On the failure of the Hörmander multiplier theorem in a limiting case

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Abstract. We discuss the Hörmander multiplier theorem for $L^p$ boundedness of Fourier multipliers in which the multiplier belongs to a fractional Sobolev space with smoothness $s$. We show that this theorem does not hold in the limiting case $|1/p - 1/2| = s/n$.

1. Introduction

Let $m$ be a bounded function on $\mathbb{R}^n$. We define the associated linear operator

$$T_m(f)(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) m(\xi) e^{2\pi i x \cdot \xi} \, d\xi, \quad x \in \mathbb{R}^n,$$

where $f$ is a Schwartz function on $\mathbb{R}^n$ and $\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} \, dx$ is the Fourier transform of $f$. The problem of characterizing the class of functions $m$ for which the operator $T_m$ admits a bounded extension from $L^p(\mathbb{R}^n)$ to itself for all $1 < p < \infty$ is one of the principal questions in harmonic analysis. We say that $m$ is an $L^p$ Fourier multiplier if the above mentioned property is satisfied. While it is a straightforward consequence of Plancherel’s identity that all bounded functions are $L^2$ Fourier multipliers, the structure of the set of $L^p$ Fourier multipliers for $p \neq 2$ turns out to be significantly more complicated.

A classical theorem of Mikhlin [10] asserts that if the condition

$$|\partial^\alpha m(\xi)| \leq C_\alpha |\xi|^{-|\alpha|}, \quad \xi \neq 0,$$

(1.1)

is satisfied for all multiindices $\alpha$ with size $|\alpha| \leq [n/2] + 1$, then $T_m$ admits a bounded extension from $L^p(\mathbb{R}^n)$ to itself for all $1 < p < \infty$. A subsequent result by Hörmander [9] showed that the pointwise estimate (1.1) can be replaced by a weaker Sobolev-type condition:

$$\sup_{R>0} R^{-n+2|\alpha|} \int_{\{\xi \in \mathbb{R}^n: R < |\xi| < 2R\}} |\partial^\alpha m(\xi)|^2 \, d\xi < \infty.$$

(1.2)

Mathematics Subject Classification (2010): Primary 42B15; Secondary 46E35.

Keywords: Hörmander multiplier theorem, Sobolev space.
Although theorems of Mikhlin and Hörmander admit a variety of applications, their substantial limitation stems from the fact that they can only be applied to functions which are $L^p$ Fourier multipliers for all values of $p \in (1, \infty)$. One can overcome this difficulty using an interpolation argument as in Calderón and Torchinsky [1] or Connett and Schwartz [2], [3]; the conclusion is, roughly speaking, that the closer $p$ is to 2, the fewer derivatives are needed in conditions (1.1) or (1.2).

To be able to formulate things precisely, let us now recall the notion of fractional Sobolev spaces. For $s > 0$ we denote by $(I - \Delta)^{s/2}$ the operator given on the Fourier transform side by multiplication by $(1 + 4\pi^2|\xi|^2)^{s/2}$. If $1 < r < \infty$, then the norm in the fractional Sobolev space $L^r_s$ is defined by

$$\|f\|_{L^r_s} = \|(I - \Delta)^{s/2} f\|_{L^r}.$$ 

The version of the Mikhlin–Hörmander multiplier theorem due to Calderón and Torchinsky ([1], Theorem 4.7) says that inequality

$$(1.3) \quad \|T_m f\|_{L^p} \leq C \sup_{D \in \mathbb{Z}} \|\phi(\xi) m(2^D \xi)\|_{L^r_s} \|f\|_{L^p}$$ 

holds provided that

$$(1.4) \quad \left|\frac{1}{p} - \frac{1}{2}\right| = \frac{1}{r} < \frac{s}{n}.$$ 

