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Quantum Topology

Categorification of level two representations of quantum sl_n via generalized arc rings

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Abstract. In this paper we construct an extension of the arc ring H^n introduced by Khovanov [4], and use it to categorify level two representations of $U_q(sl_N)$. These rings also induce invariants of tangle cobordisms.

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1. Introduction

Khovanov constructed in [4] a family of rings H^n, for $n \geq 0$, which is a categorification of Inv(n), the space of $U_q(sl_2)$-invariants in $V^{\otimes 2n}$. These rings lead to an invariant of (even) tangles which to a tangle assigns a complex of (H^n, H^m)-bimodules, up to chain homotopy equivalence. Khovanov and the author [1] built subquotients of H^n and used them to categorify the action of tangles on $V^{\otimes n}$. The same rings were also introduced by Stroppel [6].

In this paper, we extend the construction of these arc rings $A^{k,n-k}$ and give a categorification of level two representations of $U_q(sl_N)$. In Section 2 we review the definition of the arc rings $A^{n-k,k}$ and construct the rings $A_n^{k,l}$ with two platforms of arbitrary sizes k and l. We show that they lead to a tangle invariant which is functorial under tangle cobordisms. Then in Section 3 we compute the centers of $A_n^{k,l}$ and relate them to the cohomology rings of Springer varieties. Finally, in Section 4.1, we categorify level two representations of $U_q(sl_N)$ using the rings $A_n^{k,0}$.

Fix a level two representation W of $U_q(sl_N)$ with the highest weight $\omega_s + \omega_{k+s}$. There is a decomposition of W into weight spaces $W = \bigoplus_\mu W_\mu$. A weight μ is called *admissible* if it appears in the weight space decomposition of W. Denote by

[1]The author would like to thank Mikhail Khovanov for recommending this problem and too many helpful conversations and suggestions. He is also very grateful to Robin Kirby for his kindness, guidance and support.

$\mathcal{C}$ the direct sum of categories of $A^{k,0}_{m(\mu)}$-modules over admissible μ, where $m(\mu)$ is a non-negative integer depending only on μ. The Grothendieck group of the category of $A^{k,0}_{m(\mu)}$-modules is naturally isomorphic to W_μ. The exact functors $\mathcal{E}_i$, $\mathcal{F}_i$ introduced by Khovanov and Huerfano [3] naturally extend to exact functors on $\mathcal{C}$ which categorify the actions of $E_i, F_i \in U_q(sl_N)$ on W.

2. Generalization of the arc ring $A^{n-k,k}$

2.1. Arc ring $A^{n-k,k}$. We first recall the definition of H^n from [4]. Let $\mathcal{A}$ be a free abelian group of rank 2 spanned by $\mathbf{1}$ and X with $\mathbf{1}$ in degree -1 and X in degree 1. Assign to $\mathcal{A}$ a 2-dimensional TQFT $\mathcal{F}$ which associates $\mathcal{A}^{\otimes k}$ to a disjoint union of k circles. To the *pants* cobordism corresponding to merging of two circles into one, $\mathcal{F}$ associates the multiplication $m\colon \mathcal{A}\otimes\mathcal{A}\to\mathcal{A}$

$$\mathbf{1}^2=\mathbf{1},\quad \mathbf{1}X=X\mathbf{1}=X,\quad X^2=0. \tag{1}$$

To the *inverse pants* cobordism corresponding to splitting of one circle into two, $\mathcal{F}$ associates the comultiplication $\Delta\colon \mathcal{A}\to\mathcal{A}\otimes\mathcal{A}$

$$\Delta(\mathbf{1})=\mathbf{1}\otimes X+X\otimes\mathbf{1},\quad \Delta(X)=X\otimes X. \tag{2}$$

To the *cup* and *cap* cobordisms corresponding to the birth and death of a circle, $\mathcal{F}$ associates the unit map $\iota\colon \mathbb{Z}\to\mathcal{A}$ and trace map $\varepsilon\colon\mathcal{A}\to\mathbb{Z}$ respectively

$$\iota(1)=\mathbf{1},\quad \varepsilon(\mathbf{1})=0,\quad \varepsilon(X)=1.$$

Let B^n be the set of crossingless matchings of $2n$ points. For $a,b\in B^n$ denote by $W(b)$ the reflection of b about the horizontal axis, and by $W(b)a$ the closed 1-manifold obtained by closing $W(b)$ and a along their boundaries.

$\mathcal{F}(W(b)a)$ is a graded abelian group isomorphic to $\mathcal{A}^{\otimes I}$, where I is the set of circles in $W(b)a$, see Figure 1.

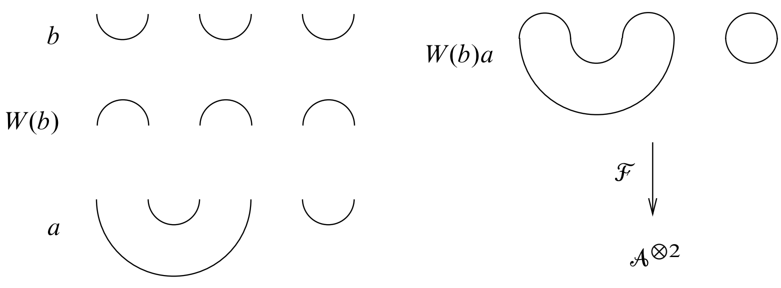


Figure 1. Gluing in B^3.

For $a, b \in B^n$ let

$$_b(H^n)_a \stackrel{\text{def}}{=} \mathcal{F}(W(b)a)\{n\}.$$

and define H^n as the direct sum

$$H^n \stackrel{\text{def}}{=} \bigoplus_{a,b} {}_b(H^n)_a,$$

where $\{n\}$ denotes the action of raising the grading up by n. Multiplication maps in H^n are defined as follows. We set $xy = 0$ if $x \in {}_b(H^n)_a$, $y \in {}_c(H^n)_d$ and $c \neq a$. Multiplication maps

$$_b(H^n)_a \otimes {}_a(H^n)_c \to {}_b(H^n)_c$$

are given by homomorphisms of abelian groups

$$\mathcal{F}(W(b)a) \otimes \mathcal{F}(W(a)c) \to \mathcal{F}(W(b)c),$$

which are induced by the *minimal* cobordism from $W(b)aW(a)c$ to $W(b)c$, see Figure 2.

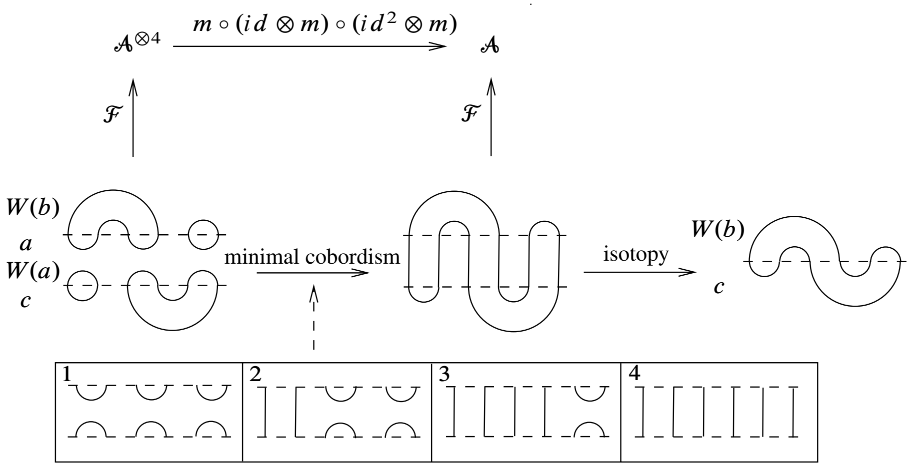


Figure 2. Multiplication in H^n.

