A p.c.f. self-similar set with no self-similar energy

Roberto Peirone

Abstract. An example of a p.c.f. (post-critically finite) self-similar set without eigenform for any set of weights, is provided. The existence of an eigenform on such sets was an important, long-standing open problem in analysis on fractals.

This problem is related to that of existence of self-similar energies on fractals, because self-similar energies on p.c.f. self-similar sets are obtained by using eigenforms (and only in this way). A general existence result of self-similar energies was previously established by the present author with respect to a weaker version of self-similarity.

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1. Introduction

An important problem in analysis on fractals is that of constructing a Laplace operator, or equivalently, a self-similar Dirichlet form. Much of analysis on fractals is, in fact, based on these notions. A Dirichlet form is a sort of energy. More precisely, it is an abstract version of the Dirichlet integral. Constructions of a self-similar Dirichlet form have been investigated especially on finitely ramified fractals. Roughly speaking, a fractal is finitely ramified if the intersection of each pair of its copies is a finite set. The Sierpiński gasket, the Vicsek set and the Lindstrøm snowflake are finitely ramified fractals, while the Sierpiński carpet is not.

More specifically, we consider the p.c.f. self similar-sets, a general class of finitely ramified fractals introduced by J. Kigami in [3]. Here, p.c.f. is short for post-critically finite. A general theory with many examples can be found in [4].

On such a class of fractals, the basic tool used to construct a self-similar Dirichlet form is a discrete Dirichlet form defined on a special finite subset $V^{(0)}$ of the fractal, which is a sort of its boundary. Such discrete Dirichlet forms have to be eigenforms, i.e., eigenvectors of a special nonlinear operator $\Lambda_r$, called renormalization operator, which depends on a set of positive numbers $r_i$, called weights, placed on the cells of the fractal.
Criteria for the existence of an eigenform with prescribed weights were discussed by T. Lindstrøm [5], C. Sabot [11], and V. Metz [6]. In particular, Lindstrøm proved that there exists an eigenform on the nested fractals with all weights equal to 1, Sabot proved a rather general criterion, and Metz improved the results of Sabot.

A slightly different point of view was investigated by B. M. Hambly, V. Metz, A. Teplyaev in [1]. Namely, instead of aiming at criteria for the existence of eigenforms with prescribed weights, they inquired into the problem of existence of an eigenform with suitable weights, which I will call \textit{G-eigenform (generalized eigenform)}. The existence of a G-eigenform on other classes of p.c.f. self-similar set was proved in [7], [8] and [9]. In particular, a G-eigenform exists on fractals with three vertices ([7]), and on fractals with connected interior ([9]). The main results in [1] and in [8] can now be seen as particular cases of those in [7] and in [9].

The problem

\begin{equation}
\text{(P) } \text{Does a G-eigenform exist on every p.c.f. self-similar set?}
\end{equation}

was a very natural, well-known and long-standing open question. For example, it was discussed in [1], and in a slightly different form in [12], Section 4.2. Moreover, in [3, p. 734], a similar question concerning the existence of harmonic structures was discussed.

In this paper, I solve the problem (P), showing an example of a p.c.f. self-similar set with no G-eigenform. It is a variant of polygaskets (see [12, Section 4.1]), in the sense that every cell $V_i$ only intersects $V_{i-1}$ and $V_{i+1}$, and has twenty vertices. The proof is based on evaluation of the effective conductances on pairs of close vertices and of far vertices.

Note that in [12] the additional property that the coefficients of the form are strictly positive was required. However, in view of Theorem 5.3 in [10], which states in particular that all G-eigenforms have the same set of 0-coefficients, there are simple examples of fractals where we cannot have eigenforms with strictly positive coefficients. In fact, on the tree-like gasket (see e.g., [9], Figure 2, p. 62), every form with coefficients $c_{1,2} > 0$, $c_{2,3} > 0$ and $c_{1,3} = 0$ is an eigenform with weights all equal to 1, as can be easily verified, thus every G-eigenform satisfies $c_{1,3} = 0$.

