On Cheeger and Sobolev differentials in metric measure spaces

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Abstract. Recently, Gigli developed a Sobolev calculus on non-smooth spaces using module theory. In this paper it is shown that his theory fits nicely into the theory of differentiability spaces initiated by Cheeger, Keith and others. A relaxation procedure for $L^p$-valued subadditive functionals is presented and a relationship between the module generated by a functional and the one generated by its relaxation is given. In the framework of differentiability spaces, which includes so called PI- and RCD($K,N$)-spaces, the Lipschitz module is pointwise finite dimensional. A general renorming theorem together with the characterization above shows that the Sobolev spaces of differentiability spaces are reflexive.

In the recent years, the analysis on metric measure spaces has made a lot of progress. Initiated by [19], [20] (see also [21], [27]), a relaxed notion of gradients, more precisely the norm of a relaxed gradient, was defined. Together with abstract Poincaré and doubling conditions a theory of minimizers resembling harmonic functions was developed (see [10], [22]). Spaces satisfying those conditions are now called PI-spaces. Those ideas were also incorporated in Cheeger’s generalized Rademacher theorem [11], where Cheeger proved existence of “chart functions" $(x_{i,n}: A_{i,n} \to \mathbb{R}^n)_{i,n=1}^\infty$, where $\{A_{i,n}\}_{i,n}$ is a Borel partition such that, for any Lipschitz function $f$, there are generalized Lipschitz differentials $Df$ with $Df|_{A_{i,n}}: A_{i,n} \to \mathbb{R}^n$ and for almost all $z_0 \in A_{i,n}$ it holds

$$f(z) = f(z_0) + \sum_{i=1}^{n} (Df(z))_j \cdot (x_{i,n}(z) - x_{i,n}(z_0)) + o(d(z,z_0)).$$

Later on, Keith [24] noticed that a weaker condition, called Lip-lip-condition, is sufficient to obtain the same structure. Bate [8] developed a dual formalism of this by constructing “sufficiently many tangent curves”, which give a more precise characterization of the differentials.

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On PI-spaces Cheeger also showed that his differentials could be used to obtain a Dirichlet form which allows for a natural definition of a linear Laplace operator and thus PDE-like harmonic functions. Combined with a form of the Bakry–Émery condition, those differentials were used to show Lipschitz continuity of harmonic functions [28]. The same structure was also used in the $L^p$-theory to generalize result from the smooth setting to the metric setting.

Independently there was a need for a PDE-like theory in metric spaces. Using the theory of optimal transport, it could be shown that lower bounds on the Ricci curvature can be identified with the natural heat flow induced by the gradient structure [23]. Ambrosio–Gigli–Savaré [4] developed for metric spaces a sufficiently strong calculus via relaxed gradients to give such an identification. The calculus was further developed in [16] and was then used in [5], [2], [14], [6] to show that on a subclass spaces behaving like generalized Riemannian manifolds, the lower Ricci curvature bounds in terms of optimal transport are equivalent to the lower bounds based on the analytic condition of Bakry–Émery. In a different paper [3] Ambrosio–Gigli–Savaré, showed that for not too bad metric spaces, their relaxed notion of gradient agrees with the weak upper gradient of Heinonen–Koskela–MacManus.

In a recent work [17], Gigli developed a Sobolev calculus which resembles the one in the smooth setting. He first constructs $L^p$-integrable “1-forms” and assigns to each Sobolev function a unique differential whose norm is given by the relaxed slope/gradient. Closedness of this assignment shows that any Sobolev function has a unique (Sobolev) differential. Using those ideas he develops a (weak) second order calculus on Riemannian-like spaces with lower Ricci curvature bounds.

In this paper, a precise description of the Sobolev differentials in terms of the Lipschitz differentials is presented. The main ingredient of Gigli’s construction was the functional

$$f \mapsto \int |Df|^p \, dm$$

which is a relaxation of

$$f \mapsto \int (\text{lip } f)^p \, dm.$$  

We show that one may construct a Lipschitz module using the latter and regard the cotangent module as a relaxed version of the Lipschitz module. With the help of a general relaxation procedure, it can be shown that the cotangent module is a quotient space of the Lipschitz module and a submodule called the Lipschitz $0$ module. This characterization immediately shows that on Lipschitz differentiable spaces the cotangent modules are locally finitely generated so that the Sobolev spaces are (super)reflexive. In PI-spaces which are infinitesimally Hilbertian, one can even show that the Cheeger differential structure is just a certain representation (with respect to some basis) of the Sobolev differentials.

This characterization can be explained as follows: the linear operator assigning to each Lipschitz map its Lipschitz differential is in general not closable. Its closure
assigns to each Sobolev map a whole affine subspace of “bounded 1-forms”. The relaxed slope is nothing but the distance of that affine subspace from the origin, which shows that the norm of the Sobolev differentials is a quotient norm. Pointwise minimality of the relaxed slope shows that the same holds for the generated module. In principle, one obtains the Sobolev space characterization more directly by looking at the pseudo-Sobolev space generated by Lipschitz functions and their Lipschitz differential. We present the module version as it translates directly into the language of Lipschitz differentiable structure.

The paper is structured as follows: in the first section normed modules are introduced and a representation theorem of locally finite dimensional modules is given. Afterwards, it is shown how to obtain an $L^p$-normed module given an $L^p$-valued functional that behaves like a pointwise norm. The next section gives a general relaxation procedure, and it is shown that a module and its relaxation can be characterized precisely if the module is weakly reflexive. The third section applies the abstract theory to Lipschitz and Sobolev functions. It is shown that on differentiability spaces the Lipschitz module is locally finite dimensional, which shows that all Sobolev spaces $W^{1,p}(M, m)$ are reflexive. In the end the relationship of Cheeger and Sobolev differentials is shown. The appendix contains an account on norms and a (unique) choice of scalar product.

Throughout the paper we have the following assumption: $(M, d, m)$ is a complete separable metric measure space, with $m$ a Radon measure. Note that by separability of $(M, d)$ and inner regularity of $m$, the measure $m$ is also $\sigma$-finite.

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1. Normed modules

In this section the theory of $L^p(m)$-normed spaces is introduced. In the context of metric spaces, $L^\infty(m)$-normed premodules appeared first in the work of Weaver [34]. Later Gigli [17] defined more general $L^p(m)$-normed modules to define generalized 1-forms and Sobolev differentials. We present Gigli’s construction independently of metric spaces and give a more precise representation of locally finite dimensional modules. That representation can be seen as an abstract version of Cheeger’s renorming theorem yielding the Cheeger differentiable structure.

**Definition 1.1 ($L^p(m)$-normed module).** A Banach space $(\mathcal{M}, \|\cdot\|_\mathcal{M})$ is an $L^p(m)$-normed module for $p \in (1, \infty)$ if there is a bilinear map $L^\infty(m) \times \mathcal{M} \to \mathcal{M}$, $(f, v) \mapsto f \cdot v$, and a map $\cdot : \mathcal{M} \to L^p(m)$, such that for every $v \in \mathcal{M}$ and $f, g \in L^\infty(m)$,

$$(fg) \cdot v = f \cdot (g \cdot v), \quad \|v\|_\mathcal{M} = \|v\|_{L^p(m)},$$

$$1 \cdot v = v, \quad |f \cdot v| = |f||v| \quad m\text{-a.e.},$$

where $1$ is the $L^\infty$-function which is $1$ everywhere.
In the definition we exclude \( p = 1 \) and \( p = \infty \) as we need uniform convexity of the \( L^p \)-norm. Note that Gigli showed ([17], 1.2.12 (ii,iv)) the conditions above imply the two additional properties:

- **(Locality)** Assume for \( v \in \mathcal{M} \) and \( \{A_n\}_{n \in \mathbb{N}} \) it holds \( \chi_{A_n} \cdot v = 0 \). Then \( \chi_{\cup A_n} \cdot v = 0 \).
- **(Gluing)** For every sequence \( (v_n)_{n \in \mathbb{N}} \) in \( \mathcal{M} \) and sequence of Borel sets \( (A_n)_{n \in \mathbb{N}} \) such that

\[
\chi_{A_i \cap A_j} \cdot v_i = \chi_{A_i \cap A_j} \cdot v_j \quad \text{and} \quad \limsup_{n \to \infty} \left\| \sum_{i=1}^{n} \chi_{A_i} \cdot v_i \right\|_{\mathcal{M}} < \infty,
\]

there is a \( v \in \mathcal{M} \) such that

\[
\chi_{A_i} \cdot v = \chi_{A_i} \cdot v_i \quad \text{and} \quad \|v\|_{\mathcal{M}} \leq \liminf_{n \to \infty} \left\| \sum_{i=1}^{n} \chi_{A_i} \cdot v_i \right\|_{\mathcal{M}}.
\]

A closed subspace \( \mathcal{N} \) of \( \mathcal{M} \) is said to be a submodule if it is also an \( L^p(\mathfrak{m}) \)-normed module. In particular, it needs to be stable under \( L^\infty(\mathfrak{m}) \)-multiplication and closed with respect to the locality and gluing constructions. In case \( \mathcal{N} \) is a submodule, we can equip the quotient space \( \mathcal{M}/\mathcal{N} = \mathcal{M}/\mathcal{N} \) with the following norm:

\[
\|v\|_{\mathcal{M}/\mathcal{N}} = \inf_{w \in \mathcal{N}} \|v + w\|_{\mathcal{M}}.
\]

Then \( \mathcal{M}/\mathcal{N} \) has natural \( L^p(\mathfrak{m}) \)-normed module structure satisfying the locality and gluing principle (see Proposition 1.2.14 in [17]).

**Example.** **(Vector-valued spaces)** The space \( L^p(M, \mathfrak{m}, \mathbb{R}^n, |\cdot|) \) of vector-valued \( L^p \)-functions such that

\[
\|v\|_{L^p}^p = \int |v(x)|^p \, dm,
\]

where \( x \mapsto |\cdot|_x \) is a measurable map into the space of norms \( \text{Norm}(\mathbb{R}^n) \) defined on \( \mathbb{R}^n \) (see appendix for properties of \( \text{Norm}(\mathbb{R}^n) \)). In case \( n = 0 \), the space is just the trivial vector space.

**\( (A\text{-submodule}) ** Let \( \mathcal{M} \) be an \( L^p(\mathfrak{m}) \)-normed module. For each measurable set \( A \subset \mathcal{M} \), the \( A \)-submodule \( \mathcal{A} \) is the submodule of all \( v \in \mathcal{M} \) with \( \chi_A \cdot v = v \), i.e., \( \{v \neq 0\} \subset A \) (compare [17], Proposition 1.4.6). It is easy to see that this is also a natural \( L^p(\mathfrak{m}_A) \)-normed module on \( A \) and that \( v \mapsto \chi_A v \) is a distance non-increasing projection onto the submodule. In case of vector-valued spaces, this means one may unambiguously write \( L^p(A, \mathfrak{m}_A, \mathbb{R}^n, |\cdot|) \leq L^p(M, \mathfrak{m}, \mathbb{R}^n, |\cdot|) \) if \( |\cdot|_x = |\cdot|_z \) for \( \mathfrak{m} \)-almost all \( x \in A \). Note that automatically \( L^p(A, \mathfrak{m}, \mathbb{R}^n, |\cdot|) = \{0\} \) if \( \mathfrak{m}(A) = 0 \).

**\( (L^p\text{-products}) ** Assume \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) are two \( L^p(\mathfrak{m}) \)-normed modules. Then the \( L^p \)-product \( \mathcal{M} \) of the modules is an \( L^p \)-normed module, i.e., for \( v = v_1 + v_2 \in \mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2 \) we set

\[
|v|^p = |v_1|^p + |v_2|^p
\]

for \( p \in [1, \infty) \) and \( |v| = \max\{|v_1|, |v_2|\} \) if \( p = \infty \).
One may take the \(L^p\)-product of countably many modules by requiring that 
\[ v = \sum_{i} v_i \in \mathcal{M} \text{ if and only if } |v| := (\sum_{i} |v_i|^p)^{1/p} \text{ is well-defined, i.e., } (|v_i|)_{i \in \mathbb{N}} \in \ell^p \text{ for } m\text{-almost all } x \in \mathcal{M}, \text{ and } |v| \text{ is in } L^p(m). \]

**Lemma 1.2** (Module partition). Assume \(\{A_n\}_{n \in \mathbb{N}}\) is a Borel partition of \(\mathcal{M}\). Then any \(L^p\)-normed module \(\mathcal{M}\) is the \(L^p\)-product of the \(A_n\)-submodules.

**Proof.** If \(v \in \mathcal{M}\), then \(v_n = \chi_{A_n} v \in \mathcal{M}_{A_n}\). Furthermore, it holds

\[
\|v\|^p_{L^p(\mathcal{M})} = \sum_{n \in \mathbb{N}} \int_{A_n} |v|^p \, dm = \sum_{n \in \mathbb{N}} \|v_n\|^p_{L^p(\mathcal{M}_{A_n})}.
\]

In particular, \(x \mapsto (|v_n(x)|)_{n \in \mathbb{N}} \in \ell^p\) for \(m\)-almost everywhere. \(\square\)

**Lemma 1.3.** If \(p \in (1, \infty)\), then any vector-valued \(L^p\)-space and any \(L^p\)-product of reflexive modules are reflexive.

**Remark.** Actually the proof shows that any vector-valued \(L^p\)-space is super-reflexive. Furthermore, the \(L^p\)-product of finitely many modules is also super-reflexive if each factor was.

**Proof.** The fact that \(L^p(M, \mathbb{R}^n, |\cdot|)\) is (super)reflexive follows from Theorem A.1. Indeed, if \(\Phi\) denotes the John scalar product selector, then \(x \mapsto |\cdot| = \Phi(|\cdot|)^{1/2}\) is also measurable and \(|\cdot| \leq |\cdot| \leq n|\cdot|\) for \(m\)-almost all \(x \in \mathcal{M}\). Thus \(L^p(M, \mathbb{R}^n, |\cdot|)\) is also an \(L^p\)-normed module and one can show that it is \(p\)-uniformly smooth if \(p \in (1, 2)\) and \(p\)-uniformly convex if \(p \in [2, \infty)\). In particular, it is (super-)reflexive.

