

On the distribution of perturbations of propagated Schrödinger eigenfunctions

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Abstract. Let (M, g_0) be a compact Riemannian manifold of dimension n . Let $P_0(h) \stackrel{\text{def}}{=} -h^2\Delta_g + V$ be the semiclassical Schrödinger operator for $h \in (0, h_0]$, and let E be a regular value of its principal symbol. Write φ_h for an L^2 -normalized eigenfunction of $P_0(h)$ with eigenvalue $E(h) \in [E - o(1), E + o(1)]$. We consider a smooth family of metric perturbations g_u of g_0 with u in the product space $B^k(\varepsilon) = (-\varepsilon, \varepsilon)^k \subset \mathbb{R}^k$ satisfying the admissibility condition in Definition 1. For $P_u(h) \stackrel{\text{def}}{=} -h^2\Delta_{g_u} + V$ and small $|t| > 0$, we define the propagated perturbed eigenfunctions

$$\varphi_{h,t}^{(u)} \stackrel{\text{def}}{=} e^{-\frac{i}{h}tP_u(h)}\varphi_h.$$

They appear in the mathematical description of the Loschmidt echo effect in physics.

Motivated by random wave conjectures in quantum chaos, we study the distribution of the real part of the perturbed eigenfunctions regarded as random variables

$$\text{Re}(\varphi_{h,t}^{(\cdot)}(x)): B^k(\varepsilon) \longrightarrow \mathbb{R}, \quad x \in M.$$

In particular, under an admissibility condition on the metric when (M, g) is chaotic, we compute the $h \rightarrow 0^+$ asymptotics of the variance $\text{Var}[\text{Re}(\varphi_{h,t}^{(\cdot)}(x))]$ and show that the odd moments vanish as $h \rightarrow 0^+$ as long as x is not on the generalized caustic set where $V(x) = E$.

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1. Introduction

Let (M, g_0) be a compact Riemannian manifold of dimension n with Laplace operator

$$\Delta_{g_0} = \delta_{g_0}d : C^\infty(M) \longrightarrow C^\infty(M),$$

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and let $V \in C^\infty(M)$ denote a smooth potential over M . For $h \in (0, h_0]$, consider the Schrödinger operator

$$P_0(h) \stackrel{\text{def}}{=} -h^2 \Delta_{g_0} + V, \quad (1)$$

and let E be a regular value of its principal symbol

$$p_0(x, \xi) \stackrel{\text{def}}{=} |\xi|_{g_0(x)}^2 + V(x).$$

Write φ_h for an L^2 -normalized eigenfunction of $P(h)$ with eigenvalue contained in a shrinking interval centered at E ; that is, $P_0(h)\varphi_h = E(h)\varphi_h$ and $E(h) \in [E - o(1), E + o(1)]$.

Consider a smooth family of perturbations g_u of the reference metric g_0 with u in the product space $B^k(\varepsilon) = (-\varepsilon, \varepsilon)^k \subset \mathbb{R}^k$. The number of parameters $k \geq n$ is chosen sufficiently large (but finite) so that the admissibility condition on the perturbation g_u in Definition 1 is satisfied. We introduce the associated perturbed Schrödinger operators

$$P_u(h) \stackrel{\text{def}}{=} -h^2 \Delta_{g_u} + V, \quad (2)$$

with principal symbol

$$p_u: T^*M \longrightarrow T^*M, \quad p_u(x, \xi) \stackrel{\text{def}}{=} |\xi|_{g_u}^2 + V(x). \quad (3)$$

Fix $t \neq 0$ small, independent of h , and define the perturbed propagated eigenfunctions

$$\varphi_{h,t}^{(u)} \stackrel{\text{def}}{=} e^{-\frac{i}{h}tP_u(h)}\varphi_h. \quad (4)$$

The perturbations satisfy $\varphi_{h,t}^{(u)} = \Phi_h^{(u)}(t)$ where $\Phi_h^{(u)}(t)$ denotes the solution at time t of the Schrödinger equation

$$\begin{cases} \left(ih \frac{\partial}{\partial s} - P_u(h) \right) \Phi_h^{(u)}(s) = 0, \\ \Phi_h^{(u)}(0) = \varphi_h. \end{cases}$$

The aim of this paper is to study the $h \rightarrow 0^+$ asymptotics of the distribution of $\varphi_{h,t}^{(u)}$, where the latter are regarded as random variables in $u \in B^k(\varepsilon)$. Specifically, we compute the variance and all odd moments in the semiclassical limit $h \rightarrow 0^+$. To state our results, we need to define an admissibility condition on the metric perturbations. Here and throughout the rest of the manuscript we adopt the notation $\delta_{u_\alpha} = \partial_{u_\alpha}|_{u=0}$.

Definition 1 (admissibility condition). Let g_u with $u \in B^k(\varepsilon)$ be a C^∞ metric perturbation of a reference metric g_0 . We say that g_u is *admissible* at $x \in M$ if the following conditions are satisfied.

- A) There exists an n -tuple of coordinates of u , $u' = (u_1, \dots, u_n)$, for which the Hessian matrices $d_{u'}d_\xi(p_u(x, \xi))$ are invertible for all $u \in B^k(\varepsilon)$ and all $\xi \in T_x^*M$ with $(x, \xi) \in p_0^{-1}(E - c\varepsilon, E + c\varepsilon)$ where the constant $c = c(\varepsilon) > 0$ is defined in (13).
- B) There exists a parameter coordinate u_α , a neighborhood \mathcal{W} of x , and a function $a \in C^\infty(\mathcal{W}, \mathbb{R} \setminus \{0\})$ such that $\delta_{u_\alpha}g_u^{-1}(x) = a(x)g_0^{-1}(x)$ for $x \in \mathcal{W}$.

Remark 1. If g_u is an admissible at x , then there exists a neighborhood $\mathcal{U} \subset M$ of x on which condition (A) holds.

We show in Section 5 that the admissibility condition in Definition 1 is satisfied by a large class of metric perturbations and we also give a geometric interpretation of the admissibility condition. The tangent space at g_0 to the space of all Riemannian metrics over M can be decomposed into the direct sum of the space of symmetric 2-tensors with the fixed volume form $dvol_{g_0}$, and the space of symmetric 2-tensors obtained by pointwise multiplication of g_0 . We show that the notion of being admissible is intrinsically related to having $n = \dim M$ volume preserving directions in which the metric g_0 is perturbed (this is condition (A)) and to having one direction in which the metric can be conformally perturbed (this is condition (B)).

As a model example, suppose one wishes to perturb the flat metric g_0 on the 2-torus \mathbb{T}^2 . Let $x_0 \in \mathbb{T}^2$ and in a neighborhood \mathcal{W} of x_0 consider any perturbation g_u with $u = (u_1, u_2, u_3, u'') \in B^k(\varepsilon)$, $k \geq 3$, of the form

$$g_u^{-1}(x) = g_0^{-1}(x) + h_{(u_1, u_2, u_3)}(x) + h_{u''}(x), \quad x \in \mathcal{W},$$

where

$$\begin{aligned} &h_{(u_1, u_2, u_3)}(x) \\ &= u_1 \begin{pmatrix} a_1(x) & b_1(x) \\ b_1(x) & -a_1(x) \end{pmatrix} + u_2 \begin{pmatrix} a_2(x) & b_2(x) \\ b_2(x) & -a_2(x) \end{pmatrix} + u_3 \begin{pmatrix} a_3(x) & 0 \\ 0 & a_3(x) \end{pmatrix} \end{aligned}$$

and $h_{u''}$ is any symmetric 2-tensor depending on the left-out variables $u'' \in B^{k-3}(\varepsilon)$ and higher powers of u_1, u_2 and u_3 . The perturbation g_u is admissible at x_0 provided ε is small, $a_2, a_2, a_3 \in C^\infty(\mathcal{W})$, $a_3 \neq 0$ in \mathcal{W} , and the vector fields $(a_1(x_0), b_1(x_0))$ and $(a_2(x_0), b_2(x_0))$ are linearly independent. We remark that this admissibility condition is satisfied on open subsets in the space of all C^∞ metric perturbations on \mathbb{T}^2 .

We next describe the sense in which the eigenfunctions $\varphi_{h,t}^{(u)}$ are regarded as random variables in the deformation parameters $u \in B^k(\varepsilon)$. Consider a cut-off

function $\chi \in C_0^\infty(B^k(\varepsilon); [0, 1])$ with $\chi(u) = 1$ for $u \in B^k(\varepsilon/2)$. We introduce the normalization constant

$$c_k(\varepsilon) \stackrel{\text{def}}{=} \left(\int_{B^k(\varepsilon)} \chi^2(u) du \right)^{-1}, \tag{5}$$

and define the probability measure ν on $B^k(\varepsilon)$ by

$$d\nu(u) \stackrel{\text{def}}{=} c_k(\varepsilon) \chi^2(u) du.$$

The introduction of the cut-off function in the definition of the probability measure is to ensure that all the integrands we consider, regarded as functions of u , are compactly supported in the interior of the ball, $B^k(\varepsilon)$. This is crucial for the h -microlocal characterization of the variance in Proposition 5.

We view the real part of the perturbed eigenfunctions $\varphi_{h,t}^{(u)}$ defined in (4) as random variables

$$\text{Re}(\varphi_{h,t}^{(\cdot)}(x)): B^k(\varepsilon) \longrightarrow \mathbb{R}$$

depending on the spatial parameters $x \in M$. Since one can study the distribution of a random variable such as $\text{Re}(\varphi_{h,t}^{(\cdot)}(x))$ by understanding its moments, we dedicate this paper to study the asymptotics of the variance $\text{Var}[\text{Re}(\varphi_{h,t}^{(\cdot)}(x))]$ and of the odd moments $\mathbb{E}[\text{Re}(\varphi_{h,t}^{(\cdot)}(x))]^p$ in the semiclassical limit $\hbar \rightarrow 0^+$.