Now we recall the definition of the subquotients of H^n from [1]. For each $n \geq 0$ and $0 \leq k \leq n$, define $B^{n-k,k}$ to be the subset of B^n where there are no matchings among the first $n-k$ points and among the last k points. Figure 3 shows $B^{1,2}$. We put two *platforms,* one on the first $n-k$ points and one on the last k points to indicate that these endpoints are special. The n points lying in between the two platforms are called *free points.*

Define $\widetilde{A}^{n-k,k}$ by

$$\widetilde{A}^{n-k,k} \stackrel{\text{def}}{=} \bigoplus_{a,b \in B^{n-k,k}} \mathcal{F}(W(b)a)\{n\}. \tag{3}$$

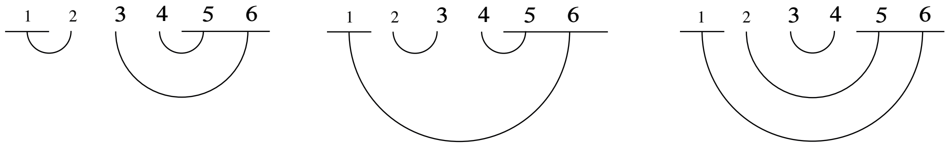


Figure 3. The 3 elements in $B^{1,2}$.

$\widetilde{A}^{n-k,k}$ sits inside H^n as a graded subring which inherits its multiplication from H^n.

For $a, b \in B^{n-k,k}$, call a circle in $W(b)a$ *type* I if it is disjoint from platforms, *type* II if it intersects at least one platform and intersect each platform at most once, and *type* III if it intersects one of the platforms at least twice (see Figure 4). An intersection point between a circle and a platform is called a *mark*.

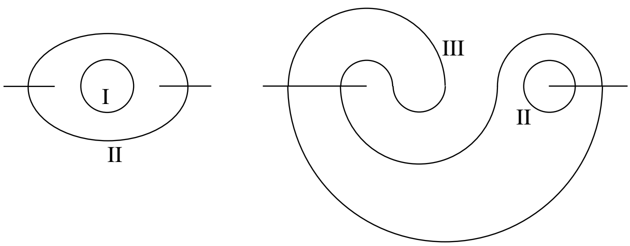


Figure 4. 3 types of circles.

The ring $\widetilde{A}^{n-k,k}$ has a two-sided graded ideal $I^{n-k,k} \subset \widetilde{A}^{n-k,k}$ (the definition of $I^{n-k,k}$ will be given in the following section). The ring $A^{n-k,k}$ is defined as the quotient of $\widetilde{A}^{n-k,k}$ by the ideal $I^{n-k,k}$

$$A^{n-k,k} \overset{\text{def}}{=} \widetilde{A}^{n-k,k} / I^{n-k,k}. \tag{4}$$

$A^{n-k,k}$ naturally decomposes into a direct sum of graded abelian groups

$$A^{n-k,k} = \bigoplus_{a,b \in B^{n-k,k}} {}_a(A^{n-k,k})_b.$$

By taking the direct product over all $0 \leq k \leq n$, we collect the rings $A^{n-k,k}$ together into a graded ring A^n

$$A^n \overset{\text{def}}{=} \prod_{0 \leq k \leq n} A^{n-k,k}.$$

As a graded abelian group, A^n is the direct sum of $A^{n-k,k}$, over $0 \leq k \leq n$.

See [4] and [1] for more details on H^n and its subquotients.

2.2. Generalization of $A^{n-k,k}$. We call the triple (n,k,l) *coherent* if $|k-l| \leq n$ and $n+k+l \equiv 0 \pmod 2$. For each coherent triple (n,k,l) denote by $B_n^{k,l}$ the subset of $B^{(n+k+l)/2}$ where there are no matchings among the first k points and among the last l points. Put one platform on the first k points and one on the last l points. Note that $B_{2n}^{0,0} = B^n$ and $B_n^{n-k,k} = B^{n-k,k}$.

Define $\widetilde{A}_n^{k,l}$ by

$$\widetilde{A}_n^{k,l} \stackrel{\text{def}}{=} \bigoplus_{a,b\in B_n^{k,l}} \mathcal{F}(W(b)a)\Big\{\frac{n+k+l}{2}\Big\}. \tag{5}$$

Just like $\widetilde{A}^{n-k,k}$, $\widetilde{A}_n^{k,l}$ is a graded subring of H^n and inherits its multiplication from H^n. The ideal $I_n^{k,l} \subset \widetilde{A}_n^{k,l}$ is defined similar to that of $\widetilde{A}^{n-k,k}$. For $a,b \in B_n^{k,l}$, if $W(b)a$ contains at least one type III circle, set ${}_b(I_n^{k,l})_a = \mathcal{F}(W(b)a)\{n\}$. If $W(b)a$ contains only circles of type I and type II, we write $\mathcal{F}(W(b)a) = \mathcal{A}^{\otimes i} \otimes \mathcal{A}^{\otimes j}$ in which type II circles correspond to the first i tensor factors, and define ${}_b(I_n^{k,l})_a$ as the span of

$$y_1 \otimes \cdots \otimes y_{t-1} \otimes X \otimes y_{t+1} \otimes \cdots \otimes y_{i+j} \in \mathcal{A}^{\otimes i} \otimes \mathcal{A}^{\otimes j} \cong \mathcal{F}(W(b)a),$$

where $1 \leq t \leq i$ and $y_s \in \{\mathbf{1}, X\}$. By taking the direct sum over all $a,b \in B_n^{k,l}$ we get a subgroup of $\widetilde{A}_n^{k,l}$

$$I_n^{k,l} \stackrel{\text{def}}{=} \bigoplus_{a,b\in B_n^{k,l}} {}_b(I_n^{k,l})_a.$$

It is easy to show that $I_n^{k,l}$ is a two-sided graded ideal of $\widetilde{A}_n^{k,l}$. Ring $A_n^{k,l}$ is defined as the quotient of $\widetilde{A}_n^{k,l}$ by the ideal $I_n^{k,l}$

$$A_n^{k,l} \stackrel{\text{def}}{=} \widetilde{A}_n^{k,l}/I_n^{k,l}. \tag{6}$$

If $W(b)a$ contains a type III circle then ${}_a(A_n^{k,l})_b = 0$. Otherwise, group ${}_a(A_n^{k,l})_b$ is free abelian of rank $2^{\#\text{ of type I circles}}$. Assuming that $W(a)b$ contains m circles in which the first i of them are of type II, ${}_a(A_n^{k,l})_b$ has a basis of the form

$$\mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes a_{i+1} \otimes \cdots \otimes a_m,$$

where $a_s \in \{\mathbf{1}, X\}$ for all $i+1 \leq s \leq m$.

