functions \( M_{f,g}(n), m_{f,g}(n) \) and \( N_{f,g}(n) \) in the case \( g \) a monomial.

Theorem 5.8. Let \( f \in \mathbb{F}_q[x] \) be an irreducible polynomial of degree \( k \), not of the form \( ax \) such that \( e = \text{ord}(f) \). Let \( D > 1 \) be a positive integer such that \( \gcd(D, p) = 1 \) and write \( D = d_1d_2 \), where \( \gcd(d_1, e) = 1 \) and each prime factor of \( d_2 \) divides \( e \). Then, for \( g(x) = x^D \), the following hold:

(i) \( M_{f,g}(n) \approx D^n \);
(ii) $m_{f,g}(n) \approx d_2^n$;

(iii) $N_{f,g}(n) \approx n^t$, where $t$ is the number of distinct prime factors of $d_1$.

Proof. From Proposition 5.7 and item (ii) of Corollary 5.4, we have that $M_{f,g}(n) \approx \varphi(D^n)$. Since

$$\frac{\varphi(D^n)}{D^n} = \frac{\varphi(D)}{D},$$

we have that $\varphi(D^n) \approx D^n$ and so $M_{f,g}(n) \approx D^n$. Similarly we obtain $m_{f,g}(n) \approx d_2^n$.

It remains to prove item (iii). Let $C_1$ and $C_2$ be the set of distinct prime factors of $d_1$ and $d_2$, respectively, and let $S$ be the set of all non-empty subsets of $C_1 \cup C_2$. For each $\mathcal{C} \in S$, let $L(\mathcal{C}, q)$ and $U(\mathcal{C}, q)$ be as in Proposition 5.7 and set $L = \min_{\mathcal{C} \in S} L(\mathcal{C}, q)$ and $U = \max_{\mathcal{C} \in S} U(\mathcal{C}, q)$. From Proposition 5.7, we have that

$$L \leq \frac{\varphi(Md_2^n)}{\text{ord}(q, Md_2^n)} \leq U,$$

where $n \geq 1$ and $M$ is any divisor of $d_1^n$. Since $\text{ord}(q, Md_2^n e) \leq \text{ord}(q, e) \cdot \text{ord}(q, Md_2^n)$, there exist $L', U' > 0$ such that, for any $n \geq 1$ and any divisor $M$ of $d_1^n$, the following holds:

$$L' \leq \frac{k \cdot \varphi(Md_2^n)}{\text{ord}(q, Md_2^n e)} \leq U'.$$

Since $M$ and $d_2$ are relatively prime, we have that

$$\varphi(Md_2^n) = \varphi(M) \cdot \varphi(d_2^n) = \varphi(M) \cdot d_2^n \frac{\varphi(d_2)}{d_2},$$

In conclusion, there exist constants $c_0, c_1 > 0$ such that

$$c_0 \leq \frac{k d_2^n \varphi(M)}{\text{ord}(q, Md_2^n e)} \leq c_1,$$

for any $n \geq 0$ and any divisor $M$ of $d_1^n$. Let $d_1 = r_1^{m_1} \cdots r_t^{m_t}$ be the prime factorization of $d_1$. In particular, $d_1^n$ has

$$(nm_1 + 1) \cdots (nm_t + 1) \approx n^t$$

distinct divisors. Therefore, from item (v) of Corollary 5.4, we have that

$$N_{f,g}(n) = \sum_{M|d_1^n} \frac{k d_2^n \varphi(M)}{\text{ord}(q, Md_2^n e)} \approx n^t.$$  

Remark 5.9. Let $f$, $g$ and $d_2$ be as in the previous theorem and let $\alpha \in \overline{\mathbb{F}_q}$ be any root of $f$. From the previous theorem, we have that $m_{f,g}(n) \to \infty$ unless $d_2 = 1$, i.e., $D$ is relatively prime with the order $e$ of $f(x)$. It is direct to verify that the latter holds if and only if $\alpha$ is not $g$-periodic, as predicted by Theorem 2.4.