For the second fact, just note that the dual of an \(L^p\)-product is the \(L^q\)-product of the duals. \(\square\)

We say that \(\{v_1, \ldots, v_n\} \subset \mathcal{M}\) is locally independent on \(A\) if for \(m\)-almost all \(x \in A\) the map \(v \mapsto |v|_x\) is a norm for the space spanned by \(\{v_1, \ldots, v_n\}\).

For \(L^p\)-normed modules this definition agrees with more general one defined in Definition 1.4.1 of [17] for general \(L^\infty(m)\)-modules.

In case of vector-valued \(L^p\)-functions, the statement

\[
\{v_1, \ldots, v_k\} \subset L^p(M, m, \mathbb{R}^n, |\cdot|)
\]

is locally independent on \(A\) is equivalent to saying that \(\{v_1(x), \ldots, v_k(x)\} \subset \mathbb{R}^n\) is linearly independent for \(m\)-almost all \(x \in A\). For the general case a similar characterization holds: for \(v \in \mathcal{M}\), the set \(\{v\}\) is locally independent on \(A\) if and only if \(|v|_x > 0\) for \(m\)-almost all \(x \in A\). Furthermore, if \(\{v_1, \ldots, v_n\}\) is locally independent on \(A\) but \(\{v_1, \ldots, v_n, v\}\) is not locally independent on \(A\), then there are a measurable set \(A' \subset A\) of positive \(m\)-measure and \(L^\infty(m)\)-functions \(\lambda_i\) such that \(|v - \sum_{i=1}^n \lambda_i v_i|_x = 0\) for all \(x \in A'\). The set \(\{v_1, \ldots, v_n, v\}\) is a maximal independent set on \(A\) if for all measurable subsets \(A' \subset A\) and all \(v \in \mathcal{M}\) the set \(\{v_1, \ldots, v_n, v\}\) is not locally independent on \(A'\).

Given that terminology, one may wonder what the maximal number of locally independent sets is and what happens if it is bounded globally. As it turns out, the vector-valued \(L^p\)-spaces are the building blocks of those modules.
Remark. (1) Reflexivity of any locally finite dimensional construction (see Section 2.2.1 in [17]).

Proof. By Lemmas 1.2 and 1.3, it suffices to show that \( \dim(M) \) is uniquely defined \( \mathbf{m} \)-almost everywhere. Using this function, we see that \( \dim(M, \cdot) \) is isometric to an infinite \( L^p \)-product of vector-valued \( L^p \)-spaces, i.e.,

\[
M \cong \oplus_{n=0}^{\infty} L^p(E_n, \mathbb{R}^n, | \cdot |).
\]

If the dimension is bounded by \( N \), then \( M \) admits a uniformly equivalent norm such that \( x \mapsto | \cdot |_x \) is almost everywhere induced by a scalar product. In particular, \( M \) is (super-) reflexive if \( p \in (1, \infty) \).

Remark. (1) Reflexivity of any locally finite dimensional \( L^p \)-normed module was already shown by Gigli (Theorem 1.4.7 in [17]) without the vector-valued representation.

(2) In an abstract language, this is exactly what Cheeger [11] did by constructing his Cheeger differential structure out of the Lipschitz differential structure. Later we will be more precise about that construction.

A general construction

In order to simplify the discussion on the relationship of the \( L^p \)-Lipschitz- and \( L^p \)-cotangent modules, we present an abstract version of Gigli’s cotangent module construction (see Section 2.2.1 in [17]).

Assume \( p \in (1, \infty) \) and denote by \( L^p_{\mathbb{R}}(\mathbf{m}) \) the set of non-negative \( L^p \)-integrable functions. Let \( L^0(M) \) be the space of measurable functions. There is a function \( F_p : D(F_p) \to L^p_{\mathbb{R}}(\mathbf{m}), f \mapsto g_f \) with \( D(F_p) \subseteq L^0(\mathbf{m}) \) such that for all \( f, h \in D(F_p) \)
and $\lambda \in \mathbb{R}$ it holds
$$g_{f+h} \leq g_f + g_h \quad \text{and} \quad g_{\lambda f} = |\lambda| g_f.$$ 

**Remark.** A similar construction also works for $p = \infty$; the evaluation is then with respect to the $L^\infty$-norm. $Pfm$ (see below) is then defined whenever $\sup_i \{\|g_i\|\} < \infty$.

Define the set $Pfm$ generated by $F_p$ as follows:
$$Pfm := \{ ((f_i, A_i))_{i \in \mathbb{N}} | (A_i)_{i \in \mathbb{N}} \text{ is a Borel partition of } M, \ f_i \in D(F_p), \text{ and } \sum_{i \in \mathbb{N}} \int_{A_i} g_i^p \, dx < \infty \}.$$ 

On $Pfm$, define the equivalence relation $\{ (f_i, A_i) \}_{i \in \mathbb{N}} \sim \{ (h_j, B_j) \}_{j \in \mathbb{N}}$ if $g_{f_i-h_j} = 0 \ \text{m-almost every on } A_i \cap B_j$.

Indeed, reflexivity of $\sim$ follows from homogeneity, symmetry follows from the definition, and transitivity from subadditivity of $f \mapsto g_f$.

Now it is easy to verify that $Pfm/\sim$ can be made into a vector space by the following definitions:
$$[(f_i, A_i)_{i \in \mathbb{N}}] + [(h_j, B_j)_{j \in \mathbb{N}}] = [(f_i + h_j, A_i \cap B_j)_{i,j \in \mathbb{N}}]$$
and
$$\lambda [(f_i, A_i)_{i \in \mathbb{N}}] = [(\lambda f_i, A_i)_{i \in \mathbb{N}}].$$

**Remark.** Below we deal with multiple $L^p$-valued functionals. In that case, we may put an $F_p$-index at the equivalence relation $[\cdot]$ to avoid confusion.

If $a \in L^\infty(m)$ is a simple function, i.e., $a = \sum_{j \in \mathbb{N}} a_j \chi_{B_j}$ with $|a_j| \leq \|a\|_\infty$, and $(B_j)_i$ is a Borel partition of $M$, then
$$a[(f_i, A_i)_{i \in \mathbb{N}}] = [(a_j f_i, A_i \cap B_j)_{i,j \in \mathbb{N}}].$$

Also define for each $v = [(f_i, A_i)_{i \in \mathbb{N}}] \in Pfm/\sim$ a measurable map $x \mapsto |v|(x)$ by
$$||(f_i, A_i)_{i \in \mathbb{N}}|| = g_{f_i} \ \text{m-almost everywhere on } A_i, \forall i \in \mathbb{N}.$$ 

The assignment $|\cdot|$ is a (pointwise) semi-norm on $Pfm/\sim$ compatible with multiplication by simple $L^\infty(m)$-functions:
$$||(f_i + h_j, A_i \cap B_j)_{i,j \in \mathbb{N}}|| \leq ||(f_i, A_i)_{i \in \mathbb{N}}|| + ||(h_j, B_j)_{j \in \mathbb{N}}||$$
and
$$|\lambda||(f_i, A_i)_{i \in \mathbb{N}}|| = |\lambda||||(f_i, A_i)_{i \in \mathbb{N}}||.$$ 

Finally, it is readily verified that $\| \cdot \|_{\mathcal{F}_p} : Pfm/\sim \to [0, \infty)$ defined by
$$|||f_i, A_i)_{i \in \mathbb{N}}||_{\mathcal{F}_p} = \||||f_i, A_i)_{i \in \mathbb{N}}||_{L^p} = \sum_{i \in \mathbb{N}} \int_{A_i} g_{f_i}^p \, dm$$
is a (possibly incomplete) norm on $Pfm/\sim$. From the properties above one can show that the completion of $Pfm$ will be an $L^p$-normed module.

**Definition 1.6** ($F_p$-module). The $L^p$-normed module $(\mathcal{M}_{\mathcal{F}_p}, \| \cdot \|_{\mathcal{F}_p})$ is the completion of $(Pfm/\sim, \| \cdot \|_{\mathcal{F}_p}).$
As the construction is essentially unique, we just say \( (\mathcal{M}_F, \| \cdot \|_F) \) is the \( F \)-module. Also define the operator \( d_{\mathcal{F}_p} : D(\mathcal{F}_p) \to \mathcal{M}_F \) by
\[
d_{\mathcal{F}_p}(f) = [f, M].
\]

**Lemma 1.7.** Assume \( F'_p \) and \( F_p \) are two functionals such that \( D(\mathcal{F}'_p) \subset D(\mathcal{F}_p) \) and \( F'_p(f) = F_p(f) \) for all \( f \in D(\mathcal{F}'_p) \). Then \( \mathcal{M}_{\mathcal{F}_p} \) can be uniquely identified with a submodule of \( \mathcal{M}_{\mathcal{F}'_p} \).

**Corollary 1.8.** If, in addition, for every \( f \in D(\mathcal{F}_p) \) there is a sequence \((f_n)_{n \in \mathbb{N}} \subset D(\mathcal{F}'_p)\) such that \( f_n \to f \) in \( L^0(\mathfrak{m}) \) and \( d_{\mathcal{F}_p} f_n \to d_{\mathcal{F}_p} f \) in \( \mathcal{M}_F \), then \( D(\mathcal{F}'_p) \) generates \( \mathcal{M}_{\mathcal{F}'_p} \).

**Proof.** The lemma is a consequence of the construction of the modules. Because
\[
\|[f, A]_{\mathcal{F}'_p} \|_{\mathcal{F}'_p} = \|[f, A]_{\mathcal{F}_p} \|_{\mathcal{F}_p}
\]
for all \( f \in D(\mathcal{F}'_p) \) and all Borel sets \( A \in \mathcal{B}(\mathfrak{m}) \), the assignment
\[
i : [f, A]_{\mathcal{F}_p} \mapsto [f, A]_{\mathcal{F}_p}
\]
is an isometric embedding of \( \text{Pfm}_{\mathcal{F}_p} \) into \( \text{Pfm}_{\mathcal{F}_p} \) which extends uniquely to an isometric embedding \( i : \mathcal{M}_{\mathcal{F}_p} \to \mathcal{M}_{\mathcal{F}'_p} \). Thus the image \( i(\mathcal{M}_{\mathcal{F}_p}) \) is a submodule of \( \mathcal{M}_{\mathcal{F}'_p} \).

To see the corollary, just note \( d_{\mathcal{F}_p} f_n \to d_{\mathcal{F}_p} f \) also implies
\[
\chi_A d_{\mathcal{F}_p} f_n \to \chi_A d_{\mathcal{F}_p} f = [f, A]_{\mathcal{F}_p}
\]
for all Borel sets \( A \in \mathcal{B}(\mathfrak{m}) \). Thus the generating set \( \{[f, A]_{\mathcal{F}_p} \mid f \in D(\mathcal{F}_p), A \in \mathcal{B}(\mathfrak{m}) \} \) of \( \mathcal{M}_{\mathcal{F}_p} \) is a subset of \( \mathcal{M}_{\mathcal{F}'_p} \), which shows that it also generates \( \mathcal{M}_{\mathcal{F}'_p} \). In particular, the two modules agree. \( \square \)

### 2. Relaxed functionals and their modules

In this section we present a general relaxation procedure of subadditive \( L^p \)-valued functionals which fits into the framework of generalized gradients. Furthermore, we give a representation of the modules generated by a functional and its relaxation. The construction here is more general and could be simplified if we look at the local Lipschitz constants (see next section). However, some of the results might be of interest in other contexts.

In the following assume \( p \in (1, \infty) \), which implies reflexivity of \( L^p(\mathfrak{m}) \). Denote by \( L^0(\mathfrak{m}) \) the space of equivalence classes of measurable real-valued functions where two functions are identified if they agree \( \mathfrak{m} \)-almost everywhere. Since \( \mathfrak{m} \) is a \( \sigma \)-finite measure, the convergence in \( \mathfrak{m} \)-measure is induced by a metrizable topology. Note also that \( f_n \to f \) in \( L^0(\mathfrak{m}) \) implies that for some subsequence \((m_n)_{n \in \mathbb{N}} \) and \( \mathfrak{m} \)-almost all \( x \in \mathfrak{M} \) it holds \( f_{m_n}(x) \to f(x) \). We frequently replace \( f_n \) with such a subsequence whenever we do \( L^0(\mathfrak{m}) \)-approximations.
A relaxation procedure

In this section we present the relaxation procedure of [4], Section 4, for general subadditive, absolutely homogeneous $L^p(m)$-valued functional $F_p$. We call this procedure the upper relaxation of $F_p$. Furthermore, we propose another relaxation procedure, called lower relaxation of $F_p$, which satisfies a “pointwise” minimality property. Whereas the upper relaxation fits nicely into vector space relaxations, the lower relaxation is suited to study relaxations of $L^p(m)$-normed modules. Note the “pointwise” minimality property is known to hold for the (minimal) relaxed slope showing that both relaxations agree, see Lemma 4.4 in [4].

The technique presented here shares similarities with the localization method in the theory of $\Gamma$-convergence. As we are only looking at $L^p$-valued functionals, we obtain a very precise description of the lower relaxation via Lemma 2.5 and Theorem 2.8.

Remark. The choice of $L^0(m)$ is not essential. Any topological vector space satisfying the conclusion of Lemma 2.3 will do. For sake of assigning names, let us call this property topological Mazur property. Any locally convex topological vector space satisfies this property, in particular, any Banach and Fréchet space.