\footnote{Note that the fractal described in this paper provides a counterexample to some open problems in the webpage \url{http://teplyaev.math.uconn.edu/POMI-2010-teplyaev-.pdf} (in particular to the first and the third).}
So, it appears to be more appropriate to consider, as here and in other papers, the more general setting of the irreducible forms.

An important remark is that, since the existence of a G-eigenform amounts to the existence of a self-similar energy on the fractal, problem (P) can be interpreted as that of the existence of a self-similar energy on every p.c.f. self-similar set. However, the self-similarity of the energy is meant to be with respect to the given set of similarities. A strictly related problem is whether there exists a self-similar energy on every p.c.f. self-similar set where the self-similarity of the energy is meant to be, more generally, with respect to a suitable set of similarities that defines the given fractal.

In fact, in [9] we have an affirmative answer to such a question. Namely, the existence of a G-eigenform is proved on every fractal at a suitable level, that is, if we consider the fractal generated by a set of similarities, which is not necessarily the given set of similarities (see [9] for the details). Note however, that in [9], as in [7], in [8] and in the present paper, the general setting was a class of p.c.f. self-similar sets with a very mild additional hypothesis, satisfied by most of the usual fractals (see Section 2 for the details). Essentially the same setting was considered also in other papers and also in [12].

Finally, note that the problem of uniqueness requires different methods (see [10] for recent results about uniqueness).

2. Definitions and notation

I will now define the fractal setting, which is based on that in [9]. This kind of approach was firstly given in [2]. A notion similar to that of a fractal triple is discussed in [3, Appendix A], and called an ancestor. In some sense, we can see a fractal triple as an ancestor with some additional (mild) assumptions.

We define a fractal by giving a fractal triple, i.e., a triple \( \mathcal{F} := (V^{(0)}, V^{(1)}, \Psi) \) where \( V^{(0)} = V \) and \( V^{(1)} \) are finite sets with \#\( V^{(0)} \) \( \geq 2 \), and \( \Psi \) is a finite set of one-to-one maps from \( V^{(0)} \) into \( V^{(1)} \) satisfying

\[
V^{(1)} = \bigcup_{\psi \in \Psi} \psi(V^{(0)}).
\]

We put \( V^{(0)} = \{P_1, \ldots, P_N\} \), and of course \( N = \#V^{(0)} \geq 2 \). A set of the form \( \psi(V^{(0)}) \) with \( \psi \in \Psi \) will be called a cell or a 1-cell. We require that

a) for each \( j = 1, \ldots, N \) there exists a (unique) map \( \psi_j \in \Psi \) such that \( \psi_j(P_j) = P_j \), and \( \Psi = \{\psi_1, \ldots, \psi_k\} \), with \( k = \#\Psi \geq N \);
b) \( P_j \notin \psi_i(V^{(0)}) \) when \( i \neq j \) (in other words, if \( \psi_i(P_h) = P_j \) with \( i = 1, \ldots, k, j, h = 1, \ldots, N \), then \( i = j = h \));

c) \( V^{(1)} \) is connected in the following sense: for every \( Q, Q' \in V^{(1)} \) there exists a sequence of points \( Q_0, \ldots, Q_n \in V^{(1)} \) such that \( Q_0 = Q, Q_n = Q' \) and for every \( h = 1, \ldots, n \) there exists \( i_h = 1, \ldots, k \) such that \( Q_{h-1}, Q_h \in \psi_{i_h}(V^{(0)}) \).

Of course, it immediately follows that \( V^{(0)} \subseteq V^{(1)} \). Let \( V_i := \psi_i(V^{(0)}) \) for each \( i = 1, \ldots, k \). Let \( J(= J(V)) = \{ \{ j_1, j_2 \}: j_1, j_2 = 1, \ldots, N, j_1 \neq j_2 \} \). Let \( \mathcal{V}(= \mathcal{V}(\mathcal{F})) = \{ 1, \ldots, k \} \). It is not difficult to see that on every fractal triple we can construct a p.c.f. self-similar set. See, for example, [3, Appendix A], for the details of such a construction. Note that, if the following property (*) holds

\[
\text{for all } j = 1, \ldots, N \text{ there exist } j = 1, \ldots, N, h, \tilde{h} \in \mathcal{V} \text{ such that } h \neq \tilde{h} \text{ and } \psi_h(P_j) = \psi_{\tilde{h}}(P_j),
\]

then it is also possible to prove that the set \( V_0 \), described in [4] amounts to \( V^{(0)} \) defined here, so that we have the same theory in both situations. It is easy to see that the fractal described in Section 3 here satisfies (*).

