Global and local structures of oscillatory bifurcation curves

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Abstract. We consider the nonlinear eigenvalue problem

\[-u''(t) = \lambda (u(t) + g(u(t))), \quad u(t) > 0, \quad t \in I := (-1, 1),\]

\[u(\pm 1) = 0,\]

where \(g(u) = u^p \sin(u^q)(0 \leq p < 1, 0 < q \leq 1)\) and \(\lambda > 0\) is a bifurcation parameter. It is known that, in this case, \(\lambda\) is parameterized by the maximum norm \(\alpha = \|u_\lambda\|_\infty\) of the solution \(u_\lambda\) associated with \(\lambda\) and is written as \(\lambda = \lambda(\alpha)\). We show that the bifurcation curve \(\lambda(\alpha)\) intersects the line \(\lambda = \pi^2/4\) infinitely many times by establishing the precise asymptotic formula for \(\lambda(\alpha)\) as \(\alpha \to \infty\) and \(\alpha \to 0\). We find that, according to the relationship between \(p\) and \(q\), there exist three types of bifurcation curves.

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1. Introduction

This paper is concerned with the following nonlinear eigenvalue problems

\[-u''(t) = \lambda (u(t) + g(u(t))), \quad t \in I := (-1, 1),\]  
\[u(t) > 0, \quad t \in I,\]  
\[u(-1) = u(1) = 0,\]  

where \(g(u)\) is an oscillatory nonlinear term and \(\lambda > 0\) is a parameter. We know from [11] that if \(u + g(u) > 0\) for \(u > 0\), then for any given \(\alpha > 0\), there exists a unique classical solution pair \((\lambda, u_\alpha)\) of (1.1)–(1.3) satisfying \(\alpha = \|u_\alpha\|_\infty\). Furthermore, \(\lambda\) is parameterized by \(\alpha\) as \(\lambda = \lambda(\alpha)\) and is continuous in \(\alpha > 0\).

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Over the past few decades, a considerable number of studies on the local and global structures of the bifurcation diagrams of the underlying differential equations have been carried out. Many topics in this area are derived from mathematical biology, engineering, etc., and have been investigated intensively by many authors. We refer to [1, 2, 3, 5, 6] and the references therein. However, little is known about the oscillatory phenomena of bifurcation curves. When bifurcation curves have the oscillatory structures, it is natural to expect that the equations include oscillatory nonlinear terms. Therefore, the oscillatory phenomena of bifurcation curves are closely connected with the inverse bifurcation problems. We refer to [7, 9, 10, 12, 13, 14] and the references therein.

As a model case of oscillatory bifurcation phenomenon, the system of equations (1.1)–(1.3) with \( g(u) = \sin \sqrt{u} \) has been considered in Cheng [4]. It has been shown there that there exists arbitrary many solutions near \( \lambda = \pi^2/4 \).

**Theorem 1.0** ([4, Theorem 6]). Let \( g(u) = \sin \sqrt{u} \) \((u \geq 0)\). Then for any integer \( r \geq 1 \), there is \( \delta > 0 \) such that if \( \lambda \in (\pi^2/4 - \delta, \pi^2/4 + \delta) \), then (1.1)–(1.3) has at least \( r \) distinct solutions.

We expect from Theorem 1.0 that \( \lambda(\alpha) \) oscillates and intersects the line \( \lambda = \pi^2/4 \) infinitely many times for \( \alpha \gg 1 \) when \( g(u) = \sin \sqrt{u} \). Motivated by this, the following asymptotic formula has been recently shown in [15].