Assume $D(F_p)$ is a linear subspace of $L^0(m)$ and let $F_p: D(F_p) \to L^p(m)$ be subadditive and absolutely homogeneous, i.e.,

$$F_p(f + g) \leq F_p(f) + F_p(g) \quad \text{and} \quad F_p(\lambda f) = |\lambda| F_p(f)$$

for all $f, g \in D(F_p)$ and $\lambda \in \mathbb{R}$. We extend $F_p$ outside of $D(F_p) \subset L^0(m)$ by setting it to $\infty$. In general $D(F_p)$ is not dense and the functional $E F_p$ not lower semicontinuous, where

$$E F_p(f) = \begin{cases} \int f \varrho_p^p \, dm & f \in D(F_p), \\ \infty & \text{otherwise}. \end{cases}$$

Define the following lower semi-continuous functional:

$$E \tilde{F}_p(f) = \inf \left\{ \liminf_{n \to \infty} \int g^p \varrho_n^p \, dm \mid D(F_p) \ni f_n \to f \text{ in } L^0(m) \right\}.$$ 

This functional is called the lower semicontinuous hull of $E F_p$. Let $D(\tilde{F}_p)$ be the set of all $f \in L^0(m)$ with $E \tilde{F}_p(f) < \infty$. We immediately obtain the following characterization of the hull.

**Lemma 2.1.** If $E: L^0(m) \to [0, \infty]$ is lower semicontinuous with $E \leq E F_p$, then $E \leq E \tilde{F}_p$. If, in addition, for every $f \in D(E)$ there is a sequence $D(F_p) \ni f_n \to f$ with $E(f) = \lim_{n \to 0} EF_p(f_n)$, then $E = E \tilde{F}_p$.

The following theorem shows that $E \tilde{F}_p$ admits an $L^p$-valued functional $\tilde{F}_p f \mapsto \tilde{g}_f$ such that $E \tilde{F}(f) = ||\tilde{g}_f||_{L^p}$, hence justifying the notation.

**Proposition 2.2 (Relaxed representation).** For any $f \in D(\tilde{F}_p)$ there is a unique $L^p(m)$-function $\tilde{F}_p(f) = \tilde{g}_f$, with $E \tilde{F}_p(f) = ||\tilde{g}_f||_{L^p}$, such that whenever $D(F_p) \ni F_p(f) \to \tilde{g}_f$.

**Proof.**
To prove this we need the following technical lemma.

Lemma 2.3 (Topological Mazur property). If \( f_n \to f \) in \( L^0(m) \), then any sequence \( (h_n)_{n \in \mathbb{N}} \) such that \( h_n \) is a (finite) convex combination of \( \{f_k\}_{k \geq n} \) converges to \( f \) in \( L^0(m) \) as well.

Proof. If \( (h_n)_{n \in \mathbb{N}} \) is a sequence of finite convex combination of \( \{f_k\}_{k \geq n} \), then \( h_n \) is of the form \( \sum_{k \geq n} \alpha_k f_k(x) \) for some \( \alpha_k \in [0, 1] \) with \( \sum_{k \geq n} \alpha_k = 1 \), and \( \{k \geq n \mid \alpha_k > 0\} \) is finite.

Let \( E \subset M \) be a measurable set with \( m(E) < \infty \). By Egorov’s theorem, for every \( \epsilon > 0 \) there is a measurable set \( E_\epsilon \) such that \( m(E_\epsilon \setminus E) < \epsilon \) and \( f_n \to f \) uniformly on \( E_\epsilon \). In particular, for each \( \delta > 0 \) there is an \( n_\delta \in \mathbb{N} \) such that \( |f_n(x) - f(x)| \leq \delta \) for all \( n \geq n_\delta \) and \( x \in E_\epsilon \). In particular, for \( n \geq n_\delta \) it holds

\[
|h_n(x) - f(x)| \leq \sum_{k \geq n_\delta} \alpha_k |f_k(x) - f(x)| \leq \delta.
\]

But this implies that \( h_n \) converges uniformly to \( f \) on \( E_\epsilon \) for all \( \epsilon > 0 \). Thus \( h_n \to f \) \( m \)-almost everywhere on \( E \). Because \( E \) was arbitrary, we see that \( h_n \to f \) in \( L^0(m) \).

Proof of Proposition 2.2. First let \( f_n \to f \) with \( E \bar{F}_p(f) = \lim_{n \to \infty} \int g_{f_n}^p \, dm \leq \infty \). Then one may replace \( (g_{f_n})_{n \in \mathbb{N}} \) with a subsequence and assume \( (g_{f_n})_{n \in \mathbb{N}} \) converges weakly in \( L^p(m) \) to \( g \). Mazur’s lemma implies that there is a sequence \( g_n = \sum_{i=0}^{N_n} \alpha_i^n g_i \) strongly converging in \( L^p \) to \( g \). By subadditivity and homogeneity of \( f \to g_f \), we also have \( g_n \leq g_n \) for \( h_n = \sum_{i=0}^{N_n} \alpha_i^n f_i \). The previous lemma shows \( h_n \to f \). But then

\[
E \bar{F}(f) \leq \lim \inf_{n \to \infty} \int g_{h_n}^p \, dm \leq \lim \sup_{n \to \infty} \int g_{h_n}^p \, dm \leq \lim_{n \to \infty} \int g_{f_n}^p \, dm = E \bar{F}(f),
\]

which implies that \( \|g_{f_n}\|_{L^p} \to \|g\|_{L^p} \) and therefore \( g_{f_n} \to g \) strongly in \( L^p(m) \).

We claim that \( g \) is the unique among all (weak) \( L^p \)-limits \( g \) of sequences \( (g_{f_n})_n \) with \( f_n \to f \) in \( L^0(m) \) and \( E \bar{F}_p(f) = \|g\|_{L^p}^p \): assume for some sequences \( f_n, f'_n \to f \) it holds \( g_{f_n} \to g \) and \( g_{f'_n} \to g' \) with \( D \bar{F}(f) = \|g\|_{L^p}^p = \|g'\|_{L^p}^p \). Then for \( h_n = (f_n + f'_n)/2 \) we have \( g_{h_n} \leq (g_{f_n} + g_{f'_n})/2 \), so that uniform convexity of the \( L^p \)-norm implies \( g = g' \), which implies uniqueness.

Using the strong approximation of \( g_f \), we obtain immediately the following corollary.

Corollary 2.4. If \( E \bar{F}_p \) is lower semicontinuous on \( D(F_p) \), then \( g_f = \bar{g}_f \) for all \( f \in D(F_p) \).

An equivalent characterization of \( \bar{g}_f \) can be given as follows (compare Definition 4.1 in [4]): we say that \( G \) is an upper relaxation at \( f \) if there is a sequence \( f_n \to f \) in \( L^0(m) \) with \( g_{f_n} \to \tilde{G} \) weakly in \( L^p(m) \) and \( \tilde{G} \leq G \) \( m \)-almost everywhere.
Denote the set of upper relaxations at $f$ by $\hat{G}_f$. One can show that the set of upper relaxations at $f$ is convex and closed. Furthermore, it is non-empty if and only if $f \in D(\mathcal{F}_p)$. In such a case uniform convexity of the $L^p$-norm implies that there is a unique element of minimal norm. By Proposition 2.2, this is given as the strong limit $\hat{g}_f$ of $(g_{f_n})$ of a sequence $D(\mathcal{F}_p) \ni f_n \to f$ in $L^0(m)$. We call $\hat{g}_f$ the \textit{minimal upper relaxation}.

Proposition 2.2 can be generalized as follows.

\textbf{Lemma 2.5.} Assume $f_n \to f$ in $L^p(m)$ with $f_n \in D(\mathcal{F}_p)$ and $g_{f_n} \to g$ weakly in $L^p(m)$. Then there is a unique $g \in L^p(m)$ such that for any sequence $(h_n)_{n \in \mathbb{N}}$ of finite convex combinations of \{$(f_k)_{k \geq n}$\} converging in $L^0(m)$ to $f$ and any $L^p$-limit $\hat{g}$ of $(g_{h_n})_{n \in \mathbb{N}}$ it holds $\|g\|_{L^p} \leq \|\hat{g}\|_{L^p}$. Furthermore, there is a sequence $(h_n)_{n \in \mathbb{N}}$ of finite convex combinations of \{$(f_k)_{k \geq n}$\} with $g_{h_n} \to g$ strongly in $L^p(m)$. In addition, for all sequences $(h_n')_{n \in \mathbb{N}}$ of finite convex combinations of \{$(h_k)_{k \geq n}$\}, it holds $g_{h_n'} \to g$ strongly in $L^p(m)$.

\textbf{Remark.} The uniqueness is with respect to the sequence $f_n$. It may happen that $g_{f_1}, g_{f_2} \to g$ with $g_1 \neq g_2$.

\textbf{Proof.} Let $\mathcal{I}_n \subset 2^n$ such that $I \in \mathcal{I}_n$ if and only if $|I| < \infty$ and $\min I \geq n$. Note that $\mathcal{I}_n$ is countable and $\mathcal{I}_n \supset \mathcal{I}_{n'}$ for $n \leq n'$. Define

$$A_n = \{(\alpha_i)_{i \in \mathbb{N}} \mid \alpha_i \geq 0, \sum \alpha_i = 1, \exists I \subset \mathcal{I}_n : \alpha_i \neq 0 \Rightarrow i \in I\}$$

and set

$$M = \inf \left\{ \liminf_{n \to 0} \int g_{h_n}^p \, d\mathbf{m} \mid h_n = \sum_{i \in \mathbb{N}} \alpha_i^n f_n \text{ for some } (\alpha_i^n)_{i \in \mathbb{N}} \in A_n \right\}.$$ 

Note that $g_{h_n}$ is well-defined by subadditivity and homogeneity.

The definition of $M$ yields a sequence $(\alpha_i^n)_{i \in \mathbb{N}} \in A_n$ such that

$$M = \lim_{n \to \infty} \int g_{h_n}^p \, d\mathbf{m}.$$ 

In particular, $(g_{h_n})_{n \in \mathbb{N}}$ is bounded in $L^p(m)$, so after choosing a subsequence and relabeling assume $g_{h_n} \to g$ weakly in $L^p(m)$. We claim $M = \|g\|_{L^p}^p$. Indeed, this claim would yield that $g_{h_n} \to g$ strongly, so that $h_n$ is the required sequence.

The proof below will also show that any finite convex combination of the tails of $(h_n)_{n \in \mathbb{N}}$ has the same property.

Let $(\beta_i^n)_{i \in \mathbb{N}} \in A_n$ be given by Mazur’s lemma applied to $(g_{h_n})$, i.e.,

$$\sum_i \beta_i^n g_{h_i} \to g$$

strongly in $L^p(m)$. Then there is a sequence $(\alpha_i^n)_{i \in \mathbb{N}} \in A_n$ such that $\sum_i \beta_i^n h_i = \sum \alpha_i^n f_n = \hat{h}_n$, and by subadditivity and homogeneity,

$$g_{\hat{h}_n} \leq \sum_{i \in \mathbb{N}} \beta_i^n g_{h_i}.$$
Putting those facts together and using convexity of \( r \mapsto |r|^p \) yields
\[
M \leq \liminf_{n \to \infty} \int g_{h_n}^p \, dm \leq \lim_{n \to \infty} \int \sum_{i \in \mathbb{N}} (\beta_i^n g_{h_i})^p \, dm \leq \lim_{n \to \infty} \sum_{i \in \mathbb{N}} (\beta_i^n)^p \int g_{h_i}^p \, dm \leq M^p.
\]
As \( \lim_{n \to \infty} \int \sum_{i \in \mathbb{N}} (\beta_i^n g_{h_i})^p \, dm = \int g^p \, dm \), we proved the claim. Observe that the proof also shows that any \( (g_{h_i})_{n \in \mathbb{N}} \) with \( (h_i^n)_{n \in \mathbb{N}} \) being a finite convex combination of \( \{h_k\}_{k \geq n} \) converges strongly to \( g \) as well.

It remains to show uniqueness. Assume \( g_1 \) and \( g_2 \) are obtained by some sequences \( (h_{1,i})_{i \in \mathbb{N}} \) and \( (h_{2,i})_{i \in \mathbb{N}} \). By subadditivity, \( g_{2,i} \leq g_{1,i} + g_{2,i} \), where \( h_{2,i} = (h_{1,i} + h_{2,i})/2 \). Then any weak accumulation point \( g_3 \) of \( (g_{2,i})_{i \in \mathbb{N}} \) must satisfy \( g_3 \leq (g_1 + g_2)/2 \). An argument as above shows \( (g_{2,i}) \) converges strongly to \( g_3 \), with \( \|g_3\|_{L^p} = M \). But \( \|g_1\| = \|g_2\| = M \), so that strict convexity of the \( L^p \)-norm shows \( g_1 = g_2 = g_3 \).

As it turns out, the set of \( g \) obtained via the lemma are essential for the characterization of relaxed modules defined below.

**Definition 2.6 (Strong upper relaxation).** Let \( \hat{G}_f^p \) be the set of \( G \in \hat{G}_f \) such that there is a sequence \( f_n \to f \) in \( L^0(m) \) so that \( g_{h_n} \to G \) strongly in \( L^p(m) \) for every finite convex combination \( h_n \) of \( \{f_k\}_{k \geq n} \).

By the previous lemma, \( \hat{G}_f^p \) is non-empty whenever \( \hat{G}_f \) is as \( \hat{g}_f \in \hat{G}_f^p \). Furthermore, every \( G \in \hat{G}_f \) admits a strong relaxation \( G' \) with \( G' \leq G \).

**Remark.** One may think of \( g_f \) as the norm of some linear assignment to a vectors like the gradient/differential of \( f \). If an upper relaxation represents the norm of a vector obtained via a closure procedure of that assignment, then it needs to be invariant under taking limits of finite convex combinations of tail of the limiting construction. In particular, their norm needs to satisfy this property. Hence the only upper relaxations which can satisfy this property are strong upper relaxations.

A priori it is not clear whether \( f_n \to f \) with \( \hat{g}_{f_n} \to G \) implies \( \hat{g}_f \leq G \) \( m \)-almost everywhere. This would be true if \( g_f \) is the local Lipschitz constant and the metric measure space is locally finite (see below). As the codomain of \( F_p \) is \( L^p(m) \), those functions enjoy certain local properties, like integrability and convergence with respect to subsets. An appropriate relaxation procedure should take this into account. However, the relaxation procedure above only requires uniform convexity of the \( L^p \)-norm and is therefore “only” a relaxation of \( DF_p \). In the following we give an alternative construction which satisfies this property and fits nicely into the module framework.