Remark 2. Throughout the paper, we write that a condition holds *locally uniformly* in a set U whenever it holds uniformly on compact subsets of U .

Our first result holds for general Riemannian manifolds (M, g_0) .

Theorem 1. *Let (M, g_0) be a compact Riemannian manifold of dimension n and let E be a regular value of p_0 . Suppose g_u is a perturbation of g_0 with $u \in B^k(\varepsilon) \subset \mathbb{R}^k$ that is admissible at every $x \in M$. Fix a positive integer $\tilde{p} \in \mathbb{Z}^+$. Then, for $\varepsilon > 0$ and $|t| > 0$ sufficiently small, depending on (M, g_0) and \tilde{p} , there is $h_0(t, \varepsilon) > 0$ such that for $h \in (0, h_0(t, \varepsilon)]$ and $x \notin V^{-1}(E)$,*

(1) *There exists a constant $C > 0$, independent of h , with*

$$\text{Var}[\text{Re}(\varphi_{h,t}^{(\cdot)}(x))] \leq C.$$

(2) *For $p \in \mathbb{Z}^+$ odd with $p \leq \tilde{p}$,*

$$\mathbb{E}[\text{Re}(\varphi_{h,t}^{(\cdot)}(x))]^p = \mathcal{O}(h^\infty).$$

These estimates are locally uniform for $x \notin V^{-1}(E)$.

Remark 3. We note that if g_u is admissible for all $x \in M$, we prove that as $h \rightarrow 0^+$,

$$\int_{B^k(\varepsilon)} |\varphi_{h,t}^{(u)}(x)|^2 d\nu(u) = \mathcal{O}(1),$$

provided $\varepsilon > 0$ and $|t| > 0$ are sufficiently small. Moreover, this estimate is uniform in $x \in M$. However, the statement of the corresponding result for the variance in Theorem 1 (1) requires the asymptotic vanishing of the mean (the first moment) for which we require the condition that $x \notin V^{-1}(E)$ (see (8)).

The assumption $x \notin V^{-1}(E)$ in Theorem 1 is used in the integration by parts argument in (33) to estimate the odd moments. At present, we do not know whether the estimates for odd moments away from these generalized turning points extend uniformly to all $x \in M$. This assumption is vacuous in the homogeneous case $V = 0$.

If the metric perturbation g_u is admissible, there exist $c > 0$ and an n -tuple of u -coordinates denoted by $u' = (u_1, \dots, u_n) \in B^n(\varepsilon)$ for which $|d_{u'} d_\xi p_{(u', u'')}(x, \eta)| \neq 0$ at $u = 0$ provided $(x, \eta) \in p_0^{-1}(E - c\varepsilon, E + c\varepsilon)$. Using this, we show via an Implicit Function Theorem argument that one can locally parametrize u' as a smooth function of $(y, \eta) \in p_0^{-1}(E)$, $u' = u'(y, \eta)$ (see (42)). We write $u'' \in B^{k-n}(\varepsilon)$ for the omitted parameters and the dependence of $u'(y, \eta)$ on (u'', x) as parameters is understood. Furthermore, without loss of generality, we assume that the coordinates of u are ordered so that $u = (u', u'')$.

Let H_{p_u} be the Hamiltonian vector field of $p_u \in C^\infty(T^*M)$ and denote by

$$G_u^s : S_{g_u}^* M \longrightarrow S_{g_u}^* M$$

the bicharacteristic flow associated to H_{p_u} at time s . In the case where the manifold (M, g_0) has an ergodic geodesic flow

$$G_0^s : S^* M \longrightarrow S^* M,$$

we get asymptotic results for the variance provided we consider quantum ergodic sequences of eigenfunctions (for a precise definition see (9)). We continue to write χ for the cut-off function in the definition of the probability measure and $c_k(\varepsilon)$ for the corresponding normalizing factor (5).

Theorem 2. *Let (M, g_0) be a compact Riemannian manifold of dimension n and let E be a regular value of p_0 . Assume the geodesic flow on $p_0^{-1}(E)$ is ergodic and that $\{\varphi_h\}_{h \in (0, h_0]}$ is a quantum ergodic sequence of L^2 -normalized eigenfunctions of $P_0(h)$. Suppose g_u with $u \in B^k(\varepsilon)$ is a perturbation of g_0 that is admissible at $x \notin V^{-1}(E)$. Fix a positive integer $\tilde{p} \in \mathbb{Z}^+$. Then, for $|t| > 0$ and $\varepsilon > 0$ sufficiently small, depending on (M, g_0) and \tilde{p} ,*

(1) we have

$$\lim_{h \rightarrow 0^+} \text{Var}[\text{Re}(\varphi_{h,t}^{(\cdot)}(x))] = \int_{B^{k-n}(\varepsilon)} \beta_{t,x}^k(u'') du'',$$

where

$$\beta_{t,x}^k : B^{k-n}(\varepsilon) \rightarrow \mathbb{R}$$

is defined by

$$\begin{aligned} & \beta_{t,x}^k(u'') \\ & \stackrel{\text{def}}{=} \frac{c_k(\varepsilon)(1 + \mathcal{O}(t))}{|t|^n |p_0^{-1}(E)|} \\ & \times \int_{p_0^{-1}(E)} \frac{|\det(d_x \pi G_{(u'(y,\eta), u'')}^{-t})(x, \eta)|}{|\det(d_{u'} d_\xi p(u'(y,\eta), u''))(x, \eta)|} \chi^2(u'(y, \eta), u'') d\omega_E(y, \eta) \end{aligned} \tag{6}$$

and $u' = u'(y, \eta)$ is defined in (42);

(2) for $p \in \mathbb{Z}^+$ odd with $p \leq \tilde{p}$,

$$\lim_{h \rightarrow 0^+} \mathbb{E}[\text{Re}(\varphi_{h,t}^{(\cdot)}(x))]^p = 0.$$

1.1. Motivation. We proceed to describe two ideas that motivate our work. We first explain how the underlying ideas in our approach are motivated by the random wave conjecture. We then relate our results to the physics notion of Loschmidt echo.

Random wave conjecture. In 1977, M. Berry conjectured that the real and imaginary parts of the eigenfunctions φ_h in the chaotic case resemble random waves; see [1]. It is also believed that the eigenfunctions φ_h of quantum mixing systems behave locally as independent Gaussian variables as $h \rightarrow 0$; see for example the discussion in [9] and references therein. One of the common issues is to define a probability model where the random functions mimic the chaotic eigenfunctions. This is the role we give to the perturbations $\varphi_{h,t}^{(u)}$.

Loschmidt echo. A natural way of measuring the noise affecting a given system is the Loschmidt echo. The idea behind this concept is to measure the sensitivity of quantum evolution to perturbations, by propagating forward an initial state ψ using the unperturbed Hamiltonian p_0 , and propagating it back via the perturbed one p_u after time t . Thus, the objects of interest in this case are the states $e^{\frac{it}{\hbar} P_u(h)} e^{-\frac{it}{\hbar} P_0(h)} \psi$ and the *Loschmidt Echo*, $M_{LE}(t)$, is defined to be the return probability to the initial state:

$$M_{LE}(t) = |\langle e^{-\frac{it}{\hbar} P_u(h)} e^{\frac{it}{\hbar} P_0(h)} \psi, \psi \rangle|^2.$$

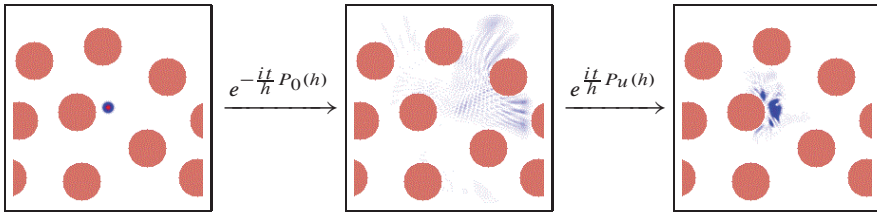


Figure 1. Illustration of the state of particle initially placed in the center of a square billiard with an irregular array of 10 circular scatterers with initial momentum pointing to the left [3].

We are interested in the case when the initial state ψ is an eigenfunction, $\psi = \varphi_h$. In this simpler case $M_{LE}(t)$ is called the *survival probability* [16] and we have

$$e^{\frac{i t}{h} P_u(h)} e^{-\frac{i t}{h} P_0(h)} \varphi_h = e^{-\frac{i t E(h)}{h}} \varphi_{h,t}^{(u)}.$$

To be precise, for an initial state φ_h the Loschmidt Echo is simply

$$M_{LE}(t) = |\langle \varphi_{h,t}^{(u)}, \varphi_h \rangle|^2.$$

As the definition shows, the fidelity $M_{LE}(t)$ can be interpreted as the decaying overlap between the evolution $\varphi_{h,t}^{(u)}$ and the unperturbed evolution φ_h ; see [11], [10], and [13].

In recent work [8], Eswarathasan and Toth have proved related results for *magnetic* deformations of the Hamiltonian $p_0(x, \xi) = |\xi|_{g_0(x)}^2 + V(x)$. We extend their upper bound results to large families of metric deformations. In addition, we characterize the asymptotic results in terms of variance and show that all odd moments are negligible. Although we do not have a rigorous argument at the moment, we hope that by further developing the methods of the present paper, we will be able to compute the higher even moments $\lim_{h \rightarrow 0^+} \mathbb{E}[\text{Re}(\varphi_{h,t}^{(\cdot)}(x))]^{2p}$ for $p \geq 2$, and compare them with the Gaussian prediction of the random wave model. We plan to return to this question elsewhere.

1.2. Outline of the paper In Section 2 we introduce the background material and notation from semiclassical analysis that we shall use to prove our results. We first show that the perturbations are semiclassically localized in $p_0^{-1}(E)$ and then explain how to microlocally cut off the propagator $e^{-\frac{i t}{h} P_u(h)}$ to obtain a localized approximation of $\varphi_{h,t}^{(u)}$.