Here, $\phi$ stands for a smooth function on $\mathbb{R}^n$ supported in the set $\{\xi \in \mathbb{R}^n : 1/2 < |\xi| < 2\}$ and satisfying $\sum_{D \in \mathbb{Z}} \phi(2^D \cdot) = 1$. Additionally, it was pointed out in [4] that the equality $|1/p - 1/2| = 1/r$ is not essential for (1.3) to be true, and (1.4) can thus be replaced by the couple of inequalities

$$(1.5) \quad \left|\frac{1}{p} - \frac{1}{2}\right| < \frac{s}{n}, \quad \frac{1}{r} < \frac{s}{n}.$$ 

Let us notice that the latter inequality in (1.5) is dictated by the embedding of $L^r_s$ into the space of essentially bounded functions. Related to this we also mention that the Sobolev-type condition in (1.3) can be further weakened by replacing the Sobolev space $L^r_s$ with the Sobolev space with smoothness $s$ built upon the Lorentz space $L^{n/s,1}$, see [7].

Let us now discuss the sharpness of the first condition in (1.5). It is well known that if inequality (1.3) holds, then we necessarily have $|1/p - 1/2| \leq s/n$, see [8], [17], [11], [12] and [4]. On the critical line $|1/p - 1/2| = s/n$, there are positive endpoint results by Seeger [13], [14], [15]. In particular, it is shown in [15] that inequality (1.3) holds when $|1/p - 1/2| = s/n$ and $r > n/s$ if the Sobolev space $L^r_s$ is replaced by the Besov space $B^s_{1,r}$, defined by

$$\|f\|_{B^s_{1,r}} = \sum_{k=0}^{\infty} 2^{ks} \|(\varphi_k f)^\vee\|_{L^r}.$$ 

Here, $\varphi_0$ stands for a Schwartz function on $\mathbb{R}^n$ such that $\varphi_0(x) = 1$ if $|x| \leq 1$ and $\varphi_0(x) = 0$ if $|x| \geq 3/2$, and $\varphi_k(x) = \varphi_0(2^{-k}x) - \varphi_0(2^{1-k}x)$ for $k \in \mathbb{N}$.
The Hörmander multiplier theorem in a limiting case

We recall that $B^{s, r}_1$ is embedded into $L^r$, thanks to the equivalence

$\|f\|_{L^r} \approx \left( \sum_{k=0}^{\infty} 2^{ks} |(\varphi_k \widehat{f}) (2^k)^r| \right)^{1/2}$

and to embeddings between sequence spaces.

In this note we show that Hörmander’s condition involving the Sobolev space $L^r_s$ fails to guarantee $L^p$ boundedness of $T_m$ in the limiting case $|1/p - 1/2| = s/n$. In fact, we can even include the more general Lorentz–Sobolev spaces $L^{r_1, r_2}_s$ in our discussion, providing thus a negative answer to the open problem A.2 raised in Appendix A of the recent paper [16]. We recall that the Lorentz–Sobolev space $L^{r_1, r_2}_s$ is defined as

$\|f\|_{L^{r_1, r_2}_s} = \|(I - \triangle)^{s/2} f\|_{L^{r_1, r_2}}$,

where

$\|g\|_{L^{r_1, r_2}} = \left\{ \begin{array}{ll}
\left( \int_0^\infty t^{r_2/r_1 - 1} (g^*(t))^{r_2} \, dt \right)^{1/r_2} & \text{if } 1 < r_1 < \infty \text{ and } 1 \leq r_2 < \infty, \\
\sup_{t > 0} t^{1/r_1} g^*(t) & \text{if } 1 < r_1 < \infty \text{ and } r_2 = \infty.
\end{array} \right.$

Here,

$g^*(t) = \inf \{ \lambda > 0 : \| \{ x \in \mathbb{R}^n : |g(x)| > \lambda \} \| \leq t \}$, \quad $t > 0$,

stands for the nonincreasing rearrangement of $g$.

Our result has the following form.

**Theorem 1.** Let $1 < p < \infty$, $p \neq 2$, and let $s > 0$ be such that

$|1/p - 1/2| = s/n$.