We denote by \( \mathcal{D}(\mathcal{F}) \) or simply \( \mathcal{D} \) the set of the Dirichlet forms on \( V \), invariant with respect to an additive constant, i.e., the set of the functionals \( E \) from \( \mathbb{R}^V \) into \( \mathbb{R} \) of the form

\[
E(u) = \sum_{\{ j_1, j_2 \} \in J} c_{\{ j_1, j_2 \}}(E)(u(P_{j_1}) - u(P_{j_2}))^2,
\]

with \( c_{\{ j_1, j_2 \}}(E) \geq 0 \). I will denote by \( \tilde{\mathcal{D}}(\mathcal{F}) \) or simply \( \tilde{\mathcal{D}} \) the set of the irreducible Dirichlet forms, i.e., \( E \in \tilde{\mathcal{D}} \) if \( E \in \mathcal{D} \) and moreover \( E(u) = 0 \) if and only if \( u \) is constant. The numbers \( c_{\{ j_1, j_2 \}}(E) \) are called coefficients of \( E \). We also say that \( c_{\{ j_1, j_2 \}}(E) \) is the conductance between \( P_{j_1} \) and \( P_{j_2} \) (with respect to \( E \)). Next, I recall the notion of effective conductance. Let \( E \in \tilde{\mathcal{D}} \), and let \( j_1, j_2 = 1, \ldots, N \), \( j_1 \neq j_2 \). Then we put

\[
\tilde{C}_{j_1, j_2}(E) := \min\{ E(u): u \in \mathcal{L}_{V:j_1, j_2} \},
\]

\[
\mathcal{L}_{V:j_1, j_2} := \{ u \in \mathbb{R}^V: u(P_{j_1}) = 0, u(P_{j_2}) = 1 \}.
\]

It can be easily proved that such a minimum exists and that \( \tilde{C}_{j_1, j_2}(E) = \tilde{C}_{j_2, j_1}(E) \). The value \( \tilde{C}_{j_1, j_2}(E) \) or for short \( \tilde{C}_{j_1, j_2} \), is called the effective conductance between \( P_{j_1} \) and \( P_{j_2} \) (with respect to \( E \)). Note that \( \tilde{C}_{j_1, j_2} > 0 \). The following remark can be easily verified (see Remark 2.9 in [9]).
Remark 2.1. Let $t_1, t_2 \in \mathbb{R}$. If $\{j_1, j_2\} \in J$ and $E \in \mathcal{D}$, then

$$\min\{E(u) : u \in \mathbb{R}^{V(0)}, u(P_{j_1}) = t_1, u(P_{j_2}) = t_2\} = (t_1 - t_2)^2 C_{j_1, j_2}.$$

Recall that for every $r \in W := ]0, +\infty[\ k$, $(r_i := r(i))$ the renormalization operator is defined as follows: for every $E \in \mathcal{D}$ and every $u \in \mathbb{R}^{V(0)},$

$$\Lambda_r(E)(u) = \inf\{S_r(E)(v) : v \in \mathcal{L}(u)\},$$

$$S_r(E)(v) := \sum_{i=1}^{k} r_i E(v \circ \psi_i), \quad \mathcal{L}(u) := \{v \in \mathbb{R}^{V(1)} : v = u \text{ on } V(0)\}.$$

It is well known that $\Lambda_r(E) \in \mathcal{D}$ and that the infimum is attained at a unique function $v := H_{1,E,r}(u)$. When $r \in W$, an element $E$ of $\mathcal{D}$ is said to be an $r$-eigenform with eigenvalue $\rho > 0$ if $\Lambda_r(E) = \rho E$. As this amounts to $\Lambda_{\rho}(E) = E$, we could also assume $\rho = 1$. The problem discussed in the present paper is that of the existence of a $G$-eigenform in $\mathcal{D}$, in other words, the existence of $E \in \mathcal{D}$ such that $\Lambda_r(E) = \rho E$ for some $\rho > 0$ and $r \in W$.

