

## Hitchin’s connection in metaplectic quantization

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**Abstract.** We give a differential geometric construction of a connection, which we call the Hitchin connection, in the bundle of quantum Hilbert spaces arising from metaplectically corrected geometric quantization of a prequantizable, symplectic manifold, endowed with a rigid family of Kähler structures, all of which give vanishing first Dolbeault cohomology groups.

This generalizes work of both Hitchin, Scheinost and Schottenloher, and Andersen, since our construction does not need that the first Chern class is proportional to the class of the symplectic form, nor do we need compactness of the symplectic manifold in question.

Furthermore, when we are in a setting similar to the moduli space, we give an explicit formula and show that this connection agrees with previous constructions.

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## 1. Introduction

Hitchin constructed in [20] a connection over Teichmüller space. This Hitchin connection is a connection in the bundle obtained from geometric quantization, with respect to the family of Kähler structures parametrized by Teichmüller space, of the moduli spaces of flat  $SU(n)$ -connections on a closed oriented surface. The significance of this connection is its relation to  $(2 + 1)$ -dimensional Reshetikhin–Turaev TQFT; see [27] and [28]. In fact, this geometric construction of these TQFT’s was proposed by Witten in [31], where he derived, via the Hamiltonian approach to quantum Chern–Simons theory, that the geometric quantization of the moduli spaces of flat connections should give the 2-dimensional part of the theory. Further, he proposed an alternative construction of the 2-dimensional part of the theory via WZW-conformal field theory. This theory has been studied intensively. In particular, the work of Tsuchiya, Ueno, and Yamada in [30] provided the major geometric constructions and results needed. In [13], their results was used to show that the category of integrable highest weight modules of level  $k$  for the affine Lie algebra associated to any simple Lie algebra is a modular tensor category. Further in [13] this result is combined with the work of Kazhdan and Lusztig, [21], [22], and [23], and the work of Finkelberg [18] to argue that this category is isomorphic to the modular tensor category associated to the corresponding quantum group, from which Reshetikhin and Turaev constructed their TQFT. Unfortunately, these results do not allow one to conclude the validity of the geometric constructions of the 2-dimensional part of the TQFT proposed by Witten. However, in joint work between Andersen and Ueno, [10], [9], [7], and [8], they give a proof, based mainly on the results of [30], that the TUY-construction of the WZW-conformal field theory after twist by a fractional power of an abelian theory, satisfies all the axioms of a modular functor. Furthermore, they have proved that the full  $(2 + 1)$ -dimensional TQFT that results from this is isomorphic to the one constructed by Blanchet, Habegger, Masbaum, and Vogel via skein theory; see [16] and [15]. Combining this with the theorem of Laszlo [26], which identifies (projectively) the representations of the mapping class groups obtained from the geometric quantization of the moduli space of flat connections with the ones obtained from the TUY-constructions, one gets a proof of the validity of the construction proposed by Witten in [31].

In [12], Axelrod, Della Pietra, and Witten gave a differential geometric construction of the Hitchin connection by using a method of symplectic reduction from the infinite-dimensional space of all  $SU(n)$ -connections. In [4] Andersen constructed the Hitchin connection in a more general setting. A corollary of the results in [4] is that the connection constructed by Axelrod, Della Pietra, and Witten in [12] is the same as Hitchin’s connection constructed in [20].

In this paper, we extend the setting from [4], in which we can construct the Hitchin connection, to include metaplectic quantization. Let us describe this setting. Consider a  $2m$ -dimensional symplectic manifold  $(M, \omega)$ , and assume that there exists a *prequantum line bundle*  $(\mathcal{L}, h, \nabla)$ , where  $\mathcal{L}$  is a complex line bundle with Hermitian

structure  $h$  and compatible connection  $\nabla$  with curvature  $-i\omega$ . In prequantization, one considers the space  $C^\infty(M, \mathcal{L}^k)$  of sections of the  $k$ -th tensor power of  $\mathcal{L}$ . This, however, does not produce a satisfying quantization, since there are in some sense too many sections. By studying simple examples from quantum mechanics, such as the harmonic oscillator, it is clear that prequantization produces wave functions which depend on both position and momentum coordinates, where they should only depend on the former.

A standard way of resolving this problem is by choosing a polarization on  $M$ , which is a certain type of integrable Lagrangian subbundle of the (complexified) tangent bundle, and then consider the subspace of polarized sections of  $C^\infty(M, \mathcal{L}^k)$ , i.e. sections which are covariant constant along the directions of the polarization; see [32]. One of the major effects of this resolution is that the quantization now depends on the auxiliary choice of polarization, which, from a physical point of view, it should not.

To study this dependence, we will focus attention on Kähler polarizations, for which the quantization procedure is rather well behaved. If we endow  $(M, \omega)$  with a Kähler structure  $J$ , we get a Kähler polarization from the splitting  $TM_{\mathbb{C}} = T \oplus \bar{T}$  given by the  $i$  and  $-i$  eigenspaces of  $J$ . By composing the connection  $\nabla$  with the projection onto  $\bar{T}$ , we get a  $\bar{\partial}$ -operator on  $C^\infty(M, \mathcal{L}^k)$ . The holomorphic sections with respect to this  $\bar{\partial}$ -operator is exactly the subspace of sections which are covariant constant along the directions of  $\bar{T}$ .

From a physical perspective, geometric quantization is still a little too crude. On basic examples from quantum mechanics, the procedure yields an energy spectrum which is off by a small shift; see [32].

To counter this effect, the notion of *metaplectic correction* must be employed. For a Kähler polarization this involves a choice of square root (if it exists)  $\delta_J$  of the canonical line bundle  $K_J = \wedge^m T^*$ , and then the metaplectically corrected quantum phase spaces are the holomorphic sections of  $\mathcal{L}^k \otimes \delta_J$ .

Suppose we have a family of Kähler structures parametrized by a manifold  $\mathcal{T}$ . We then consider the fibration  $H^{(k)}$  over  $\mathcal{T}$  with the quantum phase space for the given Kähler structure as the fiber. To do this, however, one needs make consistent choices of square roots of the canonical line bundle for *all* Kähler structures in the family at the same time. Once this is achieved, we seek to establish that  $H^{(k)}$  is a vector bundle over  $\mathcal{T}$  and find a (projectively) flat connection in  $H^{(k)}$  in order to obtain a quantization, independent of the choice of Kähler structure, given as the covariant constant sections of  $H^{(k)}$  over  $\mathcal{T}$ . Such a connection, when given by global differential operators, will be called a *Hitchin connection*. Let us now expand on this in details.

Denote by  $J : \mathcal{T} \rightarrow C^\infty(M, \text{End}(TM_{\mathbb{C}}))$  the parametrization of Kähler structures. Along any vector field  $V$  on  $\mathcal{T}$ , we can differentiate  $J$  to get a map  $V[J] : \mathcal{T} \rightarrow C^\infty(M, \text{End}(TM_{\mathbb{C}}))$ .

Define  $\tilde{G}(V) \in C^\infty(M, S^2(TM_{\mathbb{C}}))$  by

$$V[J] = \tilde{G}(V)\omega.$$

Letting  $T_\sigma$  denote the holomorphic tangent bundle on  $(M, J_\sigma)$  for any  $\sigma \in \mathcal{T}$ , we can further define  $G(V) \in C^\infty(M, S^2(T))$  by the equation

$$\tilde{G}(V) = G(V) + \bar{G}(V),$$

for all vector fields  $V$  on  $\mathcal{T}$ . We shall assume that the family  $J$  is *rigid* (cf. Definition 5.3), meaning that  $G(V)_\sigma$  is a holomorphic section of  $S^2(T_\sigma)$ . This assumption is rather restrictive, but see Appendix A for examples.

In case the second Stiefel–Whitney class vanishes, we can choose a metaplectic structure on the symplectic manifold  $(M, \omega)$  (see Section 2), which gives rise to a choice of a square root  $\delta_\sigma$  of the canonical line bundle  $K_\sigma \rightarrow M_\sigma$ , varying smoothly in the parameter  $\sigma \in \mathcal{T}$ .

The Levi-Civita connection  $\tilde{\nabla}_\sigma$ , corresponding to the Kähler metric on  $M_\sigma$ , induces a connection in the line bundle  $\delta_\sigma \rightarrow M_\sigma$ , and thus we get a connection  $\nabla_\sigma$  in  $\mathcal{L}^k \otimes \delta_\sigma \rightarrow M_\sigma$ , giving this bundle the structure of a holomorphic line bundle.

For every  $\sigma \in \mathcal{T}$ , we have the infinite-dimensional vector space  $\mathcal{H}_\sigma^{(k)} = C^\infty(M, \mathcal{L}^k \otimes \delta_\sigma)$ , and we consider the subspace of holomorphic sections

$$H_\sigma^{(k)} = H^0(M_\sigma, \mathcal{L}^k \otimes \delta_\sigma) = \{s \in C^\infty(M, \mathcal{L}^k \otimes \delta_\sigma) \mid \nabla_\sigma^{0,1}s = 0\}.$$

It is clear that the spaces  $\mathcal{H}_\sigma^{(k)}$  form a smooth vector bundle  $\mathcal{H}^{(k)} \rightarrow \mathcal{T}$ , but it is not clear that the subspaces  $H_\sigma^{(k)}$  form a smooth subbundle  $\mathcal{H}^{(k)}$ . However, it is a corollary of our construction that, under the assumptions stated in Theorem 1.2, the spaces  $H_\sigma^{(k)}$  do indeed form a smooth bundle over  $\mathcal{T}$  and that  $H^{(k)} \rightarrow \mathcal{T}$  is a smooth subbundle of  $\mathcal{H}^{(k)}$ . See Remark 5.2 below, where we establish this fact.

The spaces  $C^\infty(M, T_\sigma)$ , of smooth sections of the holomorphic tangent bundle, form a bundle  $C^\infty(M, T) \rightarrow \mathcal{T}$ , in which we have a connection  $\hat{\nabla}^T$  defined by the formula

$$\hat{\nabla}_V^T \zeta = \pi^{1,0} V[\zeta],$$

where  $\pi_\sigma^{1,0}: TM_{\mathbb{C}} \rightarrow T_\sigma$  is the projection, and  $V[\zeta]$  denotes differentiation in the trivial bundle  $\mathcal{T} \times C^\infty(M, TM_{\mathbb{C}})$ . This induces a connection in  $C^\infty(M, \delta) \rightarrow \mathcal{T}$ , and with the help of the trivial connection in  $\mathcal{T} \times C^\infty(M, \mathcal{L}^k)$  this induces a connection  $\hat{\nabla}^r$  in  $\mathcal{H}^{(k)} \rightarrow \mathcal{T}$ , which we call the reference connection (see Section 3 for further details).