**Theorem 1.1** ([15, Theorem 1.1]). Let \( g(u) = \sin \sqrt{u} \). Then as \( \alpha \to \infty \),

\[
\lambda(\alpha) = \frac{\pi^2}{4} - \pi^{3/2} \alpha^{-5/4} \sin \left( \sqrt{\alpha} - \frac{\pi}{4} \right) + o(\alpha^{-5/4}). \tag{1.4}
\]

It is quite natural to consider that oscillatory phenomena are characterized by the oscillatory term \( \sin(u^q) \) \((q = 1/2 \text{ in Theorem 1.1})\). Unfortunately, however, the proof of Theorem 1.1 depends on a very complicated time-map method, and the argument in [15] seems to be applicable only to the case \( g(u) = \sin \sqrt{u} \), and not applicable to the other relevant model equations with the nonlinear term \( g(u) = u^p \sin(u^q) \). On the other hand, for the case \( g(u) = \sin u \), it was shown in [8] that stationary phase method is useful to the analysis of the oscillatory bifurcation phenomena.

Motivated by [8] and [15], by the devised combination of time-map argument and stationary phase method, we establish the precise asymptotic formulas for \( \lambda(\alpha) \) with \( g(u) = u^p \sin(u^q) \) as \( \alpha \to \infty \). As far as the author knows, this model nonlinear term has not been investigated comprehensively yet from a viewpoint of analysis of oscillatory bifurcation curve. We note here that when we apply
the stationary phase method in the proof of Theorem 1.2 below, we have to be careful about the regularity of the functions which will be appear after the time-map argument.

Now we state our main results.

**Theorem 1.2.** Let \( g(u) = u^p \sin(u^q) \), where \( 0 \leq p < 1 \) and \( 0 < q \leq 1 \) are fixed constants. Then as \( \alpha \to \infty \),

\[
\lambda(\alpha) = \frac{\pi^2}{4} - \frac{\pi^{3/2}}{\sqrt{2q}} \alpha^{p-1-(q/2)} \sin \left( \alpha^q - \frac{\pi}{4} \right) + o(\alpha^{p-1-(q/2)}). \tag{1.5}
\]

Clearly, if \( p = 0 \) and \( q = 1/2 \), then Theorem 1.2 coincides with Theorem 1.1.

Next, to understand the whole structure of \( \lambda(\alpha) \) in detail, we establish the asymptotic formulas for \( \lambda(\alpha) \) as \( \alpha \to 0 \).

**Theorem 1.3.** Let \( g(u) = u^p \sin(u^q) \), where \( 0 \leq p < 1 \), \( 0 < q \leq 1 \) are fixed constants. Then the following asymptotic formulas hold \( \alpha \to 0 \).

(i) Assume that \( p + q > 1 \). Then

\[
\lambda(\alpha) = \frac{\pi^2}{4} - A_1 \pi \alpha^{p+q-1} + (A_1^2 + A_2 \pi) \alpha^{2(p+q-1)} + o(\alpha^{2(p+q-1)}), \tag{1.6}
\]

where

\[
A_1 = \frac{1}{p + q + 1} \int_0^1 \frac{1 - s^{p+q+1}}{(1 - s^2)^{3/2}} ds, \tag{1.7}
\]

\[
A_2 = \frac{3}{2(p + q + 1)^2} \int_0^1 \frac{(1 - s^{p+q+1})^2}{(1 - s^2)^{5/2}} ds. \tag{1.8}
\]

(ii) Assume that \( p + q = 1 \). Then

\[
\lambda(\alpha) = \frac{\pi^2}{8} + \frac{\pi}{48} B \alpha^{2q} + o(\alpha^{2q}), \tag{1.9}
\]

where

\[
B = \frac{1}{q + 1} \int_0^1 \frac{1 - s^{2q+2}}{(1 - s^2)^{3/2}} ds. \tag{1.10}
\]
(iii) Assume that $p + q < 1 < p + 3q$. Then

$$\lambda(\alpha) = \frac{p + q + 1}{2} \alpha^{1-p-q} \left\{ C_1^2 - \frac{p + q + 1}{2} C_1 C_2 \alpha^{1-p-q} + o(\alpha^{1-p-q}) \right\},$$

where

$$C_1 = \int_0^1 \frac{1}{\sqrt{1 - s^{p+q+1}}} ds,$$

$$C_2 = \int_0^1 \frac{1 - s^2}{(1 - s^{p+q+1})^{3/2}} ds.$$  \hspace{1cm} (1.12) \hspace{1cm} (1.13)