In Section 3, we study the odd moments of $\text{Re}(\varphi_{h,t}^{(\cdot)}(x))$. Provided the metric perturbation satisfies part (B) of the admissibility condition at $x_* \in M$, we prove in Lemma 4 that for fixed $\tilde{p} \in \mathbb{Z}^+$ and $\ell, q \in \mathbb{Z}^+$ with $1 \leq \ell \leq \tilde{p}$ and $2q \leq \tilde{p}$,

$$\int_{B^k(\varepsilon)} (\varphi_{h,t}^{(u)}(x))^\ell |\varphi_{h,t}^{(u)}(x)|^{2q} dv(u) = \mathcal{O}(h^\infty) \quad \text{as } h \rightarrow 0^+,$$

for $\varepsilon > 0$ and $|t| > 0$ sufficiently small depending on (M, g_0) and \tilde{p} . The error is locally uniform in $x \in \mathcal{W} \cap (V^{-1}(E))^c$ where \mathcal{W} is the open neighborhood of x_* given by part (B) of the admissibility condition.

Using Lemma 4 and the binomial expansion for $(\varphi + \bar{\varphi})^p = (2 \operatorname{Re} \varphi)^p$, we prove that for $p \in \mathbb{Z}^+$ odd,

$$\mathbb{E}[\operatorname{Re}(\varphi_{h,t}^{(\cdot)}(x))]^p = \mathcal{O}(h^\infty), \tag{7}$$

locally uniformly in $\mathcal{W} \cap (V^{-1}(E))^c$.

In Section 4 we study the variance of $\operatorname{Re}(\varphi_{h,t}^{(\cdot)}(x))$. Provided the perturbation is admissible at $x \in (V^{-1}(E))^c$, the case $p = 1$ in (7) shows that our variables are semiclassically centered in the sense

$$\mathbb{E}[\operatorname{Re}(\varphi_{h,t}^{(\cdot)}(x))] = \mathcal{O}(h^\infty).$$

Therefore,

$$\operatorname{Var}[\operatorname{Re}(\varphi_{h,t}^{(\cdot)}(x))] = \int_{B^k(\varepsilon)} |\varphi_{h,t}^{(u)}(x)|^2 d\nu(u) + \mathcal{O}(h^\infty). \tag{8}$$

It follows that studying the variance is equivalent to understanding the behavior of the right hand side in the previous equality. In Proposition 5 we compute the asymptotics of the RHS in (8) and consequently prove Theorems 1 and 2.

In Section 5 we show that there always exist large families of admissible perturbations. We show that the notion of admissibility is related to having sufficiently many volume preserving directions in which the metric tensor g_0 is perturbed (this is Condition A), and to having at least one direction in which g_0 is conformally perturbed (this is Condition B).

Remark 4. We note here that there is an easy consequence of Theorem 1 (see also Remark 3) that concerns restriction bounds of $\varphi_{h,t}^{(u)}$ to smooth submanifolds $H \subset M$ under the assumption that the family g_u is admissible for all $x \in M$. Indeed, since our upper bounds are then uniform in $x \in M$ (see Remark 3), by integrating over H and applying Fubini, one gets that for $h \in (0, h_0]$, there exist a constant $C = C(H, h_0) > 0$ with

$$\int_{B^k(\varepsilon)} \int_H |\varphi_{h,t}^{(u)}(s)|^2 d\sigma_H(s) d\nu(u) \leq C.$$

By the Tschebyshev inequality, it then follows that for any sequence $\omega(h) = o(1)$ as $h \rightarrow 0^+$, there is a measurable $D(h) \subset B^k(\varepsilon)$ with $\lim_{h \rightarrow 0^+} \frac{|D(h)|}{|B^k(\varepsilon)|} = 1$ such that for $u \in D(h)$,

$$\int_H |\varphi_{h,t}^{(u)}(s)|^2 d\sigma_H(s) = \mathcal{O}(|\omega(h)|^{-1}).$$

Therefore, the restriction bounds for most perturbed eigenfunctions are much smaller than the universal bounds for $\int_H |\varphi_{h,t}^{(0)}(s)|^2 d\sigma_H(s)$ in [2], Theorem 3, and tend to be consistent with the ergodic case; see [5] and [15].

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2. Background and Notation

In this section we introduce some background material on eigenfunction localization and semiclassically cut off propagators. Most of this is standard in semiclassical analysis, but we include it for the benefit of the reader. We refer to [17] for further details.

Let M be a compact Riemannian manifold of dimension n . We work with the class of semiclassical symbols

$$S_{\text{cl}}^{m,k}(T^*M) \stackrel{\text{def}}{=} \left\{ a \in C^\infty(T^*M \times (0, h_0]) : a(x, \xi; h) \sim_{h \rightarrow 0^+} h^{-m} \sum_{j=1}^{\infty} a_j(x, \xi) h^j \right. \\ \left. \text{with } |\partial_x^\alpha \partial_\xi^\beta a_j(x, \xi)| \leq C_{\alpha,\beta} (1 + |\xi|^2)^{\frac{k-|\beta|}{2}} \right\}.$$

For $a \in S_{\text{cl}}^{m,k}(T^*M)$, we consider the Schwartz kernel in $M \times M$ locally of the form

$$Op_h(a)(x, y) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y, \xi \rangle} a(x, \xi; h) d\xi,$$

for $(x, y) \in U \times V$ where $U, V \subset \mathbb{R}^n$ are local coordinate charts. The corresponding space of pseudodifferential operators is defined to be

$$\Psi_{\text{cl}}^{m,k}(M) \stackrel{\text{def}}{=} \{Op_h(a) : a \in S_{\text{cl}}^{m,k}(T^*M)\}.$$

Let N be another compact n -dimensional Riemannian manifold. We also consider the class of Fourier integral operators $I_{\text{cl}}^{m,k}(M \times N, \Gamma)$ with Schwartz kernels defined in the form

$$F_h(x, y) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\varphi(x,y,\xi)} a(x, y, \xi; h) d\xi$$

for $(x, y) \in U \times V$ where $U, V \subset \mathbb{R}^n$ are local coordinate charts and

$$a \in C_0^\infty(U \times V \times \mathbb{R}^n \times (0, h_0])$$

with

$$a(x, y, \xi; h) \sim_{h \rightarrow 0^+} h^{-m} \sum_{j=1}^{\infty} a_j(x, y, \xi) h^j.$$

Here φ denotes a non-degenerate phase function in the sense of Hörmander [4], Definition (2.3.10), and Γ is an immersed Lagrangian submanifold of $T^*M \times T^*M$ with

$$\Gamma = \{(x, d_x\varphi, y, -d_y\varphi) : d_\xi\varphi(x, y, \xi) = 0\} \subset T^*M \times T^*N.$$

Finally, throughout the manuscript we say that a sequence of L^2 -normalized eigenfunctions $\{\varphi_{h_j}\}_{j \geq 1}$ of $P_0(h_j)$ with $P_0(h_j)\varphi_{h_j} = E(h_j)\varphi_{h_j}$ and $E(h_j) = E + o(1)$ is quantum ergodic (QE) if for any

$$a(x, \xi, h) \sim \sum_{k=0}^{\infty} a_k(x, \xi)h^k \in S_{\text{cl}}^{0,0}(T^*M \times [0, h_0))$$

we have

$$\langle Op_{h_j}(a)\varphi_{h_j}, \varphi_{h_j} \rangle \xrightarrow{j \rightarrow \infty} \int_{p_0^{-1}(E)} a_0(x, \xi) d\omega_E(x, \xi) \tag{9}$$

where, $d\omega_E$ is normalized Liouville measure on $p_0^{-1}(E)$.

2.1. Eigenfunction localization. Fix $t \neq 0$ and let E be a regular value of p_0 . Then, for $u \in B^k(\varepsilon)$ we introduce the cut-off functions on T^*M

$$\chi_E^{(u)}(x, \xi) = \chi_0(p_0(G_u^{-t}(x, \xi)) - E), \tag{10}$$

where $\chi_0 \in C_0^\infty([-\varepsilon, \varepsilon]; [0, 1])$ equal to 1 on $[-\varepsilon/2, \varepsilon/2]$. Consequently,

$$\text{supp } \chi_E^{(0)} \subset p_0^{-1}([E - \varepsilon, E + \varepsilon]). \tag{11}$$

Since $\chi_E^{(u)}(x, \xi) = \chi_0(p_0(x, \xi) - E + \mathcal{O}(|u|))$, the support of $\chi_E^{(u)}$ remains localized near the hypersurface $p_0^{-1}(E)$ for all $u \in B^k(\varepsilon)$ (see (13) below for precise control) and that $\varphi_{h,t}^{(u)}$ is a normalized eigenfunction of the operator

$$Q_u(h) \stackrel{\text{def}}{=} e^{-\frac{i}{h}tP_u(h)} P_0(h) e^{\frac{i}{h}tP_u(h)} \in \Psi_{\text{cl}}^{0,2}(M)$$

with eigenvalue $E(h)$. By Egorov’s Theorem $Q_u(h) = Op_h(p_0 \circ G_u^{-t}) + \mathcal{O}_{L^2 \rightarrow L^2}(h)$, and since $E(h) \in [E - o(1), E + o(1)]$, it follows that $(Q_u(h) - E)\varphi_{h,t}^{(u)} = o(1)$. Using that $Q_u(h)$ is h -elliptic off $(p_0 \circ G_u^{-t})^{-1}(E)$, a parametric construction [17], Theorem 6.4, gives $\|\varphi_{h,t}^{(u)} - Op_h(\chi_E^{(u)})\varphi_{h,t}^{(u)}\|_{L^2} = \mathcal{O}(h^\infty)$ and therefore $WF_h(\varphi_{h,t}^{(u)}) \subset (p_0 \circ G_u^{-t})^{-1}(E)$. Fix $t \neq 0$ and $\varepsilon > 0$ small. Given $x \in M$, there is a coordinate chart U with $x \in U$ and such that $\pi(G_u^{-t}(x, \xi)) \subset U$ for $u \in B^k(\varepsilon)$. Then, since $p_0(G_u^{-t}(x, \xi)) = p_0(x, \xi)$, by Taylor expansion around $u = 0$,

$$p_0(G_u^{-t}(x, \xi)) = p_0(x, \xi) + R_1(x, \xi; u, t) \tag{12}$$

where

$$|R_1(x, \xi, u; t)| \leq \sqrt{k} \max_{u \in B^k(\varepsilon)} \|d_u(p_0 \circ G_u^{-t})(x, \xi)\| \|u\|.$$