Remark 2.2. It is well known that there exists a bijection between the set of the $r$-eigenforms and the set of the energies on the fractal that are self-similar with respect to $r$. By this we mean Dirichlet forms with good properties, and in particular the self-similarity property (see for example, [11, Proposition 2.11]).\footnote{In [11] in fact only regular sets of weights are considered, but the arguments in [4], Section 3.3, and in particular Lemma 3.3.7, show that we do not have energies with good properties when the set of weights is not regular. Recall that a set of weights $r$ is said to be regular if $r_i > 1$ for every $i$ (and we assume $\rho = 1$).}

The precise meaning of “good properties” of a form varies slightly for different authors. Here are the properties adopted in [11], that is, a Dirichlet form $\mathcal{E}$ that satisfies:

a) $\mathcal{E}$ is regular,

b) $\mathcal{E}(u) = 0$ if and only if $u$ is constant,

c) $\mathcal{E}$ has the spectral gap property,

d) all points of the fractal have strictly positive capacity with respect to $\mathcal{E},$

e) $\mathcal{E}$ is self-similar.

For the precise meaning of the definitions in a) to e), which would be rather long and technical to describe here, I refer to [11]. Mind, however, that in the spectral gap property there is a typo, namely the integral is of $f^2$ not of $f$.\footnote{In [11] in fact only regular sets of weights are considered, but the arguments in [4], Section 3.3, and in particular Lemma 3.3.7, show that we do not have energies with good properties when the set of weights is not regular. Recall that a set of weights $r$ is said to be regular if $r_i > 1$ for every $i$ (and we assume $\rho = 1$).}
We have defined $V^{(1)}$ as the union of all $V_i$. More generally, when $\emptyset \neq B \subseteq V$, we put

$$V(B) := \bigcup_{i \in B} V_i \subseteq V^{(1)}, \quad S_{r;B}(E)(v) := \sum_{i \in B} r_i E(v \circ \psi_i),$$

when $v \in \mathbb{R}^{V(B)}$. Also, when $v \in \mathbb{R}^{V(B)}$, and $Q, Q' \in V(B), E \in \tilde{D}$, put

$$\tilde{C}_{B,r;Q,Q'}(E) = \inf \{S_{r;B}(E)(v) : v \in \mathbb{R}^{V(B)}, v(Q) = 0, v(Q') = 1\}.$$

It can be easily proved that the infimum is in fact a minimum, and that

$$\tilde{C}_{B,r;Q,Q'}(E) = \tilde{C}_{B,r;Q',Q}(E).$$

In the next section, I will describe an example of a fractal triple where there exists no G-eigenform. To this aim, the following standard lemma will be useful (see e.g., Lemma 3.3 in [9]).

**Lemma 2.3.** For every $E \in \tilde{D}$ and $\{j_1, j_2\} \in J$ we have

$$\tilde{C}_{j_1,j_2}(\Lambda_r(E)) = \tilde{C}_{V,r;P_{j_1},P_{j_2}}(E).$$

### 3. The example

We define a fractal triple $\tilde{F} = (V^{(0)}, V^{(1)}, \Psi)$ as follows. Let $V^{(0)} = \{P_1, \ldots, P_{20}\}$ and $\Psi = \{\psi_1, \ldots, \psi_{20}\}$. Here, thus, $N = k = 20$. In the following, the indices of the points and of the maps will be meant to be mod 20. For example, $i + 9 = 2$ if $i = 13$. Suppose $V_i \cap V_{i'} = \emptyset$ if $i' \notin \{i, i - 1, i + 1\}$ and the intersection $V_i \cap V_{i+1}$ satisfy the following rules.