The following corollary is an immediate application of Theorem 5.8 and provides the proof of item (i) in Theorem 2.6.
Corollary 5.10. Let \( f \in F_q[x] \) be a polynomial of degree at least one that is not of the form \( ax^k \). Write

\[
f(x) = ax^l f_1(x) \cdots f_m(x),
\]

where \( l \geq 0, m \geq 1 \) and each \( f_i(x) \neq x \) is irreducible over \( F_q \) with order \( e_i = \text{ord}(f_i) \). Let \( C \) be the set of distinct prime divisors of the numbers \( e_i \). If \( D > 1 \) is any positive integer with \( t \) distinct prime factors, none of them being an element of \( C \) or the prime \( p \), then for \( g(x) = x^D \) the following holds:

\[
M_{f,g}(n) \approx D^n \quad \text{and} \quad N_{f,g}(n) \approx n^t.
\]

5.2. The linearized case

Here we consider \( g(x) = \sum_{i=0}^{m} a_i x^{q^i} \) a \( q \)-linearized polynomial, where \( g'(x) \) is not the zero polynomial and \( f \in F_q[x] \) is an irreducible polynomial. As pointed out in [11], we have an analogue of Theorem 5.3 for \( q \)-linearized polynomials with suitable changes. First, in contrast to the order of polynomials, we introduce the \( F_q \)-order.

Definition 5.11. Let \( \alpha \in \overline{F}_q \) and \( g \in F_q[x] \).

(a) if \( g(x) = \sum_{i=0}^{m} a_i x^{q^i} \) is the linearized \( q \)-associate of \( g \);

(b) the \( F_q \)-order of \( \alpha \) is the monic polynomial \( h \in F_q[x] \) of least degree such that its \( q \)-associate \( L_h \) vanishes at \( \alpha \), i.e., \( L_h(\alpha) = 0 \). We write \( h(x) = m_{\alpha,q}(x) \).

The following lemma compiles some basic properties of the \( q \)-associates and the \( F_q \)-order. Its proof follows from the results in Subsection 2.1 of [11] and so we omit the details.

Lemma 5.12. The following hold:

(i) the \( F_q \)-order of an element \( \alpha \) is well defined, is not divisible by \( x \) and coincides with the \( F_q \)-order of any conjugate \( \alpha^{q^i} \) of \( \alpha \);

(ii) for polynomials \( g_1, g_2 \in F_q[x] \), we have that \( L_{g_1}(L_{g_2}(x)) = L_{g_1 g_2}(x) \).

We introduce the analogue of the Euler totient function for the polynomial ring \( F_q[x] \).

Definition 5.13. The Euler totient function for polynomials over \( F_q \) is

\[
\Phi_q(f) = \left| \frac{F_q[x]}{(f)} \right|,
\]

where \( (f) \) is the ideal generated by \( f \) in \( F_q[x] \).

According to Theorem 4 of [11], we have the following result.

Theorem 5.14. Let \( f \in F_q[x] \) be an irreducible polynomial of degree \( k \) such that any of its roots has \( F_q \)-order \( h \). Let \( g \in F_q[x] \) be a monic polynomial such that
\( \gcd(g(x), x) = 1 \) and write \( g = g_1 g_2 \), where \( \gcd(g_1, h) = 1 \) and each irreducible factor of \( g_2 \) divides \( h \). If \( L_q \) denotes the \( q \)-associate of \( g \) and \( \deg(g_2) = m \), then for each monic divisor \( G \) of \( g_1 \), \( f(L_q(x)) \) has exactly

\[
\frac{k q^m \Phi_q(G)}{\mathcal{O}(x, G g_2 h)}
\]

irreducible factors of degree \( \mathcal{O}(x, G g_2 h) \) with roots of \( \mathbb{F}_q \)-order \( G g_2 h \). In addition, this describes all the irreducible factors of \( f(L_q(x)) \) over \( \mathbb{F}_q \).