Let \( \hat{G}_f \) be the set of all \( G \in L^p(m) \) such that the following holds: whenever \( f_n \to f \) in \( L^0(m) \) and \( g_{f_n} \to \hat{g} \) weakly in \( L^p(m) \), then \( G \leq \hat{G} \). It is easy to see that \( \hat{G}_f \) is a closed convex set and if \( G_1, G_2 \in \hat{G}_f \), then also \( \max\{G_1, G_2\} \in \hat{G}_f \). Furthermore, the set is always non-empty as it contains the zero element. It is bounded if the defining condition is non-trivial, i.e., if there is a sequence \( f_n \to f \) with \( (g_{f_n})_{n \in \mathbb{N}} \) bounded in \( L^p(m) \).
If $\hat{G}_f$ is bounded, we claim that it has a unique element of maximal $L^p$-norm which is given by

$$\hat{g}_f = \max_{G \in \hat{G}_f} G.$$  

Indeed, since $\hat{G}_f$ is a closed subset of $L^p(m)$ which is closed under taking the maximum-operator, $\hat{g}_f$ is a monotone $L^p$-limit of functions $G_n \in \hat{G}_f$. Note that we only required $f \in D(\hat{F}_p)$.

We call $\hat{g}_f$ the lower relaxation $F_p$ at $f \in D(\hat{F}_p)$. For notational purposes also set $D(\hat{F}_p) = D(\hat{F}_p)$ and $\hat{F}_p(f) = \hat{g}_f$.

From the characterization of the $\hat{g}_f$ and $\hat{g}_f$ we obtain the following (compare Section 4 in [4]).

**Corollary 2.7.** The lower relaxation equals a.e. the pointwise minimum of all upper relaxations at $f$. Therefore, if whenever $g_{f_n} \rightarrow G$ it holds $\hat{g}_f \leq G$, then $\hat{g}_f = \hat{g}_f$.

**Remark.** A sufficient condition to get $\hat{g}_f = \hat{g}_f$ for all $f \in D(\hat{F}_p)$ is that $\chi_B G + \chi_M D G'$ is an upper relaxation at $f$ whenever $G$ and $G'$ are upper relaxations at $f$ and $B$ a Borel set. This is known to hold for the relaxed slope, see [4], Lemma 4.4.

Obviously $f \mapsto \hat{g}_f$ is absolutely homogeneous. To see subadditivity, note that it holds

$$g_{f_n + h_n} \leq g_{f_n} + g_{h_n}.$$ 

Thus, whenever $(g_{f_n}, g_{h_n})_{n \in \mathbb{N}}$ weakly to $(G_1, G_2)$ then any weak limit $G_3$ of the sequence $(g_{f_n + h_n})_{n \in \mathbb{N}}$ must satisfy $G_3 \leq G_1 + G_2$.

From the definition we also see that

$$E_{\hat{F}_p}(f) = \begin{cases} \int \hat{g}_f^p \, dm & f \in D(\hat{F}_p), \\ \infty & \text{otherwise}, \end{cases}$$

is lower semicontinuous and $\hat{g}_{f_n} \rightarrow g$ weakly in $L^p(m)$ with $f_n \rightarrow f$ in $L^0(m)$ and $\|g\|_L^p = E_{\hat{F}_p}(f)$ implies $\hat{g}_{f_n} \rightarrow g$ strongly and $g = \hat{g}_f$.

We conclude with a local characterization of $\hat{g}_f$.

**Theorem 2.8 (Local approximation).** For any $\epsilon > 0$ and $f \in D(\hat{F}_p)$ there are a Borel partition $\{A_n\} \cup A_\infty$ with $m(A_\infty) = 0$ and a sequence $(G_n)_{n \in \mathbb{N}}$ of (strong) upper relaxations at $f$ such that

$$\sum_{n \in \mathbb{N}} \chi_{A_n} G_n - h_\epsilon \leq \hat{g}_f,$$

where $h_\epsilon$ is some non-negative $L^p$-integrable function with $\|h_\epsilon\|_{L^p}, \|h_\epsilon\|_{L^\infty} \leq \epsilon$. If $\hat{g}_f = \hat{g}_f$, we can simply choose $A_1 = M$, $G_1 = \hat{g}_f$ and $h_\epsilon \equiv 0$.

**Proof.** Note that $\hat{G}_f$ is non-empty because $f \in D(\hat{F}_p) = D(\hat{F}_p)$. Let $\{G_n\}_{n \in \mathbb{N}}$ be a countable dense set of $\hat{G}_f$. Let $\hat{G}_n = \min_{k=1}^n G_n$ so that $\hat{G}_n \geq \hat{G}_{n+1}$. (Note that $\hat{G}_n \in \hat{G}_f$ would yield $\hat{g}_f = \hat{g}_f$). From the definition, $G = \lim_{n \rightarrow \infty} \hat{G}_n$ is
measurable. As $G_0 \leq G_1^p$ with $G_1^p \in L^1(m)$, the dominated convergence theorem implies
\[
\int G_0 \, dm = \lim_{n \to \infty} \int G_n^p \, dm
\]
and also $\tilde{G}_n \to G$ strongly in $L^p(m)$.

We claim $G = \tilde{g}_f$. For this it suffices to show that $G \leq G'$ for any $G' \in \tilde{G}_f$. Suppose by contradiction there is a measurable set $A$ of positive $m$-measure and a $\delta > 0$ with $G \geq G' + \delta$. Since $(G_n)_{n \in \mathbb{N}}$ is dense in $G_f$ we may find a subsequence $(n_k)_{k \in \mathbb{N}}$ such that $G_{n_k} \to G'$ strongly in $L^p(m)$. In particular, there is a subset $A' \subseteq A$ such that $G_{n_k} |_{A'} - G' |_{A'} \leq \delta/2$ on $x \in A'$ for all $k \geq k_0$. However, we have $G_{n_k} \geq G$, so that
\[
\delta \leq G_{|A'} - G'_{|A'} \leq (G_{n_k})_{|A'} - G'_{|A'} \leq \frac{\delta}{2},
\]
which is a contradiction.

The proof is finished by a standard approximation procedure: let $A_0 \subseteq M$ be a set of positive and finite measure. As $G_n \to \tilde{g}_f$ strongly in $L^p(m)$, we can also assume it converges $m$-almost everywhere on $A_0$. Then Egorov’s theorem implies that for each $\delta > 0$ and $\epsilon > 0$ there are a measurable set $A_1 \subseteq A_0$ and $n_{\epsilon}$ such that $m(A_1 \setminus A_0) < \delta m(A)$ and $\chi_{A_1}(G_n - \tilde{g}_f) \leq \epsilon$. But then there is a finite Borel partition $\{A_{1,k}\}_{k \in \mathbb{N}}$ of $A_1$ and an increasing sequence of indices $(n_{1,k})_{k \in \mathbb{N}}$ such that $\chi_{A_{1,k}}(G_{n_{1,k}} - \tilde{g}_f) \leq \epsilon$. In particular, the result holds on $A_1$. Without loss of generality, we may replace each $G_{n_{1,k}}$ with its strong upper relaxation obtained from Lemma 2.3.

Repeating this argument, we obtain a sequence $\{A_n\}_{n \in \mathbb{N}}$ with $A_{n+1} \subseteq A_0 \setminus A_n$, partitions $\{A_{n,k}\}_{k=1}^{k_n}$ and indices $\{n_{n,k}\}_{k=1}^{k_n}$ such that the above holds on $\bigcup_{n \in \mathbb{N}} A_n$. Note that the construction shows that $m(A_0 \setminus \bigcup_{n} A_n) = 0$ and $A_n \cap A_{n'} = \emptyset$ for $n \neq n'$, so that $\{A_n\}_{n \geq 1} \cup \{A_{\infty}\}$ is the required partition.

If $m(M) < \infty$, we are done. For the non-finite $\sigma$-finite case, let $\{A^{(k)}\}_{k \in \mathbb{N}}$ be a Borel partition of $M$ with $1 \leq m(A^{(k)}) < \infty$. On $A^{(k)}$ choose $c_k^p = e^{p/2 - (k+1)m(A^{(k)})^{-1}}$ and set $h_k = \sum \chi_{A^{(k)}} \epsilon_k$. Then $\|h_k\|_{L^p} = \epsilon \geq h_\omega$. The result is now obtained by collecting the countable number of partitions and indices. \qed

**Characterization of the relaxed module**

Given $\mathcal{F}_p$ as above, let $\tilde{\mathcal{F}}_p$ be its lower relaxation. Since it is also subadditive and absolutely homogeneous, it induces the module $\mathcal{M}_{\tilde{\mathcal{F}}_p}$ which we call the relaxed module of $\mathcal{M}_{\mathcal{F}_p}$. Using Theorem 2.8, we show that there is a general relationship between $\mathcal{M}_{\mathcal{F}_p}$ and its relaxed module.

Recall $d_{\mathcal{F}_p} f = [f, M]_{\mathcal{F}_p}$. We will look at the following set:
\[
G_0 = \{ \omega \in \mathcal{M}_{\mathcal{F}_p} \mid \exists f_n \in D(\mathcal{F}_p) : f_n \to 0 \text{ in } L^0, d_{\mathcal{F}_p} f \to \omega \text{ in } \mathcal{M}_{\mathcal{F}_p} \}.
\]
One may verify that $G_0$ is a closed subspace of $\mathcal{M}_{\mathcal{F}_p}$. In the following denote by $\tilde{M}$ and $\tilde{M}_0$, the module $\mathcal{M}_{\tilde{\mathcal{F}}_p}$, its relaxed module $\mathcal{M}_{\tilde{\mathcal{F}}_p}$ and resp. the submodule generated by $G_0$. 

In general, it is not clear whether \( G_0 \) contains more than one element. However, if \( G_0 \) is trivial the operator \( d_\mathcal{F}_p : D(\mathcal{F}_p) \to \mathcal{M} \) is closable as a (partially defined) linear operator from \( L^0(\mathfrak{m}) \) to \( \mathcal{M} \) (compare also to Proposition 4.26 in [30]). In general, closability may not yield lower semicontinuity of \( E\mathcal{F}_p \). As it turns out, it is possible to characterize the relationship of the three modules if \( \mathcal{M} \) satisfies a weak form of reflexivity. First, some technical results.

**Lemma 2.9** (Closedness). Assume \( D(\hat{\mathcal{F}}_p) \ni f_n \to f \) and \( d_{\mathcal{F}_p} f_n \to \omega \in \hat{\mathcal{M}} \). Then \( f \in D(\hat{\mathcal{F}}_p) \) and \( \omega = d_{\mathcal{F}_p} f \).

**Proof.** The proof is just the abstract version of Theorem 2.2.9 in [17]: the assumptions imply \( \hat{g}_f \to |\omega| \) in \( L^p \). As \( E\mathcal{F}_p \) is lower semicontinuous we must have \( f \in D(\hat{\mathcal{F}}_p) \). Again by lower semicontinuity we have

\[
\| d_{\mathcal{F}_p} (f - f_n) \|_{\hat{\mathcal{M}}} = \int g_{f - f_n}^p \, d\mathfrak{m} \leq \liminf_{m \to \infty} \| d_{\mathcal{F}_p} (f_m - f_n) \|_{\hat{\mathcal{M}}}.
\]

However, \( d_{\mathcal{F}_p} f_n \) is Cauchy in \( \hat{\mathcal{M}} \), so that taking the lim sup as \( n \to 0 \), the right-hand side converges to zero. But this means \( d_{\mathcal{F}_p} f_n \to d_{\mathcal{F}_p} f \), i.e., \( d_{\mathcal{F}_p} f = \omega \).

**Lemma 2.10** (Mazur variant). If \( D(\mathcal{F}) \ni f_n \to f \) in \( L^0(\mathfrak{m}) \) and \( d_{\mathcal{F}_p} f_n \to \omega \) weakly in \( \mathcal{M} \), then there is a sequence \( h_n \to f \) in \( L^0(\mathfrak{m}) \) such that \( d_{\mathcal{F}_p} h_n \to \omega \) strongly and \( h_n \) is a finite convex combination of \( \{f_k\}_{k \geq n} \). A similar result holds for \( \mathcal{M} \) and \( \mathcal{M}_0 \).

**Proof.** By Mazur’s lemma, \( \omega \) is a strong limit of a sequence \( (\omega_n)_{n \in \mathbb{N}} \) which is a finite convex combination of \( \{f_k\}_{k \geq n} \). As \( f \mapsto d_{\mathcal{F}_p} f \) is linear, it holds \( \omega_n = [h_n, M] \), where \( h_n \) is a finite convex combination of \( \{f_k\}_{k \geq n} \). Lemma 2.3 shows that \( h_n \to f \) in \( L^0(\mathfrak{m}) \), proving the claim of this lemma.

The following form of reflexivity will be needed.

**Definition 2.11** (Weakly reflexive module). We say \( \mathcal{M} \) is weakly reflexive if any bounded sequence \( d_{\mathcal{F}_p} f_n \in \mathcal{M} \) with \( D(\mathcal{F}_p) \ni f_n \to f \) admits a weakly convergent subsequence.

This property and the following question already appeared in [17], Proposition 2.2.10 and Remark 2.2.11.

**Question 2.12.** Is a weakly reflexive \( L^p \)-normed module reflexive?

The next lemma shows that weak reflexivity implies that the strong upper relaxations are represented by an element in \( \mathcal{M} \).

**Lemma 2.13.** Assume \( \mathcal{M} \) is weakly reflexive. Then for any strong upper relaxation \( G \in \hat{G} \) there exist an \( \omega \in \mathcal{M} \) with \( |\omega| = G \) which is a limit of a sequence \( (d_{\mathcal{F}_p} f_n)_{n \in \mathbb{N}} \) with \( f_n \to f \). If, in addition, \( f \in D(\mathcal{F}) \), then \( (\omega - d_{\mathcal{F}_p} f) \in \mathcal{G}_0 \).