i) For every $h = 0, \ldots, 9$,

$$V_{2h+1} \cap V_{2h+2} = \{\psi_{2h+1}(P_{2h+2})\} = \{\psi_{2h+2}(P_{2h+1})\}.$$

ii) Every $V_i \cap V_{i+1}, i = 1, \ldots, 20$, is a singleton and for every $i$ the points in $V_i \cap V_{i+1}$ and in $V_i \cap V_{i-1}$ are opposite in $V_i$. Here we say that $P_h$ and $P_{h+10}$ are opposite in $V^{(0)}$ and that $\psi_i(P_h)$ and $\psi_i(P_{h+10})$ are opposite in $V_i$. In other words, we suppose

$$V_i \cap V_{i+1} = \{\tilde{Q}_i\}, \quad \tilde{Q}_i = \begin{cases} \psi_i(P_{i+1}) = \psi_{i+1}(P_i) & \text{if } i \text{ odd,} \\ \psi_i(P_{i+9}) = \psi_{i+1}(P_{i-8}) & \text{if } i \text{ even.} \end{cases}$$
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Hence,

\[ \bar{Q}_{i-1} = \psi_i(P_{\sigma(i)}), \quad \bar{Q}_i = \psi_i(P_{\sigma(i)+10}), \quad \sigma(i) := \begin{cases} i - 1 & \text{if } i \text{ even,} \\ i - 9 & \text{if } i \text{ odd.} \end{cases} \]

In Figure 1, a group of three consecutive cells of \( \bar{\mathcal{F}} \) is depicted. The reason why we need twenty vertices is that so, the two paths connecting two opposite cells are formed by nine consecutive cells, thus necessarily both contain four pairs of cells of the form \((V_{2k+1}, V_{2k+2})\). The use of this fact will be clear in the proof of Theorem 3.3. We are now going to prove that on \( \bar{\mathcal{F}} \) there exists no G-eigenform (Theorem 3.3). In order to do this, we need two Lemmas. The idea behind these two lemmas and specially, Lemma 3.2, is the rule of effective resistances in series. Recall that the reciprocal of effective conductances are called effective resistances.

**Lemma 3.1.** Let \( r \in W, E \in \bar{\mathcal{D}} \) and let \( h = 0, \ldots, 9 \). Then,

\[ \tilde{C}_{V,r;P_{2h+1},P_{2h+2}}(E) \geq \frac{1}{2} \min\{r_{2h+1},r_{2h+2}\} \tilde{C}_{2h+1,2h+2}(E). \]

**Proof.** Let \( v \in \mathbb{R}^{V^{(1)}} \) be such that \( v(P_{2h+1}) = 0, v(P_{2h+2}) = 1 \), and let \( \bar{r} = \min\{r_{2h+1},r_{2h+2}\} \). Then, since by definition,

\[ S_r(E)(v) = \sum_{i=1}^{20} r_i E(v \circ \psi_i) \geq \sum_{d=1}^{2} r_{2h+d} E(v \circ \psi_{2h+d}), \]

we have

\[ S_r(E)(v) \geq \bar{r}(E(v \circ \psi_{2h+1}) + E(v \circ \psi_{2h+2})). \quad (3.1) \]

Let \( t = v(\bar{Q}_{2h+1}) \). Let \( v_1 := v \circ \psi_{2h+1}, v_2 := v \circ \psi_{2h+2} \). Then,

\[ v_1(P_{2h+1}) = v(P_{2h+1}) = 0, \]
\[ v_1(P_{2h+2}) = v(\psi_{2h+1}(P_{2h+2})) = v(\bar{Q}_{2h+1}) = t, \]
\[ v_2(P_{2h+2}) = v(P_{2h+2}) = 1, \]
\[ v_2(P_{2h+1}) = v(\psi_{2h+2}(P_{2h+1})) = v(\bar{Q}_{2h+1}) = t. \]

By Remark 2.1 and an obvious inequality, we thus have

\[ E(v_1) + E(v_2) \geq (t^2 + (1-t)^2)\tilde{C}_{2h+1,2h+2}(E) \geq \frac{1}{2} \tilde{C}_{2h+1,2h+2}(E). \]

By (3.1) we conclude the proof. \( \square \)
Figure 1. Three consecutive cells of $\mathcal{F}$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Three consecutive cells of $\mathcal{F}$.}
\end{figure}
Lemma 3.2. Let $V_{l,h} = \{l + 1, \ldots, l + h\}$ for $l, h \in \mathbb{V}$, $E \in \mathbb{D}$ and $r \in W$. If $0 < h \leq 19$, then

$$(\bar{C}_{V_{l,h},r;\bar{Q}_{l},\bar{Q}_{l+h}}(E))^{-1} = \sum_{i=1}^{h} (r_{l+i}\bar{C}_{\sigma(l+i),\sigma(l+i)+10}(E))^{-1}. \tag{3.2}$$