Assume that $1 < r_1 < \infty$, $1 \leq r_2 \leq \infty$ and $\phi$ is a smooth function on $\mathbb{R}^n$ supported in the set $\{ \xi \in \mathbb{R}^n : 1/2 < |\xi| < 2 \}$. Then there is no finite constant $C$ such that the inequality

$\|T_m f\|_{L^p} \leq C \sup_{D \in \mathbb{Z}} \| \phi(D) m(2D) \|_{L^{r_1, r_2}} \|f\|_{L^p}$

holds for all $m$ and $f$.

The proof of Theorem 1 uses the randomization technique in the spirit of [18], Chapter 4, which has been further developed in [4] and [5].

2. Proof of Theorem 1

Let $s > 0$ and let $\Psi$ be a non-identically vanishing Schwartz function on $\mathbb{R}^n$ supported in the set $\{ \xi \in \mathbb{R}^n : |\xi| < 1/2 \}$. Then for any fixed integer $K$ and for any $t \in [0, 1]$, we define

$m_t(\xi) = \sum_{N=1}^{K} \sum_{k \in \mathbb{N}^n : N2^N \leq |k| < (N+1/2)2^N} a_N, k(t) \Psi(2^N \xi - k)$,
where $a_{N,k}(t)$ denotes the sequence of Rademacher functions indexed by the elements of the countable set $\mathbb{N} \times \mathbb{N}^n$, and $c_N = 2^{-N^s} N^{-s}$.

**Lemma 2.** Let $1 < r_1 < \infty$, $1 \leq r_2 \leq \infty$, and let $\phi$ be as in Theorem 1. Then

$$\sup_{D \in \mathbb{Z}} \| \phi(\xi) m_t(2^D \xi) \|_{L^{r_1,r_2}} \leq C,$$

with $C$ independent of $t$ and $K$.

**Proof.** We first observe that it is enough to consider the case when $r_1 = r_2$; the general case then follows by real interpolation. For simplicity of notation, we write $r = r_1$ in what follows.

One can verify that $\phi(\xi) m_t(2^D \xi) = 0$ for all $\xi$ if $D < -1$. We can thus assume that $D \geq -1$. For any such $D$, we denote

$$A_D = \left\{ \xi \in \mathbb{R}^n : \frac{1}{2} - \frac{1}{4 \cdot 2^D} < |\xi| < 2 + \frac{3}{4 \cdot 2^D} \right\}.$$

Using the version of the Kato–Ponce inequality from [6], we get

$$\| \phi(\xi) m_t(2^D \xi) \|_{L^r} = \|(I - \Delta)^{s/2} [\phi(\xi) m_t(2^D \xi)] \|_{L^r}$$

$$= \|(I - \Delta)^{s/2} [\phi(\xi) \chi_{A_D}(\xi) m_t(2^D \xi)] \|_{L^r}$$

$$\leq \|(I - \Delta)^{s/2} [\phi(\xi)] \|_{L^\infty} \| \chi_{A_D}(\xi) m_t(2^D \xi) \|_{L^r}$$

$$+ \| \phi(\xi) \|_{L^{\infty}} \| (I - \Delta)^{s/2} [\chi_{A_D}(\xi) m_t(2^D \xi)] \|_{L^r}$$

$$\leq \| \chi_{A_D}(\xi) m_t(2^D \xi) \|_{L^r} + \| (I - \Delta)^{s/2} [\chi_{A_D}(\xi) m_t(2^D \xi)] \|_{L^r},$$

since $\phi$ is a Schwartz function.