**Proof.** Let \( (f_n)_{n \in \mathbb{N}} \) be as in the definition of strong upper relaxation, i.e., \( g_{f_n} \to G \) strongly in \( L^p(\mathfrak{m}) \), where \( h_n \) is finite convex combination of \( \{f_k\}_{k \geq n} \).
Applying Theorem 2.8 to each f

Theorem 2.15 If the upper and lower relaxations at \( \hat{f} \) agree, then the right-hand side is equal to \( \inf_{\omega \in G_0} \|d_{\mathcal{F}} f + \omega\|_{\mathcal{M}}. \)

Proof. It suffices to show

\[
\|\chi_A d_{\mathcal{F}} f\|_{\mathcal{M}} = \inf_{\omega \in \mathcal{M}_0} \|\chi_A d_{\mathcal{F}} f + \omega\|_{\mathcal{M}} = \inf_{\omega \in \mathcal{M}_0} \|\chi_A d_{\mathcal{F}} f + \chi_A \omega\|_{\mathcal{M}}
\]

for all \( f \in D(\mathcal{F}_p) \) and Borel sets \( A \subset M. \)

Let \( \mathcal{P} \) be the set of all \( \omega = \sum_{i \in \mathbb{N}} \chi_{A_i} \omega_i \in \mathcal{M}_0 \) with \( \{A_i\}_{i \in \mathbb{N}} \) a Borel partition of \( M \) and \( \omega_i \in G_0. \) The set \( \mathcal{P} \) is dense in \( \mathcal{M}_0 \) so that we only need to show the above for \( \mathcal{P} \) instead of \( \mathcal{M}_0. \)

From the definition of \( G_0 \) there are \( f_{i,n} \to 0 \) with \( d_{\mathcal{F}} f_{i,n} \to \omega_i \) in \( \mathcal{M}. \) Then also \( d_{\mathcal{F}} (f + f_{i,n}) \to \omega (i) \to d_{\mathcal{F}} f + \omega i \) so that \( g_{f + f_{i,n}} \to |\omega (i)| \) in \( L^p. \) But then \( \tilde{g} f \leq |\omega (i)| \) and thus

\[
\|\chi_A d_{\mathcal{F}} f\|_{\mathcal{M}} = \sum_{i \in \mathbb{N}} \int_{A \cap A_i} \tilde{g}_f dm \leq \sum_{i \in \mathbb{N}} \int_{A \cap A_i} |\omega (i)|^p dm = \|\chi_A d_{\mathcal{F}} f + \chi_A \omega\|_{\mathcal{M}}^p.
\]

Hence, it suffices to show that for each \( \epsilon > 0 \) there is an \( \omega \in \mathcal{P} \) such that

\[
\|\chi_A d_{\mathcal{F}} f + \omega\|_{\mathcal{M}} \leq \|\chi_A d_{\mathcal{F}} f\|_{\mathcal{M}} + \epsilon.
\]

Applying Theorem 2.8 to \( f, \) we get a partition \( \{A_i\}_{i \in \mathbb{N}} \) and strong upper relaxations \( G_n \in G_f^\circ \) such that

\[
\left| \sum_{n \in \mathbb{N}} \chi_{A_n} G_n - \tilde{g} f \right| \leq f,\]
Lemma 2.16. If $\|f_\epsilon\|_{L^p} = \epsilon$. In particular,

$$\left| \sum \chi_{\mathcal{A} \cap \mathcal{A}_n} G_n \right|_{L^p} \leq \epsilon + \|\chi A \hat{g}_f\|_{L^p} = \epsilon + \|\chi A d_{\mathcal{F}_p} f\|_{\mathcal{M}}.$$ 

It remains to show that $\|\chi A \sum \chi_{\mathcal{A}_n} G_n\|_{L^p} = \|\chi A d_{\mathcal{F}_p} f + \omega_\epsilon\|_{\mathcal{M}}$ for some $\omega_\epsilon \in \mathcal{P}$.

Lemma 2.13 shows that for each $G_n$ there is an $\omega_n \in \mathcal{M}$ such that $|\omega_n| = G_n$ and $\omega_n - d_{\mathcal{F}_p} f \in \mathcal{G}_0$. Setting

$$\omega_\epsilon = \sum_{n \in \mathbb{N}} \chi_{\mathcal{A}_n} (\omega_n - d_{\mathcal{F}_p} f) \in \mathcal{P}$$

shows

$$\|\chi A d_{\mathcal{F}_p} f + \omega_\epsilon\|_{\mathcal{M}} = \left\| \sum_{n \in \mathbb{N}} \chi_{\mathcal{A} \cap \mathcal{A}_n} \omega_n \right\|_{\mathcal{M}} = \|\sum_{n \in \mathbb{N}} \chi_{\mathcal{A} \cap \mathcal{A}_n} G_n\|_{L^p},$$

which proves the claim and thus the theorem. In case $\hat{g}_f = \hat{g}_f$ and $A_1 = \mathcal{M}$ and $\omega_\epsilon \in \mathcal{M}$ independent of $\epsilon > 0$, see Lemma 2.13. \hfill $\square$

**Lemma 2.16.** If $\mathcal{M}$ is weakly reflexive and the upper and lower relaxations agree, then $\mathcal{M}' = \mathcal{M}/\mathcal{M}_0$ is weakly reflexive.

**Proof.** From proof above, there is a sequence $(f_m)_{m \in \mathbb{N}}$ in $D(\mathcal{F}_p)$ converging to $f$ such that $d_{\mathcal{F}_p} f_m \to \omega \in \mathcal{M}$ and

$$\|d_{\mathcal{F}_p} f\|_{\mathcal{M}} = \|d_{\mathcal{F}_p} f + \omega\|_{\mathcal{M}}.$$ 

Let $(f_n)_{n \in \mathbb{N}}$ be a sequence in $D(\mathcal{F}_p)$ converging in $L^0(\mathcal{M})$ to some $f \in D(\mathcal{F}_p)$ such that $(d_{\mathcal{F}_p} f_n)_{n \in \mathbb{N}}$ is bounded. We may represent $d_{\mathcal{F}_p} f_n$ by $d_{\mathcal{F}_p} f_n + \omega_n$ for some $\omega_n \in \mathcal{G}_0$. It suffices to show that $d_{\mathcal{F}_p} f_n + \omega_n$ admits a weakly convergent subsequence in $\mathcal{M}$.

For each $f_n$ there are sequences $(f_{n,m})_{m \in \mathbb{N}}$ in $D(\mathcal{F}_p)$ converging to $f_n$, with $d_{\mathcal{F}_p} f_{n,m} \to d_{\mathcal{F}_p} f_n + \omega_n$ in $\mathcal{M}$ and $(d_{\mathcal{F}_p} f_{n,m})_{m \in \mathbb{N}}$ bounded. Thus we may choose a diagonal sequence $h_n = f_{n,m}$, such that $h_n \to f$ and if for some subsequence $(h_{n'})_{n' \in \mathbb{N}}$ it holds $d_{\mathcal{F}_p} h_{n'} \to \omega$, then $d_{\mathcal{F}_p} f_{n,m} + \omega_{n'} \to \omega$. To conclude notice that since $(d_{\mathcal{F}_p} h_n)_{n \in \mathbb{N}}$ is a subsequence of $(d_{\mathcal{F}_p} f_{n,m})_{n \in \mathbb{N}}$, it is bounded and admits a weakly convergent subsequence by weak reflexivity of $\mathcal{M}$. \hfill $\square$

**Corollary 2.17.** Assume $\mathcal{M}$ is reflexive or $\mathcal{M}$ is weakly reflexive and the lower and upper relaxations agree. Then $D(\mathcal{F}_p)$ generates $\mathcal{M}$. In particular,

$$\mathcal{M} \cong \mathcal{M}/\mathcal{M}_0$$

and $\mathcal{M}$ is weakly reflexive.

**Proof.** By the previous lemma, we see that $\mathcal{M}' := \mathcal{M}/\mathcal{M}_0$ is weakly reflexive. In order to show that $\mathcal{M}'$ agrees with the relaxed module $\mathcal{M}$, we want to use Corollary 1.8: choose any $f \in D(\mathcal{F}_p)$ and let $(f_n)_{n \in \mathbb{N}}$ be a sequence in $D(\mathcal{F}_p)$
converging to \( f \) in \( L^0(\mathfrak{m}) \) such that \( (d_{\mathcal{F}_p} f_n)_{n \in \mathbb{N}} \) is bounded in \( \hat{\mathcal{M}}' \). Such a sequence exists as \( D(\mathcal{F}_p) = D(\hat{\mathcal{F}}_p) \). Weak reflexivity and Lemma 2.10 show that there are finite convex combinations \( h_n \) of \( \{f_k\}_{k \geq n} \) such that \( h_n \to f \) in \( L^0(\mathfrak{m}) \) and \( d_{\mathcal{F}_p} h_n \to \omega \in \hat{\mathcal{M}}' \). However, \( d_{\mathcal{F}_p} \) is closed so that \( \omega = d_{\mathcal{F}_p} f \). As \( f \in D(\mathcal{F}_p) \) was arbitrary we can apply Corollary 1.8 which shows that \( D(\mathcal{F}_p) \) generates \( \hat{\mathcal{M}} \). As \( D(\mathcal{F}_p) \subset D(\hat{\mathcal{F}}_p) \) also generates \( \hat{\mathcal{M}}' \), we obtain the claim.

A similar argument also shows the following vector space characterization. We only sketch the argument and leave the details to the reader. Define
\[
\mathcal{G} = \text{cl}_{\|\cdot\|_{\mathcal{M}}} \{d_{\mathcal{F}_p} f \in \mathcal{M} \mid f \in D(\mathcal{F}_p)\}, \quad \text{and}
\hat{\mathcal{G}} = \text{cl}_{\|\cdot\|_{\hat{\mathcal{M}}}} \{d_{\mathcal{F}_p} f \in \hat{\mathcal{M}} \mid f \in D(\hat{\mathcal{F}}_p)\},
\]
where \( \hat{\mathcal{F}}_p \) is the upper relaxation of \( \mathcal{F}_p \) and \( \hat{\mathcal{M}} \) its induced module.

Remark. All three sets \( \mathcal{G}, \mathcal{G}_0 \) and \( \hat{\mathcal{G}} \) only require the functional \( E_{\mathcal{F}_p} \) and its lower semicontinuous hull \( E_{\hat{\mathcal{F}}_p} \). An explicit description of the \( L^p \)-densities is not needed.

The norm is then given by \( (E_{\mathcal{F}_p}(\cdot))^{1/p} \).

**Proposition 2.18.** If \( \mathcal{G} \) is reflexive, i.e., \( \mathcal{M} \) is weakly reflexive, then
\[
\hat{\mathcal{G}} \cong \mathcal{G}/\mathcal{G}_0
\]
and \( d_{\hat{\mathcal{F}}_p} : D(\mathcal{F}_p) \to \hat{\mathcal{G}} \) is closable such that its closure is given by \( d_{\hat{\mathcal{F}}_p} : D(\hat{\mathcal{F}}_p) \to \hat{\mathcal{G}} \).

One may see this as part of a more general result: assume \( A : D \to W \) is a linear (unbounded) operator defined on some subset \( D \) of a topological vector space \( V \) with topological Mazur property. Assume \( W \) is a reflexive Banach space and let \( G \) be the closure of the graph of \( A \) in \( V \times W \). Define \( W_0 \) to be the set of all \( w \in W \) with \((0, w) \in G \). It is easy to see that \( W_0 \) is a closed subspace of \( W \). Denote by \( i : W \to W/W_0 \) the quotient map. Then there is a uniquely defined closed linear operator \( \hat{A} : D \to W/W_0 \) such that \( D \subset \hat{D} \) and \( \hat{A}v = i(Av) \) for \( v \in D \).

3. Lipschitz- and \( L^p \)-modules

In this section we will use the above construction to obtain several cotangent modules which will help to understand the analytic structure of a metric measure space better. Assume again that \( p \in (1, \infty) \). In addition, to assuming that \((\mathcal{M}, d, \mathfrak{m})\) is a complete separable metric measure space we also assume that \( \mathfrak{m} \) has full support.

**Lipschitz and Cheeger energies**

The local Lipschitz constant of a (local) Lipschitz function \( f \), also called slope of \( f \), is defined as
\[
\text{lip } f(x) = \limsup_{y \to x} \frac{|f(y) - f(x)|}{d(y, x)},
\]
with the convention \(0/0 = 0\), i.e., if \(x\) is isolated then \(\text{lip} f(x) = 0\). For a fixed Lipschitz function \(f\), it can be shown that \(x \mapsto \text{lip} f(x)\) is a measurable map. From the definition it is also easy to see that

\[ F_{\text{lip}}^p : f \mapsto \text{lip} f \]

is subadditive and absolutely homogeneous. So denote by \(D(F_{\text{lip}}^p)\) the set of all \(f \in L^p(m)\) functions such that there is a Lipschitz function \(\tilde{f}\) such that \(f = \tilde{f}\) almost everywhere and \(\text{lip} \tilde{f} \in L^p(m)\). As \(m\) has full support, the functions \(\tilde{f}\) and \(\text{lip} \tilde{f}\) are well-defined. Thus for each \(f \in D(F_{\text{lip}}^p)\) we may choose the unique Lipschitz representative, and by abuse of notation say \(f \in D(F_{\text{lip}}^p) \subset L^p(m)\) is a Lipschitz function.

**Remark.** The local Lipschitz constant depends in general on the domain, i.e., if \(f\) is Lipschitz and \(A \subset M\), then for \(x \in A\) one generally has

\[ \text{lip}_A f(x) := \limsup_{A \ni y \to x} \frac{|f(y) - f(x)|}{d(y, x)} < \text{lip} f(x). \]

However, if \(m\) is (locally) doubling then for \(m\)-almost all \(x \in A\), there is an equality as doubling measures are infinitesimally dense almost everywhere even if restricted to sets of positive measure. Thus one may replace full support assumption of \(m\) by a local doubling property in order to obtain a well-defined \(L^p\)-function \(\text{lip} \tilde{f}\). Below we will shows that without such a property the Lipschitz module will behave very badly even if the Lipschitz module is weakly reflexive.