**Remark 5.15.** We observe that, for any \( a \in \mathbb{F}_q^* \) and \( g \in \mathbb{F}_q[x] \), the following holds:

\[
a L_q(x) = L_q(ax) = L_{aq}(x).
\]

Therefore, the \( \mathbb{F}_q \)-order of \( \alpha \in \mathbb{F}_q^* \) and \( a \alpha \) coincide whenever \( a \in \mathbb{F}_q^* \). In particular, Theorem 5.14 holds without the assumption that \( g \) is monic, a fact frequently used.

From item (ii) of Lemma 5.12, we directly obtain a formula for iterates of \( q \)-linearized polynomials:

\[
L_q^{(n)}(x) = L_q^n(x), \quad n \geq 0.
\]

We observe that the condition \( \gcd(g(x), x) = 1 \) in Theorem 5.14 is compatible with our initial assumption; in fact, \( \gcd(g(x), x) = 1 \) if and only if the derivative of \( L_q(x) \) is not the zero polynomial. From Theorem 5.14 and equation (5.1), we easily obtain the following analogue of Corollary 5.4.

**Corollary 5.16.** Let \( f \in \mathbb{F}_q[x] \) be an irreducible polynomial of degree \( k \) such that any of its roots has \( \mathbb{F}_q \)-order \( h \). Let \( g \in \mathbb{F}_q[x] \) be a polynomial of degree \( D \geq 1 \) such that \( \gcd(g(x), x) = 1 \) and write \( g = g_1 g_2 \), where \( \gcd(g_1, h) = 1 \) and each irreducible factor of \( g_2 \) divides \( h \). If \( L_q \) denotes the \( q \)-associate of \( g \) and \( \deg(g_2) = m \), then for any \( n \geq 0 \), the following hold:

1. \( E_{f, L_q}(n) = e_{f, L_q}(n) = 1 \);
2. \( \Delta_{f, L_q}(n) = k \cdot q^{D n} \);
3. \( M_{f, L_q}(n) = \mathcal{O}(x, g^n h) \);
4. \( m_{f, L_q}(n) = \mathcal{O}(x, g_1^n h) \);
5. \( N_{f, L_q}(n) = \sum_{G \mid G_2} k q^m \Phi_q(G) / \mathcal{O}(x, G g_2 h) \), where \( G \) is monic and polynomial division is over \( \mathbb{F}_q \).

In analogy to the monomial case, the computation of orders \( \mathcal{O}(x, F) \) is required. This is done with the help of the following result.

**Lemma 5.17.** Let \( F \in \mathbb{F}_q[x] \) be a non constant polynomial that is not divisible by \( x \) and let \( \text{rad}(F) \) be the squarefree part of \( F \). Then the following holds:

\[
\mathcal{O}(x, F) = \mathcal{O}(x, \text{rad}(F)) \cdot p^r,
\]

where \( r = \lfloor \log_p(\nu(F)) \rfloor \).

**Proof.** For the proof of this result, see item (ii) of Lemma 5 in [11].
Proposition 5.18. Let $f \in \mathbb{F}_q[x]$ be an irreducible polynomial of degree $k$ such that any of its roots has $\mathbb{F}_q$-order $h$. Let $g \in \mathbb{F}_q[x]$ be a polynomial of degree $D \geq 1$ such that $\gcd(g(x), x) = 1$ and write $g = g_1 g_2$, where $\gcd(g_1, h) = 1$ and each irreducible factor of $g_2$ divides $h$. If $L_g$ denotes the $q$-associate of $g$, then $M_{f,L_g}(n) \approx n$. In addition, if $g_2(x) \neq 1$, then
\[ A_{f,L_g}(n) \approx n \quad \text{and} \quad N_{f,L_g}(n) \approx \frac{q^{D_n}}{n}. \]

