A Berestycki–Lions type result and applications

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Abstract. In this paper we show an abstract theorem about the existence of critical points for a functional $I$, which permits us to prove the existence of solutions for a large class of Berestycki–Lions type problems. In the proof of the abstract result we apply the deformation lemma on a special set associated with $I$ called Pohozaev set.

1. Introduction

In recent years a number of authors have devoted special attention to the existence of solutions to elliptic problems of the type

$$-\Delta u = g(u), \quad \text{in } \mathbb{R}^N,$$

where $N \geq 2$, $\Delta$ denotes the Laplacian operator and $g$ is a continuous function with some proper conditions.

The main motivation of this paper comes from a seminal paper due to Berestycki and Lions [14], in which the existence of solutions of (1.1) is considered, assuming $N \geq 3$ and the following conditions on $g$:

$$-\infty < \liminf_{s \to 0^+} \frac{g(s)}{s} \leq \limsup_{s \to 0^+} \frac{g(s)}{s} \leq -m < 0,$$

$$\limsup_{s \to +\infty} \frac{g(s)}{s^{2^*-1}} \leq 0,$$

and there is $\xi > 0$ such that $G(\xi) > 0$,

where $G(s) = \int_0^s g(t) \, dt$ and $2^* = 2N/(N-2)$.

In [15], Berestycki, Gallouet and Kavian studied the case of dimension $N = 2$, when the nonlinearity $g$ has an exponential growth of the type

$$\limsup_{s \to +\infty} \frac{g(s)}{e^{\beta s^2}} = 0, \quad \forall \beta > 0.$$

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In the two mentioned papers above, the authors used the variational method to prove the existence of solutions $u \in H^1(\mathbb{R}^N) \setminus \{0\}$ of equation (1.1). The main idea in those papers was to solve the minimization problems

$$\min \left\{ \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx : \int_{\mathbb{R}^N} G(u) \, dx = 1 \right\}$$

and

$$\min \left\{ \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx : \int_{\mathbb{R}^N} G(u) \, dx = 0 \right\}$$

for $N \geq 3$ and $N = 2$ respectively. After that, the authors showed that the minimizer functions of the above problems are, in fact, ground state solutions of (1.1). By a ground state solution, we mean a solution $u \in H^1(\mathbb{R}^N) \setminus \{0\}$ that satisfies

$$E(u) \leq E(v), \quad \text{for all nontrivial solution } v \text{ of (1.1)},$$

where $E: H^1(\mathbb{R}^N) \to \mathbb{R}$ is the energy functional associated to (1.1), given by

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \int_{\mathbb{R}^N} G(u) \, dx.$$

Later on, Jeanjean and Tanaka, in [38], showed that the mountain pass level of $E$ is a critical level and it is indeed the lowest critical level.

A version of problem (1.1) for the critical case was studied in Alves, Souto and Montenegro [7], for $N \geq 3$ and $N = 2$, see also Zhang and Zou [47], for the case $N = 3$. In those papers the function $g$ is of the type

$$g(s) = \lambda |s|^{q-2}s + |s|^{2^* - 2}s, \quad s \in \mathbb{R},$$

where $\lambda$ is a positive parameter, and $q \in (2, 2^*)$, if $N \geq 3$. Related to dimension $N = 2$, $g$ has an exponential critical growth, more precisely, there is $\alpha_0 > 0$ such that

$$\lim_{s \to +\infty} \frac{g(s)}{s^{\alpha_0}} = 0, \quad \text{if } \alpha > \alpha_0, \quad \text{and } \lim_{s \to +\infty} \frac{g(s)}{s^{\alpha_0}} = +\infty, \quad \text{if } \alpha < \alpha_0.$$

Still related to the critical case, the reader can find in Alves, Figueiredo and Siciliano [5], Chang and Wang [25] and Zhang, do Ó and Squassina [46] the same type of results involving the fractional Laplacian operator, more precisely, results for a problem like

(1.2) \hspace{1cm} (-\Delta)\alpha u = g(u), \quad \text{in } \mathbb{R}^N,

with $\alpha \in (0, 1)$ and $N \geq 1$.

We would like to point out that the method used in the above papers works well because $g$ does not depend on $x$, $-\Delta$ and $(-\Delta)^\alpha$ are homogeneous operators and there is a Pohozaev identity associated with (1.1) and (1.2). When one of these facts does not hold, it is necessary to change the arguments. In [41], Pomponio and Watanabe studied the existence of solutions for (1.1), replacing $-\Delta$ by $-\Delta_p - \Delta_q$, that is, they considered the following problem

(1.3) \hspace{1cm} -\Delta_p u - \Delta_q u = g(u), \quad \text{in } \mathbb{R}^N.
In this case, due to lack of homogeneity of the operator, the arguments used in that paper do not work anymore. To overcome this difficulty, Pomponio and Watanabe used a result found in Jeanjean [37], see Theorem 1.1, to solve the problem. However, the fact that above problem has a Pohozaev identity is crucial in their approach. In [11], Azzollini and Pomponio considered the existence of solutions for the following class of problem

\[(1.4) \quad - \Delta u + V(x)u = g(u), \quad \text{in } \mathbb{R}^N.\]

Assuming some geometric conditions on \(V\), the authors also used Theorem 1.1 in [37], as well as, the fact that there is a Pohozaev identity associated with the class of problems.

The reader is invited to see that in the papers [41] and [11] a Pohozaev identity is a key point to prove that a sequence of approximate solutions for (1.3) and (1.4) is bounded.

Motivated by the above papers, we are led to formulate the following question: If \(g\) is a discontinuous function, how can one get a solution for problems (1.1), (1.2) or (1.3)? The main difficulty in answering this question is related to the fact that the classical variational methods for \(C^1\) functionals cannot be applied. Moreover, in this situation, there is no Pohozaev identity associated with the problem. A second problem that we are interested in is a version of (1.3) involving fractional Laplacian, more precisely,

\[(1.5) \quad (-\Delta)^\alpha u + (-\Delta)^\beta u = g(u), \quad \text{in } \mathbb{R}^N,\]

where \(\alpha, \beta \in (0, 1), N > 2 \max\{\alpha, \beta\}\) and \(g\) is a continuous function. This problem is interesting, as far as we are concerned we do not know any Pohozaev identity associated with it. Finally, another problem that we are interested in is the following anisotropic problem:

\[(1.6) \quad - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left( \left| \frac{\partial u}{\partial x_i} \right|^{p_i-2} \frac{\partial u}{\partial x_i} \right) = g(u), \quad \text{in } \mathbb{R}^N,\]

where \(1 < p_1 < \cdots < p_n < N\) and \(g\) is a continuous function. Here, as in the last problem, we do not know any Pohozaev identity associated with (1.6).

From the above commentaries a new approach must be developed to ensure the existence of solutions for (1.5) and (1.6). With these problems in mind, in the present paper we show that there is an abstract theorem associated with the famous result due to Berestycki and Lions [14], which can be used to solve a lot of problems where the existence of a Pohozaev identity is not clear. An advantage of our abstract result is related to the fact that it can be applied for a lot of problems, where the nonlinearity \(g\) is continuous, and also, for some problems where the nonlinearity \(g\) is discontinuous.

Before stating our abstract theorem we need to set some notations. In the sequel \((X, \| \cdot \|)\) is a reflexive Banach space and \(\psi_1, \ldots, \psi_n, \Phi: X \rightarrow \mathbb{R}\) are continuous functionals satisfying: there is an application \(*: [0, \infty) \times X \rightarrow X\) and \(\lambda_1, \ldots, \lambda_n, \lambda_\Phi \in \mathbb{R}\) such that
(X_1) \psi_i(*(t, u)) = t^{\lambda_i} \psi_i(u);
(X_2) \phi(*(t, u)) = e^{\lambda \phi(u)};
(X_3) 0 < \max \{\lambda_1, \ldots, \lambda_p\} < \lambda \phi;
(X_4) *(0, u) = 0, \forall u \in X;
(X_5) For each fixed u \in X, the application t \mapsto *(t, u) is continuous.

There are subsets X^+, X^r \subset X that are weakly closed and a function Q: X^+ \to X^r satisfying:

(X_6) \psi_i(Q(u)) \leq \psi_i(u), \forall u \in X^+, \forall i \in \{1, \ldots, n\};
(X_7) \phi(Q(u)) \geq \phi(u), \forall u \in X^+;
(X_8) If u \in X^r, then *(t, u) \in X^r for all t \geq 0.