**Definition 1.1.** A *Hitchin connection* is a connection in the bundle  $\mathcal{H}^{(k)} \rightarrow \mathcal{T}$ , which preserves  $H^{(k)}$  and has the form

$$\nabla_V = \hat{\nabla}_V^r + u(V),$$

where  $\widehat{\nabla}^r$  is the reference connection and  $u$  is a 1-form on  $\mathcal{T}$  with values in differential operators on  $C^\infty(M, \mathcal{L}^k \otimes \delta)$ .

In our construction of a Hitchin connection,  $u(V)$  will turn out to be a second-order operator with leading order term given by the operator  $\Delta_{G(V)} = \text{Tr } \nabla G(V) \nabla$ . More precisely, we will prove the following theorem.

**Theorem 1.2.** *Let  $(M, \omega)$  be a prequantizable, symplectic manifold with vanishing second Stiefel–Whitney class. Further, let  $J$  be a rigid family of Kähler structures on  $M$ , all satisfying  $H^{0,1}(M) = 0$ . Then, there exists a 1-form  $\beta \in \Omega^1(\mathcal{T}, C^\infty(M))$  such that the connection  $\nabla$ , in the bundle  $\mathcal{H}^{(k)}$ , given by*

$$\nabla_V = \widehat{\nabla}_V^r + \frac{1}{4k}(\Delta_{G(V)} + \beta(V)),$$

*is a Hitchin connection. The connection is unique up to addition of the pullback of an ordinary 1-form on  $\mathcal{T}$ .*

We can consider rigidity of the family  $J$  as a condition on the vector fields  $V$ , rather than considering it as a condition on the family of Kähler structures. We will then get a partial connection which is defined in these rigid directions. See also Appendix A.

The theorem above can be applied to the moduli space flat  $SU(n)$  connections on a Riemann surface, and the work in this paper has been greatly inspired by previous work in this setting. As mentioned above, Witten argued [31] that geometric quantization of the moduli space would produce the 2-dimensional part of a topological quantum field theory in  $(2 + 1)$ -dimensions. The Teichmüller space of the surface parametrizes a (rigid) family of Kähler structures on the moduli space and Hitchin constructed [20] a projectively flat connection in the bundle of quantum spaces over Teichmüller space. This connection was also constructed independently by Axelrod, Della Pietra, and Witten in [12]. Hitchin’s construction was generalized by Scheinost and Schottenloher [29] to metaplectic quantization using similar methods to those applied in [20]. Later, the first author [4] generalized Hitchin’s construction to symplectic manifolds satisfying certain conditions also satisfied by the moduli space, e.g. compactness and a relationship between the symplectic structure and the first Chern class. This generalization used only differential geometric methods and produced an explicit formula in terms of Ricci potentials.

This paper uses similar techniques to produce a Hitchin connection in the metaplectic case with fewer assumptions. In particular, we do not need any compactness assumption and we no longer need the first Chern class of the symplectic manifold to be even. Also, we show that whenever the assumptions of [4] are met, we can give a completely explicit formula for the Hitchin connection (that is for  $\beta$  above) and the connection agrees with the one from [4]. We stress the fact that this is done in a purely differential geometric fashion, whereas former constructions used both algebraic geometry and the Index Theorem.

Let us be more precise about the way the Hitchin connection in metaplectic quantization, constructed in this paper, relates to the one in the non-corrected geometric quantization, constructed in [4].

Assume that  $M$  is compact with  $H^1(M, \mathbb{R}) = 0$ , and that  $J$  is a *holomorphic family* (in the sense of Definition 6.1) of Kähler structures parametrized by a *complex manifold*  $\mathcal{T}$ . That  $J$  is a holomorphic family is equivalent to the fact that  $J$  gives rise to a complex structure on  $\mathcal{T} \times M$ .

We then consider the non-corrected setting of geometric quantization of  $(M, \omega)$ , namely

$$\tilde{H}_\sigma^{(k)} = H^0(M_\sigma, \mathcal{L}^k) = \{s \in C^\infty(M, \mathcal{L}^k) \mid (\nabla^{\mathcal{L}})_\sigma^{0,1} s = 0\}.$$

Under the additional assumption that the real first Chern class of  $(M, \omega)$  is given by

$$c_1(M, \omega) = n \left[ \frac{\omega}{2\pi} \right], \quad n \in \mathbb{Z}, \quad (1)$$

there is a construction in [4] of a Hitchin connection in the trivial bundle  $\mathcal{T} \times C^\infty(M, \mathcal{L}^k)$  over  $\mathcal{T}$ , which preserves the subbundle  $\tilde{H}^{(k)} \rightarrow \mathcal{T}$ , extending Hitchin's connection constructed in [20]. Now, when (1) is satisfied, we are able to give an explicit formula for the 1-form  $\beta$ . Moreover the following theorem says, that if we choose the right normalization of the Ricci potentials, we can compare the Hitchin connection given by Theorem 1.2 with the one constructed in [4] and in fact they agree.

**Theorem 1.3.** *Let  $(M, \omega)$  be a compact, prequantizable symplectic manifold with vanishing second Stiefel–Whitney class, and  $H^1(M, \mathbb{R}) = 0$ . Further, let  $J$  be a rigid, holomorphic family of Kähler structures on  $M$  parametrized by a complex manifold  $\mathcal{T}$ . Assume that the first Chern class of  $(M, \omega)$  is divisible by an integer  $n$  and that its image in  $H^2(M, \mathbb{R})$  satisfies*

$$c_1(M, \omega) = n \left[ \frac{\omega}{2\pi} \right].$$

*Then, around every point  $\sigma \in \mathcal{T}$ , there exists an open neighborhood  $U$ , a local smooth family  $\tilde{F}$  of Ricci potentials on  $M$  over  $U$  and an isomorphism of vector bundles over  $U$*

$$\varphi: \tilde{H}^{(k-n/2)}|_U \longrightarrow H^{(k)}|_U,$$

*such that*

$$\varphi^* \nabla = \tilde{\nabla},$$

*where  $\varphi^* \nabla$  is the pullback of the Hitchin connection given by Theorem 1.2, and  $\tilde{\nabla}$  is the Hitchin connection in  $\tilde{H}^{(k-n/2)}$  constructed in [4], both of which are expressed in terms of  $\tilde{F}$ .*

We plan to address the computation of the curvature and removal of the rigidity condition in a forthcoming publication. Also, we find it interesting to analyze the relation between the connection constructed in this paper and the “ $L^2$ -induced” constructed by Charles in [17]. Further, we intend to consider this new construction in the moduli space setting, in which Hitchin originally constructed his connection, and which was applied by Andersen in [1].

We find it very interesting to explore the role of Toeplitz operators and their relation to the Hitchin connection constructed in the general setting considered in this paper. In particular, it would be interesting to understand if the results in [4], [5], [2], and [6] can be generalized to this setting. For the first steps in this direction; see also [3].

This paper is organized as follows. In Section 2, we introduce the notion of a metaplectic structure on a symplectic manifold and the construction of geometric quantization with metaplectic correction. Section 3 is devoted to the reference connection and Section 4 to the calculation of its curvature. In Section 5, we derive an equation that the Hitchin connection should satisfy. Then, we give a solution to this equation and prove Theorem 1.2. Finally, in Section 6, we study the relation between our construction and the construction of [4] in the non-corrected case, culminating with a proof of Theorem 1.3. In Appendix A we construct examples of rigid families of complex structures on symplectic manifolds.

## 2. Metaplectic structure and quantization

Consider an almost complex structure  $J$  on  $M$ , which is compatible with the symplectic structure in the sense that

$$g_J(X, Y) = \omega(X, JY)$$

defines a Riemannian metric on  $M$ . We shall denote the resulting Riemannian manifold by  $M_J$ .

The almost complex structure  $J$  induces a splitting

$$TM_{\mathbb{C}} = T_J \oplus \bar{T}_J$$

of the complexified tangent bundle into the eigenspaces of  $J$  corresponding to the eigenvalues  $i$  and  $-i$  respectively. This splitting is explicitly given by the projections onto each summand

$$\begin{aligned} \pi_J^{1,0} &= \frac{1}{2}(\text{Id} - iJ), & T_J &= \text{Im}(\pi_J^{1,0}), \\ \pi_J^{0,1} &= \frac{1}{2}(\text{Id} + iJ), & \bar{T}_J &= \text{Im}(\pi_J^{0,1}). \end{aligned} \tag{2}$$

The fact that  $T_J$  and  $\bar{T}_J$  are the eigenspaces of  $J$ , corresponding to the eigenvalues  $i$ , respectively  $-i$ , is easily verified from these formulas. Very often we shall use the notation  $X' = \pi_J^{1,0} X$  and  $X'' = \pi_J^{0,1} X$  for vector fields  $X$  on  $M$ .

Tensors, such as the symplectic form and associated metric, are extended complex linearly to  $TM_{\mathbb{C}}$ .

We recall that the first Chern class  $c_1(M_J)$  is equal to minus the first Chern class of the canonical line bundle

$$K_J = \bigwedge^m T_J^*.$$

By integrality,  $c_1(M_J)$  is independent of  $J$  since the space of compatible almost complex structures on  $(M, \omega)$  is contractible. Thus, the first Chern class is an invariant of the symplectic manifold rather than the almost complex one.

Let us assume that the second Stiefel–Whitney class  $w_2(M)$  vanishes. Since the reduction modulo 2 of the first Chern class, that is the image of  $c_1(M)$  under the map  $H^2(M, \mathbb{Z}) \rightarrow H^2(M, \mathbb{Z}_2)$ , is equal to the second Stiefel–Whitney class, this implies that the first Chern class of  $M$  is even. Thus the first Chern class of  $K_J$  is even, which is equivalent to the existence of a square root  $\delta_J$  of  $K_J$ . We shall see later that the choice of such a  $\delta_J$  determines a square root of the canonical line bundle for every other almost complex structure on  $M$ .

The metric on  $M_J$  gives rise to the Levi-Civita connection  $\tilde{\nabla}_J$ . As usual we get an induced metric and compatible connection in all tensor bundles over  $M$ , and we shall denote all of these by  $g_J$  and  $\tilde{\nabla}_J$  as well.

The metric also induces a Hermitian structure  $h_J^T$  in  $T_J$  given by

$$h_J^T(X, Y) = g_J(X, \bar{Y}),$$

for any vectors  $X$  and  $Y$  in  $T_J$ . If we further assume that  $J$  is parallel, with respect to the Levi-Civita connection  $\tilde{\nabla}_J$ , then  $J$  must be integrable and  $M_J$  Kähler. In this case  $\tilde{\nabla}_J$  preserves the holomorphic tangent bundle  $T_J$  inducing a connection  $\nabla_J^T$  compatible with  $h_J^T$ . These in turn induce a Hermitian structure  $h_J^K$  and compatible connection  $\nabla_J^K$  in the canonical line bundle  $K_J$ .