(iv) Assume that $p + 3q < 1$. Then

$$\lambda(\alpha) = \frac{p + q + 1}{2} \alpha^{1-p-q} \left\{ C_1^2 + \frac{p + q + 1}{6(p + 3q + 1)} C_1 C_3 \alpha^{2q} + o(\alpha^{2q}) \right\},$$

where

$$C_3 = \int_0^1 \frac{1 - s^{p+3q+1}}{(1 - s^{p+q+1})^{3/2}} ds.$$  \hspace{1cm} (1.14) \hspace{1cm} (1.15)

(v) Assume that $p + 3q = 1$. Then

$$\lambda(\alpha) = \frac{p + q + 1}{2} \alpha^{2q} \left\{ C_1^2 - \frac{5(p + q + 1)}{12} C_1 C_4 \alpha^{2q} + o(\alpha^{2q}) \right\},$$

where

$$C_4 = \int_0^1 \frac{1 - s^2}{(1 - s^{p+q+1})^{3/2}} ds.$$  \hspace{1cm} (1.16) \hspace{1cm} (1.17)

Note that we do not have (1.9) when we put $p + q = 1$ in (1.6) formally, because we neglect the remainder terms of the right hand side of (1.6). By Theorems 1.2 and 1.3, we understand that there exist three types of the asymptotic shapes of $\lambda(\alpha)$ (see figures below).
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Figure 1. Theorem 1.3 (i).

Figure 2. Theorem 1.3 (ii).

Figure 3. Theorem 1.3 (iii)–(v).
2. Proof of Theorem 1.2

In this section, let \( \alpha \gg 1 \). Furthermore, we denote by \( C \) the various positive constants independent of \( \alpha \). For \( u \geq 0 \), let \( g(u) = u^p \sin(u^q) \) and

\[
G(u) := \int_0^u g(s)ds. \tag{2.1}
\]

It is known that if \( (u_\alpha, \lambda(\alpha)) \in C^2(\bar{I}) \times \mathbb{R}_+ \) satisfies (1.1)–(1.3), then

\[
u_\alpha(t) = u_\alpha(-t), \quad 0 \leq t \leq 1, \tag{2.2}
\]

\[
u_\alpha(0) = \max_{-1 \leq t \leq 1} u_\alpha(t) = \alpha, \tag{2.3}
\]

\[
u_\alpha'(t) > 0, \quad -1 \leq t < 0. \tag{2.4}
\]

By (1.1), we have

\[
u_\alpha''(t) + \lambda(u_\alpha(t) + g(u_\alpha(t)))u_\alpha'(t) = 0.
\]

By this, (2.3) and putting \( t = 0 \), we obtain

\[
\frac{1}{2}u_\alpha'(t)^2 + \lambda\left(\frac{1}{2}u_\alpha(t)^2 + G(u_\alpha(t))\right) = \text{constant} = \lambda\left(\frac{1}{2}\alpha^2 + G(\alpha)\right).
\]

This along with (2.4) implies that for \(-1 \leq t \leq 0\),

\[
u_\alpha'(t) = \sqrt{\lambda}\sqrt{\alpha^2 - u_\alpha(t)^2 + 2(G(\alpha) - G(u_\alpha(t)))}. \tag{2.5}
\]

For \( 0 \leq s \leq 1 \), we have

\[
\left| \frac{G(\alpha) - G(\alpha s)}{\alpha^2(1-s^2)} \right| = \left| \frac{1}{\alpha^2(1-s^2)} \int_{\alpha s}^{\alpha} g(t)dt \right| \leq C \frac{\alpha^{p+1}(1-s^{p+1})}{\alpha^2(1-s^2)} \leq C\alpha^{p-1}. \tag{2.6}
\]