Proof. Combining item (iii) of Corollary 5.16 and Lemma 5.17, we have that
\[ M_{f,L_g}(n) = O(x, g^n h) = O(x, \text{rad}(g h)) \cdot p^{t(n)}, \quad n \geq 1, \]
where $t(n) = \lceil \log_q(\nu(g^n h)) \rceil$. If we set $t_0 = \nu(g)$, for $n$ sufficiently large, we have that $\nu(g^n h) = n t_0 + R$ for some $R \geq 0$ not depending on $n$. Therefore, $p^{t(n)} \approx p^{\log_q(n t_0)} \approx n$. In particular, since $N_{f,L_g}(n) \cdot M_{f,L_g}(n) \geq \Delta_{f,L_g}(n) = k q^{D_n}$, we have that
\[ N_{f,L_g}(n) \approx \frac{q^{D_n}}{n}. \]
If $g_2(x) \neq 1$, we follow the previous arguments and obtain that $m_{f,L_g}(n) = O(x, g_2^n h) \approx n$. Since $N_{f,L_g}(n) \cdot m_{f,L_g}(n) \leq \Delta_{f,L_g}(n) = k q^{D_n}$, we have that
\[ N_{f,L_g}(n) \ll \frac{q^{D_n}}{n}. \]
Therefore, $N_{f,L_g}(n) \approx q^{D_n}/n$. Since $A_{f,L_g}(n) \cdot N_{f,L_g}(n) = \Delta_{f,L_g}(n) \approx q^{D_n}$, we have that $A_{f,L_g}(n) \approx n$. $\square$

The following corollary is an immediate application of Proposition 5.18 and provides the proof of item (ii) in Theorem 2.6.

Corollary 5.19. Let $f \in \mathbb{F}_q[x]$ be a polynomial of degree at least one and let $g \in \mathbb{F}_q[x]$ be a polynomial of degree $D \geq 1$ such that $\gcd(g(x), x) = 1$. If $L_g$ denotes the $q$-associate of $g$, then $M_{f,L_g}(n) \approx n$.

5.3. Pairs $(f, g)$ for which $N_{f,g}(n)$ and $M_{f,g}(n)$ have exponential growth

Here we provide the proof of item (iii) in Theorem 2.6. From Lemma 3.12, it suffices to consider $f \in \mathbb{F}_q[x]$ an irreducible polynomial such that $f(x) = x^k + \sum_{i=0}^{k-1} a_i x^i$ with $a_0 \neq 0, 1$. We have the following result.

Proposition 5.20. Let $q > 2$ be a prime power and let $f \in \mathbb{F}_q[x]$ be an irreducible polynomial such that $f(x) = x^k + \sum_{i=0}^{k-1} a_i x^i$ with $a_0 \neq 0, 1$. For $g(x) = (x^q - x)^{q-1}$, the polynomial $f(g(x))$ is separable, reducible and any of its irreducible factors has degree of the form $dk$ with $d \geq 2$. In particular, $f(g(x))$ has an irreducible factor of the form $h(x) = x^{dk} + \sum_{i=0}^{d(k-1)} b_i x^i$ with $b_0 \neq 0, 1$ and $d \geq 2$.

Proof. We first prove that $f(g(x))$ is separable. For this we observe that, since $a_0 = f(0) \neq 0$, the polynomial $f(g(x))$ does not have roots in $\mathbb{F}_q$. In particular, since the formal derivative of $f(g(x))$ equals $f'(g(x)) \cdot (x^q - x)^{q-2}$, any repeated root
of \( f(g(x)) \) is also a root of \( f'(g(x)) \). Since \( f \) is irreducible and \( \mathbb{F}_q \) is a perfect field, we have that \( \gcd(f(x), f'(x)) = 1 \) and so \( \gcd(f(g(x)), f'(g(x))) = 1 \). Therefore, \( f(g(x)) \) cannot have repeated roots.

Let \( \beta \in \mathbb{F}_q \) be a root of \( f(g(x)) \) and \( B \) its degree over \( \mathbb{F}_q \). Corollary 3.7 entails that \( B \) is divisible by \( k \), the degree of \( f(x) \). If we write \( B = dk \) with \( d \geq 1 \), we just need to prove that \( d \neq 1, q(q - 1) \).