The Ricci tensor  $r_J$  on  $M_J$  is given by the following trace of the Kähler curvature

$$r_J(X, Y) = \text{Tr}(Z \mapsto \tilde{R}(Z, X)Y),$$

and the Ricci form  $\rho_J$  is the associated (1,1)-form given by

$$\rho_J(X, Y) = r(JX, Y).$$

We recall for future use that the canonical line bundle  $K_J$  has curvature  $i\rho_J$ .

Finally  $h_J^K$  and  $\nabla_J^K$  induce a Hermitian structure  $h_J^\delta$  and compatible connection  $\nabla_J^\delta$  in the line bundle  $\delta_J$ .

**Definition 2.1.** A *prequantum line bundle* over the symplectic manifold  $(M, \omega)$  is a Hermitian line bundle  $\mathcal{L}$  with a compatible connection  $\nabla^{\mathcal{L}}$  of curvature

$$R_{\nabla^{\mathcal{L}}} = -i\omega,$$



where  $R_{\nabla}(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}$ . Such a triple  $(\mathcal{L}, h^{\mathcal{L}}, \nabla^{\mathcal{L}})$  is denoted a prequantum line bundle, and we say that the symplectic manifold is *prequantizable* if it admits such a bundle.

Evidently, a necessary condition for the existence of a prequantum line bundle is that the class  $[\frac{\omega}{2\pi}]$  in  $H^2(M, \mathbb{R})$  is integral, and in fact this is also sufficient. Moreover, inequivalent choices of prequantum line bundles are parametrized by the first cohomology  $H^1(M, U(1))$  with coefficients in the circle group  $U(1) \subset \mathbb{C}$ ; see for instance [32]. We shall assume that  $M$  is prequantizable, and fix a prequantum line bundle  $(\mathcal{L}, h, \nabla^{\mathcal{L}})$

Now  $h^{\mathcal{L}}$  and  $h_J^{\delta}$  induce a Hermitian structure  $h_J$  in the line bundle  $\mathcal{L}^k \otimes \delta_J$ , and we have a compatible connection  $\nabla_J$ , induced by  $\nabla^{\mathcal{L}}$  and  $\nabla_J^{\delta}$ . Since  $\mathcal{L}^k \otimes \delta_J$  has curvature  $-ik\omega + \frac{i}{2}\rho_J$ , which is of type (1,1), the operator

$$\nabla_J^{0,1} = \pi_J^{0,1} \nabla_J$$

defines a  $\bar{\partial}$ -operator in  $\mathcal{L}^k \otimes \delta_J$ , making this a holomorphic line bundle over  $M_J$ ; see e.g. [11]. If we consider the space  $\mathcal{H}_J^{(k)} = C^\infty(M, \mathcal{L}^k \otimes \delta_J)$  of smooth sections, then the operator  $\nabla_J^{0,1}$  gives rise to the subspace  $H_J^{(k)}$  of holomorphic sections

$$H_J^{(k)} = H^0(M_J, \mathcal{L}^k \otimes \delta_J) = \{s \in C^\infty(M_J, \mathcal{L}^k \otimes \delta_J) \mid \nabla_J^{0,1} s = 0\}.$$

We can define a Hermitian inner product on this space by

$$\langle s_1, s_2 \rangle = \frac{1}{m!} \int_M h_J(s_1, s_2) \omega^m,$$

and if we consider the space of square integrable functions we obtain a Hilbert space. This is the Hilbert space resulting from the half-form corrected geometric quantization of the Kähler manifold  $M_J$ .

We will construct a connection in  $\mathcal{H}^{(k)}$  and prove that under certain conditions this connection preserves the infinitesimal condition for being contained in the subspaces  $H_J^{(k)}$ . From this we conclude, that the spaces  $H_J^{(k)}$  form a vector bundle over a manifold that parametrizes choices of  $J$ , and the fibers  $H_J^{(k)}$  are related using parallel translation of the induced connection, which we will call the Hitchin connection.

To be able to do this, we should pay closer attention to the way we choose the half-form bundle  $\delta_J$ . Clearly, there is more than one choice of a square root of  $K_J$  (when it exists), and we would like to choose  $\delta_J$  in a unified way for different  $J$ . This is where the notion of a metaplectic structure comes into the picture. We will follow the approach of [32] and do not formulate it in terms of the metaplectic group.

Consider the positive Lagrangian Grassmannian  $L^+M$  consisting of pairs  $(p, J_p)$ , where  $p \in M$  and  $J_p$  is a compatible almost complex structure on the tangent space

$T_p M$ . This space has the structure of a smooth bundle over  $M$ , with the obvious projection, and with sections corresponding precisely to almost complex structures on  $M$ .

At each point  $(p, J_p) \in L^+M$ , we can consider the 1-dimensional space  $K_{J_p} = \bigwedge^m T_{J_p}^*$ . These form a smooth bundle  $K$  over  $L^+M$ , and the pullback by a section of  $L^+M$  yields the canonical line bundle associated to the almost complex structure on  $M$  given by the section.

We want to find a square root  $\delta \rightarrow L^+M$  of the bundle  $K \rightarrow L^+M$ . Such a square root is called a metaplectic structure on  $M$ . Since  $L^+M$  has contractible fibers, we can find local trivializations of  $K$  with constant transition functions along the fibers. The construction of a metaplectic structure on  $M$  amounts to choosing square roots of these transition functions in such a way that they still satisfy the cocycle conditions. But since the transition functions are constant along the fibers, we only have to choose a square root at a single point in each fiber. In other words, a square root  $\delta_J$  of  $K_J$ , for a single almost complex structure  $J$  on  $M$ , determines a metaplectic structure. We summarize this in a proposition.

**Proposition 2.2.** *Let  $M$  be a manifold with vanishing second Stiefel–Whitney class, and let  $\omega$  be any symplectic structure on  $M$ . Then  $(M, \omega)$  admits a metaplectic structure  $\delta \rightarrow L^+M$ .*

For the rest of this paper, we shall assume that  $M$  satisfies the conditions of this proposition, and fix a metaplectic structure  $\delta$ . In this way, for every almost complex structure  $J$  on  $M$ , viewed as a section of  $L^+M$ , we have a canonical choice of square root of the canonical line bundle, given as the pullback of  $\delta$  by  $J$ .

### 3. The reference connection

Returning to the setup of the introduction, consider a manifold  $\mathcal{T}$ , and assume that we have a smooth family  $J: \mathcal{T} \rightarrow C^\infty(M, \text{End}(TM))$  of Kähler structures on  $M$ , parametrized by  $\mathcal{T}$ . More precisely,  $J$  is a smooth section of the pullback bundle  $\pi_M^* \text{End}(TM) \rightarrow \mathcal{T} \times M$ , where  $\pi_M: \mathcal{T} \times M \rightarrow M$  is the projection, such that for every  $\sigma \in \mathcal{T}$ , the endomorphism  $J_\sigma$  defines a complex structure on  $M$ , turning this into a Kähler manifold  $M_\sigma$ . As in the previous section, the Kähler metric is given by

$$g_\sigma(X, Y) = \omega(X, J_\sigma Y),$$

and  $J_\sigma$  induces a splitting  $TM_{\mathbb{C}} = T_\sigma \oplus \bar{T}_\sigma$ . Also, we write  $X'_\sigma = \pi_\sigma^{1,0} X$  and  $X''_\sigma = \pi_\sigma^{0,1} X$  for any vector field  $X$  on  $M$ .

Viewing the family  $J$  as a map  $\mathcal{T} \times M \rightarrow L^+M$ , we get a smooth bundle  $\delta \rightarrow \mathcal{T} \times M$ , by pulling back the metaplectic structure on  $M$ . For any  $\sigma \in \mathcal{T}$ , the

restriction

$$\delta_\sigma = \delta|_{\{\sigma\} \times M} \rightarrow M$$

is a square root of the canonical line bundle  $K_\sigma$  on  $M_\sigma$ . Moreover the Hermitian structure  $h_\sigma^\delta = h_{J_\sigma}^\delta$  in  $\delta_\sigma$  gives rise to a Hermitian structure  $h^\delta$  on  $\delta$ . Let

$$pi_M: \mathcal{T} \times M \longrightarrow M$$

denote the projection and define

$$\hat{\mathcal{L}} = \pi_M^* \mathcal{L} = \mathcal{T} \times \mathcal{L},$$

with Hermitian metric  $\hat{h}^{\mathcal{L}} = \pi_M^* h^{\mathcal{L}}$ . When objects are extended to the product  $\mathcal{T} \times M$ , we shall often use a hat to indicate that we are dealing with the extended object. Then,  $\hat{\mathcal{L}} \otimes \delta$  becomes a smooth line bundle over  $\mathcal{T} \times M$  with Hermitian metric  $\hat{h}$  induced by  $\hat{h}^{\mathcal{L}}$  and  $h^\delta$ .

As in the previous section, we consider the space  $\mathcal{H}_\sigma^{(k)} = C^\infty(M_\sigma, \mathcal{L}^k \otimes \delta_\sigma)$ , in which the connection  $\nabla_{J_\sigma}$ , which we shall denote by  $\nabla_\sigma$ , gives rise to the subspace of holomorphic sections

$$H_\sigma^{(k)} = H^0(M_\sigma, \mathcal{L}^k \otimes \delta_\sigma) = \{s \in \mathcal{H}_\sigma^{(k)} \mid \nabla_\sigma^{0,1} s = 0\}.$$

In fact the spaces  $\mathcal{H}_\sigma^{(k)}$  form a smooth vector bundle  $\mathcal{H}^{(k)}$  over  $\mathcal{T}$ . We will construct a connection in  $\mathcal{H}^{(k)}$  which preserves the spaces  $H_\sigma^{(k)}$ , thereby proving that these form a smooth subbundle  $H^{(k)}$  of  $\mathcal{H}^{(k)}$ , and at the same time giving a connection in  $H^{(k)}$ .

First we define a connection  $\hat{\nabla}^{\mathcal{L}}$  in  $\hat{\mathcal{L}}$  simply by extending  $\nabla^{\mathcal{L}}$  using the trivial connection in directions tangent to  $\mathcal{T}$ , i.e.  $\hat{\nabla}^{\mathcal{L}}$  is the pullback connection in the pullback bundle  $\hat{\mathcal{L}}$ . Concretely, if  $X$  is a vector field on  $\mathcal{T} \times M$ , which is tangent to  $M$ , and  $s$  is a section of  $\hat{\mathcal{L}}$ , then we define

$$(\hat{\nabla}_X^{\mathcal{L}} s)_{(\sigma,p)} = (\nabla_X^{\mathcal{L}} s)_p.$$

For any vector field  $V$  on  $\mathcal{T} \times M$ , which is tangent to  $\mathcal{T}$ , we have that

$$(\hat{\nabla}_V^{\mathcal{L}} s)_{(\sigma,p)} = V[s_p]_\sigma.$$

Here  $V[s_p]_\sigma$  denotes differentiation at  $\sigma \in \mathcal{T}$  along  $V$  of  $s_p$ , as a section of the trivial bundle  $\mathcal{T} \times \mathcal{L}_p$ .