Here, given a arbitrary matrix $A = (a_{ij})$ we write

$$\|A\| \stackrel{\text{def}}{=} \max_{i,j} |a_{ij}|.$$

For $t \neq 0$ fixed, we define the constant

$$c \stackrel{\text{def}}{=} k \max_{u \in B^k(\varepsilon)} \max_{(p_0 \circ G_u^{-t})^{-1}(E)} \|d_u(p_0 \circ G_u^{-t})(x, \xi)\|. \quad (13)$$

It then follows from (12) that

$$(p_0 \circ G_u^{-t})^{-1}(E) \subset p_0^{-1}(E - c\|u\|, E + c\|u\|) \quad (14)$$

for $c > 0$ in (13) and $u \in B^k(\varepsilon)$. Consequently, $WF_h(\varphi_{h,t}^{(u)}) \subset p_0^{-1}(E - c\varepsilon, E + c\varepsilon)$ and by a Sobolev lemma argument one can also prove $\|\varphi_{h,t}^{(u)} - Op_h(\chi_E^{(u)})\varphi_{h,t}^{(u)}\|_{C^k} = \mathcal{O}_{C^k}(h^\infty)$. It follows that

$$\varphi_{h,t}^{(u)} = Op_h(\chi_E^{(u)}) \circ e^{-\frac{it}{h} P_u(h)} \circ Op_h(\chi_E^{(0)})\varphi_h + \mathcal{O}_{C^k}(h^\infty), \quad (15)$$

and from (14) and (11), with $c > 0$ in (13),

$$\text{supp } \chi_E^{(u)} \subset p_0^{-1}([E - \varepsilon - c\|u\|, E + \varepsilon + c\|u\|]). \quad (16)$$

2.2. Semiclassically cut off propagators. Motivated by the approximation (15), for $h \in (0, h_0]$, $u \in B^k(\varepsilon)$ and $|t| > 0$ small, we define the semiclassically cut off Fourier integral operators $W_{t,u}(h) \in I_{\text{cl}}^{0,-\infty}(M \times M, \Gamma_{u,t})$,

$$W_{t,u}(h) \stackrel{\text{def}}{=} Op_h(\chi_E^{(u)}) \circ e^{-\frac{it}{h} P_u(h)} \circ Op_h(\chi_E^{(0)}), \quad (17)$$

with immersed Lagrangian,

$$\Gamma_{u,t} = \{(x, \xi; y, \eta) : (x, \xi) = G_u^{-t}(y, \eta), (y, \eta) \in \text{supp } \chi_E^{(0)}\} \subset T^*M \times T^*M.$$

We note that since G_u^{-t} is then a symplectomorphism that is close to the identity, there exists a local generating function $S(s, u, \xi; x)$ with $(x, d_x S(s, u, \eta; x)) = G_u^{-t}(d_\eta S(s, u, \eta; x), \eta)$ for s close to t . It follows that

$$\Gamma_{u,t} = \{(x, d_x S(t, u, \eta; x); d_\eta S(t, u, \eta; x), \eta) \in \text{supp } \chi_E^{(u)} \times \text{supp } \chi_E^{(0)}\} \subset T^*M \times T^*M. \quad (18)$$

The generating function $S(s, u, \eta; x)$ solves the Hamilton-Jacobi initial value problem

$$\begin{cases} \partial_s S(s, u, \eta; x) + p_u(x, d_x S(s, u, \eta; x)) = 0, \\ S(0, u, \eta; x) = \langle x, \eta \rangle, \end{cases}$$

and therefore, a Taylor expansion in s around $s = 0$ gives

$$S(s, u, \eta; x) = \langle x, \eta \rangle - sp_u(x, \eta) + \mathcal{O}(s^2). \tag{19}$$

Given local coordinate charts $U, V \subset \mathbb{R}^n$ consider the local phase function $\varphi_t \in C^\infty(V \times B^k(\varepsilon) \times \mathbb{R}^n)$,

$$\varphi_t(y, u, \xi; x) \stackrel{\text{def}}{=} S(t, u, \xi; x) - \langle y, \xi \rangle. \tag{20}$$

The Schwartz kernel of $W_{t,u}(h)$ is locally of the form

$$W_{t,u}(h)(x, y) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\varphi_t(y,u,\xi;x)} a_t(u, y, \xi; x, h) d\xi + K_x(y, u), \tag{21}$$

where $|\partial_x^\alpha \partial_y^\beta K_x(y, u)| = \mathcal{O}_{\alpha,\beta}(h^\infty)$ uniformly for $(y, x, u) \in V \times U \times B^k(\varepsilon)$.

The amplitude $a_t(u, y, \xi; x, h) \sim \sum_{j=0}^\infty a_{t,j}(u, y, \xi; x) h^j$ with

$$a_{t,j}(u, \cdot, \cdot; \cdot) \in C^\infty(B^k(\varepsilon), C_0^\infty(V \times \mathbb{R}^n \times U)).$$

From (14) it is clear that the support of $\chi_E^{(u)}$ remains localized near the hypersurface $p_0^{-1}(E)$ for all $u \in B^k(\varepsilon)$; indeed, with the constant $c > 0$ in (13),

$$\begin{aligned} \text{supp}(a_t(u, \cdot, \cdot; \cdot, h)) &\subset \{(y, \xi, x) \in T^*U \times V : \\ &(y, \xi) \in \text{supp } \chi_E^{(0)} \subset p_0^{-1}(E - \varepsilon, E + \varepsilon), \\ &y = d_\xi S(t, u, \xi; x) = x + \mathcal{O}_u(t)\}. \end{aligned} \tag{22}$$

We note that since $t \neq 0$ is a fixed small parameter and $u \in B^k(\varepsilon)$ with $\varepsilon > 0$ small, it follows from (22) that the amplitudes $a_{t,j}(u, \cdot, \cdot; \cdot)$ are supported near the set $\{(y, \xi, y) \in T^*U \times U\}$. Similarly, the Lagrangian manifolds $\Gamma_{t,u} \subset T^*M \times T^*M$ are localized near the diagonal $\Delta_{T^*M \times T^*M} = \{(x, \xi; x, \xi) \in T^*M \times T^*M\}$.

3. Odd moments

The purpose of this section is to show that provided the metric g_0 is conformally deformed in at least one direction, its odd moments are negligible for general geodesic flows. Throughout this section we continue to assume that (M, g_0) is a compact Riemannian manifold and E is a regular value of p_0 . We prove

Proposition 3. *Let g_u with $u \in B^k(\varepsilon)$ be a perturbation of g_0 that satisfies part (B) of the admissibility condition at $x_* \in M$. Fix $\tilde{p} \in \mathbb{Z}^+$ and suppose $p \in \mathbb{Z}^+$ is odd with $p \leq \tilde{p}$. Then, for $\varepsilon > 0$ and $|t| > 0$ sufficiently small depending on (M, g_0) and \tilde{p} ,*

$$\mathbb{E}[\text{Re}(\varphi_{h,t}^{(\cdot)}(x))]^p = \mathcal{O}(h^\infty) \quad \text{as } h \rightarrow 0^+, \tag{23}$$

locally uniformly for $x \in \mathcal{W} \cap (V^{-1}(E))^c$. Here \mathcal{W} is the open neighborhood of x_* given by part (B) of the admissibility condition.

Proof. Given $p \leq \tilde{p}$ odd,

$$\mathbb{E}[\operatorname{Re}(\varphi_{h,t}^{(\cdot)}(x))]^p = \int_{B^k(\varepsilon)} (\operatorname{Re}(\varphi_{h,t}^{(u)}(x)))^p d\nu(u),$$

and for any complex φ the binomial expansion of $(\varphi + \bar{\varphi})^p = (2 \operatorname{Re} \varphi)^p$ for p odd gives

$$(\operatorname{Re} \varphi)^p = \frac{1}{2^p} \sum_{0 \leq j < \frac{p}{2}} \binom{p}{j} \varphi^{p-2j} |\varphi|^{2j} + \frac{1}{2^p} \sum_{\frac{p}{2} < j \leq p} \binom{p}{j} \bar{\varphi}^{2j-p} |\varphi|^{2(p-j)}. \quad (24)$$

Therefore, to prove Proposition 3, it suffices to show that

$$\int_{B^k(\varepsilon)} (\varphi_{h,t}^{(u)}(x))^\ell |\varphi_{h,t}^{(u)}(x)|^{2q} d\nu(u) = \mathcal{O}(h^\infty), \quad 1 \leq \ell \leq \tilde{p}, 2q \leq \tilde{p}, \quad (25)$$

locally uniformly in $x \in \mathcal{W} \cap (V^{-1}(E))^c$ as $h \rightarrow 0^+$.

Since the proof of (25) is somewhat technical, we prove it separately as Lemma 4. Combining (25) with the binomial expansion (24) completes the proof. \square

We have reduced the proof of Proposition 3 to establishing the following Lemma.