An elementary calculation yields that the last two terms in (2.1) are bounded by a constant independent of $D$, $t$ and $K$. Indeed, using support properties of $\Psi$ and the definition of the set $A_D$, we deduce that

$$\chi_{A_D}(\xi) m_t(2^D \xi)$$

$$= \min_{(2^{D+1})} \sum_{N = \max(1,2D-1)} \sum_{k \in \mathbb{N}^n : N2^N < |k| < (N+1/2)2^N} c_{NA_N,k}(t) \Psi(2^{N+D} \xi - k).$$

Since $|c_N| \leq 1$ for all $N$ and the functions $\Psi(2^{N+D} \xi - k)$ have pairwise disjoint supports in $N$ and $k$ (for the fixed $D$), we obtain

$$\| \chi_{A_D}(\xi) m_t(2^D \xi) \|_{L^r} \leq C(n,r,\Psi).$$

Further, we denote $\Phi = (-\Delta)^{s/2} \Psi$ and observe that

$$(-\Delta)^{s/2} [\chi_{A_D}(\cdot) m_t(2^D \cdot)](\xi)$$

$$= \sum_{N = \max(1,2D-1)} 2^{Ds} N^{-s} \sum_{k \in \mathbb{N}^n : N2^N < |k| < (N+1/2)2^N} a_{N,k}(t) \Phi(2^{N+D} \xi - k).$$
Let $\alpha > n + n/r$ be an integer. Since $\Phi$ is a Schwartz function, we have
\[ |\Phi(\xi)| \lesssim (1 + |\xi|)^{-\alpha}. \]
This yields
\[ |(-\Delta)^{s/2} [\chi_{A_D} (\cdot) m_t(2^D \cdot)](\xi)| \lesssim \sum_{N = \max(1, 2^{D-1})}^{\min(K, 2^{D+1})} \sum_{k \in \mathbb{N}^n : N2^N < |k| < (N+1/2)2^N} (1 + |2^N + D \xi - k|)^{-\alpha} \]
\[ \approx \sum_{N = \max(1, 2^{D-1})}^{\min(K, 2^{D+1})} \int_{\{z \in \mathbb{R}^n : N2^N < |z + 2^N + D \xi| < (N+1/2)2^N\}} (1 + |z|)^{-\alpha} \, dz. \]

Now, if $z \in \mathbb{R}^n$ satisfies $N2^N < |z + 2^N + D \xi| < (N + 1/2)2^N$ then it can be verified that $|z| > 2^N$ holds for all but three values of $N$ (the exceptional $N$’s are those close to $2^D |\xi|$). In addition, if $|\xi|$ is large (say, $|\xi| \geq 6$) and $N \leq 2^{D+1}$ then $|z| > 2^{N-2} |\xi|$. This yields
\[ |(-\Delta)^{s/2} [\chi_{A_D} (\cdot) m_t(2^D \cdot)](\xi)| \leq C(n, \alpha, s, \Psi), \quad \xi \in \mathbb{R}^n, \]
and
\[ |(-\Delta)^{s/2} [\chi_{A_D} (\cdot) m_t(2^D \cdot)](\xi)| \leq C(n, \alpha, s, \Psi) |\xi|^{n-\alpha}, \quad |\xi| \geq 6. \]

A combination of estimates (2.4) and (2.5) then implies
\[ \|(-\Delta)^{s/2} [\chi_{A_D}(\cdot) m_t(2^D \xi)]\|_{L^r} \leq C(n, r, s, \Psi). \]

Finally, combining estimates (2.1), (2.3) and (2.6), we obtain the desired conclusion.

Proof of Theorem 1. We may assume, without loss of generality, that $p < 2$; the result in the case $p > 2$ will then follow by duality.

Let $t$, $K$ and $m_t$ be as described at the beginning of this section, and let $\varphi$ be a Schwartz function such that $\varphi(\xi) = 1$ if $|\xi| \leq 2$. Define a function $f$ via its Fourier transform by $\hat{f}(\xi) = \varphi(\xi/K)$. Then $\hat{f}(\xi) = 1$ if $|\xi| \leq 2K$. It is straightforward to verify that $m_t(\xi)$ is supported in the set $|\xi| < 2K$. Therefore, we have
\[ m_t(\xi) \hat{f}(\xi) = m_t(\xi), \]
and so
\[ T_{m_t} f(x) = \sum_{N = 1}^{K} c_N \sum_{k \in \mathbb{N}^n : N2^N < |k| < (N+1/2)2^N} a_{N,k}(t) 2^{-nN} |F^{-1}(\Psi)(\frac{x}{2^N})| e^{2\pi i k \cdot \frac{x}{2^N}}. \]
By Fubini’s theorem and Khintchine’s inequality, we obtain