Now  $\hat{\nabla}^{\mathcal{L}}$  is easily seen to be compatible with the Hermitian structure  $\hat{h}^{\mathcal{L}}$ , and for future reference we give the curvature, which is easily calculated.

**Lemma 3.1.** *The curvature of  $\widehat{\nabla}^{\mathcal{L}}$  is given by*

$$R_{\widehat{\nabla}^{\mathcal{L}}} = \pi_M^* R_{\nabla^{\mathcal{L}}} = -i \pi_M^* \omega,$$

where  $\pi_M: \mathcal{T} \times M \rightarrow M$  denotes the projection.

Next, we define a connection  $\widehat{\nabla}^T$  in the bundle  $T \rightarrow \mathcal{T} \times M$  in the following way. In the directions tangent to  $M$ , simply take  $\widehat{\nabla}^T$  to be the connection  $\nabla^T$  induced from the Levi-Civita connection. More explicitly we define, for any section  $Y$  of  $T$  and any vector  $X \in T_p M$ ,

$$(\widehat{\nabla}_X^T Y)_{(\sigma,p)} = ((\nabla_\sigma^T)_X Y_\sigma)_p, \tag{3}$$

where  $\nabla_\sigma^T$  denotes  $\nabla_{J_\sigma^T}$ . For the directions along  $\mathcal{T}$ , we let  $V \in T_\sigma \mathcal{T}$  be any vector on  $\mathcal{T}$  and define

$$(\widehat{\nabla}_V^T Y)_{(\sigma,p)} = \pi_\sigma^{1,0} V[Y_p]_\sigma, \tag{4}$$

for any section  $Y$  of  $T$ , where  $V[Y_p]$  denotes differentiation of  $Y_p$  in the trivial bundle  $\mathcal{T} \times T_p M_{\mathbb{C}}$ , and  $\pi_\sigma^{1,0}: \mathcal{T} \times TM_{\mathbb{C}} \rightarrow T_\sigma$  is the projection.

Now,  $\widehat{\nabla}^T$  induces a connection  $\widehat{\nabla}^K$  in  $K = \wedge^m T^*$ , which in turn induces a connection  $\widehat{\nabla}^\delta$  in the square root  $\delta$ . With the help of the connection  $\widehat{\nabla}^{\mathcal{L}}$ , this induces a connection  $\widehat{\nabla}^r$  in the line bundle  $\widehat{\mathcal{L}}^k \otimes \delta$ .

**Definition 3.2.** The connection

$$\widehat{\nabla}^r = (\widehat{\nabla}^{\mathcal{L}})^{\otimes k} \otimes \text{Id} + \text{Id} \otimes \widehat{\nabla}^\delta$$

in  $\widehat{\mathcal{L}}^k \otimes \delta \rightarrow \mathcal{T} \times M$  is called *the reference connection*.

Notice how the reference connection induces a connection in  $\mathcal{H}^{(k)} \rightarrow \mathcal{T}$ . Indeed, for any section  $s$  of  $\mathcal{H}^{(k)}$  (which is the same as a section of  $\widehat{\mathcal{L}}^k \otimes \delta$  over  $\mathcal{T} \times M$ ) and any vector field  $V$  tangent to  $\mathcal{T}$ , it is simply given by  $\widehat{\nabla}_V^r s$ . Moreover, if we restrict to a point  $\sigma \in \mathcal{T}$  and take  $X$  to be a vector field tangent to  $M$ , then  $(\widehat{\nabla}_X^r s)_\sigma = (\nabla_\sigma)_X s_\sigma$ , so the reference connection is a unified description of a connection in  $\mathcal{H}^{(k)}$  and the connections in the bundles  $\mathcal{L}^k \otimes \delta_\sigma \rightarrow M$ .

#### 4. Curvature of the reference connection

Later, we shall have need for the curvature of the reference connection, which is given by Propositions 4.1, 4.2, and 4.3 below.

**Proposition 4.1.** *For vector fields  $X$  and  $Y$  tangent to  $M$ , we have*

$$R_{\widehat{\nabla}^r}(X, Y) = -ik\omega(X, Y) + \frac{i}{2}\rho(X, Y), \tag{5}$$

where  $\rho_\sigma$  denotes the Ricci form on  $M_\sigma$ .

*Proof.* This follows immediately by the curvature of prequantum line bundles and the fact that the canonical line bundle  $K_\sigma$  over  $M_\sigma$  has curvature  $i\rho_\sigma$ .  $\square$

Before giving the curvature in the mixed directions, we introduce some more notation. Since the symplectic form is non-degenerate, it induces an isomorphism

$$i_\omega : TM_{\mathbb{C}} \longrightarrow TM_{\mathbb{C}}^*,$$

by contraction in the first entry. Moreover  $\omega$  is  $J$ -invariant, or equivalently of type (1,1), which implies that  $i_\omega$  interchanges types. Similarly the metric induces a type-interchanging isomorphism  $i_g : TM_{\mathbb{C}} \rightarrow TM_{\mathbb{C}}^*$ , and the two are related by  $i_g = -Ji_\omega$ .

For any vector field  $V$  tangent to  $\mathcal{T}$ , we can differentiate the family of complex structures in the direction of  $V$  and obtain

$$V[J] : \mathcal{T} \longrightarrow C^\infty(M, \text{End}(TM_{\mathbb{C}})).$$

By differentiation of the identity  $J^2 = -\text{Id}$ , we see that  $V[J]$  anticommutes with  $J$ . This in turn implies that  $V[J]_\sigma$  interchanges types on  $M_\sigma$ , whence it decomposes as

$$V[J]_\sigma = V[J]'_\sigma + V[J]''_\sigma,$$

where  $V[J]'_\sigma \in C^\infty(M, \bar{T}_\sigma^* \otimes T_\sigma)$  and  $V[J]''_\sigma \in C^\infty(M, T_\sigma^* \otimes \bar{T}_\sigma)$ .

Now define  $\tilde{G}(V) \in C^\infty(M, TM_{\mathbb{C}} \otimes TM_{\mathbb{C}})$  by the relation

$$V[J] = (\text{Id} \otimes i_\omega)(\tilde{G}(V))$$

for all vector fields  $V$ . We use the notation

$$\tilde{G}(V)\omega = (\text{Id} \otimes i_\omega)(\tilde{G}(V)).$$

The way to interpret this, is to contract the right contravariant part of  $\tilde{G}(V)$  with the left covariant part of  $\omega$ , as prescribed by  $(\text{Id} \otimes i_\omega)(\tilde{G}(V))$ . Now observe, that the combined types of  $V[J]$  and  $\omega$  yield a decomposition

$$\tilde{G}(V) = G(V) + \bar{G}(V), \tag{6}$$

for all real vector fields  $V$  on  $\mathcal{T}$ , where  $G(V)_\sigma \in C^\infty(M, T_\sigma \otimes T_\sigma)$  and  $\bar{G}(V)_\sigma \in C^\infty(M, \bar{T}_\sigma \otimes \bar{T}_\sigma)$ . Differentiating the relation  $g = \omega J$  along  $V$ , we have

$$V[g] = \omega V[J] = \omega \tilde{G}(V)\omega = -(i_\omega \otimes i_\omega)(\tilde{G}(V)). \tag{7}$$

Once again, notice how the notation  $\omega \tilde{G}(V)\omega$  is used to denote tracing the right covariant part of  $\omega$  with the left contravariant part of  $\tilde{G}(V)$ , as well as tracing the right contravariant part of  $\tilde{G}(V)$  with the left covariant part of  $\omega$ . Since  $g$  is symmetric, so is  $V[g]$ , which implies that  $G(V)_\sigma \in C^\infty(M, S^2(T_\sigma))$  and  $\bar{G}(V)_\sigma \in C^\infty(M, S^2(\bar{T}_\sigma))$ .

Also, we introduce a 2-form  $\theta$  on  $T$ , with values in  $C^\infty(M)$ . For any vector fields  $V$  and  $W$  on  $\mathcal{T}$  we define

$$\theta(V, W) = -\frac{i}{4} \text{Tr}[V[J], W[J]]\pi^{1,0}, \tag{8}$$

where the outer brackets denote the commutator. In other words, we restrict the commutator to the holomorphic tangent bundle and take the trace of this restriction. Evidently, this yields an anti-symmetric and real 2-form on  $\mathcal{T}$ .

By a small calculation, we obtain another useful formula for the connection  $\hat{\nabla}^T$  in the directions tangent to  $\mathcal{T}$ . Indeed, we have that

$$\hat{\nabla}_V^T Y = V[\pi^{1,0} Y] - V[\pi^{1,0}]Y = V[Y] + \frac{i}{2}V[J]Y, \tag{9}$$

for any section  $Y$  of  $T$ .

Now we are ready to calculate the curvature of the reference connection in the remaining directions. To do this, we recall the general fact, which was already implicitly used to find the curvature of the half-form bundle, that the curvature of  $\hat{\nabla}^\delta$  is given by

$$R_{\hat{\nabla}^\delta} = -\frac{1}{2} \text{Tr} R_{\hat{\nabla}^T}, \tag{10}$$

where we take the trace of the endomorphism part of  $R_{\hat{\nabla}^T} \in \Omega^2(T \times M, \text{End}(T))$ . The change of sign appears when we induce  $\hat{\nabla}^T$  in  $T^*$ , the trace appears when we induce in  $K = \bigwedge^m T^*$ , and the division by two appears when we induce in  $\delta$ . Then we have the following result.

**Proposition 4.2.** *For vector fields  $V$  and  $W$  tangent to  $\mathcal{T}$  we have*

$$R_{\hat{\nabla}^r}(V, W) = \frac{i}{2}\theta(V, W). \tag{11}$$

*Proof.* Take  $V$  and  $W$  to be pullbacks of vector fields on  $\mathcal{T}$  such that  $[V, W] = 0$ .

Then using (9), we find that

$$\begin{aligned} \widehat{\nabla}_V^T \widehat{\nabla}_W^T Y &= \widehat{\nabla}_V^T \left( W[Y] + \frac{i}{2} W[J] \right) \\ &= V W[Y] + \frac{i}{2} V W[J] Y + \frac{i}{2} W[J] V[Y] \\ &\quad + \frac{i}{2} V[J] W[Y] - \frac{1}{4} V[J] W[J] Y. \end{aligned}$$

Using that  $V$  and  $W$  commute we get

$$\begin{aligned} R_{\widehat{\nabla}^T}(V, W)Y &= \widehat{\nabla}_V^T \widehat{\nabla}_W^T Y - \widehat{\nabla}_W^T \widehat{\nabla}_V^T Y \\ &= -\frac{1}{4}(V[J]W[J] - W[J]V[J])Y \\ &= -\frac{1}{4}[V[J], W[J]]Y, \end{aligned}$$

and so by (10) we get

$$R_{\widehat{\nabla}^T}(V, W) = R_{\widehat{\nabla}^{\mathcal{E}}}^{(k)}(V, W) - \frac{1}{2} \text{Tr} R_{\widehat{\nabla}^T}(V, W) = \frac{i}{2} \theta(V, W),$$

as desired, since  $R_{\widehat{\nabla}^{\mathcal{E}}}(V, W) = 0$ . □

Now, we calculate the curvature of the reference connection in the mixed directions.