Before writing out our next assumptions, we would like to fix the following notations:

\begin{align*}
  u_t &:= *(t, u), \forall t \geq 0 \text{ and } u \in X, \\
  J(u) &:= \sum_{i=1}^{n} \psi_i(u), \forall u \in X \\
  I(u) &= J(u) - \phi(u), \forall u \in X.
\end{align*}

and

Moreover, we also assume the following:

\begin{enumerate}
  \item[(F_1)] \phi(0) = 0 and there is u \in X such that \phi(u) > 0.
  \item[(F_2)] \psi_i(u) \geq 0 for all i \in \{1, \ldots, n\} and u \in X. Moreover, J(u) = 0 \Leftrightarrow u = 0.
  \item[(F_3)] There exists r > 0 such that if 0 < \|u\| < r, then
  \begin{equation*}
  \sum_{i=1}^{n} \lambda_i \psi_i(u) > \lambda \phi \phi(u).
  \end{equation*}
  \item[(F_4)] For any sequence (u_k) satisfying \phi(u_k) \geq 0 and J(u_k) \to 0, we have \|u_k\| \to 0. Moreover, if (J(u_k)) is bounded, then (u_k) is also bounded.
  \item[(F_5)] If (u_k) \subset X^r is weakly convergent sequence for u in X, then
  \begin{equation*}
  \limsup_{k \to +\infty} \phi(u_k) \leq \phi(u).
  \end{equation*}
  \item[(F_6)] If (u_k) is weakly convergent sequence for u in X, then
  \begin{equation*}
  \psi_i(u) \leq \liminf_{k \to +\infty} \psi_i(u_k), \forall i \in \{1, 2, \ldots, n\}.
  \end{equation*}
\end{enumerate}

Throughout this article, we denote by \mathcal{P} and \mathcal{P}^+ the sets

\begin{align*}
  \mathcal{P} &= \{u \in X \setminus \{0\} : \lambda_1 \psi_1(u) + \cdots + \lambda_n \psi_n(u) = \lambda \phi(u)\} \\
  \mathcal{P}^+ &= \{u \in X^+ \setminus \{0\} : \lambda_1 \psi_1(u) + \cdots + \lambda_n \psi_n(u) = \lambda \phi(u)\}.
\end{align*}
The set $\mathcal{P}$ will be called the Pohozaev set, and associated with it, we have the operator
$$K(u) = \lambda_1 \psi_1(u) + \cdots + \lambda_n \psi_n(u) - \lambda_\Phi \Phi(u),$$
which will be called the Pohozaev operator. Note that $K^{-1}(\{0\}) = \mathcal{P} \cup \{0\}$.

Now, we are ready to state our main result.

**Theorem 1.1.** Let $X$, $\Phi$, $\psi_1, \ldots, \psi_n$ satisfying $(X_1)$–$(X_8)$ and $(F_1)$–$(F_6)$. If
$$\inf_{w \in \mathcal{P}} I(w) = \inf_{w \in \mathcal{P}^+} I(w),$$
then there is $u \in \mathcal{P}$ such that $I(u) = \inf_{w \in \mathcal{P}} I(w) > 0$. If $I$ is locally Lipschitz, then $u$ is a critical point of $I$ in $X$, that is, $0 \in \partial I(u)$.

The plan of the paper is as follows: In Section 2 we prove some preliminary results that will be used in Section 3 to prove Theorem 1.1. In Section 4 we study the existence of solutions for a large class of problem, which includes the problem
$$(P_1) \quad (-\Delta)^\alpha u + (-\Delta)^\beta u = g(u), \quad \text{in } \mathbb{R}^N,$$
where $\alpha, \beta \in (0, 1)$, $(-\Delta)^\alpha$ and $(-\Delta)^\beta$ denote the fractional Laplacian of order $\alpha$ and $\beta$ respectively, $N > 2 \max\{\alpha, \beta\}$ and $g(s) = f(s) - s$ is a continuous function. In Section 5, we consider the existence of solutions for an anisotropic problem like
$$(P_2) \quad -\sum_{i=1}^n \frac{\partial}{\partial x_i} \left( | \frac{\partial u}{\partial x_i} |^{p_i-2} \frac{\partial u}{\partial x_i} \right) = g(u), \quad \text{in } \mathbb{R}^N,$$
where $1 < p_1 < \cdots < p_n < N$ and $g = f(s) - |s|^{p_1-2}s$ is a continuous function. Finally, in Section 6, we establish the existence of solutions for a class of discontinuous problem of the type
$$(P_3) \quad -\Delta u(x) \in \partial G(u(x)), \quad \text{a.e. in } \mathbb{R}^N,$$
where $N \geq 1$, $G$ is the primitive of a function $g(s) = f(s) - s$, which can have a finite number of discontinuity and $\partial G(s)$ is the generalized gradient of $G$ at $s \in \mathbb{R}$.

2. Abstract framework

In this section, we show some technical lemmas that will be used in the next section to prove Theorem 1.1.

**Lemma 2.1.** Let $u \in X$ satisfying $\Phi(u) > 0$. Then, there exists a unique $t^* > 0$ such that $u_{t^*} \in \mathcal{P}$. Hence,
$$I(u_{t^*}) = \max_{t \geq 0} I(u_t)$$
and for $u \in \mathcal{P}$,
$$I(u) = \max_{t \geq 0} I(u_t).$$
Proof. Let \( u \in X \) satisfying \( \Phi(u) > 0 \). By \((F_1)\), \( u \neq 0 \). For each \( t \geq 0 \), we set
\[
h(t) := I(u_t), \quad \forall t \in [0, +\infty).
\]
By \((X_1)-(X_2)\),
\[
h(t) = t^{\lambda_1} \psi_1(u) + \cdots + t^{\lambda_n} \psi_n(u) - t^{\lambda\Phi} \Phi(u).
\]
Now, from \((X_3)\) and \((F_2)\), \( h(t) > 0 \) for small \( t > 0 \) and
\[
\lim_{t \to +\infty} h(t) = -\infty.
\]
This ensures that \( h \) has a maximum at some \( t^* \in (0, +\infty) \), that is,
\[
I(u_{t^*}) = \max_{t > 0} I(u_t).
\]
Since \( h'(t^*) = 0 \), we derive that \( u_{t^*} \in P \). To show the uniqueness of \( t^* \), we first recall that \( u_{t^*} \in P \) if, and only if,
\[
\lambda_1 t^{\lambda_1} \psi_1(u) + \cdots + \lambda_n t^{\lambda_n} \psi_n(u) = \lambda_\Phi t^{\lambda\Phi} \Phi(u).
\]
In the sequel, without loss of generality, we may assume that \( \lambda_1 = \max \{\lambda_i\}_{i=1}^n \).
Then, \( u_t \in P \) if, and only if,
\[
(2.1) \quad \lambda_1 \psi_1(u) = -\sum_{i=2}^n \lambda_i t^{(\lambda_i - \lambda_1)} \psi_i(u) + \lambda_\Phi t^{(\lambda_\Phi - \lambda_1)} \Phi(u).
\]
Combining \((X_3)\) with \((F_2)\) and using the fact that \( \Phi(u) > 0 \), it follows that the function
\[
m(t) = -\sum_{i=2}^n \lambda_i t^{(\lambda_i - \lambda_1)} \psi_i(u) + \lambda_\Phi t^{(\lambda_\Phi - \lambda_1)} \Phi(u),
\]
has a positive derivative in the interval \((0, +\infty)\). Therefore, \( m(0) = 0, m \) is increasing, and \( m(t) \to +\infty \) as \( t \to +\infty \). These facts guarantee the existence of a unique \( t > 0 \) satisfying \((2.1)\). \( \square \)

**Corollary 2.2.** The Pohozaev set \( P \) is not empty.

**Proof.** The result follows combining the last lemma with hypothesis \((F_1)\). \( \square \)

**Lemma 2.3.** There exists \( r > 0 \) such that
\[
\|u\| \geq r, \quad \forall u \in P.
\]

**Proof.** For all \( u \in P \),
\[
K(u) = \sum_{i=1}^n \lambda_i \psi_i(u) - \lambda_\Phi \Phi(u) = 0.
\]
On the other hand, by \((F_3)\), there exists \( r > 0 \) such that
\[
K(u) = \sum_{i=1}^n \lambda_i \psi_i(u) - \lambda_\Phi \Phi(u) > 0, \quad \forall 0 < \|u\| < r.
\]
Thus,
\[
\|u\| \geq r, \quad \forall u \in P.
\]
\( \square \)
Proposition 2.4. Suppose that \( \inf_{w \in \mathcal{P}} I(w) = \inf_{w \in \mathcal{P}^+} I(w) \). Then the functional \( I \) is bounded from below on \( \mathcal{P} \), and there exists \( u_0 \in \mathcal{P} \) satisfying
\[
I(u_0) = \inf_{u \in \mathcal{P}} I(u).
\]
Moreover, \( \inf_{u \in \mathcal{P}} I(u) > 0 \).