There is a natural decomposition of $A_n^{k,l}$ into a direct sum of graded abelian groups

$$A_n^{k,l} = \bigoplus_{a,b\in B_n^{k,l}} {}_a(A_n^{k,l})_b,$$

where

$${}_a(A_n^{k,l})_b = \mathcal{F}(W(a)b)\Big\{\frac{n+k+l}{2}\Big\}/{}_a(I_n^{k,l})_b.$$

Let $P_n^{k,l}(a)$, or simply P(a), for $a \in B_n^{k,l}$, be a left $A_n^{k,l}$-module given by

$$P(a) = \bigoplus_{b \in B_n^{k,l}} {}_b(A^{n-k,k})_a.$$

$A_n^{k,l}$ decomposes into a direct sum of left $A_n^{k,l}$-modules

$$A_n^{k,l} = \bigoplus_{a \in B_n^{k,l}} P(a).$$

$P(a)$ is left projective since it is a direct summand of the free module $A_n^{k,l}$. Actually, any indecomposable left projective $A_n^{k,l}$-module is isomorphic to $P(a)\{s\}$ for some $a \in B_n^{k,l}$ and $s \in \mathbb{Z}$.

Here are some basic facts about the ring $A_n^{k,l}$.

- $A_n^{k,l} \cong A_n^{l,k}$. Reflecting a diagram in $B_n^{k,l}$ about a vertical axis produces a diagram in $B_n^{l,k}$. It leads to an isomorphism of sets $B_n^{k,l} \cong B_n^{l,k}$ which induces an isomorphism of rings $\widetilde{A}_n^{k,l} \cong \widetilde{A}_n^{l,k}$ and of the quotient rings $A_n^{k,l} \cong A_n^{l,k}$.
- The minimal idempotents in $A_n^{k,l}$ are $1_a \stackrel{\text{def}}{=} \mathbf{1}^{\otimes(n+k+l)/2} \in {}_a(A_n^{k,l})_a$. The unit element 1 of $A_n^{k,l}$ is the sum of 1_a over all $a \in B_n^{k,l}$: $1 \stackrel{\text{def}}{=} \sum_{a \in B_n^{k,l}} 1_a$.
- $A_n^{k,l}$ sits inside $A_n^{k+1,l+1}$ as a subring. This inclusion stabilizes when $k+l > n$. In particular, we have $A_n^{k,n-k} \cong A_n^{k+1,n-k+1} \cong A_n^{k+2,n-k+2} \cong \cdots$.

Proposition 1. *The rings $A_n^{0,l}$ are symmetric and, therefore, Frobenius over $\mathbb{Z}$.*

Proof. The proof is similar to [4], Proposition 32. □

2.3. Flat tangles and bimodules. Denote by $\widehat{B}_n^m$ the space of flat tangles with m top endpoints and n bottom endpoints. For simplicity we assume that the top and bottom endpoints lie on $\mathbb{R} \times \{1\}$ and $\mathbb{R} \times \{0\}$, and have integer coefficients $1, 2, \dots, m$ and $1, 2, \dots, n$ respectively. Figure 5 shows two elements in $\widehat{B}_6^4$.

To a flat tangle $T \in \widehat{B}_n^m$ we would like to assign a bimodule over algebras $A_n^{k,l}$ and $A_m^{s,t}$ where both (n,k,l) and (m,s,t) are coherent triples and $k-l = s-t$. Define a graded $(\widetilde{A}_m^{s,t}, \widetilde{A}_n^{k,l})$-bimodule $\widetilde{\mathcal{F}}(T)$ by

$$\widetilde{\mathcal{F}}(T) = \bigoplus_{b \in B_n^{k,l}, c \in B_m^{s,t}} {}_c\widetilde{\mathcal{F}}(T)_b,$$

where

$${}_c\widetilde{\mathcal{F}}(T)_b \stackrel{\text{def}}{=} \mathcal{F}(W(c)Tb)\Big\{\frac{n+k+l}{2}\Big\}.$$

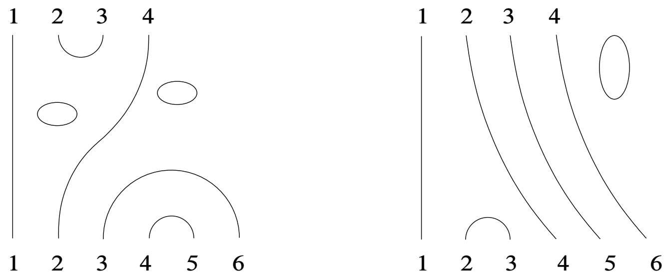


Figure 5. Two flat tangles in $\widehat{B}_6^4$.

The plane diagram $W(c)Tb$ is not a union of circles if $k \neq s$. In that case we close it in the obvious way before applying the functor $\mathcal{F}$ (see Figure 6). The left action $\widetilde{A}_m^{s,t} \times \widetilde{\mathcal{F}}(T) \to \widetilde{\mathcal{F}}(T)$ comes from maps

$$\mathcal{F}(W(a)c) \times {}_c\widetilde{\mathcal{F}}(T)_b \to {}_a\widetilde{\mathcal{F}}(T)_b,$$

and the right action $\widetilde{\mathcal{F}}(T) \times \widetilde{A}_n^{k,l} \to \widetilde{\mathcal{F}}(T)$ comes from maps

$${}_c\widetilde{\mathcal{F}}(T)_b \times \mathcal{F}(W(b)a) \to {}_c\widetilde{\mathcal{F}}(T)_a.$$

Both maps are induced by the obvious *minimal cobordism* (see Figure 2).

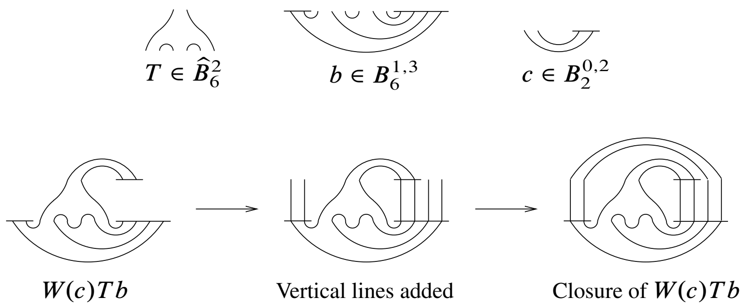


Figure 6. Closing $W(c)Tb$.

Now define a subgroup ${}_bI(T)_a$ of ${}_b\widetilde{\mathcal{F}}(T)_a$ as follows. Set ${}_bI(T)_a = {}_b\widetilde{\mathcal{F}}(T)_a$ if $W(b)Ta$ contains a type III arc. Otherwise, assuming that $\mathcal{F}(W(b)Ta) \cong \mathcal{A}^{\otimes r}$ in which type II circles correspond to the first i tensor factors, set ${}_bI(T)_a$ to be the span of

$$u_1 \otimes \cdots \otimes a_{j-1} \otimes X \otimes u_{j+1} \otimes \cdots \otimes u_r \in \mathcal{F}(W(b)Ta) \cong \mathcal{A}^{\otimes r},$$

where $1 \leq j \leq i$ and $u_e \in \{\mathbf{1}, X\}$ for each $1 \leq e \leq r$, $e \neq j$. By taking the direct sum we get a subgroup

$$I(T) \stackrel{\text{def}}{=} \bigoplus_{a\in B_m^{s,t},\ b\in B_n^{k,l}} {}_aI(T)_b.$$

$I(T)$ is in fact a subbimodule of $\widetilde{\mathcal{F}}(T)$ and we can define $\mathcal{F}(T)$ to be the quotient bimodule

$$\mathcal{F}(T) \stackrel{\text{def}}{=} \widetilde{\mathcal{F}}(T)/I(T).$$

It is easy to show that the action of $I_n^{k,l}$ on $\mathcal{F}(T)$ is trivial (see [1]), thus the $(\widetilde{A}_m^{s,t}, \widetilde{A}_n^{k,l})$-bimodule structure on $\mathcal{F}(T)$ descends to an $(A_m^{s,t}, A_n^{k,l})$-bimodule structure.