**Proposition 4.3.** *For vector fields  $V$  and  $X$ , tangent to  $\mathcal{T}$  and  $M$  respectively, we have*

$$R_{\widehat{\nabla}^T}(V, X) = \frac{i}{4} \text{Tr} \widetilde{\nabla}(\widetilde{G}(V))\omega X. \tag{12}$$

*Proof.* First we calculate the curvature of  $\widehat{\nabla}^T$ . Let  $X$  and  $V$  be pullbacks of real vector fields on  $M$  and  $\mathcal{T}$  respectively, and let  $Y$  be any section of  $T$ . Then we get

$$\begin{aligned} R_{\widehat{\nabla}^T}(V, X)Y &= \widehat{\nabla}_V^T \widehat{\nabla}_X^T Y - \widehat{\nabla}_X^T \widehat{\nabla}_V^T Y \\ &= \pi^{1,0} V[\widetilde{\nabla}_X Y] - \widetilde{\nabla}_X \pi^{1,0} V[Y] \\ &= \pi^{1,0} V[\widetilde{\nabla}_X Y] - \pi^{1,0} \widetilde{\nabla}_X V[Y] \\ &= \pi^{1,0} V[\widetilde{\nabla}]_X Y. \end{aligned}$$

By Theorem 1.174 in [14], we get that the variation of the Levi-Civita connection in the tangent bundle is a symmetric (2,1)-tensor given by

$$\begin{aligned} g(V[\widetilde{\nabla}]_X Y, Z) &= \frac{1}{2}(\widetilde{\nabla}_X(V[g])(Y, Z) + \widetilde{\nabla}_Y(V[g])(X, Z) - \widetilde{\nabla}_Z(V[g])(X, Y)) \end{aligned} \tag{13}$$

for vector fields  $X, Y$  and  $Z$  on  $M$  and  $V$  on  $\mathcal{T}$ . We focus our attention on a point  $p \in M$ , and let  $e_1, \dots, e_m$  be a basis of  $T_p M$  satisfying the orthogonality condition that  $g(e'_j, e''_l) = \delta_{jl}$ . Then

$$\text{Tr } R_{\widehat{\nabla}^r}(V, X) = \text{Tr } \pi^{1,0} V[\widetilde{\nabla}]_X \pi^{1,0} = \sum_{\nu} g(V[\widetilde{\nabla}]_X e'_\nu, e''_\nu).$$

But taking into account the type of  $V[g]$ , and the fact that  $\widetilde{\nabla}$  preserves types, we get

$$\begin{aligned} g(V[\widetilde{\nabla}]_X e'_\nu, e''_\nu) &= \frac{1}{2} \widetilde{\nabla}_{e'_\nu}(V[g])(X, e''_\nu) - \frac{1}{2} \widetilde{\nabla}_{e''_\nu}(V[g])(X, e'_\nu) \\ &= \frac{1}{2} X\omega \widetilde{\nabla}_{e'_\nu}(\widetilde{G}(V))\omega e''_\nu - \frac{1}{2} X\omega \widetilde{\nabla}_{e''_\nu}(\widetilde{G}(V))\omega e'_\nu \\ &= \frac{i}{2} X\omega \widetilde{\nabla}_{e'_\nu}(G(V))g e''_\nu + \frac{i}{2} X\omega \widetilde{\nabla}_{e''_\nu}(\overline{G}(V))g e'_\nu \\ &= -\frac{i}{2} g(\widetilde{\nabla}_{e'_\nu}(G(V))\omega X, e''_\nu) - \frac{i}{2} g(\widetilde{\nabla}_{e''_\nu}(\overline{G}(V))\omega X, e'_\nu). \end{aligned}$$

Summing over  $\nu$ , we conclude that

$$\begin{aligned} \text{Tr } R_{\widehat{\nabla}^r}(V, X) &= -\frac{i}{2} \text{Tr } \widetilde{\nabla}(G(V))\omega X - \frac{i}{2} \text{Tr } \widetilde{\nabla}(\overline{G}(V))\omega X \\ &= -\frac{i}{2} \text{Tr } \widetilde{\nabla}(\widetilde{G}(V))\omega X, \end{aligned}$$

at the point  $p$  which was arbitrary. Finally we get by 3.1 and (10) that

$$\begin{aligned} R_{\widehat{\nabla}^r}(V, X) &= R_{\widehat{\nabla}^x}^{(k)}(V, X) - \frac{1}{2} \text{Tr } R_{\widehat{\nabla}^r}(V, X) \\ &= \frac{i}{4} \text{Tr } \widetilde{\nabla}(\widetilde{G}(V))\omega X, \end{aligned}$$

which was the claim. □

### 5. The Hitchin connection

Let  $\mathcal{D}(M_\sigma, \mathcal{L}^k \otimes \delta_\sigma)$  denote the space of differential operators on  $\mathcal{H}^{(k)} = C^\infty(M_\sigma, \mathcal{L}^k \otimes \delta_\sigma)$ , and consider the bundle  $\mathcal{D}(M, \widehat{\mathcal{L}}^k \otimes \delta)$  over  $\mathcal{T}$  having these spaces as fibers. One could think of  $\mathcal{D}(M, \widehat{\mathcal{L}}^k \otimes \delta)$  as the space of differential operators on sections of  $\widehat{\mathcal{L}}^k \otimes \delta$ , which are of order zero in the directions tangent to  $\mathcal{T}$ . Then, for any 1-form  $u$  on  $\mathcal{T}$  with values in  $\mathcal{D}(M, \widehat{\mathcal{L}}^k \otimes \delta)$ , we have a connection  $\nabla$  in the bundle  $\mathcal{H}^{(k)} = C^\infty(M, \mathcal{L}^k \otimes \delta)$  over  $\mathcal{T}$  given by

$$\nabla_V = \widehat{\nabla}_V^r + u(V),$$



for any vector field  $V$  on  $\mathcal{T}$ . Now we wish to find a  $u$  such that  $\nabla$  preserves the subspaces  $H_\sigma^{(k)}$ , thereby proving that these form a subbundle and inducing a connection in this subbundle.

**Lemma 5.1.** *The connection  $\nabla$  preserves  $H^{(k)}$  if and only if*

$$\nabla^{0,1}u(V)s = \frac{i}{2}V[J]\nabla s + \frac{i}{4}\text{Tr}\tilde{\nabla}(G(V))\omega s, \tag{14}$$

for all vector fields  $V$  on  $\mathcal{T}$ , and all  $s \in H^{(k)}$ .

*Proof.* Let  $X$  and  $V$  be the pullbacks of a vector field on  $M$  and  $\mathcal{T}$  respectively. Then we see that

$$[V, X''] = \frac{i}{2}V[J]X. \tag{15}$$

Now, assume that  $s \in H_\sigma^{(k)}$  and consider any extension of  $s$  to a smooth section of  $\mathcal{H}^{(k)} \rightarrow \mathcal{T}$ . Then we get

$$\begin{aligned} \nabla_{X''}\nabla_V s &= \widehat{\nabla}_{X''}^r \widehat{\nabla}_V^r s + \nabla_{X''}u(V)s \\ &= \widehat{\nabla}_V^r \widehat{\nabla}_{X''}^r s - R_{\widehat{\nabla}^r}(V, X'')s - \widehat{\nabla}_{[V, X'']}^r s + \nabla_{X''}u(V)s \\ &= -\frac{i}{2}\nabla_{V[J]X}s - \frac{i}{4}\text{Tr}(\tilde{\nabla}(G(V))\omega X)s + \nabla_{X''}u(V)s, \end{aligned}$$

at the point  $\sigma \in \mathcal{T}$ , where we used (15) and Proposition 4.3 for the last equality. This tells us, that  $\nabla$  preserves  $H^{(k)}$  if and only if  $u$  satisfies the equation in the lemma.  $\square$

**Remark 5.2.** Once we have a  $u$  which satisfies (14), we get induced a connection  $\nabla$  in  $\mathcal{H}^{(k)}$ . Its parallel transport then produce a local trivialization of the collection of subspaces  $\{H_\sigma^{(k)}\}_{\sigma \in \mathcal{T}}$ , and thereby establishes that  $H^{(k)}$  is a subbundle of  $\mathcal{H}^{(k)}$ .

For any vector field  $V$  tangent to  $\mathcal{T}$ , the tensor  $G(V)_\sigma \in C^\infty(M_\sigma, S^2(T_\sigma))$  induces a linear map  $G(V)_\sigma : TM_\mathbb{C}^* \rightarrow TM_\mathbb{C}$ , by the formula

$$\alpha \longmapsto \text{Tr}(G(V)_\sigma \otimes \alpha) = G(V)_\sigma \alpha.$$

Obviously this is in fact a map  $G(V)_\sigma : T_\sigma^* \rightarrow T_\sigma$ . We then define a second-order operator  $\Delta_{G(V)_\sigma} \in \mathcal{D}(M, \mathcal{L}^k \otimes \delta_\sigma)$  by  $\Delta_{G(V)_\sigma} = \text{Tr}\nabla_\sigma G(V)_\sigma \nabla_\sigma$ , or more explicitly

by the diagram

$$\begin{array}{ccc}
 & & C^\infty(M_\sigma, TM_{\mathbb{C}}^* \otimes \mathcal{L}^k \otimes \delta_\sigma) \\
 & \nearrow \nabla_\sigma & \downarrow G(V)_\sigma \otimes \text{Id} \otimes \text{Id} \\
 C^\infty(M_\sigma, \mathcal{L}^k \otimes \delta_\sigma) & & C^\infty(M_\sigma, T_\sigma \otimes \mathcal{L}^k \otimes \delta_\sigma) \\
 & \nwarrow \text{Tr} & \downarrow \tilde{\nabla}_\sigma \otimes \text{Id} + \text{Id} \otimes \nabla_\sigma \\
 & & C^\infty(M_\sigma, TM_{\mathbb{C}}^* \otimes T_\sigma \otimes \mathcal{L}^k \otimes \delta_\sigma).
 \end{array} \tag{16}$$

We shall make the additional assumption, that the family  $J$  is rigid in the sense that  $G(V)_\sigma$  is a holomorphic section of  $S^2(T_\sigma)$  over  $M_\sigma$ .