Proof. If \( u \in \mathcal{P} \), \((F_2)\) combined with \((X_3)\) gives
\[
(I(u) = J(u) - \Phi(u) = \sum_{i=1}^{n} \psi_i(u) - \sum_{i=1}^{n} \frac{\lambda_i}{\lambda^*} \psi_i(u) = \sum_{i=1}^{n} \left(1 - \frac{\lambda_i}{\lambda^*}\right) \psi_i(u) \geq 0,
\]
assuring the boundedness from below of \( I \) on \( \mathcal{P} \). In what follows,
\[
I_{\infty} = \inf_{u \in \mathcal{P}} I(u) = \inf_{u \in \mathcal{P}^+} I(u)
\]
and \((u_k)\) is a minimizing sequence associated with \( I_{\infty} \), that is, \((u_k) \subset \mathcal{P}^+ \) and
\[
I(u_k) \to I_{\infty}.
\]
From \((2.2)\), \((J(u_k))\) is a bounded sequence. Hence, by \((F_4)\), the sequence \((u_k)\) is also bounded in \( X \). On the other hand, by conditions \((X_6)-(X_7)\), we know that \((Q(u_k)) \subset X^r\),
\[
J(Q(u_k)) \leq J(u_k), \quad \forall k \in \mathbb{N}
\]
and
\[
\Phi(Q(u_k)) \geq \Phi(u_k) > 0, \quad \forall k \in \mathbb{N}.
\]
By Lemma 2.1, there exits \( t_k^* > 0 \) such that \((Q[u_k])_{t_k^*} \subset \mathcal{P}\). Therefore,
\[
I_{\infty} \leq I ((Q[u_k])_{t_k^*}) \leq I ((u_k)_{t_k^*}) \leq \max_{t>0} I ((u_k)_t) = I(u_k).
\]
The last inequality yields
\[
I ((Q[u_k])_{t_k^*}) \to I_{\infty}.
\]
From this, without loss of generality, we can assume that \((u_k) \subset X^r\). As \( X \) is reflexive and \( X^r \) is weakly closed, we can suppose that for some subsequence, \((u_k)\) is weakly convergent for some \( u \in X^r \). Since, \( u_k \in \mathcal{P} \), we must have \( \Phi(u_k) > 0 \), and so, by \((F_5)\),
\[
\Phi(u) \geq 0.
\]
We claim that \( \Phi(u) > 0 \). Indeed, assume by contradiction that \( \Phi(u) = 0 \). Then,
\[
\Phi(u_k) \to 0 \quad \text{as } k \to +\infty.
\]
Using the fact that \( u_k \in \mathcal{P} \), we derive that
\[
J(u_k) \to 0 \quad \text{as } k \to +\infty.
\]
Now, using \((F_1)\) we get
\[
\|u_k\| \to 0,
\]
which contradicts Lemma 2.3. Thereby, $\Phi(u) > 0$, and so, $u \neq 0$. Consequently, Lemma 2.1 guarantees the existence of $t^* > 0$ such that $u_{t^*} \in \mathcal{P}$. From $(X_1)-(X_2)$,

$$I(u_k) = \max_{t > 0} I((u_k)_t) \geq I((u_k)_{t^*}) = \sum_{i=1}^n \psi_i((u_k)_{t^*}) - \Phi((u_k)_{t^*})$$

$$= \sum_{i=1}^n (t^*)^{\lambda_i} \psi_i(u_k) - (t^*)^{\lambda_i} \Phi(u_k).$$

The last inequality combined with $(F_5)-(F_6)$ result in

$$I_\infty \geq \liminf_{k \to \infty} \left( \sum_{i=1}^n (t^*)^{\lambda_i} \psi_i(u_k) - (t^*)^{\lambda_i} \Phi(u_k) \right)$$

$$\geq \sum_{i=1}^n (t^*)^{\lambda_i} \psi_i(u) - (t^*)^{\lambda_i} \Phi(u) = I(u_{t^*}).$$

Recalling that $u_{t^*} \in \mathcal{P}$, we deduce that

$$I_\infty = I(u_{t^*}).$$

Now, we are going to show that $I_\infty = \inf_{u \in \mathcal{P}} I(u) > 0$. In fact, by (2.2), $I_\infty \geq 0$. If $I_\infty = 0$, we can argue as above to find a minimizing sequence $(u_k) \subset \mathcal{P}^+$ satisfying $\Phi(u_k) \geq 0$ and $J(u_k) \to 0$. However this, together with $(F_4)$, leads to $\|u_k\| \to 0$, contradicting Lemma 2.3. Therefore, $\inf_{u \in \mathcal{P}} I(u) > 0$, and the proof is finished. \(\square\)

3. Proof of Theorem 1.1

In this section our main goal is to show Theorem 1.1. To do so, we need to prove some preliminary lemmas.

**Lemma 3.1.** Let $u \in \mathcal{P}$ and set $\gamma(t) := u_t$. Then,

$$\lim_{t \to +\infty} I(\gamma(t)) = -\infty.$$  

**Proof.** First of all, note that

$$I(\gamma(t)) = \sum_{i=1}^n t^{\lambda_i} \psi_i(u) - t^{\lambda_i} \Phi(u), \quad \forall t \in [0, +\infty).$$

As $u \in \mathcal{P}$, we have $\Phi(u) > 0$. This combined with $(X_3)$ gives the desired result. \(\square\)

**Lemma 3.2.** Let $\gamma: \mathbb{R} \to X$ be a continuous path satisfying

$$\gamma(0) = 0 \quad \text{and} \quad \lim_{t \to +\infty} I(\gamma(t)) = -\infty.$$  

Then, there exists $t_0 > 0$ such that $\gamma(t_0) \in \mathcal{P}$.

**Proof.** We begin by supposing that $\gamma(t) \neq 0$ for all $t > 0$. Since

$$K(u) = \sum_{i=1}^n \lambda_i \psi_i(u) - \lambda \Phi(u),$$
by \((F_3)\), there exists \(r > 0\) such that
\[ K(u) > 0, \quad \text{for } ||u|| < r. \]
As \(\gamma(0) = 0\) and \(\gamma, K\) are continuous functions, for \(t\) small enough we must have
\[ K(\gamma(t)) > 0. \]
On the other hand, the definitions of \(I, (X_3)\) and \((F_3)\) lead to
\[ K(u) = \lambda \Phi I(u) + \sum_{i=1}^{n} (\lambda_i - \lambda \Phi) \psi_i(u) \leq \lambda \Phi I(u), \quad \forall u \in X. \]
Hence,
\[ K(\gamma(t)) \leq \lambda \Phi I(\gamma(t)), \quad \forall t \in [0, +\infty). \]
Then, \(K(\gamma(t)) < 0\), for \(t\) large enough. From this, there is \(\bar{t} > 0\) satisfying
\[ K(\gamma(\bar{t})) = 0, \]
implying that \(\gamma(\bar{t}) \in P\). For the general case, let us fix \(\bar{t} > 0\) satisfying
\[ \bar{t} = \sup \{t \in [0, +\infty); \gamma(t) = 0\}. \]
By continuity, \(\gamma(\bar{t}) = 0\). Setting
\[ \beta(t) = \gamma(t + \bar{t}) \quad \forall t \in [0, +\infty), \]
we have that \(\beta(0) = 0, \lim_{t \rightarrow \infty} I(\beta(t)) = -\infty\) and \(\beta(t) \neq 0\), for all \(t > 0\). By the above arguments, there exists \(t_0 > 0\) such that
\[ \beta(t_0) \in P, \]
showing that \(\gamma(t_0 + \bar{t}) \in P\). \(\square\)

Next, we recall the deformation lemma for locally Lipschitz functional by Figueiredo and Pimenta [33], that will be used in the proof of Theorem 1.1.