By (2.5) and (2.6), putting \( s := u_\alpha(t)/\alpha \) and Taylor expansion, we obtain

\[
\sqrt{\lambda} = \int_{-1}^0 \frac{u_\alpha'(t)}{\sqrt{\alpha^2 - u_\alpha(t)^2 + 2(G(\alpha) - G(u_\alpha(t)))}}dt = \int_1^0 \frac{1}{\sqrt{1-s^2 + 2(G(\alpha) - G(\alpha s))}/\alpha^2}ds = \int_1^0 \frac{1}{\sqrt{1-s^2}} \frac{1}{\sqrt{1 + 2(G(\alpha) - G(\alpha s))/(\alpha^2(1-s^2))}}ds = \int_1^0 \frac{1}{\sqrt{1-s^2}} \left\{1 - \frac{G(\alpha) - G(\alpha s)}{\alpha^2(1-s^2)}(1 + o(1))\right\}ds = \pi - \frac{1}{\alpha^2 (1 + o(1))} \int_0^1 \frac{G(\alpha) - G(\alpha s)}{(1-s^2)^{3/2}}ds. \tag{2.7}
\]
We put

\[ K(\alpha) := \int_0^1 \frac{G(\alpha) - G(\alpha s)}{(1 - s^2)^{3/2}} ds. \]  

(2.8)

By combining \cite[Lemma 2]{8} and \cite[Lemma 2.24]{10}, we have following equalities.

**Lemma 2.1** \cite[Lemma 2]{8} and \cite[Lemma 2.24]{10}. Assume that the function \( f(r) \in C^2[0, 1] \), and \( h(r) = \cos(\pi r/2) \). Then as \( \mu \to \infty \),

\[
\int_0^1 f(r) e^{i \mu h(r)} dr = e^{i(\mu - (\pi/4))} \sqrt{\frac{2}{\pi \mu}} f(0) + O\left(\frac{1}{\mu}\right). \tag{2.9}
\]

In particular, by taking the imaginary part of (2.9),

\[
\int_0^1 f(r) \sin(\mu h(r)) dr = \sqrt{\frac{2}{\pi \mu}} f(0) \sin \left(\mu - \frac{\pi}{4}\right) + O\left(\frac{1}{\mu}\right). \tag{2.10}
\]

**Lemma 2.2.** As \( \alpha \to \infty \),

\[
K(\alpha) = \sqrt{\frac{\pi}{2q}} \alpha^{p+1-(q/2)} \sin \left(\alpha^q - \frac{\pi}{4}\right) + o(\alpha^{p-1-(q/2)}). \tag{2.11}
\]

**Proof.** We put \( s = \sin \theta \) in (2.8). Then by integration by parts, we obtain

\[
K(\alpha) = \int_0^{\pi/2} \frac{1}{\cos^2 \theta} (G(\alpha) - G(\alpha \sin \theta)) d\theta
= \int_0^{\pi/2} (\tan \theta)'(G(\alpha) - G(\alpha \sin \theta)) d\theta
= [\tan \theta (G(\alpha) - G(\alpha \sin \theta))]_0^{\pi/2}
+ \alpha \int_0^{\pi/2} \tan \theta (\cos \theta (\alpha \sin \theta)^p \sin((\alpha \sin \theta)^q)) d\theta.
\]

(2.12)

By l’Hôpital’s rule, we obtain

\[
\lim_{\theta \to \pi/2} \frac{1}{\cos \theta} \int_0^\alpha y^p \sin(y^q) dy = \lim_{\theta \to \pi/2} \frac{\alpha \cos \theta (\alpha \sin \theta)^p \sin((\alpha \sin \theta)^q)}{\sin \theta} = 0.
\]
We put $m = 1/q$, $\sin^q \theta = \sin x$, $x = (\pi/2) - y$ and $y = (\pi/2)r$. By this and (2.12), we obtain

\[
K(\alpha) = \alpha^{p+1} \int_0^{\pi/2} \sin^{p+1} \theta \sin(\alpha^q \sin^q \theta) \, d\theta
\]

\[
= \frac{1}{q} \alpha^{p+1} \int_0^{\pi/2} \sin^{(p+2-q)/q} x \frac{\cos x}{\sqrt{1 - \sin^2 m x}} \sin(\alpha^q \sin x) \, dx
\]