**Definition 5.3.** The family  $J$  of Kähler structures on  $(M, \omega)$  is called *rigid* if

$$\tilde{\nabla}_\sigma^{0,1}(G(V)_\sigma) = 0$$

for all vector fields  $V$  tangent to  $\mathcal{T}$  and  $\sigma \in \mathcal{T}$ .

This is a rather restrictive condition, but see Appendix A for constructions of rigid families of complex structures.

From now on, we will for simplicity often suppress the subscription  $\sigma$  from the notation. Assuming that  $J$  is rigid, we have the following lemma.

**Lemma 5.4.** At every point  $\sigma \in \mathcal{T}$ , the operator  $\Delta_{G(V)}$  satisfies

$$\nabla^{0,1} \Delta_{G(V)} s = -2ik\omega G(V) \nabla s + ik \text{Tr} \tilde{\nabla}(G(V)) \omega s - \frac{i}{2} \text{Tr} \tilde{\nabla}(G(V)) \rho s$$

for all vector fields  $V$  on  $\mathcal{T}$  and all (local) holomorphic sections  $s$  of the line bundle  $\mathcal{L}^k \otimes \delta \rightarrow M$ .

*Proof.* The proof is by direct calculation. Letting  $G$  denote  $G(V)$  we have

$$\nabla^{0,1} \Delta_G s = \nabla^{0,1} \text{Tr} \nabla G \nabla s = \text{Tr} \nabla^{0,1} \nabla G \nabla s.$$

Working further on the right side we commute the two connections, giving as extra terms the curvature of  $M_\sigma$  and of the line bundle  $\mathcal{L}^k \otimes \delta_\sigma$ ,

$$\nabla^{0,1} \Delta_G s = \text{Tr} \nabla \nabla^{0,1} G \nabla s - ik\omega G \nabla^{1,0} s + \frac{i}{2} \rho G \nabla s - i\rho G \nabla s.$$

Collecting the last two terms, and using the fact that  $J$  is rigid on the first, we obtain

$$\nabla^{0,1} \Delta_G s = \text{Tr} \nabla G \nabla^{0,1} \nabla s - ik\omega G \nabla s - \frac{i}{2} \rho G \nabla s.$$

Commuting the two connections, and using that  $s$  is holomorphic, we get

$$\nabla^{0,1} \Delta_G s = ik \operatorname{Tr} \nabla G \omega s - \frac{i}{2} \operatorname{Tr} \nabla G \rho s - ik \omega G \nabla s - \frac{i}{2} \rho G \nabla s.$$

Expanding the covariant derivatives in the first two terms by the Leibniz rule, and using the fact that  $\omega$  is parallel, we get the following, after collecting and cancelling terms,

$$\nabla^{0,1} \Delta_G s = ik \operatorname{Tr} \tilde{\nabla}(G) \omega s - 2ik \omega G \nabla s - \frac{i}{2} \operatorname{Tr} \tilde{\nabla}(G \rho) s.$$

This was the desired expression. Moreover we notice, that the above is a local computation, so that the identity is valid for local holomorphic sections of  $\mathcal{L}^k \otimes \delta$  as well.  $\square$

**Corollary 5.5.** *Provided that  $H^{0,1}(M) = 0$ , we have that  $\operatorname{Tr} \tilde{\nabla}(G(V)\rho)$  is exact with respect to the  $\bar{\partial}$ -operator on  $M$ .*

*Proof.* By appealing to Lemma 5.4 in the case where  $k = 0$ , we get for any local holomorphic section  $s$  of  $\mathcal{L}^k \otimes \delta_\sigma \rightarrow M_\sigma$  that

$$0 = \frac{i}{2} \nabla_\sigma^{0,1} \operatorname{Tr} \tilde{\nabla}_\sigma(G(V)_\sigma \rho_\sigma) s = \frac{i}{2} \bar{\partial}_\sigma (\operatorname{Tr} \tilde{\nabla}_\sigma(G(V)_\sigma \rho_\sigma)) s.$$

This immediately implies that

$$0 = \bar{\partial}_\sigma (\operatorname{Tr} \tilde{\nabla}_\sigma(G(V)_\sigma \rho_\sigma)),$$

and since  $H^{0,1}(M) = 0$ , the corollary follows.  $\square$

We remark that, by the Hodge decomposition theorem, the assumption  $H^{0,1}(M) = 0$  is satisfied for any compact Kähler manifold with  $H^1(M, \mathbb{R}) = 0$ .

By Corollary 5.5, we choose any smooth 1-form  $\beta \in \Omega^1(\mathcal{T}, C^\infty(M))$  such that

$$\bar{\partial} \beta(V) = \frac{i}{2} \operatorname{Tr} \tilde{\nabla}(G(V)\rho), \tag{17}$$

for any vector field  $V$  on  $\mathcal{T}$ . Then finally, we define

$$u(V) = \frac{1}{4k} (\Delta_{G(V)} + \beta(V)), \tag{18}$$

which clearly solves eq. (14). Thus, we have proved Theorem 1.2.

## 6. Relation to non-corrected quantization

We now impose the same assumptions as in [4] in order to give an explicit formula for the Hitchin connection and eventually compare with the one previously constructed in [4].

Thus, from now on  $M$  is assumed to be compact with  $H^1(M, \mathbb{R}) = 0$ . The real first Chern class of  $(M, \omega)$ , that is the image of the first Chern class in  $H^2(M, \mathbb{R})$ , is assumed to satisfy

$$c_1(M, \omega) = n \left[ \frac{\omega}{2\pi} \right], \quad (19)$$

where  $n \in \mathbb{Z}$  is some integer, which must be even by our assumption on the second Stiefel–Whitney class of  $M$ . Finally,  $\mathcal{T}$  is assumed to be a complex manifold and the map  $J$  to be *holomorphic* in the following sense.

**Definition 6.1.** The family  $J$ , of Kähler structures on  $M$  parametrized by  $\mathcal{T}$ , is called *holomorphic* if it satisfies

$$V'[J] = V[J]' \quad \text{and} \quad V''[J] = V[J]'',$$

for every vector field  $V$  tangent to  $\mathcal{T}$ .

These assumptions have a number of consequences which we shall explore in the following. First we give an alternative characterization of holomorphic families of Kähler structures.

Let  $I$  denote the integrable almost complex structure on  $\mathcal{T}$  induced by its complex structure. Then we have an almost complex structure  $\hat{J}$  on  $\mathcal{T} \times M$  defined by

$$\hat{J}(V \oplus X) = IV \oplus J_\sigma X, \quad \text{and} \quad V \oplus X \in T_{(\sigma,p)}(\mathcal{T} \times M). \quad (20)$$

The following gives another characterization of holomorphic families.

**Proposition 6.2.** *The family  $J$  is holomorphic if and only if  $\hat{J}$  is integrable.*

*Proof.* We show that  $J$  is holomorphic if and only if the Nijenhuis tensor for  $\hat{J}$  vanishes. By the Newlander–Nirenberg Theorem this will imply the proposition; see e.g. [24].

Clearly the Nijenhuis tensor vanishes, when evaluated only on vectors tangent to  $\mathcal{T}$ , since  $I$  is integrable. Likewise it will vanish when evaluated only on vectors tangent to  $M$ , since  $J$  is a family of integrable almost complex structures. Thus we are left with the case of mixed directions.

Let  $X$  and  $V$  be pullbacks to  $\mathcal{T} \times M$  of vector fields on  $M$  and  $\mathcal{T}$  respectively. Then since  $X$  is constant along  $\mathcal{T}$  and  $V$  is constant along  $M$  we find that

$$[V, JX] = V[J]X. \quad (21)$$

Now consider the following evaluation of the Nijenhuis tensor

$$\begin{aligned} N(V', X) &= [IV', JX] - [V', X] - \hat{J}[IV', X] - \hat{J}[V', JX] \\ &= i[V', JX] - \hat{J}[V', JX] \\ &= iV'[J]X - JV'[J]X \\ &= 2i\pi^{0,1}V'[J]X. \end{aligned}$$

Similarly one shows, that  $N(V'', X) = -2i\pi^{1,0}V''[J]X$ . Thus we see that  $N(V, X)$  vanishes if and only if

$$\pi^{0,1}V'[J]X = 0 \quad \text{and} \quad \pi^{1,0}V''[J]X = 0.$$

This proves the proposition. □

We shall denote by  $\hat{d}$  the differential on  $\mathcal{T} \times M$ , which splits as

$$\hat{d} = d_{\mathcal{T}} + d_M$$

into the sum of the differentials on  $\mathcal{T}$  and  $M$  respectively. Similar notation is used for  $\partial$  and  $\bar{\partial}$ .

**6.1. Explicit formula for  $\beta(V)$ .** As a first consequence of our additional assumptions we are able to give an explicit formula for the 1-form  $\beta$  in (18).

Since the curvature of the canonical line bundle  $K_{\sigma}$  is  $i\rho_{\sigma}$ , the real first Chern class of  $M_{\sigma}$  is represented by  $\frac{\rho_{\sigma}}{2\pi}$ . Since the Kähler form is harmonic, assumption (19) is then equivalent to  $\rho_{\sigma}^H = n\omega$ , where  $\rho_{\sigma}^H$  denotes the harmonic part of the Ricci form.

Since any real exact (1,1)-form on a Kähler manifold is  $\partial\bar{\partial}$ -exact, there exists, for any  $\sigma \in \mathcal{T}$ , a real function  $F_{\sigma}$ , called a *Ricci potential*, satisfying

$$\rho_{\sigma} = \rho_{\sigma}^H + 2i\partial_{\sigma}\bar{\partial}_{\sigma}F_{\sigma}.$$

By compactness of  $M$ , any two Ricci potentials on  $M_{\sigma}$  differ by a constant. Thus, choosing a particular normalization, such as

$$\int_M F_{\sigma}\omega^m = 0, \tag{22}$$

would yield a real smooth function  $F \in C^{\infty}(\mathcal{T} \times M)$ , with  $F_{\sigma}$  a Ricci potential on  $M_{\sigma}$  for every  $\sigma \in \mathcal{T}$ . In fact, we shall call any smooth real function  $F \in C^{\infty}(\mathcal{T} \times M)$  satisfying

$$\rho = n\omega + 2i\partial_M\bar{\partial}_M F \tag{23}$$

a *smooth family of Ricci potentials over  $\mathcal{T}$* , and the normalization (22) is just one way to single out a particular among such functions. Later, we will need to work with another family of Ricci potentials.

We will need the following lemma, the proof of which is given in [4].

**Lemma 6.3.** *Any smooth family  $F$  of Ricci potentials satisfies*

$$\bar{\partial}_M V'[F] = -\frac{i}{4} \operatorname{Tr} \tilde{\nabla}(G(V))\omega - \frac{i}{2} \partial_M F G(V)\omega, \quad (24)$$

for any vector field  $V$  tangent to  $\mathcal{T}$ .