Lemma 4. *Let g_u with $u \in B^k(\varepsilon)$ be a perturbation of g_0 that satisfies part (B) of the admissibility condition at $x_* \in M$. Fix $\tilde{p} \in \mathbb{Z}^+$ and suppose $\ell, q \in \mathbb{Z}^+$ with $1 \leq \ell \leq \tilde{p}$ and $2q \leq \tilde{p}$. Then, for $\varepsilon > 0$ and $|t| > 0$ sufficiently small depending on (M, g_0) and \tilde{p} ,*

$$\int_{B^k(\varepsilon)} (\varphi_{h,t}^{(u)}(x))^\ell |\varphi_{h,t}^{(u)}(x)|^{2q} d\nu(u) = \mathcal{O}(h^\infty) \quad \text{as } h \rightarrow 0^+,$$

locally uniformly for $x \in \mathcal{W} \cap (V^{-1}(E))^c$. Here \mathcal{W} is the open neighborhood of x_* given by part (B) of the admissibility condition.

Proof. We identify the product manifold $M^{(\ell+2q)}$ with $M^{(\ell)} \times M^{(q)} \times M^{(q)}$ and write

$$(\tilde{y}, \tilde{z}, \tilde{z}') \stackrel{\text{def}}{=} (y^{(1)}, \dots, y^{(\ell)}, z^{(1)}, \dots, z^{(q)}, z'^{(1)}, \dots, z'^{(q)}) \in V^{(\ell+2q)}$$

for the local coordinates.

By assumption, there exists $a \in C^\infty(M)$ so that $\delta_{u_\alpha} g_u^{-1}(x) = a(x) g_0^{-1}(x)$ with $a(x) \neq 0$ for all $x \in \mathcal{W}$. Let us continue to write χ for the cut-off function appearing in the definition of the probability measure ν , and $c_k(\varepsilon)$ for the corresponding normalizing factor in (5). Since from (15), $\varphi_{h,t}^{(u)}(x) = [W_{t,u}(h)\varphi_h](x) + \mathcal{O}(h^\infty)$,

writing $W_{t,u}(x, y)$ for the kernel of $W_{t,u}$ we get

$$\begin{aligned} & \int_{B^k(\varepsilon)} (\varphi_{h,t}^{(u)}(x))^\ell |\varphi_{h,t}^{(u)}(x)|^{2q} dv(u) \\ &= \int_{B^k(\varepsilon)} (|W_{t,u}(h)\varphi_h|(x))^\ell |W_{t,u}(h)\varphi_h|(x)|^{2q} dv(u) + \mathcal{O}(h^\infty) \\ &= c_k(\varepsilon) \int_{B^k(\varepsilon)} \int_{M^{\ell+2q}} B_{t,u}^{[\ell,q]}(\tilde{y}, \tilde{z}, \tilde{z}'; x, h) \varphi_h(y^{(i)}) \varphi_h(z^{(j)}) \overline{\varphi_h(z'^{(j)})} \\ & \quad \times \chi^2(u) d\tilde{y} d\tilde{z} d\tilde{z}' du + \mathcal{O}(h^\infty), \end{aligned} \tag{26}$$

where $B_{t,u}^{[\ell,q]} \in C^\infty(V^{\ell+2q} \times U \times [0, h_0))$ is defined by the formula

$$B_{t,u}^{[\ell,q]}(\tilde{y}, \tilde{z}, \tilde{z}'; x, h) \stackrel{\text{def}}{=} \prod_{\substack{1 \leq i \leq \ell \\ 1 \leq j \leq q}} W_{t,u}(x, y^{(i)}) W_{t,u}(x, z^{(j)}) \overline{W_{t,u}(x, z'^{(j)})}.$$

From (21) we deduce the kernel expansion

$$\begin{aligned} & B_{t,u}^{[\ell,q]}(\tilde{y}, \tilde{z}, \tilde{z}'; x, h) \\ &= \frac{1}{(2\pi h)^{n(2q+\ell)}} \int_{\mathbb{R}^{nq}} \int_{\mathbb{R}^{nq}} \int_{\mathbb{R}^{n\ell}} e^{\frac{i}{h} \Phi_t^{[\ell,q]}(\tilde{y}, \tilde{z}, \tilde{z}', u, \tilde{\xi}, \tilde{\eta}, \tilde{\eta}'; x)} \\ & \quad \times c_t^{[\ell,q]}(u, \tilde{y}, \tilde{z}, \tilde{z}', \tilde{\xi}, \tilde{\eta}, \tilde{\eta}'; x, h) d\tilde{\xi} d\tilde{\eta} d\tilde{\eta}' \\ & \quad + K_x(\tilde{y}, \tilde{z}, \tilde{z}', u), \end{aligned} \tag{27}$$

for $\Phi_t^{[\ell,q]}$, $c_t^{[\ell,q]}$ and K_x as follows.

(1) The phase function $\Phi_t^{[\ell,q]}$ is defined by

$$\begin{aligned} & \Phi_t^{[\ell,q]}(\tilde{y}, \tilde{z}, \tilde{z}', u, \tilde{\xi}, \tilde{\eta}, \tilde{\eta}'; x) \\ & \stackrel{\text{def}}{=} \sum_{j=1}^{\ell} \varphi_t(y^{(j)}, u, \xi^{(j)}; x) + \sum_{j=1}^q \varphi_t(z^{(j)}, u, \eta^{(j)}; x) - \varphi_t(z'^{(j)}, u, \eta'^{(j)}; x), \end{aligned} \tag{28}$$

where φ_t is given in (20).

(2) The amplitude $c_t^{[\ell,q]}$ satisfies

$$c_t^{[\ell,q]}(u, \tilde{y}, \tilde{z}, \tilde{z}', \tilde{\xi}, \tilde{\eta}, \tilde{\eta}'; x, h) \sim_{h \rightarrow 0^+} \sum_{j=0}^{\infty} c_{t,j}^{[\ell,q]}(u, \tilde{y}, \tilde{z}, \tilde{z}', \tilde{\xi}, \tilde{\eta}, \tilde{\eta}'; x) h^j$$

with $c_{i,j}^{[\ell,q]}(u, \cdot, \cdot; \cdot) \in C^\infty(B^k(\varepsilon), C_0^\infty(V^{(\ell+2q)} \times \mathbb{R}^{n(\ell+2q)} \times U))$ for $U, V \subset \mathbb{R}^n$ local coordinate charts as in (20). Moreover,

$$\begin{aligned} & \text{supp}(c_i^{[\ell,q]}(u, \tilde{y}, \tilde{z}, \tilde{z}', \cdot; x, h)) \\ & \subset \{(\tilde{\xi}, \tilde{\eta}, \tilde{\eta}') : (x, \xi^{(i)}), (x, \eta^{(j)}), (x, \eta'^{(j)}) \in p_0^{-1}(E - c\varepsilon, E + c\varepsilon), \\ & \quad x \in U, i \leq \ell, j \leq q\} \\ & \subset \mathbb{R}^{n(\ell+2q)}. \end{aligned} \tag{29}$$

(3) The residual operator K_x satisfies

$$|\partial_x^\alpha \partial_{(\tilde{y}, \tilde{z}, \tilde{z}')}^\beta K_x(\tilde{y}, \tilde{z}, \tilde{z}', u)| = \mathcal{O}_{\alpha,\beta}(h^\infty)$$

locally uniformly in $(\tilde{y}, \tilde{z}, \tilde{z}', u) \in V^{(\ell+2q)} \times B^k(\varepsilon)$.

Claim. For $\varepsilon > 0$ and $|t| > 0$ sufficiently small, there exists $C = C(t, \varepsilon, E, g_0) > 0$ such that for $(\tilde{\xi}, \tilde{\eta}, \tilde{\eta}') \in \text{supp}(c_i^{[\ell,q]}(u, \tilde{y}, \tilde{z}, \tilde{z}', \cdot; x, h))$,

$$|\partial_{u_\alpha} \Phi_t^{[\ell,q]}(\tilde{y}, \tilde{z}, \tilde{z}', u, \tilde{\xi}, \tilde{\xi}', \tilde{\eta}, \tilde{\eta}'; x)| \geq C > 0, \tag{30}$$

where this bound holds locally uniformly for $(\tilde{y}, \tilde{z}, \tilde{z}') \in M^{(\ell+2q)}$, $u \in B^k(\varepsilon)$, and $x \in \mathcal{W} \cap (V^{-1}(E))^c$.

To prove this claim we first observe that since $\delta_{u_\alpha} g_u^{-1}(x) = a(x) g_0^{-1}(x)$ for $x \in \mathcal{W}$,

$$\delta_{u_\alpha} p_u(x, \xi) = a(x) |\xi|_{g_0(x)}^2. \tag{31}$$

Also, from the Taylor expansion of the generating function (19) around $s = 0$, together with (20), we know that for $x \in \mathcal{W}$

$$\varphi_t(y, u, \eta; x) = \langle x - y, \eta \rangle - t p_u(x, \eta) + \mathcal{O}(t^2), \tag{32}$$

where in (32), the error $\mathcal{O}(t^2)$ depends on \tilde{p} . Combining (28) with (32) and (31), for $x \in \mathcal{W}$ we get

$$\begin{aligned} & \partial_{u_\alpha} \Phi_t^{[\ell,q]}(\tilde{y}, \tilde{z}, \tilde{z}', u, \tilde{\xi}, \tilde{\eta}, \tilde{\eta}'; x) = \\ & = -t a(x) \left(\sum_{i=1}^{\ell} |\xi^{(i)}|_{g_0(x)}^2 + \sum_{j=1}^q |\eta^{(j)}|_{g_0(x)}^2 - \sum_{j=1}^q |\eta'^{(j)}|_{g_0(x)}^2 + \mathcal{O}(|u|) \right) \\ & \quad + \mathcal{O}(t^2). \end{aligned}$$