In general the local Lipschitz constant \(\text{lip} f\) and the global Lipschitz constant defined by

\[ \text{Lip} f = \sup_{x \neq y} \frac{|f(y) - f(x)|}{d(y, x)} \]

are not related. Indeed, if \(d_{S^1}\) is the intrinsic distance on the unit circle, then the function \((x, y) \mapsto d_{S^1}((x, y), (1, 0))\) on \(S^1 \subset \mathbb{R}^2\) equipped with the extrinsic Euclidean distance has local Lipschitz constant equal to 1 everywhere, but \(f(-1, 0) = \pi\) and \(\|(−1, 0) − (1, 0)\| = 2\), showing that it is not 1-Lipschitz.

Let \(F_p\) be the lower relaxation of \(F_{\text{lip}}^p\). Denote lower relaxation at \(f\) by \(|Df|_{*,p}\) and call it the relaxed slope of \(f\). As \(m\) is a Radon measure, we obtain the following proposition.

**Proposition 3.1.** The following holds:

- the minimal upper and lower relaxations agree.
- For any Lebesgue null set \(N \subset \mathbb{R}\) it holds \(|Df|_{*,p} = 0\) on \(m\)-a.e. on \(f^{-1}(N)\).
- \([Df]_{*,p} = [Dg]_{*,p}\) \(m\)-a.e. on \(\{f = g\}\).
- For any Lipschitz function \(\varphi : \mathbb{R} \to \mathbb{R}\), it holds \(|D\varphi(f)|_{*,p} = |\varphi'(f)||Df|_{*,p}\) \(m\)-a.e.

**Proof.** With minor adjustments, the proofs of Lemmas 4.4 and 4.8 in [4] still work if \(f\) is assumed to be bounded. For equality in the last case, also see
Remark. As the relaxed slope obtained by \( f \) converges weakly to some \( m \in L^p(M) \), a neighborhood of their support has finite measure. In particular, \( \text{lip} \) \( f \) is a sequence of Lipschitz functions with bounded Lipschitz constants such that \( |Df|_{*,p} \to |Df|_{*,p} \) in \( L^p(M) \), which yields the claims. \( \square \)

**Corollary 3.2.** Assume \( \text{Lip} g \leq C \| \text{lip} g \| \infty \) for all local Lipschitz functions \( g \). Then for every \( f \in L^p(M) \) admitting a relaxed slope there is sequence \( f_n \to f \) in \( L^p(M) \) such that \( |Df_n|_{*,p} \to |Df|_{*,p} \) in \( L^p(M) \).

**Remark.** As the relaxed slope obtained by \( L^p \)-approximations is a priori larger than the one obtained by \( L^0 \)-approximation, the statement is equivalent to showing that the two notions agree. This was already pointed out in Remark 4.4 of [3].

**Proof.** Denote by \( |D^s f|_{*,p} \) the relaxed slope obtained by \( L^p \)-approximation, and assume \( |Df|_{*,p} < |D^s f|_{*,p} \) on a set \( A \) of positive measure. Without loss of generality, assume \( A \) is bounded.

Let \( f_n \to f \) in \( L^0(M) \) with \( \text{lip} f_n \to |Df|_{*,p} \). By Egorov’s theorem, for every \( \epsilon > 0 \) there is a \( A_\epsilon \) such that \( m(A \setminus A_\epsilon) < \epsilon \), and \( f_n \to f \) and \( \text{lip} f_n \to |Df|_{*,p} \) uniformly on \( A_\epsilon \), and each of the functions is uniformly bounded by \( D \) on \( A_\epsilon \).

From the assumption we see that \( (f_n)_A \) is Lipschitz continuous when restricted to \( A \) with Lipschitz constant bounded by \( C \cdot D \). Using the MacShane’s extension theorem, we obtain a sequence \( (g_n) \) of Lipschitz functions with Lipschitz constants bounded by \( C \cdot D \) which agree with \( f_n \) on \( A \). Using a cut-off we can assume \( (g_n)_{n \in \mathbb{N}} \) is a sequence of Lipschitz functions with bounded Lipschitz constants such that a neighborhood of their support has finite measure. In particular, \( (\text{lip} g_n)_{n \in \mathbb{N}} \) is uniformly bounded in \( L^p(M) \). So by reflexivity and Arzelà–Ascoli we can assume \( (g_n)_{n \in \mathbb{N}} \) converges uniformly to some \( g \in \text{Lip}(M) \) with \( g|_A = f|_A \) and \( (\text{lip} g_n)_{n \in \mathbb{N}} \) converges weakly to some \( G \in L^p(M) \). But then \( |D^s g|_{*,p} \leq G \) on \( A \), as uniform convergence implies \( L^p \)-convergence. Also note that \( (G)|_A = (|D^s f|_{*,p})|_A \). However, \( (|D^s f|_{*,p})|_A = (|D^s g|_{*,p})|_A \leq G|_A = (|Df|_{*,p})|_A \), contradicting our assumption. \( \square \)

**Lipschitz and Sobolev modules**

We call the module generated by \( F^\text{lip}_p \) the \( L^p \)-**Lipschitz module** and denote it by \( L^p_\text{lip}(TM) \). As \( F_p : f \mapsto |Df|_{*,p} \) is the lower relaxation of \( F^\text{lip}_p \), it also generates an \( L^p \)-normed module \( L^p(TM) \) which we call \( L^p \)-**cotangent module**.

In both cases, the set of generators \( D(F^\text{lip}_p) \) and \( D(F_p) \) have a uniquely defined objects in those spaces. We call those objects differentials as they agree with the usual notion of a differential when \( M \) is a smooth Riemannian/Finsler manifold.

**Definition 3.3** (Differentials). The \( (L^p) \)-**Lipschitz differential** of a function \( f \in D(F^\text{lip}_p) \) is defined as

\[
d_\text{lip} f = d_{F^\text{lip}_p} f \in L^p_\text{lip}(T^*M).
\]

If \( f \in D(F_p) \), then the \( L^p \)-**differential** (sometimes \( L^p \)-Sobolev differential) is defined as

\[
d_p f = d_{F_p} f \in L^p(T^*M).
\]
Remark. The differential $d_{\text{lip}}$ does not really depend on $p$ as it is uniquely defined only depending on the local Lipschitz constant which needs to be in $L^p(m)$. The $L^p$-differential, however, depends in general on $p$ (see [13]).

The following shows that the $L^p$-differentials admit a nice calculus. We refer to [17] as those properties are not needed in the course of this paper.

**Lemma 3.4** ($L^p$-calculus Theorem 2.2.6 in [17]). The operator $d_p$ satisfies the following:

- **(Leibniz rule)** For any $f,g \in D(F_p) \cap L^\infty(m)$, it holds
  \[ d_p(fg) = f \cdot d_p g + g \cdot d_p f \mbox{ } m\text{-almost everywhere}. \]

- **(Chain rule)** For any Lebesgue null set $N \subset \mathbb{R}$ and any $f \in D(F_p)$,
  \[ d_p f = 0 \mbox{ } m\text{-almost everywhere on } f^{-1}(N), \]
  and for any open $I \subset \mathbb{R}$ such that $m(f^{-1}(\mathbb{R}\setminus I)) = 0$ and any Lipschitz function $\varphi : I \to \mathbb{R}$, it holds
  \[ d_p(\varphi \circ f) = \varphi'(f) \cdot d_p f \mbox{ } m\text{-almost everywhere}. \]

- **(Locality)** For any $f \in D(F_p)$, it holds
  \[ d_p f = d_p g \mbox{ } m\text{-almost everywhere on } \{ f = g \}. \]

We have the following, which is just Lemma 2.9 in terms of the cotangent module.

**Proposition 3.5** (Closedness of $d_p$). Assume $f \in D(F_p)$ is converging almost everywhere to a measurable function $f \in L^0(m)$. If $d_p f_n \to \omega \in L^p(T^*M)$, then $f \in D(F_p)$ and $d_p f = \omega$.

Because $d_p$ is a closed operator, we can define a Sobolev space via the graph norm of $d_p$.

**Definition 3.6** (Sobolev space). The Sobolev space $W^{1,p}(M, m)$ is defined as the set of $L^p$-functions with $f \in D(F_p)$. The norm of $W^{1,p}(M, m)$ is given by

\[ \| f \|^p_{W^{1,p}} = \| f \|^p_{L^p} + \| d_p f \|^p_{L^p(T^*M)}. \]

By the previous proposition, $W^{1,p}(M, m)$ is a Banach space.

The operator $d_{\text{lip}}$ is in general not closed as an operator from $L^0(m)$ to the module $L^p(T^*M)$. Nevertheless, one might ask whether it is closable. From the above it is clear that $d_{\text{lip}}$ is closable if $\text{lip } f = |Df|_{\ast,p}$ for all Lipschitz functions in $D(F_p)$. In this case $D(F_{\text{lip}}^p)$ also generates $\mathcal{M}_{\mathcal{F}_p}$. If Lipschitz functions are dense in $W^{1,p}$, then by Lemma 1.7 and its corollary we have $\mathcal{M}_{\mathcal{F}_p} = \mathcal{M}_{\mathcal{F}_{\text{lip}}^p}$. However, it is not clear if closability of $d_{\text{lip}}$ would imply lower semicontinuity of $f \mapsto \int (\text{lip } f)^p \,dm$ on $D(F_{\text{lip}}^p)$.
Corollary 3.8. If \( \text{ability of } d \)\(^2\)\( \text{W} \) laxations of the local Lipschitz constant agree. We give a separate argument for \( \omega \) and a restatement of Theorem 2.15. The modules \( \text{As a subspace of } \text{reflexivity of } \text{This space agrees with a certain Sobolev space defined in [15], [30].} \)

In addition, \( d \) is closable as an operator from \( L^p_m(T^*M) \) to \( L^p_m(T^*M) \) if and only if \( L^p_m(T^*M) \) is trivial, in which case \( L^p(T^*M) \approx L^p_m(T^*M) \) and \( |Df|_{*,p} = \text{lip } f \) for all \( f \in \text{Lip}(M,d) \cap L^p(T^*M) \).

**Proof.** See Theorem 2.15. \( \square \)

**Corollary 3.8.** If \( L^p_{\text{lip}}(T^*M) \) is weakly reflexive, then \( W^{1,p}(M, m) \) and \( L^p(T^*M) \) are (weakly) reflexive.

**Proof.** This could be deduced from Lemma 2.18 since the lower and upper relaxations of the local Lipschitz constant agree. We give a separate argument for \( W^{1,p}(M, m) \). Define the pseudo-Sobolev space \( W^{1,p}_{\text{lip}}(M, m) \) as the closure of all \( L^p \)-integrable Lipschitz functions in \( D(F^p_{\text{lip}}) \), where the norm (on \( D(F^p_{\text{lip}}) \)) is given by

\[
\|f\|_{W^{1,p}_{\text{lip}}}^p = \|f\|_{L^p}^p + \|d_{\text{lip}}f\|_{L^p_{\text{lip}}(T^*M)}^p.
\]

This space agrees with a certain Sobolev space defined in [15], [30].

In the following we identify \( W^{1,p}_{\text{lip}}(M, m) \) as a closed convex subset of \( L^p(m) \times L^p_{\text{lip}}(T^*M) \) equipped with the \( L^p \)-product norm, and to keep the notation simple, we set \( \mathcal{V} = W^{1,p}_{\text{lip}}(M, m) \).

The closed subspace measuring the lack of closability of \( d_{\text{lip}} \) at the origin is the following:

\[
\mathcal{V}_0 = \{ v \in W^{1,p}_{\text{lip}} : \exists f_n \in D(F^p_{\text{lip}}) \cap L^p(m) : f_n \xrightarrow{L^p} 0, f_n \xrightarrow{W^{1,p}_{\text{lip}}} v \}.
\]

As a subspace of \( L^p(m) \times L^p_{\text{lip}}(T^*M) \) this translates to \( \mathcal{V}_0 = \{0 \} \times \mathcal{G}_0 \).

We claim that \( \mathcal{V} \) and thus \( \mathcal{V}_0 \) are reflexive: if \( (v_n)_{n \in \mathbb{N}} \) is bounded in \( \mathcal{V} \) then there are Lipschitz functions \( f_{n,m} \in \mathcal{V} \) such that \( f_{n,m} \rightarrow v_n \). In particular, choosing a diagonal sequence we obtain a sequence \( (f_{n,m})_{n \in \mathbb{N}} \subseteq \mathcal{V} \) of Lipschitz functions such that \( f_{n,m} \rightarrow v \) weakly in \( \mathcal{V} \) if and only if \( v_n \rightarrow v \) weakly in \( \mathcal{V} \). Weak reflexivity of \( L^p_{\text{lip}}(T^*M) \) and reflexivity of \( L^p(m) \) show that there are \( f \in L^p(m) \) and \( \omega \in L^p_{\text{lip}}(T^*M) \) such that \( (f_{n,m}, d_{\text{lip}}f_{n,m}) \rightarrow (f, \omega) \). But then there is a \( v \in \mathcal{V} \) represented by \( (f, \omega) \) with \( f_{n,m} \rightarrow v \), proving the claim.
As $|Df|_{*,p}$ is also an upper relaxation, the proof of Theorem 2.15 shows
\[
\|d_pf\|_{L^p(T^*M)} = \inf_{\omega \in \mathcal{G}_0} \|d_{\text{lip}}f + \omega\|_{L^p_{\text{lip}}(T^*M)}
\]
for all Lipschitz functions $f \in W^{1,p}(M,m)$. But then
\[
W^{1,p}(M,m) \cong \mathcal{V}/\mathcal{V}_0,
\]
which shows reflexivity of $W^{1,p}(M,m)$.

By Corollary 7.5 in [1], if $m$ is doubling and finite then every Sobolev space $W^{1,p}(M,m)$, $1 < p < \infty$, is reflexive so that, in the light of Proposition 2.2.10 in [17], the theorem and its corollary applies to those spaces. Note that, by [12], the Lipschitz modules cannot be locally finite dimension if $m$ is a doubling measure on a convex subset of $\mathbb{R}^n$ which is not absolutely continuous with respect to the Lebesgue measure, see below for more on spaces with local finite dimensional Lipschitz modules. It is likely that in most such cases the Lipschitz module is non-trivial and agrees with the Lipschitz module.