Then we have the following

**Lemma 6.4.** *Let  $F$  be a smooth family of Ricci potentials. Then the 1-form  $\beta \in \Omega^1(\mathcal{T}, C^\infty(M))$  given by*

$$\beta(V) = -2nV'[F] - \partial_M F G(V)\partial_M F - \operatorname{Tr} \tilde{\nabla}(G(V)\partial_M F)$$

satisfies  $\bar{\partial}_M \beta(V) = \frac{i}{2} \operatorname{Tr} \tilde{\nabla}(G(V)\rho)$ .

*Proof.* Throughout this proof we shall denote  $\partial_M$  and  $\bar{\partial}_M$  for short by  $\partial$  and  $\bar{\partial}$  respectively. Since  $\omega$  is parallel, with respect to the Levi-Civita connection  $\tilde{\nabla}$ , we get

$$\begin{aligned} \operatorname{Tr} \tilde{\nabla}(G(V)\rho) &= \operatorname{Tr} \tilde{\nabla}(G(V)(n\omega + 2i\partial\bar{\partial}F)) \\ &= n \operatorname{Tr} \tilde{\nabla}(G(V))\omega + 2i \operatorname{Tr} \tilde{\nabla}(G(V)\partial\bar{\partial}F). \end{aligned}$$

Moreover, it is easily verified that

$$\begin{aligned} \operatorname{Tr} \tilde{\nabla}(G(V)\partial\bar{\partial}F) &= -i\partial F G(V)\rho + \bar{\partial} \operatorname{Tr} \tilde{\nabla}(G(V)\partial F) \\ &= -in\partial F G(V)\omega + 2\partial F G(V)\partial\bar{\partial}F + \bar{\partial} \operatorname{Tr} \tilde{\nabla}(G(V)\partial F). \end{aligned}$$

Then the lemma follows by Lemma 6.3 and the identity

$$\bar{\partial}(\partial F G(V)\partial F) = 2\partial F G(V)\partial\bar{\partial}F,$$

which is easily verified, using the symmetry of  $G(V)$ . □

Thus, under the assumptions of this section, we have a completely explicit formula for the Hitchin connection.

**6.2. Curvature of the reference connection revisited.** Notice that the type of  $\omega$ , and the fact that  $J$  is holomorphic, implies

$$V'[J] = V[J]' = G(V)\omega,$$

which in turn gives  $G(V) = G(V')$ . Then, having calculated the curvature of the reference connection in all directions, we see that it is of type (1,1) over  $\mathcal{T} \times M$  and

thus the (0,2)-part of the curvature vanishes. This means that the reference connection defines a holomorphic structure on the line bundle  $\hat{\mathcal{L}}^k \otimes \delta$ , over the complex manifold  $\mathcal{T} \times M$ . Moreover, we observe that  $(\hat{\nabla}^r)^{0,1}$  preserves the bundle  $H^{(k)} \rightarrow \mathcal{T}$ , since  $u(V'') = 0$  solves (14). Thus the reference connection defines a holomorphic structure on the bundle  $H^{(k)} \rightarrow \mathcal{T}$ .

We now prove that, at least locally over  $\mathcal{T}$ , the curvature of the reference connection can be expressed in terms of a certain family of Ricci potentials.

First we have the following lemma, which is an immediate consequence of Lemma 6.3 by direct verification.

**Proposition 6.5.** *For any smooth family  $F$  of Ricci potentials and any vector fields  $V$  on  $\mathcal{T}$  and  $X$  on  $M$ , the curvature of the reference connection satisfies*

$$R_{\hat{\nabla}^r}(V, X) = -\hat{\partial}\bar{\partial}F(V, X).$$

*Proof.* Let  $V$  and  $X$  be pullbacks of real vector fields on  $\mathcal{T}$  and  $M$  respectively. Then, we have

$$\begin{aligned} \bar{\partial}\hat{\partial}F(X'', V') &= \hat{d}\hat{\partial}F(X'', V') \\ &= X''(\hat{\partial}F(V')) - V'(\hat{\partial}F(X'')) - \hat{\partial}F([X'', V']) \\ &= X''V'[F] + \frac{i}{2}\hat{\partial}F(V'[J]X) \\ &= X''V'[F] + \frac{i}{2}\partial_M FG(V)\omega X'' \\ &= -\frac{i}{4}\text{Tr } \tilde{\nabla}(G(V))\omega X'' \\ &= -R_{\hat{\nabla}^r}(V', X''), \end{aligned}$$

where we use Lemma 6.3 and Proposition 4.3 for the last two equalities. The case of  $X'$  and  $V''$  is similar by conjugation of the identity in Lemma 6.3.  $\square$

To express the 2-form  $\theta$  in terms of Ricci potentials, we first prove the following lemma.

**Lemma 6.6.** *For any smooth family  $F$  of Ricci potentials, the expression*

$$\theta - 2i\partial_{\mathcal{T}}\bar{\partial}_{\mathcal{T}}F \tag{25}$$

*defines an ordinary 2-form on  $\mathcal{T}$ .*

*Proof.* Take  $V$ ,  $W$ , and  $X$  to be commuting vector fields so that  $V$  and  $W$  are tangent to  $\mathcal{T}$  and  $X$  is tangent to  $M$ . We must prove that (25) takes values in constant functions on  $M$ , i.e. that

$$0 = X[\theta(V, W) - \hat{\partial}\bar{\partial}F(V, W)].$$

Now, by the differential Bianchi identity and by Proposition 4.2 we have

$$\begin{aligned} 0 &= d^{\nabla^r} R_{\nabla^r}(X, V, W) \\ &= \nabla_X^r R_{\nabla^r}(V, W) - \nabla_V^r R_{\nabla^r}(X, W) + \nabla_W^r R_{\nabla^r}(X, V) \\ &= \frac{i}{2} X[\theta(V, W)] + \nabla_W^r R_{\nabla^r}(X, V) - \nabla_V^r R_{\nabla^r}(X, W). \end{aligned}$$

Then Proposition 6.5 yields

$$\begin{aligned} \frac{i}{2}\theta(V, W) &= W[\hat{\partial}\bar{\partial}F(X, V)] - V[\hat{\partial}\bar{\partial}F(X, W)] \\ &= WXV''[F] - WVX''[F] - VXM''[F] + VWX''[F] \\ &= XWV''[F] - XVW''[F] \\ &= -X[\hat{\partial}\bar{\partial}F(V, W)] \end{aligned}$$

as desired. □

This allows us to prove

**Proposition 6.7.** *Over any open subset  $U$  of  $\mathcal{T}$  with  $H^1(U, \mathbb{R}) = 0$ , we can find a family  $\tilde{F}$  of Ricci potentials satisfying*

$$\theta = 2i \partial_{\mathcal{T}} \bar{\partial}_{\mathcal{T}} \tilde{F}. \quad (26)$$

*Proof.* Let  $\sigma \in \mathcal{T}$  and fix a smooth family  $F$  of Ricci potentials, say the one satisfying (22). Let  $V$  and  $W$  be vector fields tangent to  $\mathcal{T}$ . Then, by Lemma 6.6, we can define a 2-form  $\alpha \in \Omega^{1,1}(\mathcal{T})$  by

$$\alpha = \theta - 2i \partial_{\mathcal{T}} \bar{\partial}_{\mathcal{T}} F. \quad (27)$$

By applying the Bianchi identity to the reference connection it follows that  $\theta$  is closed on  $\mathcal{T}$ . Thus, we see that  $\alpha$  is a closed 2-form on  $\mathcal{T}$ . Since  $\theta$  is real, so is  $\alpha$ , and therefore we can find a real function  $A$  on  $U$  such that

$$\alpha|_U = 2i \partial_{\mathcal{T}} \bar{\partial}_{\mathcal{T}} A.$$

But then  $\tilde{F} = F|_U + A$  defines a new smooth family of Ricci potentials with the desired property. □



The previous two propositions can be combined with Proposition 4.1 to prove

**Theorem 6.8.** *Let  $(M, \omega)$  be a compact, prequantizable, symplectic manifold with the real first Chern class satisfying  $c_1(M, \omega) = n[\frac{\omega}{2\pi}]$ ,  $H^1(M, \mathbb{R}) = 0$  and vanishing second Stiefel–Whitney class. Let  $J$  be a rigid, holomorphic family of Kähler structures on  $M$ , parametrized by a complex manifold  $\mathcal{T}$ . Then, for any open subset  $U$  of  $\mathcal{T}$  with  $H^1(U, \mathbb{R}) = 0$  there exists a family of Ricci potentials  $\tilde{F}$  over  $U$  such that*

$$R_{\hat{\nabla}_r}^{(k)} = R_{\hat{\nabla}_{\mathcal{L}}}^{(k-n/2)} - \hat{\partial}\bar{\partial}\tilde{F}, \tag{28}$$

where  $R_{\hat{\nabla}_r}^{(k)}$  denotes curvature of the reference connection in  $\hat{\mathcal{L}}^k \otimes \delta$  and  $R_{\hat{\nabla}_{\mathcal{L}}}^{(k-n/2)}$  denotes the curvature of  $\hat{\nabla}_{\mathcal{L}}$  in  $\hat{\mathcal{L}}^{k-n/2}$ .

*Proof.* Let  $X$  and  $Y$  be vector fields tangent to  $M$ , and let  $V$  and  $W$  be vector fields tangent to  $\mathcal{T}$ . Use Proposition 6.7 to find a family of Ricci potentials over  $U$  satisfying (26). Then, by Proposition 4.1 and (23) we have that

$$\begin{aligned} R_{\hat{\nabla}_r}(X, Y) &= -ik\omega(X, Y) + \frac{i}{2}\rho(X, Y) \\ &= -i(k - \frac{n}{2})\omega(X, Y) - \partial_M\bar{\partial}_M\tilde{F}(X, Y) \\ &= R_{\hat{\nabla}_{\mathcal{L}}}^{(k-n/2)}(X, Y) - \hat{\partial}\bar{\partial}\tilde{F}(X, Y). \end{aligned}$$

By Lemma 3.1, the curvature  $R_{\hat{\nabla}_{\mathcal{L}}}^{(k-n/2)}$  vanishes in the remaining directions, and so the theorem follows from Proposition 6.5 and (26). □

Using this result, we are able to relate our construction of the Hitchin connection to the construction of Andersen [4] in the non-corrected setting.

**6.3. Hitchin’s connection in non-corrected quantization.** We wish to relate the quantum spaces of half-form corrected quantization to the spaces of non-corrected geometric quantization, with the intent to describe, in the non-corrected setting, our construction of a Hitchin connection and relate it to the construction in [4].