\[
= \frac{1}{q} \alpha^{p+1} \int_0^{\pi/2} \sin^{(p+2-q)/q} x \frac{\cos x}{\sqrt{1 - \sin^2 m x}} \sin(\alpha^q \sin x) \, dx
\]

\[
= \frac{1}{q} \alpha^{p+1} \int_0^{\pi/2} \cos^{(p+2-q)/q} y \frac{\cos x}{\sqrt{1 - \cos^2 m y}} \sin(\alpha^q \cos y) \, dy
\]

\[
= \frac{\pi}{2q} \alpha^{p+1} \int_0^1 \cos^{(p+2-q)/q} \left( \frac{\pi}{2} r \right) \sqrt{1 - \cos^2 \left( \frac{\pi}{2} r \right)} \sin \left( \alpha^q \cos \left( \frac{\pi}{2} r \right) \right) \, dr.
\]

(2.13)

We put

\[
f(r) = \cos^{(p+2-q)/q} \left( \frac{\pi}{2} r \right) \sqrt{1 - \cos^2 \left( \frac{\pi}{2} r \right)} \sin \left( \alpha^q \cos \left( \frac{\pi}{2} r \right) \right), \quad \mu = \alpha^q
\]

(2.14)

and put $f$ and $h(r) = \cos(\pi r/2)$ in (2.10) when $f \in C^2[0,1]$. Since $f(0) = \sqrt{q}$, we obtain

\[
K(\alpha) = \sqrt{\frac{\pi}{2q}} \alpha^{p+1-q/2} \sin \left( \alpha^q - \frac{\pi}{4} \right) + o(\alpha^{p+1-q/2}).
\]

(2.15)

This implies (2.11). Finally, we consider the case $f \not\in C^2[0,1]$. Fortunately, we are still able to apply Lemma 2.1 to this case by modifying the proof of Lemma 2.1, and obtain (2.11). For completeness, the argument will be given in the appendix. Thus the proof is complete. \[\square\]

Now Theorem 1.2 follows from (2.7) and Lemma 2.2.

3. Proof of Theorem 1.3

In this section, let $0 < \alpha \ll 1$.

Proof of Theorem 1.3 (i). By (2.7),

\[
\sqrt{\lambda} = \int_0^1 \frac{1}{\sqrt{1 - s^2} + \frac{2}{\alpha^2} \int_0^{\alpha} \theta^p \sin(\theta^q) \, d\theta} \, ds
\]

\[
= \int_0^1 \frac{1}{\sqrt{1 - s^2} \left( 1 + \frac{2}{\alpha^2 (1-s^2)} \right)} \frac{1}{\int_0^{\alpha} \theta^p \sin(\theta^q) \, d\theta} \, ds.
\]

(3.1)
Note that \( p + q > 1 \). By Taylor expansion, for \( 0 \leq s \leq 1 \), we have

\[
M := \int_{as}^{a} \theta^p \sin(\theta^q) d\theta \\
= \int_{as}^{a} \left( \theta^{p+q} - \frac{1}{6} \theta^{p+3q} (1 + o(1)) \right) d\theta \\
= \frac{1}{p + q + 1} \alpha^{p+q+1} (1 - s^{p+q+1}) \\
- \frac{1}{6(p + 3q + 1)} \alpha^{p+3q+1} (1 + o(1))(1 - s^{p+3q+1}). \tag{3.2}
\]

By this, Taylor expansion and (3.1), we obtain

\[
\sqrt{\lambda} = \int_{0}^{1} \frac{1}{\sqrt{1 - s^2}} \left[ 1 - \frac{1}{\alpha^2(1 - s^2)} M + \frac{3}{2\alpha^4(1 - s^2)^2} M^2 (1 + o(1)) \right] ds. \tag{3.3}
\]

Since \( 2(p + q - 1) < p + 3q - 1 \), by (3.2) and (3.3), we obtain

\[
\sqrt{\lambda} = \frac{\pi}{2} - \alpha^{p+q-1} \frac{1}{p + q + 1} \int_{0}^{1} \frac{1 - s^{p+q+1}}{(1 - s^2)^{3/2}} ds \\
+ \frac{3}{2(p + q + 1)^2} \alpha^{2(p+q-1)} \int_{0}^{1} (1 - s^{p+q+1})^2 \frac{1}{(1 - s^2)^{5/2}} ds \tag{3.4}
\]

By this, we obtain (1.6).