**Theorem 3.3 (Deformation lemma).** Let \(X\) be a Banach space and let \(I: X \rightarrow \mathbb{R}\) be a locally Lipschitz functional. Assume that there are \(c, S \subset E\), \(\alpha, \delta, \epsilon_0 > 0\) satisfying
\[ \beta(x) := \min \{||z||_X; z \in \partial I(x)\} \geq \alpha, \quad \forall x \in I^{-1}([c - \epsilon_0, c + \epsilon_0]) \cap S_{2\delta}. \]
where \(S_{2\delta}\) is a \(2\delta\)-neighborhood of \(S\). Then, for each \(0 < \epsilon < \min\{\delta \alpha / 2, \epsilon_0\}\) there exists a homeomorphism \(\eta: X \rightarrow X\) satisfying
- \(\eta(u) = u, \quad \text{for } u \notin I^{-1}([c - \epsilon_0, c + \epsilon_0]) \cap S_{2\delta}\);
- \(\eta(I^{c+\epsilon} \cap S) \subset I^{c-\epsilon}\);
- \(I(\eta(u)) \leq I(u)\) for all \(u \in X\),
where \(I^a = \{u \in X; I(u) \leq a\}\) for all \(a \in \mathbb{R}\).
Now, we are ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** By Proposition 2.4, there is \( u \in P \) with

\[
I(u) = I_\infty = \inf_{w \in P} I(w) > 0.
\]

Assume by contradiction that \( 0 \notin \partial I(u) \). Then, there is \( \alpha > 0 \) such that

\[
|x - u| < \alpha \Rightarrow \beta(x) = \min \{ \|z\|_{X^*} ; z \in \partial I(x) \} > \alpha.
\]

Indeed, otherwise for each \( \alpha = 1/k \), it would exist \( x_k \in X \) with

\[
|u - x_k| < \frac{1}{k} \quad \text{and} \quad \beta(x_k) < \frac{1}{k}, \quad \forall k \in \mathbb{N}.
\]

Consequently, it would exist \( z_k \in \partial I(x_k) \) with

\[
\|z_k\|_{X^*} < \frac{1}{k}.
\]

By the definition and properties of \( I_0(u,v) \) (see [24]), we must have

\[
I_0(u,v) \geq \limsup_{k \to \infty} I_0(x_k,v) \geq \limsup_{k \to \infty} \langle z_k, v \rangle = \langle 0, v \rangle, \quad \forall v \in X,
\]

showing that \( 0 \in \partial I(u) \), which is a contradiction. This proves (3.2).

Applying the deformation lemma for \( c = I_\infty, \delta = \alpha/4, \epsilon_0 = I_\infty/2 \) and \( S = B_{\alpha/2}(u) \), we get a homeomorphism \( \eta : X \to X \) satisfying

(i) \( \eta(u) = u, \) if \( u \notin I^{-1}([c - \epsilon_0, c + \epsilon_0]) \cap S_{2\delta} \);

(ii) \( \eta(I^{c-\epsilon} \cap S) \subset I^{c-\epsilon} \);

(iii) \( I(\eta(u)) \leq I(u) \) for all \( u \in X \).

Fix

\[
\beta(t) := \eta(\gamma(t))
\]

where \( \gamma(t) = u_t \). By \( (X_4) \), the function \( \beta \) is continuous. Moreover, from \( (X_4) \) and the fact that \( I(0) < c - \epsilon_0 \), we obtain

\[
\beta(0) = \eta(\gamma(0)) = \eta(0) = 0.
\]

Now, by Lemma 3.1 and (iii),

\[
\lim_{t \to +\infty} I(\beta(t)) = \lim_{t \to +\infty} I(\eta(\gamma(t))) \leq \lim_{t \to +\infty} I(\gamma(t)) = -\infty.
\]

The above analysis permits us to apply Lemma 3.2 to find \( t^* > 0 \) such that \( \beta(t^*) \in P \). Hence,

\[
c \leq I(\beta(t^*)) \leq \max_{t > 0} I(\beta(t)).
\]
On the other hand, as $I$ and $\gamma$ are continuous, $I(u) = c < c + \epsilon$ and $\gamma(1) = u$, it is possible to pick $\tau > 0$ in such a way that
\[ \gamma(t) \in I^{c+\epsilon} \cap S, \quad \forall t \in [1 - \tau, 1 + \tau]. \]
Thereby, if $t \in [1 - \tau, 1 + \tau]$, the deformation lemma yields
\[ I(\beta(t)) = I(\eta(\gamma(t))) \leq c - \epsilon. \]
Now, we will analyze the case $t \notin [1 - \tau, 1 + \tau]$. In this case, by Lemma 2.1 and (iii),
\[ I(\beta(t)) = I(\eta(\gamma(t))) \leq I(\gamma(t)) < \max_{t > 0} I(\gamma(t)) = I(\gamma(1)) = c. \]
In any case, we deduce that
\[ \max_{t > 0} I(\beta(t)) < c, \]
which contradicts (3.4). Therefore, $0 \in \partial I(u)$, proving the theorem.

The corollary below is a version of Theorem 1.1 when the functional $I$ is a $C^1$ functional.

**Corollary 3.4.** Let $X, \psi_1, \ldots, \psi_n$ and $\Phi$ satisfying $(X_1)-(X_k)$ and $(F_1)-(F_h)$. Assuming that $\Phi, \psi_1, \ldots, \psi_n \in C^1(X, \mathbb{R})$ and
\[ \inf_{w \in \mathcal{P}} I(w) = \inf_{w \in \mathcal{P}^+} I(w). \]
Then there exists $u \in \mathcal{P}$ such that
\[ I(u) = \inf_{w \in \mathcal{P}} I(w). \]
Moreover, $u$ is a critical point of $I$ in $X$, that is, $I'(u) = 0$.

**Proof.** Since $I \in C^1(X, \mathbb{R})$, we have that
\[ \partial I(u) = \{I'(u)\}, \quad \forall u \in X. \]
Therefore, the corollary is an immediate consequence of Theorem 1.1.

### 4. A problem involving $s$ and $t$ fractional Laplacian

As mentioned in the introduction, in this section we intend to prove the existence of nonnegative solution for the following problem:

\[ (P_1) \quad \sum_{j=1}^{n} (-\Delta)^{s_j} u = g(u), \quad \text{in} \ \mathbb{R}^N, \]

where $0 < s_1 \leq s_2 \leq \cdots \leq s_n < 1$, $N > 2s_n$, $(-\Delta)^{s_j}$ denotes the $s_j$-fractional Laplacian and
\[ g(s) = f(s) - s, \quad \forall s \in \mathbb{R}, \]
with $f: \mathbb{R} \to \mathbb{R}$ being a continuous function satisfying:
(f_1) \lim_{s \to 0} f(s)/s = 0.
(f_2) \limsup_{|s| \to +\infty} \frac{|f(s)|}{|s|^q} < \infty, \text{ for some } q \in (2, 2s_n^*) \text{ where } 2s_n^* = 2N/(N - 2s_n).
(f_3) f(s) > 0, \quad \forall s > 0.
(f_4) \text{There is } \tau > 0 \text{ such that } G(\tau) = \int_0^\tau g(s) \, ds > 0.

Since we intend to find a nonnegative solution, in what follows we assume that
\[ f(s) = 0, \quad \forall s < 0, \]
and denote by \( F \) the primitive of \( f \), that is,
\[ F(s) = \int_0^s f(t) \, dt. \]

The energy functional associated with \( (P_1) \) is given by \( I_1 : H^{s_n}(\mathbb{R}^N) \to \mathbb{R} \) with
\[ I_1(u) = \frac{1}{2} \sum_{i=1}^n \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy - \int_{\mathbb{R}^N} G(u) \, dx. \]

It is easy to see that \( I_1 \in C^1(H^{s_n}(\mathbb{R}^N), \mathbb{R}) \), and that its critical points are weak solutions of \( (P_1) \).