From the support conditions on the amplitude $c_i^{[\ell,q]}$ in (29), we have that

$$\begin{aligned} |\xi^{(j)}|_{g_0(x)}^2 + V(x) &= E + \mathcal{O}(\varepsilon), \\ |\eta^{(j)}|_{g_0(x)}^2 + V(x) &= E + \mathcal{O}(\varepsilon), \end{aligned}$$

and

$$|\eta^{(j)}|_{g_0(x)}^2 + V(x) = E + \mathcal{O}(\varepsilon),$$

for $i \leq \ell$ and $j \leq q$. Therefore, for $x \in \mathcal{W}$,

$$\begin{aligned} \partial_{u_\alpha} \Phi_t^{[\ell, q]}(\tilde{y}, \tilde{z}, \tilde{z}', u, \tilde{\xi}, \tilde{\eta}, \tilde{\eta}'; x) \\ = -ta(x)(\ell(E - V(x)) + (\ell + 2q + 1)\mathcal{O}(\varepsilon)) + \mathcal{O}(t^2), \end{aligned} \tag{33}$$

and so

$$\begin{aligned} |\partial_{u_\alpha} \Phi_t^{[\ell, q]}(\tilde{y}, \tilde{z}, \tilde{z}', u, \tilde{\xi}, \tilde{\xi}', \tilde{\eta}, \tilde{\eta}'; x)| \\ \geq |ta(x)| \cdot |\ell(E - V(x)) + 2\tilde{p}\mathcal{O}(\varepsilon)| + \mathcal{O}(t^2), \end{aligned} \tag{34}$$

uniformly in all variables. Given a compact subset $K \subset \mathcal{W} \cap (V^{-1}(E))^c$, for $x \in K$, one has $|V(x) - E| \geq \frac{1}{C_0} > 0$ for some constant $C_0 > 0$. Since $a(x) \neq 0$ for all $x \in \mathcal{W}$, from (34) it follows that for such a given compact set K and number of odd moments $\tilde{p} \in \mathbb{Z}^+$, we can choose $\varepsilon > 0$ and $t \neq 0$ sufficiently small (depending on \tilde{p} and K) so that the RHS of (34) is uniformly bounded away from zero. We conclude that the claim in (30) holds for $x \in \mathcal{W} \cap (V^{-1}(E))^c$.

We then use the operator $\left(\frac{h}{i\partial_{u_\alpha} \Phi_t^{[\ell, q]}}\right) \frac{\partial}{\partial u_\alpha}$, to repeatedly integrate by parts in (26) and obtain

$$B_{t,u}^{[\ell, q]}(h)(\tilde{y}, \tilde{z}, \tilde{z}'; x, h) = \mathcal{O}(h^\infty)$$

locally uniformly for $(\tilde{y}, \tilde{z}, \tilde{z}') \in M^{(\ell+2q)}$, $u \in B^k(\varepsilon)$ and $x \in \mathcal{W} \cap (V^{-1}(E))^c$. We note that there are no boundary term contributions arising from the integration by parts since $\chi(u) = 0$ for $u \in \partial B^k(\varepsilon)$. From (26) it follows that

$$\int_{B^k(\varepsilon)} (\varphi_{h,t}^{(u)}(x))^\ell |\varphi_{h,t}^{(u)}(x)|^{2q} dv(u) = \mathcal{O}(h^\infty),$$

locally uniformly in $x \in \mathcal{W} \cap (V^{-1}(E))^c$. □

4. Variance

As explained in the Introduction (see (8)), provided the perturbation is admissible at $x \in (V^{-1}(E))^c$, the case $p = 1$ in Proposition 3 shows that our random variables are semiclassically centered in the sense

$$\mathbb{E}[\text{Re}(\varphi_{h,t}^{(\cdot)}(x))] = \mathcal{O}(h^\infty).$$

Therefore,

$$\text{Var}[\text{Re}(\varphi_{h,t}^{(\cdot)}(x))] = \int_{B^k(\varepsilon)} |\varphi_{h,t}^{(u)}(x)|^2 dv(u) + \mathcal{O}(h^\infty).$$

It then follows that studying the variance is equivalent to understanding the behavior of the right hand side in the previous equality. We compute the asymptotics of the RHS in the next Proposition.

Proposition 5. *Let g_u be admissible at $x_* \in M$ and let \mathcal{U} be the neighborhood of x_* given in Remark 1. For $\varepsilon > 0$ and $|t| > 0$ sufficiently small, there exist a choice of coordinates $u'' \in B^{k-n}(\varepsilon)$ and corresponding operators $A_{t,x,u''}(h) \in \Psi_{cl}^{0,-\infty}(M)$ defined for all $(x, u'') \in \mathcal{U} \times B^{k-n}(\varepsilon)$, such that*

$$\int_{B^k(\varepsilon)} |\varphi_{h,t}^{(u)}(x)|^2 d\nu(u) = c_k(\varepsilon) \int_{B^{k-n}(\varepsilon)} \langle A_{t,x,u''}(h)\varphi_{\hbar}, \varphi_{\hbar} \rangle_{L^2(M)} du'' + \mathcal{O}(h^\infty). \tag{35}$$

Proof. By Remark 1, we choose $\mathcal{U} \subset M$ to be an open neighborhood of x_* so that the admissibility condition (A) holds on \mathcal{U} . That is, given the constant $c > 0$ in (13) and some subset of n coordinates of u , which we denote $u' \in B^n(\varepsilon)$, so that the matrix

$$d_{u'} d_{\xi}(p_u(x, \xi)) \text{ is invertible for } (x, \xi) \in p_0^{-1}(E - c\varepsilon, E + c\varepsilon), \tag{36}$$

$$(x, u) \in \mathcal{U} \times B^k(\varepsilon).$$

We write $u'' \in B^{k-n}(\varepsilon)$ for the omitted variables and assume that the coordinates of u are ordered so that $u = (u', u'')$.

Write $W_{t,u}(h)(x, y)$ for the Schwartz kernel of $W_{t,u}(h)$. Then, for $u'' \in B^{k-n}(\varepsilon)$ and $x \in \mathcal{U}$, we define a new family of operators

$$\widehat{W}_{t,x,u''}(h) : C^\infty(M) \longrightarrow C_0^\infty(B^n(\varepsilon)), \tag{37}$$

with Schwartz kernels

$$\widehat{W}_{t,x,u''}(h)(u', y) \stackrel{\text{def}}{=} \chi(u) \cdot W_{t,u}(h)(x, y), \quad u = (u', u'') \in B^k(\varepsilon),$$

where we continue to write χ for the cut-off function appearing in the definition of the probability measure ν in (5). By (15),

$$\chi(u)\varphi_{h,t}^{(u)}(x) = \chi(u)[W_{t,u}(h)\varphi_{\hbar}](x) + \mathcal{O}(h^\infty) = [\widehat{W}_{t,x,u''}(h)\varphi_{\hbar}](u') + \mathcal{O}(h^\infty),$$

and so,

$$\begin{aligned} & \int_{B^k(\varepsilon)} |\varphi_{h,t}^{(u)}(x)|^2 d\nu(u) \\ &= c_k(\varepsilon) \int_{B^k(\varepsilon)} |\widehat{W}_{t,x,u''}(h)\varphi_{\hbar}(u')|^2 du + \mathcal{O}(h^\infty) \\ &= c_k(\varepsilon) \int_{B^{k-n}(\varepsilon)} \langle \widehat{W}_{t,x,u''}(h)\varphi_{\hbar}, \widehat{W}_{t,x,u''}(h)\varphi_{\hbar} \rangle_{L^2(B^n(\varepsilon))} du'' + \mathcal{O}(h^\infty). \end{aligned} \tag{38}$$

From (21), the Schwartz kernel of $\widehat{W}_{t,x,u''}(h)$ is given by

$$\begin{aligned} &\widehat{W}_{t,x,u''}(h)(u', y) \\ &= \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\varphi_t(y,u',u'',\xi;x)} a_t(u, y, \xi; x, h) \chi(u) d\xi + K_x(y, u), \end{aligned} \tag{39}$$

where $|\partial_x^\alpha \partial_y^\beta K_x(y, u)| = \mathcal{O}_{\alpha,\beta}(h^\infty)$ uniformly in $(x, y, u) \in U \times V \times B^k(\varepsilon)$ for $\varepsilon > 0$ small, where $U, V \subset \mathbb{R}^n$ are local coordinate charts with $U \subset \mathcal{U}$. The amplitude

$$a_t(u, y, \xi; x, h) \sim \sum_{j=0}^{\infty} a_j(u, y, \xi; x) h^j$$

with

$$a_j(u, \cdot, \cdot; \cdot) \in C^\infty(B^k(\varepsilon), C_0^\infty(V \times \mathbb{R}^n \times U)).$$

Moreover, we recall from (22) that

$$\begin{aligned} \text{supp}(a_t(u, \cdot, \cdot; \cdot, h)) &\subset \{(y, \xi, x) \in T^*U \times V : (x, \xi) \in p_0^{-1}(E - c\varepsilon, E + c\varepsilon), \\ &y = d_\xi S(t, u, \xi; x) = x + \mathcal{O}_u(t)\}. \end{aligned}$$

By the same argument presented in [8], Proposition 4.1, it can be shown that for $x \in U$, $\varepsilon > 0$ and $|t| \neq 0$ small enough, $\widehat{W}_{t,x,u''}(h) \in I_{\text{cl}}^{0,-\infty}(M \times B^n(\varepsilon); \Gamma_{t,x,u''})$ with

$$\begin{aligned} &\Gamma_{t,x,u''} \\ &\stackrel{\text{def}}{=} \{(u', d_{u'} S(t, u, \eta; x)), d_\eta S(t, u, \eta; x), \eta) : (d_\eta S(t, u, \eta; x), \eta) \in \text{supp } \chi_E^{(0)}\} \\ &\subset T^*B^n(\varepsilon) \times T^*M. \end{aligned} \tag{40}$$

where $u \stackrel{\text{def}}{=} (u', u'') \in B^k(\varepsilon)$ for ε small, and $x \in U$. It remains to show that $\Gamma_{t,x,u''}$ is a canonical graph.