**Proof of Theorem 1.3 (ii).** Since \( p + q = 1 \), by Taylor expansion and (3.2), we obtain

\[
\frac{1}{2} u'_\alpha(t)^2 + \lambda \left[ u_\alpha(t)^2 - \frac{1}{12(q + 1)} u_\alpha(t)^{2(q+1)} + \frac{1}{5!(4q + 2)} (1 + o(1)) u_\alpha(t)^{4q+2} \right] \\
= \lambda \left[ \alpha^2 - \frac{1}{12(q + 1)} \alpha^{2(q+1)} + \frac{1}{5!(4q + 2)} (1 + o(1)) \alpha^{4q+2} \right]. \tag{3.5}
\]

This implies that for \(-1 \leq t \leq 0\),

\[
u'_\alpha(t) = \sqrt{2\lambda R(\alpha, u_\alpha(t))}, \tag{3.6}
\]

where

\[
R(\alpha, u) := \alpha^2 - u^2 - \frac{1}{12(q + 1)} (1 + o(1))(\alpha^{2q+2} - u^{2q+2}). \tag{3.7}
\]
By this, Taylor expansion and direct calculation, we obtain

\[
\sqrt{\lambda} = \frac{1}{\sqrt{2}} \int_{-1}^{0} \frac{u'_\alpha(t)}{\sqrt{R(\alpha, u_\alpha(t))}} dt
\]

\[
= \frac{1}{\sqrt{2}} \int_{0}^{1} \frac{1}{\sqrt{1-s^2}} \left(1 + \frac{\alpha^{2q} - s^{2q+2}}{2q+1} \right) ds + o(\alpha^{2q}) \tag{3.8}
\]

\[
= \frac{1}{\sqrt{2}} \left( \frac{\pi}{2} + \frac{1}{24} B\alpha^{2q} + o(\alpha^{2q}) \right).
\]

By this, we obtain (1.9).

\( \square \)

**Proof of Theorem 1.3 (iii).** Assume that \( p + q < 1 < p + 3q \). Then by (3.1) and (3.2), we obtain

\[
\sqrt{\lambda} = \int_{0}^{1} \frac{\alpha}{\sqrt{L(\alpha, s)}} ds, \tag{3.9}
\]

where

\[
L(\alpha, s) = \alpha^{2}(1-s^{2}) + \frac{2}{p+q+1} \alpha^{p+q+1}(1-s^{p+q+1}) - \frac{1}{3(p+3q+1)} \alpha^{p+3q+1}(1-s^{p+3q+1})(1+o(1)). \tag{3.10}
\]

By (3.9), (3.10), Taylor expansion, and direct calculation we obtain

\[
\sqrt{\lambda} = \alpha^{(1-p-q)/2} \sqrt{\frac{p+q+1}{2}} \int_{0}^{1} \frac{1}{\sqrt{1-s^{p+q+1}}} ds
\]

\[
= \alpha^{(1-p-q)/2} \sqrt{\frac{p+q+1}{2}} \int_{0}^{1} \frac{1}{\sqrt{1-s^{p+q+1}}} \left\{1 - \frac{p+q+1}{4} \alpha^{p+q+1}(1-s^{2})(1+o(1)) \right\} ds
\]

\[
= \alpha^{(1-p-q)/2} \sqrt{\frac{p+q+1}{2}} \left\{ C_1 - \frac{p+q+1}{4} \alpha^{p+q+1} C_2 (1+o(1)) \right\}. \tag{3.11}
\]

By this, we obtain (1.11).