Proposition 2. *An isotopy between $T_1, T_2 \in \widehat{B}_n^m$ induces an isomorphism of bimodules $\mathcal{F}(T_1) \cong \mathcal{F}(T_2)$. Two isotopies between T_1 and T_2 induce equal isomorphisms iff the bijections from circle components of T_1 to circle components of T_2 induced by the two isotopies coincide.*

Proof. The proof is similar to that in [4]. □

Cobordisms between flat tangles induce bimodule maps (see Figure 7).

Proposition 3. *Let $T_1, T_2 \in \widehat{B}_n^m$ and S a cobordism between T_1 and T_2. Then S induces a degree $\frac{n+m}{2} - \chi(S)$ homomorphism of $(A_m^{s,t}, A_n^{k,l})$-bimodules*

$$\mathcal{F}(S)\colon \mathcal{F}(T_1) \to \mathcal{F}(T_2),$$

where $\chi(S)$ is the Euler characteristic of S.

Proof. It follows from the definition that $\widetilde{\mathcal{F}}(T_1) = \bigoplus_{a,b} \mathcal{F}(W(b)T_1a)\{\frac{n+k+l}{2}\}$ and $\widetilde{\mathcal{F}}(T_2) = \bigoplus_{a,b} \mathcal{F}(W(b)T_2a)\{\frac{n+k+l}{2}\}$, where the sum is over all $a \in B_n^{k,l}$ and $b \in B_m^{s,t}$. The surface S induces a homogeneous map of graded abelian groups $\mathcal{F}(W(b)T_1a) \to \mathcal{F}(W(b)T_2a)$. Summing over all a and b we get a homomorphism of $(\widetilde{A}_n^{k,l}, \widetilde{A}_m^{s,t})$-bimodules $\widetilde{\mathcal{F}}(S)\colon \widetilde{\mathcal{F}}(T_1) \to \widetilde{\mathcal{F}}(T_2)$. The grading assertion follows from the fact that $\chi(S') = \chi(S) - \frac{n+m}{2}$. It is easy to show that $\widetilde{\mathcal{F}}(S)$ takes $I(T_1)$ into $I(T_2)$. See [1] for details. □

Proposition 4. *Isotopic (rel boundary) surfaces induce equal bimodule maps.*

Proposition 5. *Let $T_1, T_2, T_3 \in \widehat{B}_n^m$ and S_1, S_2 be cobordisms from T_1 to T_2 and from T_2 to T_3 respectively. Then $\mathcal{F}(S_2)\mathcal{F}(S_1) = \mathcal{F}(S_2 \circ S_1)$.*

Proof. Proofs of the above two propositions are similar to those in [4]. □

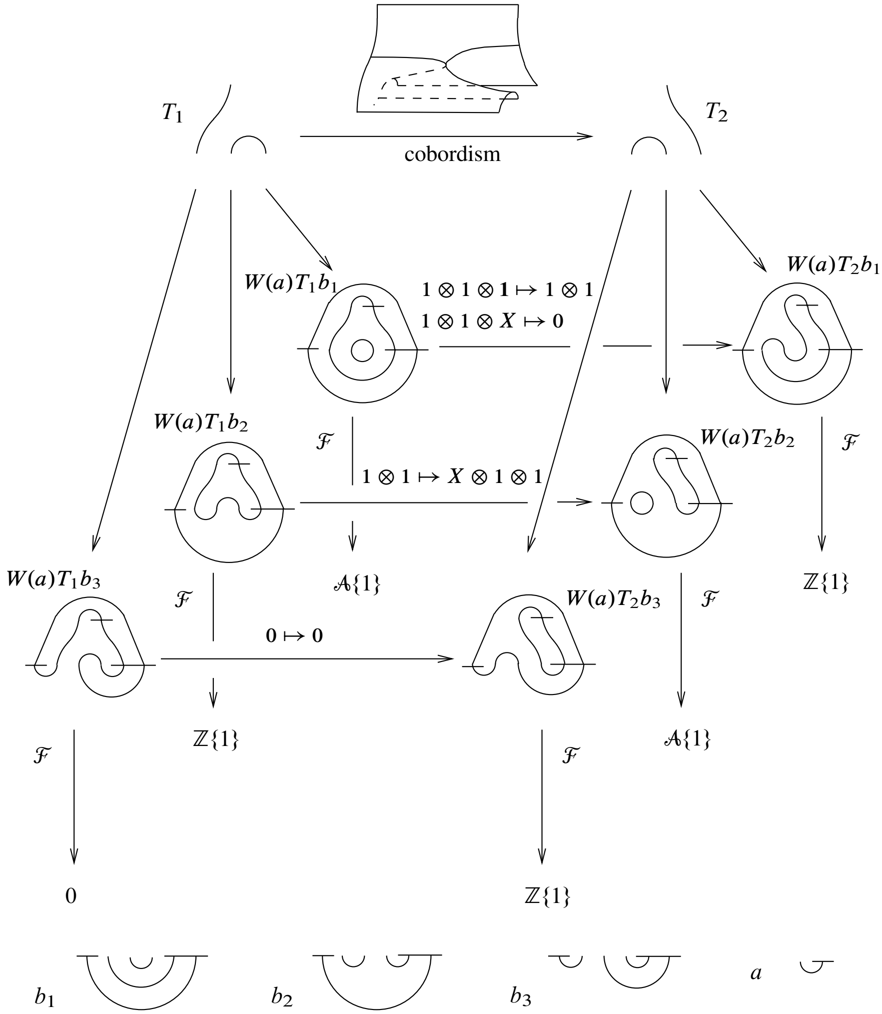


Figure 7. Cobordism induces bimodule map.

Two coherent triples (n, k, l) and (m, s, t) are called *T-compatible* if $k + l = n$, $s + t = m$, and $t = l + \frac{m-n}{2}$. On the other hand, they are called *F-compatible* if $k = s$ and $l = t$. A $(A_m^{s,t}, A_n^{k,l})$-bimodule is called *T-compatible* (*F-compatible*) if (n, k, l) and (m, s, t) are *T-compatible* (*F-compatible*).

Proposition 6. *Let $T \in \widehat{B}_n^m$, bimodule $\mathcal{F}(T)$ is projective as a left $A_m^{s,t}$-module and as a right $A_n^{k,l}$-module if (n, k, l) and (m, s, t) are compatible.*

Proof. Ignore all the grading shifts. The bimodule $\mathcal{F}(T)$ is isomorphic, as a left

$A_m^{s,t}$-module, to the direct sum $\bigoplus_{a\in B_n^{k,l}} \mathcal{F}(Ta)$. To prove $\mathcal{F}(T)$ is left projective it suffices to prove that $\mathcal{F}(Ta)$ is left projective for all $a \in B_n^{k,l}$. Fix any $a \in B_n^{k,l}$. In general, Ta is a union of circles and arcs. With all circles removed, Ta is isotopic to some $a' \in B_m^{k,l}$ (see Figure 8).