We recall that, for any \( s \in (0, 1) \), the fractional Sobolev space \( H^s(\mathbb{R}^N) \) is defined by
\[ H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy < \infty \right\}, \]
endowed with the norm
\[ \|u\| = \left( \|u\|^2_{L^2(\mathbb{R}^N)} + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy \right)^{1/2}. \]

The s-Laplacian, \((-\Delta)^s u\), of a smooth function \( u : \mathbb{R}^N \to \mathbb{R} \) is defined by
\[ \mathcal{F}((-\Delta)^s u)(\xi) = |\xi|^{2s} \mathcal{F}(u)(\xi), \quad \xi \in \mathbb{R}^N, \]
where \( \mathcal{F} \) denotes the Fourier transform, that is,
\[ \mathcal{F}(\phi)(\xi) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{-i\xi \cdot x} \phi(x) \, dx \equiv \hat{\phi}(\xi), \]
for functions \( \phi \) in the Schwartz class. As mentioned in [27], see Lemma 3.2, \((-\Delta)^s u\) can be equivalently represented by
\[ (-\Delta)^s u(x) = -\frac{1}{2} C(N, s) \int_{\mathbb{R}^N} \frac{(u(x + y) + u(x - y) - 2u(x))}{|y|^{N+2s}} \, dy, \quad \forall x \in \mathbb{R}^N, \]
where
\[ C(N, s) = \left( \int_{\mathbb{R}^N} \frac{(1 - \cos \xi_1)}{|\xi|^{N+2s}} \, d\xi \right)^{-1}, \quad \xi = (\xi_1, \xi_2, \ldots, \xi_N). \]
A Berestycki–Lions type result and applications

(4.1) \((|(-Δ)^{s/2} u|^2_{L^2(\mathbb{R}^N)})_R = \int_{\mathbb{R}^N} |\xi|^{2s} |\hat{u}|^2 d\xi = \frac{1}{2} C(N, s) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} dx dy,
\)

for all \(u \in H^s(\mathbb{R}^N)\). For \(N > 2s\), from Theorem 6.5 in [27] we also know that, for any \(p \in [2, 2^*_s]\), there exists \(C_p > 0\) such that

(4.2) \(|u|_{L^p(\mathbb{R}^N)} \leq C_p \|u\|, \quad \text{for all } u \in H^s(\mathbb{R}^N).
\)

In the sequel, we will show that all conditions of Theorem 1.1 hold, which ensures that functional \(I_1\) has a nontrivial critical point, and hence \((P_1)\) has a nontrivial solution. First of all, note that

The reader is invited to observe that \(X^+\) and \(X^\ast\) are weakly closed in \(H^s(\mathbb{R}^N)\). Moreover, we would like to point out that \(X^\ast\) is compactly embedded into \(L^q(\mathbb{R}^N)\), for all \(q \in (2, 2^*_s)\), that is, if \((u_k) \subset X^\ast\) is a bounded sequence in \(H^s(\mathbb{R}^N)\), then there are a subsequence of \((u_k)\), still denoted by itself, and \(u \in X^\ast\) such that

\[ u_k \to u \quad \text{in } L^q(\mathbb{R}^N), \quad \forall q \in (2, 2^*_s). \]

The proof of this fact follows from the radial Lemma A.IV in [14].

The main result in this section has the following statement.

\textbf{Theorem 4.1.} Assume that conditions \((f_1)-(f_4)\) hold. Then, problem \((P_1)\) has a nontrivial solution.

\textbf{Proof.} In the sequel, we will show that all conditions of Theorem 1.1 hold, which ensures that functional \(I_1\) has a nontrivial critical point, and hence \((P_1)\) has a nontrivial solution. First of all, note that

\[ I_1(u) = J(u) - \Phi(u), \quad \forall u \in X = H^s(\mathbb{R}^N). \]

In what follows, we set \(*((t, u)) := u_t : \mathbb{R}^N \to \mathbb{R}\) defined by

\[ u_t(x) = \begin{cases} u(x/t), & \text{for } t > 0, \\ 0, & \text{for } t = 0. \end{cases} \]
It is easy to check that
\[ \psi_1, \psi_2, \ldots, \psi_n, \Phi \in C^1(H^{s_n}(\mathbb{R}^N), \mathbb{R}), \]
\[ u \in X^r \Rightarrow u_t \in X^r, \quad \forall t \geq 0, \]
\[ \Phi(u_t) = t^N \Phi(u), \quad \forall t \geq 0 \text{ and } \forall u \in H^{s_n}(\mathbb{R}^N) \]
and
\[ \psi_i(u_t) = t^{N-2s_i} \psi_i(u), \quad \forall t \geq 0, \quad \forall i \in \{1, 2, \ldots, n\} \quad \text{and} \quad \forall u \in H^{s_n}(\mathbb{R}^N). \]

Thus, the conditions \((X_1)-(X_4)\) and \((F_2)\) occur. Moreover, a simple computation shows that for each \(u \in H^{s_n}(\mathbb{R}^N)\), the application \(t \mapsto u_t\) is continuous, and so, \((X_5)\) is also proved.

The conditions \((X_6)-(X_8)\) are satisfied by considering \(Q : H^{s_n}(\mathbb{R}^N) \rightarrow X^r, u \mapsto (u^+)\) where \((u^+)\) is the Schwartz symmetrization of \(u^+ = \max\{u, 0\}\).

Now, we are going to prove the conditions \((F_1)\) and \((F_3)-(F_6)\).

**Claim 4.2** (Proof of \((F_1)\)). The equality \(\Phi(0) = 0\) holds and there exists \(u \in H^{s_n}(\mathbb{R}^N)\) such that \(\Phi(u) > 0\).

**Proof.** By definition of \(\Phi\), we have \(\Phi(0) = 0\). For each \(k \in \mathbb{N}\), take \(\phi_k = \tau\) in \(B_1(0)\) and \(|\phi_k| \leq \tau\). Note that
\[ \int_{\mathbb{R}^N} G(\phi_k) \, dx = \int_{B_1(0)} G(\phi_k) \, dx + \int_{B_1+1/k(0) \setminus B_1} G(\phi_k) \, dx \]
and
\[ \left| \int_{B_1+1/k(0) \setminus B_1(0)} G(\phi_k) \, dx \right| \leq \left( \sup_{\{|t| \leq \tau\}} G(t) \right) |B_{1+1/k} \setminus B_1(0)|. \]

Since \(|B_{1+1/k} \setminus B_1(0)| \rightarrow 0\) as \(k \rightarrow +\infty\), and 
\(G(\tau) > 0\), we can fix large \(k \in \mathbb{N}\) such that
\[ \int_{\mathbb{R}^N} G(\phi_k) \, dx \geq G(\tau)|B_1(0)| - \left( \sup_{\{|t| \leq \tau\}} G(t) \right) |B_{1+1/k} \setminus B_1(0)| > 0, \]
proving the claim. \(\square\)

**Claim 4.3** (Proof of \((F_3)\)). There exists \(r > 0\) such that
\[ \sum_{i=1}^n \lambda_i \psi_i(u) > \lambda \Phi(u), \quad \text{for } 0 < \|u\| < r. \]
Claim 4.4 (Proof of \(F_4\)). Let \((u_k)\) be a sequence in \(H^{s_n}(\mathbb{R}^N)\) with \(\Phi(u_k) \geq 0\), for all \(k \in \mathbb{N}\). If \((J(u_k))\) is bounded, we have that \((u_k)\) is also bounded. Moreover, if \(J(u_k) \to 0\), then \(\|u_k\| \to 0\).