Proof. By definition, $\bar{C}_{V_{l,h},r;\bar{Q}_{l},\bar{Q}_{l+h}}(E)$ is the minimum of $S_{r,V_{l,h}}(E)(v)$ with $v \in H_{l,h}$ where $H_{l,h} := \{v \in \mathbb{R}^{V_{l,h}} : v(\bar{Q}_{l}) = 0, v(\bar{Q}_{l+h}) = 1\}$. We can characterize $H_{l,h}$ in this way. Let $Y_h := \{x \in \mathbb{R}^h : \sum_{i=1}^{h} x_i = 1\}$, and for $x \in Y_h$ let $s(x)_n = s_n$ be defined for $n = 0, \ldots, h$ as $s_n = \sum_{i=1}^{n} x_i$. Then $v \in H_{l,h}$ if and only if $v \in \mathbb{R}^{V_{l,h}}$ and $v(\bar{Q}_{l+i}) = s_i$ for some $x \in Y_h$ and every $i = 0, \ldots, h$. Next, recall that the points $\bar{Q}_i$ are the only points belonging to two different cells. Thus, the functions $v_i := v \circ \psi_{l+i}$, $i = 1, \ldots, h$ are arbitrary functions in $\mathbb{R}^{V_{l,h}}$ such that

$$v_i(P_{\sigma(l+i)}) = v(\psi_{l+i}(P_{\sigma(l+i)})) = v(\bar{Q}_{l+i-1}) = s_{i-1},$$

$$v_i(P_{\sigma(l+i)+10}) = v(\psi_{l+i}(P_{\sigma(l+i)+10})) = v(\bar{Q}_{l+i}) = s_i. \tag{3.3}$$

In view of Remark 2.1, the minimum value of $E(v_i)$ is

$$(s_i - s_{i-1})^2 \bar{C}_{\sigma(l+i),\sigma(l+i)+10}(E) = \bar{C}_{\sigma(l+i),\sigma(l+i)+10}(E) x_i^2. \tag{3.4}$$

Thus, we have to minimize $\sum_{i=1}^{h} r_{l+i} E(v_i)$, i.e.,

$$\sum_{i=1}^{h} r_{l+i} \bar{C}_{\sigma(l+i),\sigma(l+i)+10}(E) x_i^2, \quad x \in Y_h. \tag{3.5}$$

A simple use of Lagrange’s multiplier rule concludes the proof. \hfill \square

Theorem 3.3. On $\mathbb{F}$ there exists no $G$-eigenform.

Proof. Suppose by contradiction there exist $E \in \mathbb{D}$ and $r \in W$ such that $\Lambda_r(E) = E$. Of course, in view of Lemma 2.3, this implies

$$\bar{C}_{j_1,j_2}(E) = \bar{C}_{V,r;P_{j_1},P_{j_2}}(E) \quad \text{for all } \{j_1, j_2\} \in J. \tag{3.6}$$

Now, let $\bar{r} = \max\{\min\{r_{2h+1}, r_{2h+2} : h = 0, \ldots, 9\} : h = 0, \ldots, 9\}. \tag{3.7}$

Thus,

there exists $\tilde{h} = 0, \ldots, 9$ such that for all $d = 1, 2$, $r_{2\tilde{h}+d} \geq \bar{r}. \tag{3.8}$