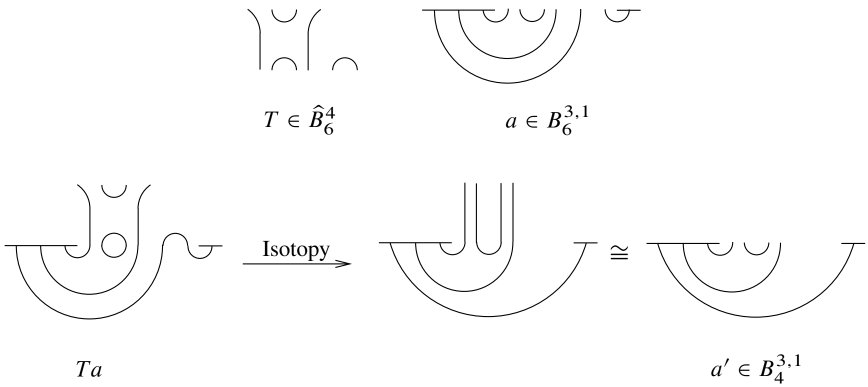


Figure 8. Deformation of Ta.

Case 1: $k = s$ and $l = t$. In this case, assuming there are c circles in Ta,

$$\mathcal{F}(Ta) = \Big(\bigoplus_{b\in A_m^{s,t}} \mathcal{F}(W(b)Ta)\Big) \otimes \mathcal{A}^{\otimes c} \cong \Big(\bigoplus_{b\in A_m^{s,t}} \mathcal{F}(W(b)a')\Big) \otimes \mathcal{A}^{\otimes c}.$$

By definition $\bigoplus_{b\in A_m^{s,t}} \mathcal{F}(W(b)a') = P_m^{s,t}(a')$, therefore $\mathcal{F}(Ta)$ is left projective.

Case 2: $k + l = n$, $s + t = m$, $t = l + \frac{m-n}{2}$. Without loss of generality we assume that $m \leq n$. Let $\Theta = \frac{n-m}{2}$. The case $\Theta = 0$ is proved in case 1. Suppose the statement is true when $\Theta \leq d$. Consider any $T \in \hat{B}_n^m$ and $a \in B_n^{k,l}$ such that $\frac{n-m}{2} = d + 1$. There exists at least one *cap* in T which connects two bottom endpoints since $n > m$. Pick a cap c which has no other bottom endpoints of T between its two feet. After gluing a to T, either there is an arc in a connecting the two platforms, or both feet of c is connected to the platforms since $k + l = n$. Therefore there is always an arc connecting the two platforms in Ta. By definition of the arc ring, the two far ends of the two platforms are then connected by an arc e. When closing the graph $W(b)Ta$ for some $b \in B_m^{s,t}$ we need to add $d + 1$ arcs since $t < l$ (see Figure 6). Denote the topmost added arc by f. The arcs e and f form a type II circle g which encloses the rest of $W(b)Ta$. We can remove g from $W(b)Ta$ for any $b \in B_m^{s,t}$ since it contributes nothing to $\mathcal{F}(W(b)Ta)$, and then reduce to the case $\Theta = d$. The proposition follows by induction. $\square$

Proposition 7. *Let $T_1 \in \hat{B}_n^p$, $T_2 \in \hat{B}_p^m$, $\mathcal{F}(T_1)$ be a compatible $(A_p^{q,r}, A_n^{k,l})$-bimodule, and $\mathcal{F}(T_2)$ be a compatible $(A_m^{s,t}, A_p^{q,r})$-bimodule. Then there is a canon-*

ical isomorphism of $(A_m^{s,t}, A_n^{k,l})$*-bimodules*

$$\mathcal{F}(T_2T_1) \cong \mathcal{F}(T_2) \otimes_{A_p^{q,r}} \mathcal{F}(T_1).$$

Proof. It follows from Proposition 6 that $W(a)T_2$ is a projective right $A_p^{q,r}$-module and T_1b is a projective left $A_p^{q,r}$-module for $a \in B_m^{s,t}$ and $b \in B_n^{k,l}$. The proof in [4], Theorem 1, therefore works in our case without any changes. □

Remark. If two flat tangles T_1 and T_2 belong to the same type, T_2T_1 is then compatible and also belongs to that type. Therefore we can compose as many flat tangles as we want within the same type. However, if T_1 and T_2 belong to different types their composition T_2T_1 may not be compatible.

Now consider only F-compatible triples and bimodules for the rest of the section. For each n such that (n, k, l) is coherent, denote by $A_n^{k,l}$-mod the category of finitely-generated graded left $A_n^{k,l}$-modules and module maps. For each $T \in \widehat{B}_n^m$, tensoring with the $(A_m^{k,l}, A_n^{k,l})$-bimodule $\mathcal{F}(T)$ is an exact functor from $A_n^{k,l}$-mod to $A_m^{k,l}$-mod. A cobordism S between two flat tangles $T_1, T_2 \in \widehat{B}_n^m$ induces a homomorphism $\mathcal{F}(S)$ of $(A_m^{k,l}, A_n^{k,l})$-bimodules. The following proposition is a summary of this section.

Proposition 8. *For each pair* (k, l)*, bimodules* $\mathcal{F}(T)$ *and homomorphisms* $\mathcal{F}(S)$ *assemble into a* 2*-functor from the* 2*-category of flat tangle cobordisms to the* 2*-category of natural transformations between exact functors between* $A_n^{k,l}$*-mod.*

2.4. Tangles and complexes of bimodules. A (m, n)*-tangle* L is a proper embedding of $\frac{n+m}{2}$ oriented arcs and a finite number of oriented circles into $\mathbb{R}^2 \times [0, 1]$ such that the boundary points of arcs map to

$$\{1, 2, \ldots, n\} \times \{0\} \times \{0\}, \{1, 2, \ldots, m\} \times \{0\} \times \{1\}.$$

A plane diagram of a tangle is a generic projection of the tangle onto $\mathbb{R} \times [0, 1]$.

Fix k and l throughout the rest of this section. We would like to define a tangle invariant using the rings $A_n^{k,l}$. The construction follows the same line as in [1]. The sizes of the platforms do not matter. We will state the results here for completeness and refer readers to [1] and [4] for details.

Fix a diagram D of an oriented (m, n)-tangle L. We define the complex of $(A_m^{k,l}, A_n^{k,l})$-bimodules $\mathcal{F}(D)$ associated to D inductively as follows.

- If D has no crossings (therefore a flat tangle), $\overline{\mathcal{F}}(D)$ is just the complex

  $$0 \to \mathcal{F}(D) \to 0,$$

  where $\mathcal{F}(D)$, sitting in cohomological degree zero, is the bimodule associated to the flat tangle D.

- If D has one crossing, consider the complex $\overline{\mathcal{F}}(D)$ of $(A_m^{k,l}, A_n^{k,l})$-bimodules
$$0 \to \mathcal{F}(D(0)) \xrightarrow{\partial} \mathcal{F}(D(1))\{-1\} \to 0$$
where $D(i), i = 0, 1$ denotes the i-smoothing of the crossing, ∂ is induced by the *saddle* cobordism (see Figure 9), and $\mathcal{F}(D(0))$ sits in the cohomological degree zero.
- To a diagram with $t + 1$ crossings we associate the total complex $\overline{\mathcal{F}}(D)$ of the bicomplex
$$0 \to \mathcal{F}(D(c_0)) \xrightarrow{\partial} \mathcal{F}(D(c_1))\{-1\} \to 0$$
where $D(c_i), i = 0, 1$ denotes the i-smoothing of a crossing c of D.
- Finally, define $\mathcal{F}(D)$ to be $\overline{\mathcal{F}}(D)$ shifted by $[x(D)]\{2x(D) - y(D)\}$, where $x(D)$ counts the number of negative crossings and $y(D)$ counts the number of positive crossings (see Figure 9).