It might well happen that the $L^p$-Lipschitz module is not weakly reflexive but $d_{\text{lip}}$ is closable. If no non-trivial upper relaxations we still obtain the following.

**Theorem 3.9.** Assume $m$ is finite on bounded sets. If 0 admits no non-trivial strong upper relaxation of the local Lipschitz constant at 0, then $f \mapsto \int (\text{lip} f)^p \, dm$ is lower semicontinuous on $\text{D}(\text{F}_{\text{lip}}^p)$, $d_{\text{lip}}$ is closable, and the $L^p$-Lipschitz module is a submodule of the $L^p$-cotangent module. If Sobolev differentials represented by Lipschitz functions are dense in the set of Sobolev differentials, then the two modules agree.

**Proof.** The proof is similar to the proof of Theorem 8.4 in [1]. Assume $f_n \to f$ in $L^0(m)$ with $f_n, f \in D(F_{\text{lip}}^p)$ and $(\text{lip} f_n)_{n \in \mathbb{N}}$ is bounded in $L^p(m)$. By choosing a subsequence we may assume
\[
\liminf_{n \to \infty} \int (\text{lip} f_n)^p \, dm = \lim_{n \to \infty} \int (\text{lip} f_n)^p \, dm.
\]

Because $\text{lip}(f_n - f)$ is also bounded in $L^p(m)$, we may further replace $(f_n)_{n \in \mathbb{N}}$ by one of its subsequences and assume $\text{lip}(f_n - f) \to G$ weakly in $L^p(m)$. Then Lemma 2.5, together with triviality of strong upper gradients at 0, shows there is a sequence $h_n \in D(F_{\text{lip}}^p)$ of finite convex combinations of $\{f_k\}_{k \geq n}$ with $(f - h_n) \to 0$ in $L^0(m)$ and $(\text{lip}(f - h_n))_{n \in \mathbb{N}}$ converges strongly in $L^p(m)$ to some strong upper relaxation $G' \leq G$ at 0. Since 0 does not admit a non-trivial strong upper relaxation, we have $G' = 0$.

Observe that subadditivity of $f \mapsto \text{lip} f$ implies
\[
\int (\text{lip} h_n)^p \, dm \leq \sup_{k \geq n} \int (\text{lip} f_n)^p \, dm.
\]
Therefore,
\[ \left( \int (\text{lip } f)^p \, dm \right)^{1/p} \leq \liminf_{n \to \infty} \left\{ \left( \int (\text{lip } h_n)^p \, dm \right)^{1/p} + \left( \int (f - h_n)^p \, dm \right)^{1/p} \right\} \]
\[ \leq \limsup_{n \to \infty} \left( \int (\text{lip } f_n)^p \, dm \right)^{1/p} + \lim_{n \to \infty} \left( \int (f - h_n)^p \, dm \right)^{1/p} \]
\[ \leq \lim_{n \to \infty} \left( \int (\text{lip } f_n)^p \, dm \right)^{1/p}. \]

Thus the upper relaxation of \( \mathcal{F}_{lip}^p : f \mapsto \int (\text{lip } f)^p \, dm \) is trivial on \( D(\mathcal{F}_{lip}^p) \) and Lemma 1.7 and its corollary imply the theorem. \( \square \)

One can show that a lack of closability implies that there are non-trivial strong upper relaxations at 0. However, whether closability implies absence of non-trivial upper relaxations is not clear.

**Question.** Is there a (compact/doubling) metric measure space such that \( d_{lip} \) is closable but \( f \mapsto \int (\text{lip } f)^p \, dm \) not lower semicontinuous?

Furthermore, one may ask whether the lack of lower semicontinuity implies that there are non-constant Lipschitz functions with trivial Sobolev differential.

**Question.** Does \( L_{lip0}^p (T^*M) \neq \{0\} \) imply that there is a non-trivial Lipschitz function with \( d_{lip} f = 0 \)? Vice versa, assume that \( f \equiv \text{const} \) whenever \( f \in \text{Lip}(M, d) \) and \( d_{lip} f = 0 \). Does this imply that \( L_{lip0}^p (T^*M) \) is trivial?

The known (non-product) examples show that the space has either trivial \( L^p \)-structure or the trivial Lipschitz\(_0\)-module. An example of a space between these two extremes might help to answer that question. Even in \( \mathbb{R}^n \), the structure of space of Lipschitz differential for measures which are not absolutely continuous is not understood.

Before studying spaces with local finite dimensional Lipschitz module, we show that the Lipschitz\(_0\) module is non-trivial without a denseness assumption of the measure.

**Definition 3.10** (Infinitesimal Lipschitz dense). A measure \( m \) on a complete separable metric measure space is infinitesimally dense if, for all Borel set \( A \subset M \) and for all Lipschitz functions \( f : M \to \mathbb{R} \) with \( \text{lip}_A f = 0 \) on \( A \), it holds \( \text{lip} f = 0 \) \( m \)-almost everywhere on \( A \), where
\[ \text{lip}_A f(x) := \begin{cases} \limsup_{A \ni y \to x} \frac{|f(y) - f(x)|}{d(y, x)} & x \in A, \\ \text{lip} f(x) & x \notin A. \end{cases} \]

The property can be interpreted as saying any function \( f \) defined on a measurable set \( A \) which is Lipschitz when restricted to \( A \) has well-defined local Lipschitz constant on \( A \) which does not depend on the chosen Lipschitz extension.

The following lemma shows there are plenty of measures which are infinitesimally Lipschitz dense. A slight adjustment of Keith’s proof also shows that it holds for pointwise doubling measures, see Definition 1.5 in [9].
Lemma 3.11 (Proposition 3.5 in [25]). If $\mathfrak{m}$ is doubling, then it is infinitesimally Lipschitz dense.

Lemma 3.12. If $\operatorname{lip}_A f < \operatorname{Lip} f$ on a measurable set $A$ of positive $\mathfrak{m}$-measure, then there are a Lipschitz function $\tilde{f}$ with $\operatorname{Lip} \tilde{f} < \operatorname{Lip} f$, and a set $A' \subset A$ of positive $\mathfrak{m}$-measure with $\tilde{f}|A' = f|A$ and $\operatorname{lip} \tilde{f} < \operatorname{lip} f$. In particular, there is a Lipschitz function $g$ with $g|A' = 0$ and $\operatorname{lip} g > 0$ $\mathfrak{m}$-almost everywhere on $A'$.

Proof. We may find $\epsilon > 0$ and a set $A' \subset A$ of positive measure with $\operatorname{Lip} A' f + \epsilon/2 \leq \operatorname{Lip} f$ on $A'$, where

$$\operatorname{Lip} A' f = \sup_{y, x \in A'} \frac{|f(y) - f(x)|}{d(y, x)}.$$ 

Any Lipschitz extension $\tilde{f}$ of $f|A'$ with $\operatorname{Lip} \tilde{f} = \operatorname{Lip} A' f$ gives the desired function. \qed

Proposition 3.13. If $\mathfrak{m}$ is not infinitesimally Lipschitz dense, then the $L^p$-Lipschitz module is non-trivial provided that the $L^p$-Lipschitz module is weakly reflexive.

Proof. By the previous lemma, we may assume there is a Lipschitz function $f$ with $f = 0$ on a set $A$ of positive $\mathfrak{m}$-measure and $\operatorname{lip} f > 0$ $\mathfrak{m}$-almost everywhere on $A$. Via cut-off functions, we may assume $\operatorname{lip} f$ is $L^p$-integrable. But have $\chi_A d_{\operatorname{lip}} f \neq 0$. By Lemma 3.4, it holds $\chi_A d_{\operatorname{lip}} f = 0$. From Theorem 3.7, we obtain

$$0 = \|\chi_A d_{\operatorname{lip}} f\|_{L^p(T^* M)} = \inf_{\omega \in \mathcal{M}_0} \|\chi_A d_{\operatorname{lip}} f + \omega\|_{L^p_{\operatorname{lip}}(T^* M)},$$

implying there is an $\omega_n \in \mathcal{M}_0$ with $-\omega_n \to \chi_A d_{\operatorname{lip}} f$. Since $\mathcal{M}_0$ is a submodule, we must have $\chi_A d_{\operatorname{lip}} f \in \mathcal{M}_0$. \qed

Lipschitz differentiable structure and Cheeger differentials

In this section we combine the abstract theory above with the theory of metric spaces admitting a Lipschitz differentiable structure. It turns out that that theory fits nicely into the abstract structure of the previous section. For further reference, see [11], [24], [8].

The following can be deduced from the fact that $L^p_{\operatorname{lip}}(T^* M)$ is generated by Lipschitz functions.

Lemma 3.14. There is a Borel partition $\{A_\infty\} \cup \{A_{i, n}\}_{i, n \in \mathbb{N}}$ of $M$ such that for each $i, n \in \mathbb{N}$ there are Lipschitz functions $\{f_1, \ldots, f_n\}$ such that their Lipschitz differentials generate $L^p_{\operatorname{lip}}(T^* A_{i, n})$. In particular, we can assume $A_\infty = E_\infty$ and $A_{i, n} \subset E_n$, where $E_n$ is given by the dimensional decomposition of $L^p_{\operatorname{lip}}(T^* M)$.

By further partitioning $A_{i, n}$, we can assume each $A_{i, n}$ is a bounded and each element $f_i$ of the basis has bounded support. In particular, $d_{\operatorname{lip}} f_i \in L^p(T^* M)$ for all $p \in (1, \infty)$ if $\mathfrak{m}(\operatorname{supp} f_i) < \infty$. 
Corollary 3.15. Assume \( \mathfrak{m} \) is finite on bounded subsets. Then the dimensional decomposition above is valid independently of \( p \in (1, \infty) \). In addition, \( L^p_{\text{lip}}(T^*M) \) is locally finite dimensional (bounded by \( N \)) for some \( p \in (1, \infty) \) if and only if it is locally finite dimensional (bounded by \( N \)) for all \( p \in (1, \infty) \).

The corollary resembles exactly what is known in case “\( p = \infty \)” : the assignment \( f \mapsto \text{lip } f(x) \) is a semi-norm for all \( x \in M \). A finite set \( \{ f_i \}_{i=1}^n \) of Lipschitz functions is said to be independent on a set \( A \) if for \( \mathfrak{m} \)-almost all \( x \in A \) the assignment \( f \mapsto \text{lip } f(x) \) is a norm when restricted to span\( \{ f_i \}_{i=1}^n \).

Definition 3.16 (Differentiability space). A metric measure space \((M, d, \mathfrak{m})\) is called a (Lipschitz) differentiability space if there is a uniform bound on the maximal number of Lipschitz functions which can be independent on a set of positive measure.

By a cut-off argument the decomposition is local so that we obtain the following.

Corollary 3.17. A metric measure space which assigns finite measure to each bounded set is a differentiability space with constant \( N \in \mathbb{N} \) if and only if \( L^p_{\text{lip}}(T^*M) \) is locally finite dimensional and bounded by the same constant \( N \) for some (and thus all) \( p \in (1, \infty) \).

Choose a basis \( \{ f_{1}^{i,n}, \ldots, f_{n}^{i,n} \} \) of Lipschitz functions on each \( A_{i,n} \) such that for each Lipschitz map \( f \) there is a linear map \( f \mapsto \alpha_j^{i,n} \in L^\infty(A_{i,n}) \), \( j = 1, \ldots, n, \) with \( \text{lip } f = \sum_{j=1}^{n} \alpha_j^{i,n} \text{lip } f_j^{i,n} \). The definition implies

\[
\text{lip } \left( f - \sum_{j=1}^{n} \alpha_j^{i,n}(x_0)f_j^{i,n}(x_0) \right)(x_0) = 0 \quad \text{for } \mathfrak{m} \text{-almost all } x_0 \in A_{i,n},
\]
or equivalently

\[
f(x) = f(x_0) + \sum_{j=1}^{n} \alpha_j^{i,n}(x_0)(f_j^{i,n}(x) - f_j^{i,n}(x_0)) + o(d(x, x_0)).
\]

One may verify that \( \text{lip } f(x_0) = \text{lip } \left( \sum_{j=1}^{n} \alpha_j^{i,n}(x_0)(f_j^{i,n}(\cdot) - f_j^{i,n}(x_0)) \right)(x_0) \).

We call the operator

\[
D : f \mapsto \sum_{i,n \in \mathbb{N}} \chi_{A_{i,n}} \alpha_j^{i,n} \in \bigoplus_{n \in \mathbb{N}} \mathbb{R}^n.
\]

the Cheeger differential of \( f \) (with respect to the structure induced by \( \{ A_{i,n} \} \) and \( \{ f_{j}^{i,n} \} \)). Note that this is just a form of the representation theorem (Theorem 1.5) with respect to the chosen basis element.

Remark. The Lipschitz differentiable structure only depends on the measure class of \( \mathfrak{m} \), i.e., if \( \mathfrak{m}' = \mathfrak{m} \) is a measure with \( \mathfrak{m}' = f \mathfrak{m} \) and \( f(x) \in (0, \infty) \) \( \mathfrak{m} \)-almost everywhere then \((M, d, \mathfrak{m}')\) is Lipschitz differentiable as well. This follows from the fact that \( \mathfrak{m}(M \setminus \Omega_N) = 0 \) where \( \Omega_N = \{ x \in M \mid f(x) \in (N^{-1}, N) \} \).

Using the characterization theorem we get a more precise description of the modules and their dimensions.
Definition 3.18 (Dimension). The $L^p$-Lipschitz dimension $\dim_{lips} A$ of a Borel set $A$ is defined as

$$\dim_{lips} A = \text{ess sup}_{x \in A} \dim(L^p_{lip}(T^*M), x).$$

Similarly, define the $L^p$-analytic dimension $\dim_p A$ of $A$ as the essential supremum of the dimension of the $L^p$-cotangent module.

Remark. If $\mu$ is finite on bounded sets, then $\dim_{lips}$ does not depend on $p$.

If the $L^p$-Lipschitz module is locally finite, Theorem 1.5 yields the following.