1. If \( d = q(q - 1) \), \( f((x^q - x)^{q - 1}) \) is an irreducible polynomial. However, using Theorem 5 and Corollary 1 of [11], we easily see that any polynomial of the form \( f((x^q - x)^{q - 1}) \) is reducible over \( \mathbb{F}_q \) if \( q > 2 \).
2. If \( d = 1 \), we have that \( \beta \in \mathbb{F}_{q^k} \) and so \( \beta^q - \beta \in \mathbb{F}_{q^k} \). Since \( \beta \) is a root of \( f(g(x)) \), there exists a root \( \alpha \in \mathbb{F}_{q^k} \) of \( f(x) \) such that \( (\beta^q - \beta)^{q - 1} = \alpha \). However, \( \alpha^{(q - 1)(q^{-1})} = a_0 \neq 1, 0 \) and so \( \alpha \) cannot be of the form \( \gamma^{q - 1} \) for any \( \gamma \in \mathbb{F}_{q^k} \).

The statement regarding the existence of an irreducible factor \( h(x) \) as above follows from the fact that \( a_0 \neq 0, 1 \).

The previous proposition immediately gives the following result.

**Corollary 5.21.** Let \( q > 2 \) be a prime power and let \( f \in \mathbb{F}_q[x] \) be an irreducible polynomial of degree \( k \) such that \( f(x) = x^k + \sum_{i=0}^{k-1} a_i x^i \) with \( a_0 \neq 0, 1 \). For \( g(x) = (x^q - x)^{q - 1} \) and any \( n \geq 0 \), we have that

\[
N_{f,g}(n) \geq 2^n \quad \text{and} \quad M_{f,g}(n) \geq k \cdot 2^n.
\]

6. Conclusions and open problems

This paper provides a study on the growth of some arithmetic functions related to the factorization of iterated polynomials \( f(g^{(n)}(x)) \) over finite fields, such as the number and the degree of the irreducible factors. This study extends and enhances many results of [5], where the case \( f = g \) is considered; for more details, see Theorem 1.1 and Corollary 2.5. Here we provide some open problems and conjectures based on theoretical and computational considerations.

Throughout this section we consider arbitrary pairs \((f, g)\) of polynomials over \( \mathbb{F}_q \) that are neither critical nor \( p \)-critical, where \( f \) is irreducible and \( g \) has degree at least two. Theorem 2.6 shows that \( N_{f,g}(n) \) may have polynomial growth of any degree, while \( M_{f,g}(n) \) may reach the linear growth (showing that the bound \( M_{f,g}(n) \gg n \) in Theorem 2.4 is optimal for a generic polynomial \( f \in \mathbb{F}_q[x] \)).

**Problem 6.1.** For an irreducible polynomial \( f \in \mathbb{F}_q[x] \), find all the polynomials \( g \in \mathbb{F}_q[x] \) such that \( M_{f,g}(n) \approx n \).

**Conjecture 6.2.** One of the following holds:

(i) \( M_{f,g}(n) \approx n \);
(ii) \( \log M_{f,g}(n) \gg n \).

Conjecture 6.2 entails that \( M_{f,g}(n) \) does not have polynomial growth of high degree (quadratic, cubic, etc). We believe that any proof (or disproof) of Conjecture 6.2 contains a solution for Problem 6.1.
Remark 2.3 implies that there exists $c > 0$ such that, for any $n \geq 0$, either $N_{f,g}(n) \geq c \cdot d^{n/2}$ or $M_{f,g}(n) \geq c \cdot d^{n/2}$. However, this is not sufficient to conclude that $N_{f,g}(n)$ or $M_{f,g}(n)$ have exponential growth. Nevertheless, we believe that this is always the case.

**Conjecture 6.3.** Either $\log(N_{f,g}(n)) \gg n$ or $\log(M_{f,g}(n)) \gg n$.

**Problem 6.4.** Prove or disprove: $A_{f,g}(n) \gg n$.

We comment that if the bound $A_{f,g}(n) \gg n$ holds, then it is optimal, since we have provided examples where $A_{f,g}(n) \approx n$ (see Proposition 5.18).

**References**


Received October 19, 2018; revised August 1, 2019. Published online March 16, 2020.

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The author was supported by FAPESP 2018/03038-2, Brazil.