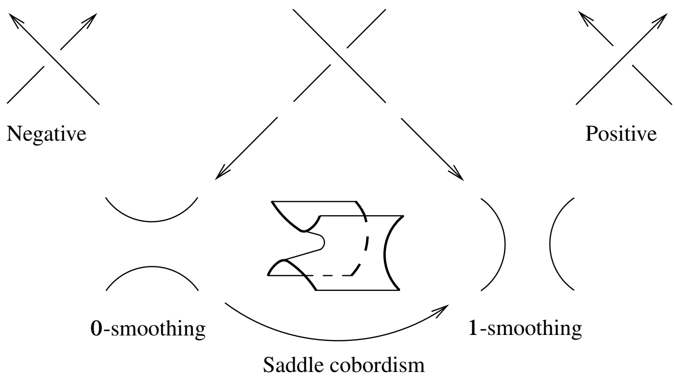


Figure 9. Two smoothings of a crossing.

Figure 10 shows a complex of bimodules associated to a $(2, 2)$-tangle. Each arrow is induced by the saddle cobordism and the sign on each arrow indicates the sign of each map in the total complex.

Theorem 1. *If D_1 and D_2 are two diagrams of an oriented (m, n)-tangle L, then the complexes $\mathcal{F}(D_1)$ and $\mathcal{F}(D_2)$ of graded $(A_m^{k,l}, A_n^{k,l})$-bimodules are chain homotopy equivalent.*

The following proposition is a special case of the more general Theorem 3 in Section 3.

Proposition 9. *The only invertible degree zero central elements in $A_n^{k,l}$ are ± 1*
$$Z_0^*(A_n^{k,l}) \cong \{\pm 1\}.$$

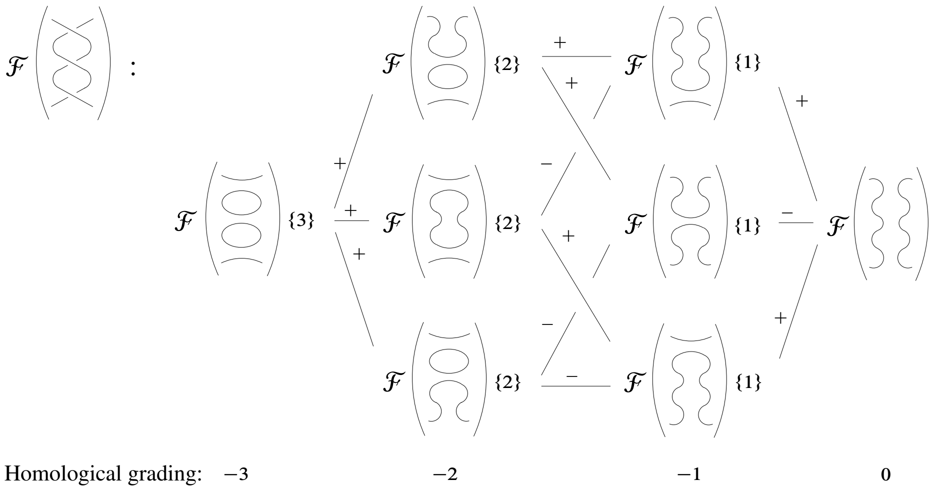


Figure 10. A total complex associated to a $(2, 2)$-tangle.

We now extend our invariant to oriented tangle cobordisms. Let S be a movie presentation of a cobordism between two (m, n)-tangles. S is thus a sequence of Reidemeister moves and handle moves. Each consequent pair of frames corresponds to a homomorphism which is an isomorphism for each Reidemeister move, and is induced by ι, ε, m, or Δ for each handle move. The composition of these homomorphisms gives us a homomorphism

$$\mathcal{F}(S)\colon \mathcal{F}(D) \longrightarrow \mathcal{F}(D'),$$

where D and D' are the first and the last frame in the movie S. It follows from Proposition 9 that $\mathcal{F}(S) = \pm\mathcal{F}(S')$ if S and S' are two different presentations of the same cobordism.

Denote by $\mathcal{C}_{\mathrm{cob}}$ the 2-category of oriented tangle cobordisms and by $\mathcal{C}_{A_n^{k,l}}$ the 2-category of natural transformations of exact functors between homotopy categories of complexes of graded $A_n^{k,l}$-modules. We have the following theorem.

Theorem 2. *Complexes $\mathcal{F}(T)$ of bimodules and homomorphisms $\pm\mathcal{F}(S)$ assigned to diagrams of tangle cobordisms assemble into a projective 2-functor from $\mathcal{C}_{\mathrm{cob}}$ to $\mathcal{C}_{A_n^{k,l}}$.*

Remark. The projective Grothendieck group $K_p(A_n^{k,l} - \mathrm{gmod})$ of the category of finitely-generated graded projective $A_n^{k,l}$-modules is a free $\mathbb{Z}[q, q^{-1}]$-module with a basis $[P_n^{k,l}(a)]$, $a \in B_n^{k,l}$. There is a natural way to identify $K_p(A_n^{k,l} - \mathrm{gmod})$ with a $\mathbb{Z}[q, q^{-1}]$-lattice of $\mathrm{Hom}(V_k \otimes V_l, V^{\otimes n})$

$$K_p(A_n^{k,l} - \mathrm{gmod}) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C} \cong \mathrm{Hom}(V_k \otimes V_l, V^{\otimes n}), \tag{7}$$

where V is the fundamental 2-dimensional representation of $U_q(sl_2)$ and V_i is the irreducible $(i+1)$-dimensional representation of $U_q(sl_2)$. Under this isomorphism the basis $[P_n^{k,l}(a)]$ goes to dual canonical basis of

$$\mathrm{Hom}(V_k \otimes V_l, V^{\otimes n}) \cong \mathrm{Inv}(V_k^* \otimes V_l^* \otimes V^{\otimes n}) \cong \mathrm{Inv}(V_k \otimes V_l \otimes V^{\otimes n}).$$

Denote by $\mathcal{K}(\mathcal{W})$ the category of bounded complexes of objects of an abelian category $\mathcal{W}$ up to chain homotopies. For each (m,n)-tangle T, it follows from Proposition 6 that the complex of bimodules $\mathcal{F}(T)$ consists of right projective bimodules. Therefore the tensor product with $\mathcal{F}(T)$ is an exact functor from $\mathcal{K}(A_n^{k,l}\text{-gmod})$ to $\mathcal{K}(A_m^{k,l}\text{-gmod})$ which induces a homomorphism $[\mathcal{F}(T)]$ of $\mathbb{Z}[q,q^{-1}]$-modules

$$K_p(A_n^{k,l}\text{-gmod}) \longrightarrow K_p(A_m^{k,l}\text{-gmod}).$$

Direct computation shows that under the isomorphism (7) they give the standard action of tangles on $\mathrm{Inv}(V_k \otimes V_l \otimes V^{\otimes n})$.