\( \square \)

The proofs of Theorem 1.3 (iv) and (v) are the same as that of Theorem 1.3 (iii). So we omit the proofs. Thus the proof of Theorem 1.3 is complete.
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4. Appendix

In this section, we show that (2.9) in Lemma 2.1 holds for $0 \leq p < 1$ and $0 < q \leq 1$ for completeness. For $m = 1/q$ and $0 \leq x \leq 1$, we put

$$f(x) = f_1(x) f_2(x) := \cos^{p+2-q/q} \left( \frac{\pi}{2} x \right) \frac{1 - \cos^{2m} \left( \frac{\pi}{2} x \right)}{1 - \cos^{2m} \left( \frac{\pi}{2} x \right)}.$$  \hspace{1cm} (4.1)

The essential point of the proof of (2.9) in this case is to show \cite{one.prop, two.prop, four.prop}, Lemma \cite{two.prop, two.prop, five.prop}. Namely, as $\mu \to \infty$,

$$\Phi(\mu) := \int_0^1 f(x) e^{-i\mu x^2} dx = \frac{1}{2} \sqrt{\frac{\pi}{\mu}} e^{-i\pi/4} f(0) + O\left( \frac{1}{\mu} \right).$$ \hspace{1cm} (4.2)

We put $w(x) = (f(x) - f(0))/x$. Then we have $f(x) = f(0) + x w(x)$. We know from \cite{ten, Lemma 2.24} that

$$\int_0^1 e^{-i\mu x^2} dx = \frac{1}{2} \sqrt{\frac{\pi}{\mu}} e^{-i\pi/4} + O\left( \frac{1}{\mu} \right).$$ \hspace{1cm} (4.3)

Since $f(0) = \sqrt{q}$, by (4.3), we obtain

$$\Phi(\mu) = f(0) \int_0^1 e^{-i\mu x^2} dx + \int_0^1 x e^{-i\mu x^2} w(x) dx$$

$$= \frac{1}{2} \sqrt{\frac{\pi}{\mu}} e^{-i\pi/4} \sqrt{q} + O\left( \frac{1}{\mu} \right) + \int_0^1 x e^{-i\mu x^2} w(x) dx.$$ \hspace{1cm} (4.4)

We put

$$\Phi_1(\mu) := \int_0^1 x e^{-i\mu x^2} w(x) dx.$$ \hspace{1cm} (4.5)

Now we prove that $w(x) \in C^1[0, 1]$, because if it is proved, then by integration by parts, we easily show that $\Phi_1(\mu) = O(1/\mu)$ and our conclusion (4.2) follows immediately from (4.4) and (4.5). To do this, there are several cases to consider. We note that, by direct calculation, we can show that if $q > 0$, namely, $m > 1$, then $f_2(x) \in C^2[0, 1]$.

Case 1 Assume that $p = 0$ and $q = 1$. Then we have $f(x) = \cos \left( \frac{\pi}{2} x \right)$ and $f \in C^2[0, 1]$.

Case 2 Assume that $0 < q < 1$ and $p + 2 \geq 3q$. Then $(p + 2 - q)/q \geq 2$ and $f_1(x) \in C^2[0, 1]$. Consequently, $f \in C^2[0, 1]$ in this case.
Case 3 Assume that $0 < p < 1$ and $q = 1$. Then $f(x) = \cos^{p+1} \left( \frac{x}{2} \right) \not\in C^2[0, 1]$. However, by direct calculation, we can show that $w(x) = (f(x) - f(0))/x \in C^1[0, 1]$. It is reasonable, because by Taylor expansion, for $0 < x \ll 1$, we have

$$w(x) = -\frac{(p + 1)\pi^2}{8}x + O(x^3). \quad (4.6)$$

Case 4 Assume that $0 < q < 1$ and $p + 2 < 3q$. Then $1 < m < 3/2$ and $m(p + 2 - (1/m)) = mp + 2(m - 1) + 1 := \eta + 1 > 1$, $0 < \eta < 1$ and $f_1(x) = \cos^{q+1} x$. Since $f_2 \in C^2[0, 1]$, by the consequence of Case 3 above, we find that $w \in C^1[0, 1]$.

Thus the proof is complete.

References


Global and local structures of oscillatory bifurcation curves


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