Corollary 3.19. If the $L^p$-Lipschitz module is locally finite dimensional (bounded by $N$), then $L^p_{lip}(T^*M)$ is reflexive and Theorem 3.7 above applies. Furthermore,

$$\dim_{lips}(M, x) = \dim_p(M, x) - \dim(L^p_{lip}(T^*M), x) \text{ m-almost everywhere.}$$

In particular, $L^p(T^*M)$ is locally finite dimensional (bounded by $N$), and $L^p(T^*M)$ and $W^{1,p}(M, \mu)$ are reflexive.

Further, $d_{lip}$ is closable if and only if $\dim_{lips}(M, x) = \dim_p(M, x)$ for $m$-almost all $x \in M$, and either of the conditions is equivalent to lower semicontinuity of $f \mapsto \frac{1}{p} \int (\text{lip } f)^p \, d\mu$.

Remark. The fact that a $L^p$-Lipschitz module, which is locally finite dimensional bounded by $N$, is weakly reflexive was already observed by Schioppa (Theorem 4.16 in [30]).

Proof. Since any closed submodule of a locally finite dimensional $L^p(\mu)$-normed module is also locally finite dimensional with the same bound, the only thing that needs to be proven is the dimension formula. For this assume $E$ is a set where all dimensions are constant. Then

$$L^p_{lip}(T^*E) \cong L^p(E, \mathbb{R}^{n_1}, | \cdot |), \quad L^p_{lip}(T^*E) \cong L^p(E, \mathbb{R}^{n_2}, | \cdot |),$$

and

$$L^p(T^*E) \cong L^p(E, \mathbb{R}^{n_3}, | \cdot |').$$

The theorem implies

$$L^p(E, \mathbb{R}^{n_3}, | \cdot |') \cong L^p(E, \mathbb{R}^{n_1}, | \cdot |)/L^p(E, \mathbb{R}^{n_2}, | \cdot |).$$

But then

$$(\mathbb{R}^{n_3}, | \cdot |') = (\mathbb{R}^{n_1}, | \cdot |)/(\mathbb{R}^{n_2}, | \cdot |_x),$$

which implies $n_3 = n_1 - n_2$. \qed

Cheeger differential structure and infinitesimally Hilbertian spaces

On PI$^p$-spaces, i.e., those satisfying a doubling and $(1, p)$-Poincaré condition, one frequently “renorms” the $L^p$-cotangent module to obtain a (pointwise) inner product $(f, g) \mapsto Df \cdot Dg$ with $Df$ the Cheeger differential of $f$. This structure is then called Cheeger differential structure. The choice of scalar product is, however, highly non-unique. Using Theorem 1.5 one may choose the John or Binet–Legendre
scalar product given by Theorem A.1 to obtain unique scalar product so that $Df$ only depends on the charts and basis elements.

For a certain class of spaces the renorming is actually trivial: recall that a space is called \textit{infinitesimally Hilbertian} (see [5], [16]) if $E_{F_2}$, the lower relaxation of $f \mapsto |Df|^2 \, dm$, is a quadratic form and thus a strongly local, closed and Markovian Dirichlet form. Then one can show that it is equivalent to $f \mapsto |Df|^2_{L^2}$ being a quadratic form $m$-almost everywhere. But then the $L^2$-cotangent module is a Hilbert module and therefore reflexive. More generally, we may say that the space is \textit{infinitesimally Riemannian (with respect to $m$)} if $f \mapsto (\text{lip } f)^2$ is quadratic $m$-almost everywhere.

Remark. The terminology infinitesimally Hilbertian stems from the fact the Sobolev space $W^{1,2}(M, m)$ is a Hilbert space if and only if the space is infinitesimally Hilbertian. We choose infinitesimally Riemannian as it relates more directly to Lipschitz functions. This class includes sub-Riemannian manifolds, like the class of Heisenberg groups with Carnot–Carathéodory metric induced by an inner product on the horizontal bundle.

The characterization of the Lipschitz and cotangent modules via Theorem 3.7 also shows the following.

\textbf{Proposition 3.20.} Assume $(M, d, m)$ is infinitesimally Riemannian and $m$ is finite on bounded sets. Then $(M, d, m)$ is infinitesimally Hilbertian and $L^p(T^* M)$ is reflexive for $p \in (1, \infty)$.

Remark. One can also show that infinitesimally Riemannian spaces are $p$-infinitesimal strict convex in the sense of Gigli [16], because the pointwise norms of dual of $L^p(T^* M)$ are induced by a scalar product making them strictly convex.

\textbf{Proof.} If $x \mapsto |\cdot|^2$ is quadratic then $x \mapsto |\cdot|$ is 2-uniformly convex and 2-uniformly smooth. Thus $x \mapsto |\cdot|$ is $p$-uniformly convex for $p \geq 2$ and $p$-uniformly smooth if $p \leq 2$ with convexity and resp. smoothness constants only depending on $p$. But then

$f \mapsto \|\chi_A d_{\text{lip}}(f)\|_{L^p(T^* M)}$

is $p$-uniformly convex/smooth with the same constant. In particular, $L^p_{\text{lip}}(T^* M)$ is (super)reflexive. As $L^p(T^* M)$ is a quotient space of a reflexive space, it is itself reflexive.

In case $p = 2$ one sees that

$v \mapsto \|v\|_{L^2_{\text{lip}}(T^* M)}^2 = \int |v|^2 \, dm$

is quadratic and thus $L^2_{\text{lip}}(T^* M)$ a Hilbert module. But in Hilbert spaces quotient spaces are isometric to orthogonal subspaces, i.e.,

$L^2(T^* M) \cong (L^2_{\text{lip}}(T^* M))^\perp$,

showing that $L^2(T^* M)$ is a Hilbert module and $E_{F_2}$ quadratic. \hfill $\square$

If it is a priori known that the Lipschitz$_{0}$-module $L^p_{\text{lip}_{0}}(T^* M)$ is trivial then the converse is true as well.
Corollary 3.21. Assume \( \mathbf{m} \) is bounded on finite subsets and \( L^2_{\text{lip}}(T^* M) \) is trivial. Then \( (M, d, \mathbf{m}) \) is infinitesimally Riemannian if and only if it is infinitesimally Hilbertian. In either case, the Sobolev spaces are reflexive.

Example. (1) Every infinitesimally Hilbertian PI\(_p\)-space with \( p \leq 2 \) is infinitesimally Riemannian and its Cheeger differential structure gives a representation of the Sobolev differentials. If \( p > 2 \), the \( L^2 \)-cotangent module might be trivial and thus trivially infinitesimally Hilbertian.

(2) By [18], on RCD\((K, \infty)\)-spaces the relaxed slope/weak upper gradient is independent of \( p \) and \( |Df|_{s,p} = \text{lip} f \) for all \( f \in D(F^p_{\text{lip}}) \). For those spaces the Lipschitz\(_N\)-modules are trivial but the \( L^2 \)-cotangent module might not be locally finite dimensional. Note that this result only needs the Bakry–Émery condition and is independent of the proof of a \((1, 1)\)-Poincaré condition on RCD\((K, N)\)-spaces with \( N \in (1, \infty) \).

Further results

To finish this section, let us see what happens in the finite measure case. For this assume for simplicity that \( \mathbf{m} \) is a probability measure. Using cut-off functions and the chain rule, one can also show the results below for \( \sigma \)-finite measures.

Lemma 3.22. Assume \( \mathbf{m} \) is finite. Then non-triviality of \( L^p_{\text{lip}}(T^* M) \) implies that \( L^{p'}_{\text{lip}}(T^* M) \) is non-trivial for all \( 1 < p' \leq p \). Or equivalently, closability of \( d_{\text{lip}} \) in \( L^p_{\text{lip}}(T^* M) \) implies closability of \( d_{\text{lip}} \) in \( L^{p'}_{\text{lip}}(T^* M) \) for all \( p' \leq p < \infty \).

Proof. First note that \( D(F^p_{\text{lip}}) = \text{Lip}(M, d) \) for \( p \in (1, \infty) \). From Hölder’s inequality we get

\[
\|d_{\text{lip}} f\|_{L^{p'}_{\text{lip}}(T^* M)} = \left( \int (\text{lip } f)^{p'} \, d\mathbf{m} \right)^{1/p'} \\
\leq \mathbf{m}(M) \left( \int (\text{lip } f)^p \, d\mathbf{m} \right)^{1/p} = \mathbf{m}(M)\|d_{\text{lip}} f\|_{L^p_{\text{lip}}(T^* M)}.
\]

So if \( (d_{\text{lip}} f_n)_{n \in \mathbb{N}} \) is Cauchy in \( L^p_{\text{lip}}(T^* M) \) then it is also Cauchy in \( L^{p'}_{\text{lip}}(T^* M) \). Thus \( L^p_{\text{lip}}(T^* M) \subset L^{p'}_{\text{lip}}(T^* M) \) as the spaces are defined via completion, i.e., “space of Cauchy sequences”. In particular, if \( d_{\text{lip}} f_n \to \omega \in L^p_{\text{lip}}(T^* M) \) then also \( d_{\text{lip}} f_n \to \omega \in L^{p'}_{\text{lip}}(T^* M) \). But if \( |\omega| \neq 0 \) on a set of positive \( \mathbf{m} \)-measure then \( \omega \neq 0 \in L^{p'}_{\text{lip}}(T^* M) \). In particular, if \( d_{\text{lip}} \) is closable in \( L^{p'}_{\text{lip}}(T^* M) \) then it is also closable in \( L^p_{\text{lip}}(T^* M) \). \( \square \)

The result is a generalized version of what happens if a \((1, p')\)-Poincaré inequality holds: in such a case, one can show that \( |Df|_{s,p'} = \text{lip } f \) so that \( d_{\text{lip}} \) is closable in all \( L^p_{\text{lip}}(T^* M) \) with \( p \in [p', \infty) \). Furthermore, it is known there is a \( p_\infty \in (1, \infty) \) such that for \( p > p_\infty \) the \((1, p)\)-Poincaré condition holds but not the \((1, p_\infty)\)-Poincaré condition (see [26]). In this general setting we cannot prove this fact yet.
Question 3.23 (Open ended condition). Assume $\mathbf{m}$ is finite and $L^p_{\text{lip}}(T^*M)$ weakly reflexive for all $p \in (1, \infty)$. Is $L^p_{\text{lip}}(T^*M)$ non-trivial for the $p_\infty$ given above?

Generally one may wonder whether the condition here is strictly weaker even on spaces with Poincaré condition. However, two recent results by Schioppa show the following.

Lemma 3.24. For any $p_0 \in (1, \infty)$ there is a metric measure space such that the modules $L^p_{\text{lip}}(T^*M)$, $p \in (1, \infty)$, are trivial and the $(1, p)$-Poincaré condition holds if and only if $p > p_0$.

Proof. Let $(M, d, \mathbf{m})$ be a space constructed in [32] with $\beta = p_0$. Then all tangents $(Y, d, \mathbf{n})$ satisfy the $(1, p)$-Poincaré inequality if and only if $p > p_0$. Then Lemma 5.7 in [31] shows that the iterated tangents are (blow-up) tangents of $(M, d, \mathbf{m})$ for $\mathbf{m}$-almost all $x \in M$ and are regular in the sense of [31], Section 5. Let $(Y, d, \mathbf{n})$ be such an iterated tangent. By Theorem 5.8 in [31], it holds $\text{lip} f = |df|_p$ for all $p \in (1, \infty)$. In particular, $L^p_{\text{lip}}(T^*M)$ is trivial. $\square$

A. Selection of scalar products

Let $\text{Norm}(\mathbb{R}^n)$ be the space of norms on $\mathbb{R}^n$. We can equip this space with the following intrinsic metric:

$$d(F, F') = \sup_{v \in \mathbb{R}^n \setminus \{0\}} \left\{ \ln \frac{F(v)}{F'(v)} \right\}.$$  
Indeed, $d(F, F') = d(F, F') \geq 0$, with equality if and only if $F(v) = F'(v)$ for all $v \in \mathbb{R}^n$. Choose $v \in \mathbb{R}^n \setminus \{0\}$. Then

$$\ln \frac{F(v)}{F'(v)} = \ln \left( \frac{F(v)}{F'(v)} \right) = \ln \frac{F(v)}{F'(v)} + \ln \frac{F'(v)}{F(v)},$$

which immediately yields the triangle inequality. The interested reader may verify that $(\text{Norm}(\mathbb{R}^n), d)$ is a proper metric space and its topology agrees with the $C^0$-topology. In particular, the space is separable.

Remark. An equivalent characterization is via the constant of uniform equivalency: $d(F, F')$ is the infimum over all $C > 0$ such that $C^{-1} F \leq F' \leq FC$.

Every scalar product induces a norm on $\mathbb{R}^n$ that satisfies pointwise the parallelogram equations. As $d$-convergence also implies pointwise convergence, the space of scalar products $\text{Scalar}(\mathbb{R}^n)$ can be seen as a closed subspace in $\text{Norm}(\mathbb{R}^n)$.

Theorem A.1 (John ellipsoid). For any $F \in \text{Norm}(\mathbb{R}^n)$, there is a unique scalar product $g^F_J$ such that for all $v \in \mathbb{R}^n$,

$$g^F_J(v, v) \leq F^2(v) \leq n g^F_J(v, v).$$  
In particular, $F \mapsto g^F_J$ is continuous from the space of norm $\text{Norm}(\mathbb{R}^n)$ to the space of scalar products $\text{Scalar}(\mathbb{R}^n)$. 
Proof. Existence is well known [7]. For continuity, just observe that the metric space \((\text{Norm}(\mathbb{R}^n), d)\) is proper and by the characterization \(d(F, g^F_J) \leq \sqrt{n}\). Continuity then follows from uniqueness of the John ellipsoid.

Alternatively one may use the Binet ellipsoid \(g^F_{BL}\) of the unit sphere with respect to \(F\) which induces a scalar product called the Binet–Legendre scalar product. The assignment satisfies \(d(g^F_{BL}, g^F_{BL}') \leq nd(F, F')\) and \(d(F, g_{BL}) \leq \sqrt{n(n+2)}\). We refer to [29] for more details on this construction and its use in Finsler geometry.

References


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