3. The center of $A_n^{k,l}$

Let $\mathcal{B}_{\sigma_1,\sigma_2}$ be the Springer variety of complete flags in $\mathbb{C}^n$ stabilized by a fixed nilpotent operator with two Jordan blocks of sizes σ_1 and σ_2 respectively. We prove in this section that the center of the ring $A_n^{k,l}$ is isomorphic to the cohomology ring of $\mathcal{B}_{\sigma_1,\sigma_2}$. Following from Khovanov's construction in [5], we introduce the space $\widetilde{S}$ and use it as a bridge to link the center of $A_n^{k,l}$ and the cohomology rings of Springer varieties. Without loss of generality, we assume throughout this section that $n \geq m$, $n+m \equiv 0 \bmod 2$, and $0 \leq l-k \leq n$ (note that $A_n^{k,l}$ is trivial if $l-k > n$). The proofs in this section rely heavily on [5].

Theorem 3. *The center of $A_n^{k,l}$ is isomorphic to the cohomology ring of $\mathcal{B}_{\sigma_1,\sigma_2}$*

$$Z(A_n^{k,l}) \cong H^*(\mathcal{B}_{\sigma_1,\sigma_2}),$$

where $\sigma_1 = \frac{n+l-k}{2}$ and $\sigma_2 = \frac{n-l+k}{2}$.

Denote by S the 2-sphere S^2 and let p be the north pole of S. Let $S^{\times n}$ be the direct product of n spheres

$$S^{\times n} \stackrel{\text{def}}{=} \underbrace{S \times S \times \cdots \times S}_{n}.$$

Label the n free points of $B_n^{k,l}$ by $1,2,\ldots,n$ from left to right. For each $a \in B_n^{k,l}$ define a submanifold $S_a \in S^{\times n}$ consisting of sequences $(x_1,\ldots x_n)$, $x_i \in S$, such that $x_i = x_j$ whenever (i, j) is a type I arc in a, and $x_s = p$ if s is connected to a platform. Let $\widetilde{S}_n^{k,l}$ be the subspace of $S^{\times n}$ which is the union of all S_a

$$\widetilde{S}_n^{k,l} \overset{\text{def}}{=} \bigcup_{a \in B_n^{k,l}} S_a.$$

When there is no confusion we write $\widetilde{S}$ instead of $\widetilde{S}_n^{k,l}$.

Note that the cohomology ring of S_a is isomorphic to the ring ${}_a(A_n^{k,l})_a$, and the cohomology ring of $S_a \cap S_b$, viewed as abelian group, is isomorphic to ${}_a(A_n^{k,l})_b$. These observations lead to the following theorem.

Theorem 4. *The center of $A_n^{k,l}$ is isomorphic to the cohomology ring of $\widetilde{S}$*

$$Z(A_n^{k,l}) \cong H^*(\widetilde{S}, \mathbb{Z}).$$

Proof. Denote by $H(Y)$ the cohomology ring of the space Y with integer coefficients. As noted above, we have $H(S_a) \cong {}_a(A_n^{k,l})_a$ and $H(S_a \cap S_b) \cong {}_a(A_n^{k,l})_b$. The second isomorphism allows us to make ${}_a(A_n^{k,l})_b$ into a ring with unit $1 \overset{\text{def}}{=} 1^s \in \mathcal{F}(W(b)a) \cong \mathcal{A}^{\otimes s}$.

We have natural ring homomorphisms induced by inclusions

$$\psi_{a;a,b} \colon H(S_a) \to H(S_a \cap S_b), \quad \psi_{b;a,b} \colon H(S_b) \to H(S_a \cap S_b),$$

and also

$$\gamma_{a;a,b} \colon {}_a(A_n^{k,l})_a \to {}_a(A_n^{k,l})_b, \quad \gamma_{b;a,b} \colon {}_b(A_n^{k,l})_b \to {}_a(A_n^{k,l})_b,$$

which are given by $x \mapsto x\,{}_a1_b$ and $x \mapsto {}_a1_b x$. Assemble all these together we get a commutative diagram of ring homomorphisms

$$\begin{tikzcd}
H(\widetilde{S}) \arrow[r, "\tau"] & \mathrm{Eq}(\psi) \arrow[r] \arrow[d, "\cong"] & \prod_a H(S_a) \arrow[r, "\psi"] \arrow[d, "\cong"] & \prod_{a \neq b} H(S_a \cap S_b) \arrow[d, "\cong"] \\
Z(A_n^{k,l}) \arrow[r, "\cong"] & \mathrm{Eq}(\gamma) \arrow[r] & \prod_a {}_a(A_n^{k,l})_a \arrow[r, "\gamma"] & \prod_{a \neq b} {}_a(A_n^{k,l})_b
\end{tikzcd}$$

where

$$\psi = \sum_{a \neq b} (\psi_{a;a,b} + \psi_{b;a,b}) \quad \text{and} \quad \gamma = \sum_{a \neq b} (\gamma_{a;a,b} + \gamma_{b;a,b}).$$

$\mathrm{Eq}(\alpha)$ is the *equalizer* of the map α (see [5]). For example, $\mathrm{Eq}(\psi)$ is a subring of $\prod_a H(S_a)$ consisting of $\times_a h_a$ such that if $h_a \in H(S_a)$ and $h_b \in H(S_b)$ then their images in $H(S_a \cap S_b)$ under ψ are equal.

For all $x \in A_n^{k,l}$ write x as $\sum_{a,b \in B_n^{k,l}} {}_a x_b$. Assuming x is central we have $x 1_b 1_a = 1_b x 1_a = {}_a x_b$. Therefore ${}_a x_b = 0$ if $a \neq b$. So $x = \sum_a {}_a x_a$ is central if and only if $({}_a x_a)({}_a 1_b) = ({}_a 1_b)({}_b x_b)$, which means $Z(A_n^{k,l}) \cong \mathrm{Eq}(\gamma)$. The ring homomorphism $H(\widetilde{S}) \to \prod_a H(S_a)$ factors through $\mathrm{Eq}(\psi)$. To prove Theorem 2 it suffices to show that τ is an isomorphism.

For $a, b \in B_n^{k,l}$ write $a \to b$ if there exists a *horizontal merging* of two arcs (see Figure 11).

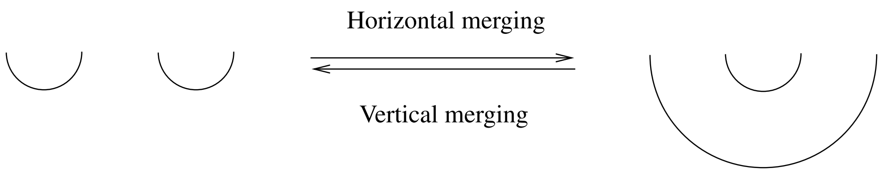


Figure 11. Horizontal and vertical mergings of two arcs.

Introduce a partial order on $B_n^{k,l}$ by setting $a \prec b$ if there is a chain of arrows $a \to a_1 \to \cdots \to a_m \to b$. Extend this partial order arbitrarily to a total order $<$ on $B_n^{k,l}$. See Figure 12 for arrow relations and ordering of $B_5^{0,1}$ ($a_i < a_j$ if and only if $i < j$).

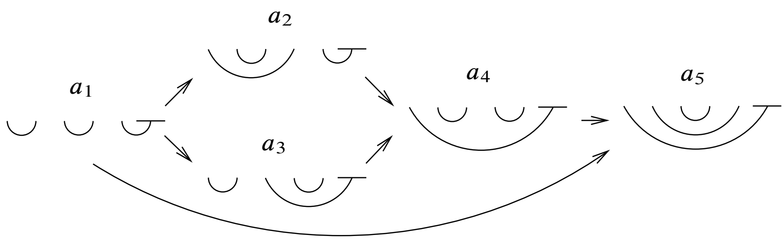


Figure 12. Arrow relations and ordering of $B_5^{0,1}$.