Proof. Let us fix \(\epsilon < 1/2\), we find \(C, C_1 > 0\) satisfying
\[
\lambda_n \psi_n(u) - \lambda \Phi(u) = \lambda_n \psi_n(u) + \frac{\lambda \phi_k}{2} \int_{\mathbb{R}^N} |u|^2 \, dx - \lambda \Phi \int_{\mathbb{R}^N} F(u) \, dx \\
\geq \lambda_n \psi_n(u) + \lambda \phi \left( \frac{1}{2} - \epsilon \right) \int_{\mathbb{R}^N} |u|^2 \, dx - \lambda \Phi \int_{\mathbb{R}^N} |u|^q \, dx \\
\geq C \|u\|^2 - C_1 \|u\|^q,
\]
where \(C, C_1 > 0\). As \(\epsilon > 2\), we obtain the desired result. \(\square\)

Claim 4.5 (Proof of \(F_5\)). If \((u_k)\) is weakly convergent for \(u\) in \(X^r\), then
\[
\limsup_{k \to +\infty} \Phi(u_k) \leq \Phi(u).
\]

Proof. The assumptions on \(f\) ensure that for each \(\epsilon > 0\), there is \(C_\epsilon > 0\) satisfying
\[
F(s) \leq \frac{\epsilon}{6L} \|s\|^2 + C_\epsilon |s|^p,
\]
where \(L = \sup_{k \in \mathbb{N}} \|u_k\|^2\). Then, for \(R > 0\)
\[
\int_{B_R} |F(u_k)| \, dx \leq \frac{\epsilon}{6} + C_\epsilon \int_{B_R} |u_k|^p \, dx.
\]
Since \(X^r\) is compactly embedded into \(L^q(\mathbb{R}^N)\), for \(q \in (2, 2^*)\), there are large \(R\) and \(k_0\) such that
\[
\int_{B_R} |F(u)| \, dx < \frac{\epsilon}{3} \quad \text{and} \quad \int_{B_R} |F(u_k)| \, dx < \frac{\epsilon}{3}, \quad \forall k \geq k_0.
\]
Now, as $H^s(\mathbb{R}^N)$ is compactly embedded into $L^q(B_R)$, for $q \in [1, 2^*_s)$ and since $f$ has subcritical growth, we have
\[
\int_{B_R} |F(u_k) - F(u)| \, dx \to 0.
\]
From this,
\[
\limsup_{k \to +\infty} \int_{\mathbb{R}^N} |F(u_k) - F(u)| \, dx \leq \epsilon,
\]
implying that
\[
\int_{\mathbb{R}^N} F(u_k) \, dx \to \int_{\mathbb{R}^N} F(u) \, dx.
\]
The last limit leads to
\[
\liminf_{k \to +\infty} \left( \frac{1}{2} \int_{\mathbb{R}^N} |u_k|^2 \, dx - \int_{\mathbb{R}^N} F(u_k) \, dx \right) 
\geq \liminf_{k \to +\infty} \left( \frac{1}{2} \int_{\mathbb{R}^N} |u|^2 \, dx \right) - \int_{\mathbb{R}^N} F(u) \, dx 
\geq \frac{1}{2} \int_{\mathbb{R}^N} |u|^2 \, dx - \int_{\mathbb{R}^N} F(u) \, dx = -\Phi(u),
\]
that is, \( \liminf_{k \to +\infty} (-\Phi(u_k)) \geq -\Phi(u) \), proving the result.

**Claim 4.6** (Proof of (F6)). Suppose \((u_k)\) is weakly convergent to \(u\) in $H^s(\mathbb{R}^N)$, then
\[
\psi_i(u) \leq \liminf_{k \to +\infty} \psi_i(u_k), \quad \forall i \in \{1, 2, \ldots, n\}.
\]
**Proof.** The result is an immediate consequence of the fact that \(\psi_i\) is a convex function on $H^s(\mathbb{R}^n)$, for \(i \in \{1, 2, \ldots, n\}\). \(\square\)

**Claim 4.7.** The functional $I_1$ satisfies the equality
\[
\inf_{w \in \mathcal{P}} I_1(w) = \inf_{w \in \mathcal{P}^+} I_1(w).
\]
**Proof.** By definition of $I_1$, it is easy to see that
\[
I_1(u) \geq I_1(u^+), \quad \forall u \in H^s(\mathbb{R}^N),
\]
where $u^+ = \max\{u, 0\}$. For each $u \in \mathcal{P}$, we know that $u^+ \neq 0$, thus there is $t^+ > 0$ such that $(u_t^+)^+ \in \mathcal{P}$. Then,
\[
\inf_{w \in \mathcal{P}^+} I_1(w) \leq I_1((u_t^+)^+) \leq I_1((u_t^+)) \leq \max_{t > 0} I_1(u_t) = I_1(u), \quad \forall u \in \mathcal{P},
\]
showing the desired result. \(\square\)

The above claims permit us to conclude that $I_1$ satisfies the assumptions of Theorem 1.1, more precisely, those of Corollary 3.4. Hence, $I_1$ has a nontrivial critical point, and so, problem (P1) has a nontrivial solution. \(\square\)

Before concluding this section, we would like point out that the reader can find recent results involving fractional Laplacian in Barrios, Colorado, de Pablo and Sánchez [13], Brändle, Colorado and Sánchez [16], Caffé and Sire [17], Caffarelli and Silvestre [18], Fall, Mahmoudi and Valdinoci [31], Felmer, Quass and Tan [32], Secchi [45] and their references.
5. Existence of solutions for a class of anisotropic problem

In this section we study the existence of solutions for the following anisotropic problem:

\[(P_2)\]

\[-\sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2} \frac{\partial u}{\partial x_i}\right) = g(u), \text{ in } \mathbb{R}^N,\]

where \(1 < p_1 \leq \cdots \leq p_N < N\) and \(g: \mathbb{R} \to \mathbb{R}\) is a function given by

\[g(s) = f(s) - |s|^{p_1-2}s, \quad \forall s \in \mathbb{R},\]

with \(f: \mathbb{R} \to \mathbb{R}\) being a continuous function satisfying \((f_3)-(f_4)\) and the conditions:

\[(f_5) \lim_{s \to 0} \frac{f(s)}{|s|^{p_1-2}} = 0.\]

\[(f_6) \limsup_{|s| \to +\infty} \frac{|f(s)|}{|s|^q} < \infty, \text{ for some } q \in (p_1, p^*), \text{ where}\]

\[p^* = \frac{N}{\sum_{i=1}^{N} \frac{1}{p_i} - 1}.\]

As in the previous section, we will assume that

\[f(s) = 0, \quad \forall s < 0,\]

and denote by \(F\) the primitive of \(f\), that is,

\[F(s) = \int_0^s f(t) \, dt.\]

The main theorem in this section is the following

**Theorem 5.1.** Assume the conditions \((f_3)-(f_6)\) Then, \((P_2)\) has a nontrivial solution.

The reader can find some results associated with anisotropic problems in Alves and El Hamidi [3], El Hamidi and Rakotoson, [28], [29], [30], Fragala, Gazzola, and Kawohl [34] and their references.

Hereafter, we fix \(\vec{p} = (p_1, \ldots, p_N)\) and define the anisotropic Sobolev space \(W^{1,\vec{p}}(\mathbb{R}^N)\) by

\[W^{1,\vec{p}}(\mathbb{R}^N) = \left\{ u \in L^{p_i}(\mathbb{R}^N) : \frac{\partial u}{\partial x_i} \in L^{p_i}(\mathbb{R}^N), \, i = 1, 2, \ldots, N \right\}\]

endowed with the norm

\[\|u\| := \left( \int_{\mathbb{R}^N} |u|^{p_i} \, dx \right)^{1/p_i} + \sum_{i=1}^{N} \left( \int_{\mathbb{R}^N} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} \, dx \right)^{1/p_i}.\]
Related to the space \((W^{1,\frac{q}{p}}(\mathbb{R}^N), \| \cdot \|)\), it is possible to prove that it is a reflexive Banach space and that \(C_0^\infty(\mathbb{R}^N)\) is dense in \(W^{1,\frac{q}{p}}(\mathbb{R}^N)\). Moreover, the space \(W^{1,\frac{q}{p}}(\mathbb{R}^N)\) is continuously embedded into \(L^q(\mathbb{R}^N)\) for all \(q \in [p_1,p^*]\). For more details about this subject see Nikol’skii [40] and Rakosnik [43], [44].

The proof of the next lemma follows the same ideas explored in Lemma 2.5 and Theorem 3.1 in [35], and therefore its proof will be omit.

**Lemma 5.2.** If \(u \in W^{1,\frac{q}{p}}(\mathbb{R}^N)\) is a nonnegative function, then

\[
\left\| \frac{\partial u^*}{\partial x_i} \right\|_{p_i} \leq \left\| \frac{\partial u}{\partial x_i} \right\|_{p_i}, \quad \forall i \in \{1, 2, \ldots, n\},
\]

where \(u^*\) is the Schwartz symmetrization of \(u\).