We recall that if $(d_\eta S(t, u, \eta; x), \eta) \in \text{supp } \chi_E^{(0)}$ for $u \in B^k(\varepsilon)$ then one gets that $(x, d_x S(t, u, \eta; x)) \in \text{supp } \chi_E^{(u)} \subset p_0^{-1}((E - (c + 1)\varepsilon, E + (c + 1)\varepsilon))$ with $c > 0$ as in (13). By (40) and the admissibility assumption (B) it follows that by possibly shrinking $t \neq 0$, we can ensure that there is a constant $C_0 > 0$, such that the non-degeneracy condition

$$\det(d_{u'} d_\eta \varphi_t(y, u', u'', \eta; x)) = |t|^n (\det(d_{u'} d_\eta p_u(x, \eta)) + \mathcal{O}(t^2)) \geq C_0 |t|^n, \tag{41}$$

holds uniformly for (u, y, η) with $(y, \eta) \in \text{supp } \chi_E^{(0)}$ and $y = d_\eta S(t, u, \eta; x)$. Now, for u'' fixed and $x \in U$, consider the map

$$(u', y, \eta) \longmapsto d_\eta \varphi_t(y, u', u'', \eta; x), \quad (u', \tau; y, \eta) \in \Gamma_{t,x,u''}.$$

We claim that due to the non-degeneracy condition (41), the Lagrangian (40) is a canonical graph. Indeed, equation (41) allows us to apply the Implicit Function Theorem and locally write $u' = u'(y, \eta)$ satisfying

$$u' = u'(y, \eta) \quad \text{when } d_\eta \varphi_t(y, u', u'', \eta; x) = 0, \tag{42}$$

for $x \in U$. Then, taking into account that for $x \in U$

$$d_\eta \varphi_t(y, u', u'', \eta; x) = 0 \quad \text{when } y = d_\eta S(t, u', u'', \eta; x),$$

we write $(y, \eta) \in V \times \mathbb{R}^n$ as local parametrizing variables for $\Gamma_{t,x,u''}$ as in (40) and get

$$\begin{aligned} \Gamma_{t,x,u''} = \{ & (u'(y, \eta), d_{u'} S(t, u'(y, \eta), u'', \eta; x); y, \eta) : \\ & (y, \eta) \in \text{supp } \chi_E^{(0)}, y = d_\eta S(t, u', u'', \eta; x) \}. \end{aligned} \tag{43}$$

For $u'' \in B^{k-n}(\varepsilon)$ and $x \in U$ define the operators

$$A_{t,x,u''}(h) : C^\infty(M) \longrightarrow C^\infty(M), \quad A_{t,x,u''}(h) \stackrel{\text{def}}{=} (\widehat{W}_{t,x,u''}(h))^* \circ (\widehat{W}_{t,x,u''}(h)). \tag{44}$$

Since $\widehat{W}_{t,x,u''}(h) \in I_{\text{cl}}^{0,-\infty}(M \times B^n(\varepsilon); \Gamma_{t,x,u''})$ and the immersed Lagrangian $\Gamma_{t,x,u''}$ is a canonical graph, the operator

$$A_{t,x,u''}(h) \in \Psi_{\text{cl}}^{0,-\infty}(M),$$

for $x \in U$ and $u'' \in B^{k-n}(\varepsilon)$. From (38) and (44) it follows that

$$\int_{B^k(\varepsilon)} |\varphi_{h,t}^{(u)}(x)|^2 dv(u) = c_k(\varepsilon) \int_{B^{k-n}(\varepsilon)} \langle A_{t,x,u''}(h) \varphi_h, \varphi_h \rangle_{L^2(M)} du'' + \mathcal{O}(h^\infty).$$

□

4.1. Proof of Theorem 1. Since M is compact we choose a finite covering

$$M \subset \bigcup_{j=1}^N \mathcal{V}_{x_j}$$

where $x_j \in M$ and $\mathcal{V}_{x_j} = \mathcal{W}_{x_j} \cap \mathcal{U}_{x_j}$. Here \mathcal{W}_{x_j} is given by part (B) of the admissibility condition at x_j , and \mathcal{U}_{x_j} is given in Remark 1.

Fix $j \in \{1, \dots, N\}$ and let $x \in \mathcal{V}_{x_j}$. To prove the first part of Theorem 1 we note that Proposition 5 gives $A_{t,x,u''}(h) \in \Psi_{\text{cl}}^{0,-\infty}(M)$. Thus, by L^2 boundedness there exists a constant $C_j = C_j(\varepsilon, t, E, g_0) > 0$ such that

$$\langle A_{t,x,u''}(h) \varphi_h, \varphi_h \rangle_{L^2(M)} \leq C_j$$

uniformly in $(x, u'', h) \in \mathcal{V}_{x_j} \times B^{k-n}(\varepsilon) \times (0, h_0]$. Therefore, from (8) and (35) one can choose a positive constant $C > 0$ so that the first part of the statement of Theorem 1 holds uniformly for $x \in K$, where $K \subset (V^{-1}(E))^c$ is any compact subset.

To prove the second part of Theorem 1 regarding the odd moments we simply apply Proposition 3 in each neighborhood \mathcal{V}_{x_j} .

4.2. Proof of Theorem 2. From Proposition 5 and equation (8),

$$\begin{aligned} \lim_{h \rightarrow 0^+} \text{Var}[\text{Re}(\varphi_{h,t}^{(\cdot)}(x))] &= \lim_{h \rightarrow 0^+} \int_{B^k(\varepsilon)} |\varphi_{h,t}^{(u)}(x)|^2 d\nu(u) \\ &= \lim_{h \rightarrow 0^+} c_k(\varepsilon) \int_{B^{k-n}(\varepsilon)} \langle A_{t,x,u''}(h)\varphi_{\hbar}, \varphi_{\hbar} \rangle_{L^2(M)} du''. \end{aligned} \tag{45}$$

Since (φ_h) is a quantum ergodic sequence,

$$\begin{aligned} \lim_{h \rightarrow 0^+} \langle A_{t,x,u''}(h)\varphi_{\hbar}, \varphi_{\hbar} \rangle_{L^2(M)} \\ = \frac{1}{|p_0^{-1}(E)|} \int_{p_0^{-1}(E)} \sigma_0(A_{t,x,u''}(h))(y, \eta) d\omega_E(y, \eta). \end{aligned} \tag{46}$$

In addition, following the same argument presented in Corollary 4.2 of [8], the principal symbol can be locally written as

$$\begin{aligned} \sigma_0(A_{t,x,u''}(h))(y, \eta) \\ = |\chi_E^{(u',u'')}(x, \eta)|^2 \frac{|\det(d_x \pi G_{(u',u'')}^{-t}(x, \eta))|}{|\det(d_{u'} d_{\eta} S(t, x, \eta; u', u''))|} \chi^2(u', u'') \\ = |\chi_E^{(u',u'')}(x, \eta)|^2 \frac{|\det(d_x \pi G_{(u',u'')}^{-t}(x, \eta))|}{|t|^n |\det(d_{u'} d_{\eta} p_{(u',u'')}(x, \eta))|} (1 + \mathcal{O}(t)) \chi^2(u', u'') \end{aligned} \tag{47}$$

for $u' = u'(y, \eta)$ parametrizing the Lagrangian $\Gamma_{t,x,u''}$ regarded as a canonical graph. The first statement of Theorem 2 then follows by combining (45), (46) and the expression for the principal symbol (47).

The second statement of Theorem 2 about odd moments is a direct application of the second part of Theorem 1.

5. Admissible perturbations

In this section we study the geometry behind the admissibility condition and show that perturbations satisfying such conditions always exist. It is clear that one can always have perturbations satisfying part (B) of the admissibility condition. We therefore

focus on proving the existence of metric perturbations satisfying condition (A). The symbol $p_u: T^*M \rightarrow T^*M$ defined in (3) has the form

$$p_u(x, \xi) = \sum_{i,j=1}^n g_u^{ij}(x) \xi_i \xi_j + V(x).$$

Write \mathcal{M} for the space of Riemannian metrics on M . For each coordinate u_s of u define the symmetric tensor $h_{u_s} \stackrel{\text{def}}{=} \delta_{u_s} g_u^{-1}$ and write in local coordinates

$$h_{u_s} = h_{u_s}^{ij} dx_i \otimes dx_j, \quad h_{u_s}^{ij} \stackrel{\text{def}}{=} \delta_{u_s} g_u^{ij}. \tag{48}$$

It is straight forward to check that $\partial_{u_s} \partial_{\xi_i} p_u(x, \xi)|_{u=0} = 2 \sum_{l=1}^n h_{u_s}^{li}(x) \xi_l$. Thereby, a metric perturbation satisfies condition (A) provided there exist $c > 0$ and an n -tuple $u' = (u_1, \dots, u_n)$ of coordinates of u so that for all $(x, \xi) \in p_0^{-1}(E - c\varepsilon, E + c\varepsilon)$, the matrix $(\sum_{l=1}^n h_{u'_j}^{li}(x) \xi_l)_{i,j=1, \dots, n}$ is invertible. By definition, the notion of admissibility depends on the direction, inside the space of symmetric tensors, in which g_0 is deformed. In what follows we show that the admissibility condition is directly related to performing the deformation g_u in sufficiently many *volume preserving* directions, described below.