for all $h = 0, \ldots, 9$ there exists $d = 1, 2$ such that $r_{2h+d} \leq \bar{r}. \tag{3.9}$
By Lemma 3.1 and (3.2) we have
\[ \tilde{C}_{2k+1,2k+2}(E) \geq \frac{\tilde{r}}{2} \tilde{C}_{2k+1,2k+2}(E), \]
thus
\[ \tilde{r} \leq 2. \quad (3.5) \]
Let now \( l = 1, \ldots, 20 \) be so that
\[ \tilde{C}_{l,l+10}(E) \geq \tilde{C}_{l,l+10}(E) \quad \text{for all } l = 1, \ldots, 20. \quad (3.6) \]
By Lemma 3.2 there exist \( v_1 \in \mathbb{R}^{\hat{v}_l,9} \) and \( v_2 \in \mathbb{R}^{\hat{v}_l+10,9} \) such that
\[ v_1(\tilde{Q}_l) = 0, \quad v_1(\tilde{Q}_{l+9}) = 1, \quad v_2(\tilde{Q}_{l+10}) = 1, \quad v_2(\tilde{Q}_{l+19}) = 0, \]
and
\[ S_{r,\hat{v}_l,9}(E)(v_1) = \left( \sum_{i=1}^{9} (r_{i+l} \tilde{C}_{i+l,i+10})^{-1} \right)^{-1}, \]
\[ S_{r,\hat{v}_l+10,9}(E)(v_2) = \left( \sum_{i=1}^{9} (r_{i+l+10} \tilde{C}_{i+l+10,i+10})^{-1} \right)^{-1}. \]
Let \( v \in \mathbb{R}^{\hat{v}(1)} \) be so that
\[ v = \begin{cases} 0 & \text{on } V_{\hat{l}}, \\ 1 & \text{on } V_{\hat{l}+10}, \\ v_1 & \text{on } V(\hat{v}_{\hat{l},9}), \\ v_2 & \text{on } V(\hat{v}_{\hat{l}+10,9}). \end{cases} \]
Since \( E(v \circ \psi_{\hat{l}}) = E(v \circ \psi_{\hat{l}+10}) = 0, \)
\[ S_{r}(E)(v) = \sum_{i=\hat{l}}^{\hat{l}+19} r_i E(v \circ \psi_i) = S_{r,\hat{v}_{\hat{l},9}}(E)(v_1) + S_{r,\hat{v}_{\hat{l}+10,9}}(E)(v_2). \]
By (3.6) and (3.7),
\[ S_{r,\hat{v}_{\hat{l},9}}(E)(v_1) \leq \tilde{C}_{\hat{l},\hat{l}+10}(E) \left( \sum_{i=1}^{9} (r_{i+l})^{-1} \right)^{-1} < \frac{\tilde{r}}{4} \tilde{C}_{\hat{l},\hat{l}+10}(E) \]
where the last inequality holds, since by (3.4), for at least four \( i = 1, \ldots, 9 \) we have \( r_{i+l} \leq \tilde{r} \). Similarly, \( S_{r,\hat{v}_{\hat{l}+10,9}}(E)(v_2) < \frac{\tilde{r}}{4} \tilde{C}_{\hat{l},\hat{l}+10}(E) \) by (3.8), (3.6), and (3.4). Since \( v(P_l) = 0, \) \( v(P_{l+10}) = 1, \) by the definition of \( \tilde{C}_{\hat{v},r,P_l,P_{l+10}}(E) \) and (3.2) we have \( \tilde{C}_{\hat{l},\hat{l}+10}(E) \leq S_{r}(E)(v) < \frac{\tilde{r}}{2} \tilde{C}_{\hat{l},\hat{l}+10}(E). \) Thus, \( \tilde{r} > 2, \) which contradicts (3.5).
We can consider two natural open questions.

(A) What is the maximum $N$ such that on every fractal triple with $N$ vertices there exists at least one $G$-eigenform?

In view of the result of [7] and the example here, such maximum $N$ is between 3 and 19.

(B) Does there exist a level $n$ such that on every fractal triple there exists a $G$-eigenform at the level $n$?

A theorem in [9] states that on every fractal triple there exists a $G$-eigenform at a suitable level, sufficiently large, but possibly depending on the fractal triple. In (B) it is asked whether such a level can be chosen to be independent of the fractal triple.

A natural question, related to (B), also suggested by the referee, is what is the minimal level at which there exists a $G$-eigenform on the fractal $\tilde{F}$ described in the present paper.

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References


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Roberto Peirone, Dipartimento di Matematica, Università di Roma “Tor Vergata”, via della Ricerca Scientifica, 00133 Roma, Italy
e-mail: peirone@mat.uniroma2.it