Our intention is to show that the energy functional \(I_2: W^{1,\frac{q}{p}}(\mathbb{R}^N) \to \mathbb{R}\) given by

\[
I_2(u) = \sum_{i=1}^{N} \frac{1}{p_i} \int_{\mathbb{R}^N} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} dx + \int_{\mathbb{R}^N} |u|^{p_1} dx - \int_{\mathbb{R}^N} F(u) dx
\]

satisfies the conditions of Theorem 1.1, because the critical points of \(I_2\) are weak solutions of \((P_2)\). Since \(I_2\) belongs to \(C^1(W^{1,\frac{q}{p}}(\mathbb{R}^N), \mathbb{R})\), we will prove that \(I_2\) satisfies the conditions of Corollary 3.4. Having this in mind, in the sequel we define \(\psi_i, \Phi: W^{1,\frac{q}{p}}(\mathbb{R}^N) \to \mathbb{R}\) by

\[
\psi_i(u) = \frac{1}{p_i} \int_{\mathbb{R}^N} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} dx, \quad \text{for } i \in \{1, 2, \ldots, N\}
\]

\[
\Phi(u) = \int_{\mathbb{R}^N} G(u) dx = \int_{\mathbb{R}^N} F(u) dx - \frac{1}{p_1} \int_{\mathbb{R}^N} |u|^{p_1} dx
\]

\[
X = W^{1,\frac{q}{p}}(\mathbb{R}^N), \quad X^+ = \{u \in W^{1,\frac{q}{p}}(\mathbb{R}^N) : u(x) \geq 0 \text{ a.e. in } \mathbb{R}^N\}
\]

and

\[
X^r = \{u \in W^{1,\frac{q}{p}}(\mathbb{R}^N) \cap X^+ : 0 \leq u(x) \leq u(y) \text{ if } 0 < |y| \leq |x|\}.
\]

Arguing as Radial Lemma A.IV in [14], it is possible to prove that \(X^r\) is compactly embedded into \(L^q(\mathbb{R}^N)\), for all \(q \in (p_1,p^*)\).

Using the above notations

\[
I_2(u) = J(u) - \Phi(u), \quad \forall u \in X = W^{1,\frac{q}{p}}(\mathbb{R}^N).
\]

As in the previous section, we also consider \(*\mathcal{t} u, u) := u_t: \mathbb{R}^N \to \mathbb{R}\) by

\[
u_t(x) = \begin{cases} u(x/t), & \text{for } t > 0, \\ 0, & \text{for } t = 0. \end{cases}
\]

A simple computation yields

\[
\psi_i(u_t) = t^{N-p_i} \psi_i(u) \quad \text{and} \quad \Phi(u_t) = t^N \Phi(u), \quad \forall t \geq 0 \text{ and } \forall u \in W^{1,\frac{q}{p}}(\mathbb{R}^N).
\]
Moreover, the application $t \mapsto u_t$ is a continuous function for $t \in [0, +\infty)$, for all $u \in W^{1,\tilde{p}}(\mathbb{R}^N)$ and $u_t \in X^r \quad \forall t \in \mathbb{R}$, when $u \in X^r$.

In the sequel, we set $Q: W^{1,\tilde{p}}(\mathbb{R}^N) \rightarrow X^r$ by $Q(u) = (u^+)^*$. Using Lemma 5.2 and properties of radial functions, it follows that

$$\psi_i(Q(u)) \leq \psi_i(u) \quad \text{and} \quad \Phi(u) = \Phi(Q(u)), \quad \forall u \in X^r.$$

Moreover, with few modifications, we can argue as in Section 4 to prove the claims below:

1. There exists $u \in W^{1,\tilde{p}}(\mathbb{R}^N)$ such that $\Phi(u) > 0$.
2. $\psi_i(u) \geq 0, \forall i \in \{1, 2, \ldots, N\}$ and $J(u) = 0 \Leftrightarrow u = 0$.
3. There is $r > 0$ such that

$$\sum_{i=1}^{N} \frac{(N - p_i)}{p_i} \int_{\mathbb{R}^N} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} dx - N\Phi(u) > 0 \quad \text{for} \quad 0 < ||u|| < r.$$

4. By Sobolev embeddings, there is a positive $C_1$ such that

$$\int_{\mathbb{R}^N} |u|^{p_1} dx \leq C_1 \left( \sum_{i=1}^{N} (\psi_i(u))^{1/p_i} \right)^{p^*}, \quad \forall u \in W^{1,\tilde{p}}(\mathbb{R}^N) \text{ with } \Phi(u) > 0.$$

5. If $(u_k) \subset X^r$ is weakly convergent to $u$ in $W^{1,\tilde{p}}(\mathbb{R}^N)$, by the compact embedding of $X^r$ into $L^q(\mathbb{R}^N)$, for all $q \in (p_0, p^*)$, we have

$$\limsup_{k \rightarrow +\infty} \Phi(u_k) \leq \Phi(u).$$

6. If $(u_k) \subset W^{1,\tilde{p}}(\mathbb{R}^N)$ is weakly convergent to $u$ in $W^{1,\tilde{p}}(\mathbb{R}^N)$, then

$$\liminf_{k \rightarrow +\infty} \left( \sum_{i=1}^{N} \psi_i(u_k) \right) \geq \sum_{i=1}^{N} \psi_i(u).$$

7. $\inf_{w \in P} I_2(w) = \inf_{w \in P^+} I_2(w)$.

From the above commentaries, Theorem 5.1 is proved, because all the conditions of Theorem 1.1 are satisfied.
6. An application involving discontinuous nonlinearity

In this section we consider the existence of solutions for the problem

\[(P_3) \quad -\Delta u(x) \in \partial G(u(x)), \quad \text{a.e. in } \mathbb{R}^N,\]

where \( N \geq 1 \), \( G \) is the primitive of a function \( g: \mathbb{R} \to \mathbb{R} \) given by \( g(s) = f(s) - s \), that is,

\[G(s) = \int_0^s g(t) \, dt = \int_0^s f(t) \, dt - \frac{1}{2}|s|^2 = F(s) - \frac{1}{2}|s|^2\]

and \( \partial G(s) \) is the generalized gradient of \( G \) at \( s \in \mathbb{R} \), given by

\[\partial G(s) = [g(s), \overline{g}(s)]\]

where

\[g(s) = \lim_{r \downarrow 0} \text{ess inf}_{|s-t|<r} \{g(t); |s-t|<r\} \quad \text{and} \quad \overline{g}(s) = \lim_{r \downarrow 0} \text{ess sup}_{|s-t|<r} \{g(t); |s-t|<r\}.

When \( g \) is a continuous function, which is equivalent to say that \( f \) is continuous, we know that \( G \in C^1(\mathbb{R}, \mathbb{R}) \) and in this case

\[\partial G(s) = \{g(s)\}, \quad \forall s \in \mathbb{R}.

In this section, we assume that \( f \) can have a finite number of discontinuity points \( a_1, a_2, \ldots, a_p \in \mathbb{R} \setminus \{0\} \). Moreover, \( f \) satisfies \((f_1), (f_3)\) and the condition:

\[(f_7) \quad \text{There are } A, B > 0 \text{ such that } |f(s)| \leq A|s| + B|s|^q, \quad \forall s \in \mathbb{R},\]

for some \( q \in (1, 2^* - 1) \), where \( 2^* = 2N/(N-2) \) if \( N \geq 3 \), and \( 2^* = +\infty \) if \( N = 1, 2 \).

Since we intend to find a nonnegative solution, in what follows we assume that

\[f(s) = 0, \quad \forall s < 0.

Hereafter, by a solution, we understand as a function \( u \in W^{2,(q+1)/q}(\mathbb{R}^N) \cap H^1(\mathbb{R}^N) \) that satisfies \((P_3)\), or equivalently, if \( u \) is a solution of the problem below

\[\text{For the case where } f \text{ is a continuous function, the above solution must satisfy the equation}\]

\[-\Delta u(x) + u(x) = \left[ f(u(x)), \overline{f}(u(x)) \right], \quad \text{a.e. in } \mathbb{R}^N.