Let \mathcal{P} denote the multiplicative group of positive smooth functions on M , which we refer to as *pointwise conformal deformations*. \mathcal{P} acts on \mathcal{M} by multiplication

$$\mathcal{P} \times \mathcal{M} \longrightarrow \mathcal{M}, \quad (p, g) \longmapsto pg.$$

Given $g_0 \in \mathcal{M}$, the orbit of g_0 under \mathcal{P} denoted by $\mathcal{P} \cdot g_0$, is a closed submanifold of \mathcal{M} with tangent space at g_0 given by

$$T_{g_0}(\mathcal{P} \cdot g_0) = \{v \in S^2(M) : v = fg_0, f \in C^\infty(M, \mathbb{R})\}. \tag{49}$$

Let μ be a volume form on M and define

$$\mathcal{N}_\mu \stackrel{\text{def}}{=} \{g \in \mathcal{M} : \mu = \mu_g\}$$

where μ_g denotes the Riemannian volume measure associated to g . Pointwise conformal transformations $g_0 \mapsto fg_0$ multiply the volume form $\mu_{g_0}(x)$ at a point x by $(f(x))^{n/2}$. Transverse to the orbit of g_0 by the action of pointwise conformal transformations is the sub manifold $\mathcal{N}_{\mu_{g_0}}$ of all metrics g on M with the fixed volume form; equivalently, the determinant $\det(g_{ij}(x))$ is preserved for all x . It is well-known that the tangent space to the space of symmetric matrices with fixed determinant consists of symmetric traceless matrices. Accordingly, it can be shown (cf. [6]) that the tangent space $T_{g_0} \mathcal{N}_{\mu_{g_0}}$ is given by

$$T_{g_0}(\mathcal{N}_{\mu_{g_0}}) = \{v \in S^2(M) : (\text{tr}_{g_0} v)(x) = 0, x \in M\}. \tag{50}$$

For every metric $g_0 \in \mathcal{M}$ the space of symmetric tensors has the pointwise orthogonal splitting

$$T_{g_0}\mathcal{M} = T_{g_0}(\mathcal{N}_{\mu_{g_0}}) \oplus T_{g_0}(\mathcal{P} \cdot g_0)$$

where every $v \in S^2(M)$ is decomposed as

$$v = (v - \frac{\text{tr}_{g_0} v}{n} g_0) + \frac{1}{n} (\text{tr}_{g_0} v) g_0.$$

Let g_u be a metric deformation of g_0 ; we continue to write $h_{u_s} = \delta_{u_s} g_u^{-1}$. Working in geodesic normal coordinates at x_* , it is not difficult to show that volume-preserving deformations are characterized by the condition $\text{tr}_{g_0^{-1}}(h_{u_s}(x)) = 0$ for all s . We shall show below that the admissibility condition holds for such deformations.

5.1. Surfaces. On surfaces, we claim that perturbations g_u that have two linearly independent u -derivatives in the volume preserving directions are admissible.

Proposition 6. *Let (M, g_0) be a compact Riemannian surface. Let E be a regular value of p_0 . Suppose g_u with $u \in B^k(\varepsilon)$ is a perturbation of g_0 such that there exist two coordinates $u' = (u_1, u_2)$ of u for which $h_{u_1}(x)$ and $h_{u_2}(x)$, as defined in (48), are linearly independent tensors with $\text{tr}_{g_0^{-1}}(h_{u_1})(x) = \text{tr}_{g_0^{-1}}(h_{u_2})(x) = 0$ at some $x \notin V^{-1}(E)$. Then, for ε small enough, the perturbation g_u satisfies part (A) of the admissibility condition at x .*

Proof. Let $x_* \in M$ be such that x belongs to a geodesic ball centered at x_* , and consider normal coordinates at x_* . In these coordinates, $g_{0ij}(x) = \delta_{ij} + \mathcal{O}(|x|^2)$ for x being at a small distance $|x|$ from x_* . Therefore, since $\text{tr}_{g_0^{-1}}(h_{u_s})(x) = 0$ for $s = 1, 2$, we have $h_{u_s}^{11}(x) = -h_{u_s}^{22}(x) + \mathcal{O}(|x|^2)$ for $s = 1, 2$. It is straight forward to check

$$\det \left(\sum_{j=1}^2 h_{u_s}^{ij}(x) \xi_j \right)_{s,i=1,2} = |\xi|_{g_0(x)}^2 \left(\det \begin{pmatrix} h_{u_1}^{11}(x) & h_{u_2}^{11}(x) \\ h_{u_1}^{12}(x) & h_{u_2}^{12}(x) \end{pmatrix} + \mathcal{O}(|x|^2) \right).$$

Since we are only interested in what happens when $|\xi|_{g_0(x)}^2 + V(x) = E + \mathcal{O}(\varepsilon)$, the result follows from the assumption $V(x) \neq E$ and the fact that $h_{u_1}(x)$ and $h_{u_2}(x)$ are linearly independent tensors. \square

5.2. Manifolds. In what follows we show that on an n -dimensional manifold we can always have admissible perturbations.

Let M be an n -dimensional compact manifold and fix $x_* \in M$. Consider a geodesic normal coordinate system at x_* . We shall consider deformations of the reference metric g_0 that, as in the surface case, preserve the volume form. Infinitesimally, as explained in (50), the corresponding quadratic form is given by a traceless symmetric matrix. The space of traceless symmetric tensors at $x \in M$ has dimension

$$\kappa_n \stackrel{\text{def}}{=} \frac{n^2 + n - 2}{2}, \tag{51}$$

and the basis of the space of such forms is given by

$$\xi_1^2 - \xi_i^2, 2 \leq i \leq n; \quad \text{and} \quad \xi_j \xi_k, 1 \leq j < k \leq n,$$

for $\xi = (\xi_1, \dots, \xi_n) \in T_{x_*}^* M$; we denote these polynomials evaluated at $x = x_*$ by $q_j(\xi)$ for $j = 1, \dots, \kappa_n$.

Note that since we are using normal coordinates centered at x_* then $|\xi|_{g_0(x_*)}^2 = \sum_i \xi_i^2$, and remark that the polynomials $q_j(\xi)$ form a basis in the space of spherical harmonics of degree two on the sphere $S_{x_*}^{n-1} = \{\xi \in T_{x_*}^* M : |\xi|_{g_0(x_*)} = 1\}$ with the round metric $g_{S_{x_*}^{n-1}}$. In the proof we shall use a computation showing that the round metric on $S_{x_*}^{n-1}$ is an extremal for the second eigenvalue of the Laplacian in the space of nearby Riemannian metrics. We refer to [7], [12], and [14] and references therein for a description of general theory of such metrics.

Below we summarize several well-known facts about extremal metrics. Let g_0 be an extremal metric on a compact d -dimensional manifold N . Then

- a) if (N, g_0) is a homogeneous space (e.g. a round S^{n-1}), then the metric g_0 is extremal for *all* eigenvalues of the Laplacian Δ_{g_0} ;
- b) if g_0 is extremal for an eigenvalue λ (of multiplicity m), and $\varphi_1, \dots, \varphi_m$ form an orthonormal basis of the corresponding eigenspace E_λ , then

$$\sum_{j=1}^m d\varphi_j \otimes d\varphi_j = (\lambda/2)g_0. \tag{52}$$

We remark that one can show that $m > d$ in (52).

Claim 1. Assume that $\{\varphi_1, \dots, \varphi_m\}$ satisfy (52). Then the (Jacobian) matrix

$$\frac{\partial(\varphi_1, \dots, \varphi_m)}{\partial(x_1, \dots, x_d)}$$

has the maximal possible rank d .

Proof. Assume that the rank is less than d . Then the quadratic form $\sum_{j=1}^m d\varphi_j \otimes d\varphi_j$ cannot be positive-definite; however by assumption it is proportional to the positive-definite Riemannian metric g_0 on N . Contradiction finishes the proof. \square

The existence of admissible perturbations will follow from Claim 1.

We shall show that a metric perturbation is admissible if it is of the form

$$g_u^{-1}(x) = g_0^{-1}(x) + \sum_{j=1}^{\kappa_n} u_j h_j(x), \quad (53)$$

where $h_j(x)(\xi, \xi)$ are homogeneous of degree 2 in the ξ variables and are required to satisfy the following conditions:

- (a) $h_j(x_*)(\xi, \xi) = q_j(\xi)$, $1 \leq j \leq \kappa_n$;
- (b) h_j are C^2 tensors.

Proposition 7. *Let a perturbation of g_u of g_0 have the form (53) with h_j satisfying conditions (a) and (b). Then there exist $\varepsilon > 0$ and $\delta > 0$ such that for all $u \in B^n(\varepsilon)$ the perturbation g_u satisfies part (A) of the admissibility condition at x for $x \in B(x_*, \delta)$.*

Proof. Clearly, $\partial_{u_j}(|\xi|_{g_u(x_*)}) = q_j(\xi)$, and hence the j -th column (say) of the mixed hessian matrix $d_\xi d_u p_u(x_*, \xi)$ corresponds to the gradient $d_\xi q_j(\xi)$.

Now, since the sphere $S_{x_*}^{n-1}$ is a homogeneous space, the round metric $g_{S_{x_*}^{n-1}}$ is a critical metric for the corresponding eigenvalue functional $g \mapsto \lambda(g) \cdot \text{Vol}(g)^{2/n}$, where λ denotes the second positive eigenvalue (without multiplicity) of the Laplacian.

By (52), see [7], [12], and [14], the L^2 -normalized basis of the eigenspace $E(\lambda)$ (which can be chosen as $\{q_1(\xi), \dots, q_{\kappa_n}(\xi)\}$ in our case) satisfies

$$\sum_{j=1}^{\kappa_n} d_\xi q_j \otimes d_\xi q_j = c \lambda g_{S_{x_*}^{n-1}}, \quad c \neq 0.$$

By Claim 1, the subspace spanned by $d_\xi q_1(\xi), \dots, d_\xi q_{\kappa_n}(\xi)$ has the full dimension $n - 1$ in $T_\xi^*(S_{x_*}^{n-1})$ at any point $\xi \in S_{x_*}^{n-1}$. This shows that $\{d_\xi q_j(\xi) : 1 \leq j \leq \kappa_n\}$ span the full $T_\xi^*(S_{x_*}^{n-1})$, which proves the required non-degeneracy condition.

Next, since h_j are C^2 in (x, ξ) by condition (b), the rank of $d_\xi d_u p_u(x, \xi)$ changes continuously in x and so is equal to $n - 1$ for $x \in B(x_*, \delta)$ on M for some $\delta > 0$. \square

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