It turns out that the choice of prequantum line bundle plays a role in this. This is because of the choice of metaplectic structure we made. We note that what we really chose was a half of  $c_1(M, \omega)$ , so all we know is that  $\delta$  is a line bundle satisfying  $2c_1(\delta) = -c_1(M, \omega)$ . Thus, if we impose on  $(M, \omega)$  that  $n$  divides  $c_1(M, \omega)$ , then we get that  $\frac{n}{2}$  divides  $c_1(\delta)$ . We will need that the prequantum line bundle is related to the metaplectic structure in a certain way, and the following lemma ensures that this is possible.

**Lemma 6.9.** *If  $c_1(M, \omega)$  is divisible by  $n$  in  $H^2(M, \mathbb{Z})$ , there exists a prequantum line bundle  $\mathcal{L}$  over  $M$  such that*

$$\frac{n}{2}c_1(\mathcal{L}) = -c_1(\delta).$$

*Proof.* Let  $\mathcal{L}_0$  be any prequantum line bundle on  $M$  and pick an auxiliary Kähler structure  $J$  on  $M$ . Let  $F_J$  be a Ricci potential on  $M$  and consider the line bundles  $(\mathcal{L}_0^{-n/2}, e^{F_J} h^{\mathcal{L}_0})$  and  $(\delta_J, h_J^\delta)$  over  $M$ . Then it is easily calculated, that the line bundles have the same curvature. Thus, the tensor product of the former with the dual of the latter yields a flat Hermitian line bundle  $L_1$ . Since  $c_1(\delta)$  is divisible by  $\frac{n}{2}$ , there exists a flat Hermitian line bundle  $L_2$  such that  $L_2^{n/2} \cong L_1$ . Finally, the line bundle  $\mathcal{L} = \mathcal{L}_0 \otimes L_2$  has the structure of a prequantum line bundle, and  $\frac{n}{2}c_1(\mathcal{L}) = c_1(\mathcal{L}^{n/2}) = -c_1(\delta)$ . Thus  $\mathcal{L}$  is the desired prequantum line bundle.  $\square$

From now on, we will assume that our prequantum line bundle satisfies  $\frac{n}{2}c_1(\mathcal{L}) = -c_1(\delta)$ . We note, that only when  $H^2(M, \mathbb{Z})$  has torsion, is the assumption a further restriction on  $(M, \omega)$ , as otherwise the curvature determines the line bundle completely.

Next, let  $\tilde{F}$  be a family of Ricci potentials over  $U$ , with  $H^1(U, \mathbb{R}) = 0$ , such that (28) is satisfied. We wish to construct an isomorphism  $\hat{\varphi}$  of holomorphic Hermitian line bundles over  $U \times M$

$$\hat{\varphi}: (\hat{\mathcal{L}}^{k-n/2}, e^{\tilde{F}} \hat{h}^{\mathcal{L}}) \longrightarrow (\hat{\mathcal{L}}^k \otimes \delta, \hat{h}). \tag{29}$$

Since  $\frac{n}{2}c_1(\mathcal{L}) = -c_1(\delta)$ , the line bundles are isomorphic as complex line bundles, and with the given Hermitian structures, a simple calculation and application of (28) reveals that they have the same curvature. Thus, the obstruction to finding the structure preserving isomorphism  $\hat{\varphi}$  lies in the first cohomology of  $U \times M$ . But this is trivial by the Künneth formula, since  $H^1(U, \mathbb{R}) = 0$  and  $H^1(M, \mathbb{R}) = 0$  by assumption.

Moreover, it is easily seen that the pullback under  $\hat{\varphi}$  of the reference connection is given by

$$\hat{\varphi}^* \hat{\nabla}^r = \hat{\nabla}^{\mathcal{L}} + \hat{\partial} \tilde{F}, \tag{30}$$

since the right hand side is the unique Hermitian connection compatible with the holomorphic structure of  $\hat{\mathcal{L}}^{k-n/2}$ .

In the paper [4], Andersen constructs a Hitchin connection in  $\mathcal{T} \times C^\infty(M, \mathcal{L}^k)$ , preserving the subbundle of holomorphic sections. His construction is valid for any rigid, holomorphic family of Kähler structures on  $M$  parametrized by  $\mathcal{T}$ , provided that  $H^1(M, \mathbb{R}) = 0$  and  $c_1(M, \omega) = n[\frac{\omega}{2\pi}]$ .

Now, the existence of the isomorphism (29) enables us to compare his construction to the one presented in this paper. Thus, we shall briefly recall that the Hitchin connection constructed in [4] is given by

$$\tilde{\nabla}_V = \hat{\nabla}_V^{\mathcal{L}} + \tilde{u}(V), \tag{31}$$

where

$$\tilde{u}(V) = \frac{1}{4k + 2n} (\Delta_{G(V)}^{\mathcal{L}} + 2\nabla_{G(V)\partial\tilde{F}}^{\mathcal{L}} + 4kV'[\tilde{F}]), \quad (32)$$

and  $\Delta_{G(V)}^{\mathcal{L}}$  is the operator given by the diagram

$$\begin{array}{ccc}
 & & C^\infty(M_\sigma, TM_{\mathbb{C}}^* \otimes \mathcal{L}^k) \\
 & \nearrow \nabla^{\mathcal{L}} & \downarrow G(V)_\sigma \otimes \text{Id} \\
 C^\infty(M_\sigma, \mathcal{L}^k) & & C^\infty(M_\sigma, T_\sigma \otimes \mathcal{L}^k) \\
 & \nwarrow \text{Tr} & \downarrow \tilde{\nabla}_\sigma \otimes \text{Id} + \text{Id} \otimes \nabla^{\mathcal{L}} \\
 & & C^\infty(M_\sigma, TM_{\mathbb{C}}^* \otimes T_\sigma \otimes \mathcal{L}^k).
 \end{array} \quad (33)$$

We leave it to the reader to verify, using (30), that the pullback by  $\hat{\varphi}$  of the operator  $\Delta_{G(V)}$ , acting on sections of  $\hat{\mathcal{L}}^k \otimes \delta$ , is given by

$$\hat{\varphi}^* \Delta_{G(V)} = \Delta_{G(V)}^{\mathcal{L}} + 2\nabla_{G(V)\partial_M \tilde{F}}^{\mathcal{L}} - \beta(V) - 2nV'[\tilde{F}], \quad (34)$$

where  $\beta(V)$  is given by the expression in Lemma 6.4, but in terms of  $\tilde{F}$ .

Furthermore, in the bundle  $\hat{\mathcal{L}}^{k-n/2}$ , the formula (32) becomes

$$\begin{aligned}
 \tilde{u}(V) &= \frac{1}{4k} (\Delta_{G(V)}^{\mathcal{L}} + 2\nabla_{G(V)\partial_M \tilde{F}}^{\mathcal{L}} - 2nV'[\tilde{F}]) + V'[\tilde{F}] \\
 &= \frac{1}{4k} (\hat{\varphi}^* \Delta_{G(V)} + \beta(V)) + V'[\tilde{F}] \\
 &= \hat{\varphi}^* u(V) + V'[\tilde{F}].
 \end{aligned} \quad (35)$$

But this means, that the pullback of our Hitchin connection by  $\hat{\varphi}$  is given by

$$\begin{aligned}
 \hat{\varphi}^* \nabla_V &= \hat{\varphi}^* \hat{\nabla}_V^r + \hat{\varphi}^* u(V) \\
 &= \hat{\nabla}_V^{\mathcal{L}} + V'[\tilde{F}] + \hat{\varphi}^* u(V) \\
 &= \hat{\nabla}_V^{\mathcal{L}} + \tilde{u}(V) \\
 &= \tilde{\nabla}_V.
 \end{aligned} \quad (36)$$

Thus the two connections agree, and we have proved Theorem 1.3.

### Appendix A. Examples of rigid families of complex structures on symplectic manifolds

Hitchin showed that the family of complex structures on moduli spaces of semi-stable bundles over Riemann surfaces parametrized by Teichüller space is Rigid. Scheinost and Schottenloher generalized this to moduli space of semi-stable bundles on arbitrary families of Kähler manifolds. In general the Rigid condition gives a distribution on the space of complex structures on a symplectic manifold. Any sub-family which is tangent to this distribution satisfies the rigidity condition. – In general the condition is rather restrictive. – Let us however show that we can construct local examples in all dimensions.

Let  $(M, \omega)$  be an open subset of  $\mathbb{R}^2$  with the standard symplectic form  $\omega = dx \wedge dy$  and let  $\mathcal{T} = \mathbb{R}^l$ . We want to analyze a family of complex structures,

$$J_\sigma \left( \frac{\partial}{\partial x} \right) = A(\sigma, x, y) \frac{\partial}{\partial x} + B(\sigma, x, y) \frac{\partial}{\partial y},$$

given by functions  $A, B \in C^\infty(\mathcal{T} \times M)$ . Then, the identity  $J^2 = -\text{Id}$  yields

$$J_\sigma \left( \frac{\partial}{\partial y} \right) = - \left( \frac{1}{B} + \frac{A^2}{B} \right) \frac{\partial}{\partial x} - A \frac{\partial}{\partial y}.$$

It is clear that  $\omega$  is  $J_\sigma$  invariant and that  $g_\sigma$  is positive definite when  $B > 0$ .

For simplicity, suppose that  $B$  is constant along  $\mathcal{T}$ . Given a vector field  $V$  on  $\mathcal{T}$ , the variation of  $J_\sigma$  is then given by

$$V[J] = V[A] \frac{\partial}{\partial x} dx - \left( \frac{2AV[A]}{B} \frac{\partial}{\partial x} + V[A] \frac{\partial}{\partial y} \right) dy$$

and the identity  $\tilde{G}(V)\omega = V[J]$  gives the formula

$$\tilde{G}(V) = -2V[A] \frac{\partial}{\partial x} \frac{\partial}{\partial y} - \frac{2AV[A]}{B} \frac{\partial^2}{\partial x^2}.$$

From this we can calculate  $G(V)$  as

$$G(V) = \frac{-2iV[A]}{B} \frac{\partial^2}{\partial z^2},$$

which implies that  $J_\sigma$  is rigid if

$$0 = -V[A] \frac{\partial B}{\partial y} + B \frac{\partial V[A]}{\partial y} = V[A] \frac{\partial B}{\partial x} - B \frac{\partial V[A]}{\partial x}.$$

These equations have solutions  $B(x, y) = B_0(x, y)$  and  $A(\sigma, x, y) = A_0(x, y) + \sum_{i=1}^l \sigma_i B_0(x, y)$  where  $A_0$  and  $B_0$  are arbitrary functions on  $(M, \omega)$ . This means

that given any initial complex structure

$$J_0\left(\frac{\partial}{\partial x}\right) = A_0(x, y)\frac{\partial}{\partial x} + B_0(x, y)\frac{\partial}{\partial y}$$

we have obtained a rigid family of deformations parametrized by  $\mathbb{R}^l$ .

By taking cross products of this construction with itself  $m$  times, one gets examples on any open subset of  $\mathbb{R}^{2m}$  with the standard symplectic form.

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