A rich literature is available on problems with discontinuous nonlinearities. We refer the reader to Alves, Bertone and Gonçalves [2], Alves and Bertone [1], Alves, Gonçalves and Santos [6], Alves and Nascimento [8], Ambrosetti and Turner [10], Ambrosetti, Calahorrano and Dobarro [9], Badiale and Tarantello [12], Carl, Le and Motreanu [22], Clarke [24], Chang [24], Carl and Dietrich [20], Carl and Heikkila [21], Carl [19], Cerami [23], Hu, Kourogenis and Papageorgiou [36],
Montreanu and Vargas [39], Radulescu [42] and references therein. Several techniques have been developed or applied in the study of problems with discontinuous nonlinearity, such as variational methods for nondifferentiable functionals, lower and upper solutions, global branching, fixed point theorem, and the theory of multivalued mappings.

In the sequel, we will show the existence of a nontrivial critical point of the functional energy associated with problem $(P_3)$, the functional $I_3: H^1(\mathbb{R}^N) \to \mathbb{R}$ given by

$$I_3(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} |u|^2 \, dx - \int_{\mathbb{R}^N} F(u) \, dx.$$ 

Since $I_3$ is locally Lipschitz, a function $u \in H^1(\mathbb{R}^N)$ is a critical point of $I_3$, if $0 \in \partial I_3(u)$, where $\partial I_3(u)$ denotes the generalized gradient of $I_3$. We recall the generalized gradient of $I_3$ at $u$ is the set

$$\partial I_3(u) = \{ \mu \in (H^1(\mathbb{R}^N))^*: I_0^3(u,v) \geq \langle \mu, v \rangle, \forall v \in H^1(\mathbb{R}^N) \}$$

where $I_0^3(u,v)$ denotes the directional derivative of $I_3$ on $u$ in the direction of $v \in H^1(\mathbb{R}^N)$, which is defined by

$$I_0^3(u,v) = \limsup_{h \to 0, \lambda \downarrow 0} \frac{I_3(u+h+\lambda v) - I_3(u+h)}{\lambda}.$$

The reader can find more details about this subject in Chang [24] and Clarke [26].

The main result in this section is the following

**Theorem 6.1.** Assume the conditions $(f_1)$–$(f_3)$ and $(f_7)$. Then, $(P_3)$ has a nontrivial solution.

As in the previous section, we will show that functional $I_3$ satisfies the conditions of Theorem 1.1. In what follows,

$$X = H^1(\mathbb{R}^N), \quad X^+ = \{ u \in H^1(\mathbb{R}^N) : u(x) \geq 0 \text{ a.e. in } \mathbb{R}^N \},$$

$$X^r = \{ u \in H^1_{\text{rad}}(\mathbb{R}^N) \cap X^+ : 0 \leq u(x) \leq u(y) \text{ if } 0 < |y| \leq |x| \},$$

$$J(u) = \psi(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx$$

and

$$\Phi(u) = \int_{\mathbb{R}^N} G(u) \, dx.$$ 

Using these notations, we derive that

$$I_3(u) = J(u) - \Phi(u), \quad \forall u \in X = H^1(\mathbb{R}^N).$$

A well known argument shows that $\Phi$ is locally Lipschitz, $J \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$ and $I_3$ is locally Lipschitz.

Moreover, as in the previous sections, we set $* (t,u) := u_t : \mathbb{R}^N \to \mathbb{R}$ by

$$u_t(x) = \begin{cases} u \left( \frac{x}{t} \right), & \text{for } t \neq 0, \\ 0, & \text{for } t = 0. \end{cases}$$
By definition of $\psi$ and $\Phi$, it follows that
\[ \Phi(u_t) = t^N \Phi(u) \quad \text{and} \quad \psi(u_t) = t^{N-2} \psi(u), \quad \forall t \geq 0 \quad \text{and} \quad \forall u \in H^1(\mathbb{R}^N). \]

From the above commentaries, we have proved $(X_1)$–$(X_4)$ and $(F_2)$ of Theorem 1.1. Moreover, as in the previous sections, for each $u \in H^{s_n}(\mathbb{R}^N)$, the application $t \mapsto u_t$ is continuous, and so, $(X_5)$ is also proved.

Next, as in Section 4, we will prove that the other conditions of Theorem 1.1 also holds. Setting
\[ Q : H^1(\mathbb{R}^N) \longrightarrow X^r, \quad u \mapsto (u^+)^*, \]
where $(u^+)^*$ is the Schwartz symmetrization of $u^+$, standard arguments assure the validity of the statements $(X_6)$–$(X_8)$.

Claim 6.2 (Proof of $(F_1)$). There is $u \in H^1(\mathbb{R}^N)$ such that $\Phi(u) > 0$ and $\Phi(0) = 0$.
Proof. The claim follows with the same arguments employed to prove Claim 4.2. \qed

Claim 6.3 (Proof of $(F_3)$). There exists $r > 0$ such that
\[ (N - 2) \psi(u) > N \Phi(u), \quad \text{for} \quad 0 < \|u\| < r. \]
Proof. See the proof of Claim 4.3. \qed

Claim 6.4 (Proof of $(F_4)$). Let $(u_k)$ be a sequence in $H^1(\mathbb{R}^N)$ with $\Phi(u_k) \geq 0$ for all $k \in \mathbb{N}$. If $(J(u_k))$ is bounded, we have that $(u_k)$ is also bounded. Moreover, if $J(u_k) \to 0$, then $\|u_k\| \to 0$.
Proof. An immediate consequence of the properties of $J$. \qed

Claim 6.5 (Proof of $(F_5)$). If $(u_k)$ is weakly convergent to $u$ in $H^{1}_{rad}(\mathbb{R}^N)$, then
\[ \limsup_{k \to +\infty} \Phi(u_k) \leq \Phi(u). \]
Proof. See the proof of Claim 4.5. \qed

Claim 6.6 (Proof of $(F_6)$). Suppose $(u_k)$ is weakly convergent to $u$ in $H^1(\mathbb{R}^N)$, then
\[ \psi(u) \leq \liminf_{k \to +\infty} \psi(u_k). \]
Proof. See the proof of Claim 4.6. \qed

Claim 6.7. The functional $I_3$ satisfies the equality
\[ \inf_{w \in P} I_3(w) = \inf_{w \in P^+} I_3(w). \]
Proof. See the proof of Claim 4.7. \qed
From the statements we may deduce that $I_3$ satisfies the conditions of Theorem 1.1. From this, there is $u_0 \in H^1(\mathbb{R}^N)$ such that

$$0 \in \partial I_3(u_0) \quad \text{and} \quad I_3(u_0) = \inf_{w \in P} I_3(w) > 0.$$  

As $J$ is $C^1(H^1(\mathbb{R}^N), \mathbb{R})$, we have

$$J'(u_0) \in \partial \Phi(u_0).$$

On the other hand, since

$$\Phi(u) = \Phi_1(u) + \Phi_2(u), \quad \forall u \in H^1(\mathbb{R}^N)$$

with

$$\Phi_1(u) = \int_{\mathbb{R}^N} F(u) \, dx \quad \text{and} \quad \Phi_2(u) = \frac{1}{2} \int_{\mathbb{R}^N} |u|^2 \, dx,$$

it follows that

$$J'(u_0) + \Phi_2'(u_0) \in \partial \Phi_1(u_0).$$

Here, we used that $\Phi_2 \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$.

Using a well-known result found in [4], we know that

$$\partial \Phi_1(u_0) \subset [f_1(u_0), f_2(u_0)].$$  

This fact, combined with (6.2), guarantees the existence of a measurable function $\rho: \mathbb{R}^N \to \mathbb{R}$ satisfying

$$\rho(x) \in [\underline{f}(u_0(x)), \overline{f}(u_0(x))], \quad \text{a.e. in } \mathbb{R}^N,$$

such that $u_0$ is a weak solution of the problem

$$-\Delta u_0 + u_0 = \rho, \quad \text{in } \mathbb{R}^N.$$  

By (6.3), $\rho \in L^{(p+1)/p}_{\text{loc}}(\mathbb{R}^N)$, then the elliptic regularity implies that

$$u_0 \in W^{2,(q+1)/q}_{\text{loc}}(\mathbb{R}^N) \cap H^1(\mathbb{R}^N),$$

showing the $u_0$ is a nontrivial solution of (6.1).

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