\[
\int_0^1 \|T_m f(x)\|_{L^p}^p \, dt = \int_{\mathbb{R}^n} \int_0^1 |T_m f(x)|^p \, dt \, dx
\]
\[
\approx \int_{\mathbb{R}^n} \left( \sum_{N=1}^K c_N^2 2^{-2nN} \left( \frac{x}{2^N} \right)^2 \right)^{p/2} \, dx
\]
\[
\approx \int_{\mathbb{R}^n} \left( \sum_{N=1}^K c_N^2 N^{n-1} 2^{-nN} \left( \frac{x}{2^N} \right)^2 \right)^{p/2} \, dx.
\]

Let \( A > 0 \) be such that \( F^{-1} \psi \) does not vanish in \( \{ y \in \mathbb{R}^n : A \leq |y| < 2A \} \). Then

\[
\int_0^1 \|T_m f(x)\|_{L^p}^p \, dt
\]
\[
\approx \int_{\mathbb{R}^n} \left( \sum_{N=1}^K c_N^p N^{(n-1)p/2} 2^{-nN} \right) \left( \frac{x}{2^N} \right)^p \chi_{\{ x : A \leq |x|/2^N < 2A \}}(x) \, dx
\]
\[
\approx \sum_{N=1}^K c_N^p N^{(n-1)p/2} 2^{nN(1-p/2)} \int_{\{ y : A \leq |y| < 2A \}} \left( \frac{x}{2^N} \right)^p \, dy
\]
\[
\approx \sum_{N=1}^K c_N^p N^{(n-1)p/2} 2^{nN(1-p/2)} \sum_{N=1}^K N^{(n-1)p/2 - sp} 2^{n(1-np/2 - sp)}
\]
\[
= \sum_{N=1}^K N^{np-n-p/2},
\]

where the last equality follows from (1.6). We observe that \( np - n - p/2 > -1 \) as

\[
p > 1 > \frac{n-1}{n-1/2}.
\]

Thus,

(2.7) \[ \int_0^1 \|T_m f(x)\|_{L^p}^p \, dt \gtrsim K^{n^p-n-p/2+1}. \]

Let us now estimate the \( L^p \)-norm of \( f \). Since \( f(x) = K^n (F^{-1} \varphi)(Kx) \), we obtain

(2.8) \[ \|f\|_{L^p}^p = K^{np} \int_{\mathbb{R}^n} |(F^{-1} \varphi)(Kx)|^p \, dx \]
\[ = K^{np-n} \int_{\mathbb{R}^n} |(F^{-1} \varphi)(y)|^p \, dy \approx K^{np-n}. \]
Assume that inequality (1.7) is satisfied. Then, applying (1.7) with \(m = m_t\), integrating with respect to \(t\) and using Lemma 2, we get
\[
\int_0^1 \|T_{m_t}f(x)\|_{L^p}^p \, dt \leq C \|f\|_{L^p}^p,
\]
which implies, via (2.7) and (2.8), that
\[
K^{np-n-p/2+1} \leq C K^{np-n},
\]
or, equivalently,
\[
(2.9) \quad K^{1-p/2} \leq C.
\]
As \(p < 2\), we have \(\lim_{K \to \infty} K^{1-p/2} = \infty\), which contradicts (2.9). The proof is complete.

Acknowledgments. I am grateful to Andreas Seeger for useful discussions and to Loukas Grafakos for careful reading of this paper and valuable comments. I also thank the referees for their comments which helped to improve the paper.

References


Received June 11, 2018. Published online December 17, 2019.

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