We would like to construct a cell decomposition of S_a. Associate a decorated graph Γ to $a \in B_n^{0,m} \subset B^{\frac{n+m}{2}}$ as follows (see Figure 13 for an example).

- Each type I arc x_i in a corresponds to a *hollow* vertex i in $\widetilde{\Gamma}$.
- Each type II arc x_j in a corresponds to a *solid* vertex j in $\widetilde{\Gamma}$.
- Two vertices i, j in $\widetilde{\Gamma}$ are connected by an edge iff the result of merging x_i and x_j *vertically* still lies in $B^{\frac{n+m}{2}}$.
- Γ is obtained from $\widetilde{\Gamma}$ by contracting all edges with two *solid* ends.
- Mark a vertex in each connected component of Γ without *solid* vertices.

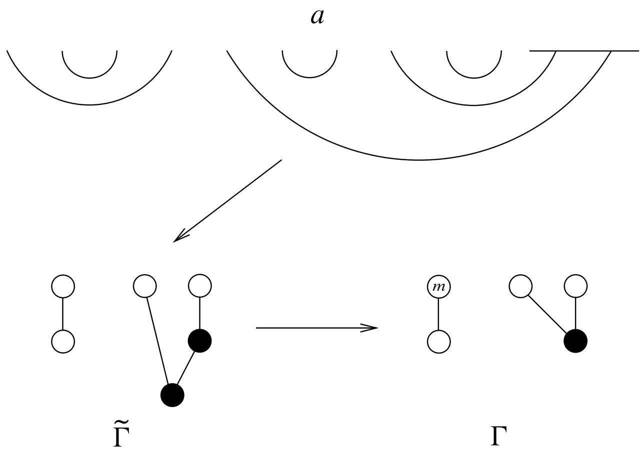


Figure 13. An element $a \in \widetilde{B}^{0,2}_{10}$ and its associated graph Γ.

Let E be the set of edges, M be the set of marked points, and I be the set of vertices in Γ. For each $J \subset (E \sqcup M)$ let $c(J)$ be the subset of $S^{\times I}$ consisting of points $\{y_i\}_{i\in I}, y_i \in S$ such that

$$\begin{aligned} y_i &= y_j \quad \text{if } (i,j) \in J, \\ y_i &\neq y_j \quad \text{if } (i,j) \notin J, \\ y_i &= p \quad \text{if } i \in M \cap J, \\ y_i &= p \quad \text{if } i \text{ is } \textit{solid,} \\ y_i &\neq p \quad \text{if } i \in M, i \notin J, \end{aligned}$$

where (i,j) denotes the edge connecting i and j. Ignore the first two conditions if (i,j) does not exist. Clearly, $S^{\times I} = \bigsqcup_J c(J)$ and $c(J)$ is an open cell of dimension $2(|I| - |J| - \#$ of solid vertices). We thus obtain a decomposition of S_a into even dimensional cells.

Lemma 1. *$S_{<a} \cap S_a = (\cup_{b\to a} S_b) \cap S_a$, where $S_{<a} = \bigcup_{b<a} S_b$.*

The next lemma follows from Lemma 1 and the above construction.

Lemma 2. *The cell decomposition constructed above restricts to a cell decomposition of $S_a \backslash S_{<a}$, which is a union of cells $c(J)$ such that $J \cap E = \emptyset$.*

We thus obtain a *cell partition* of $\widetilde{S}$ by adding cells in $S_a \backslash S_{<a}$ following the total order. Since there are only even-dimensional cells in the partition, the rank of $H(\widetilde{S})$ is equal to the number of cells.

By induction on a with respect to the total order $<$ we get the following result, see [5].

Proposition 10. *$S_{\leq a}$ has cohomology in even degrees only and the following sequence is exact*

$$0 \to H(S_{\leq a}) \xrightarrow{\varphi} \bigoplus_{b \leq a} H(S_b) \xrightarrow{\psi^-} \bigoplus_{b<c\leq a} H(S_b \cap S_c), \tag{8}$$

where φ is induced by inclusions $S_b \subset S_{\leq a}$, while

$$\psi^- \stackrel{\text{def}}{=} \sum_{b<c\leq a} (\psi_{b,c} - \psi_{c,b}),$$

where

$$\psi_{b,c} \colon H(S_b) \to H(S_b \cap S_c)$$

is induced by the inclusion $(S_b \cap S_c) \subset S_b$.

When a is maximal with respect to the total order $<$, the exact sequence (8) becomes

$$0 \to H(\widetilde{S}) \xrightarrow{\varphi} \bigoplus_{b} H(S_b) \xrightarrow{\psi^-} \bigoplus_{b<c} H(S_b \cap S_c),$$

which means that $H(\widetilde{S})$ is isomorphic to the equalizer of ψ. □

Lemma 3. *The center of $A_n^{k,l}$ is isomorphic to the center of $A_n^{0,l-k}$,*

$$Z(A_n^{k,l}) \cong Z(A_n^{0,l-k}).$$

Proof. It follows from the definition of $\widetilde{S}$ that

$$\widetilde{S} \cong \bigcup_{a \in \widetilde{B}_n^{k,l}} S_a,$$

where $\widetilde{B}_n^{k,l} \subset B_n^{k,l}$ is the set of crossingless matchings with all points on the left platform connected to the right platform. On the other hand, since the bottommost type II arcs contribute nothing, we have (see Figure 14)

$$\bigcup_{a \in \widetilde{B}_n^{k,l}} S_a \cong \bigcup_{a \in B_n^{0,l-k}} S_a.$$

Theorem 2 and the above observations imply that

$$Z(A_n^{k,l}) \cong H(\widetilde{S}_n^{k,l}) \cong H(\widetilde{S}_n^{0,l-k}) \cong Z(A_n^{0,l-k}).$$ □

$\widetilde{B}_3^{2,3}$:

$B_3^{0,1}$:

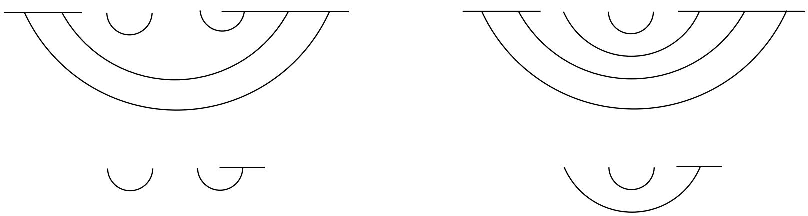


Figure 14. Removing bottommost type II arcs.

Proposition 11. *$\widetilde{S}_n^{0,m}$ has $\binom{n}{\frac{n-m}{2}}$ cells in the partition constructed above.*

Proof. The proposition is equivalent to the statement that $\widetilde{S}_{2s-k}^{0,k}$ has $\binom{2s-k}{s-k}$ cells. Fix the total number of points $2s$ (including marked and free). Induction on the size of the right platform k.

Induction base $k = 0$ is proved in [5], Lemma 4.1. Assuming the statement is true up to k, it suffices to prove that extending the size of the platform by 1 eliminates $\binom{2s-k}{s-k}$ - $\binom{2s-k-1}{s-k-1}$ cells in $\widetilde{S}_{2s-k}^{0,k}$. If we label the $2s$ points by $1, 2, 3, \ldots, 2s$ from right to left, the vanished cells are exactly those in S_a where a has a type II arc connecting k and $k+1$ (see Figure 15). Denote the set of those a by $a(k, k+1)$.

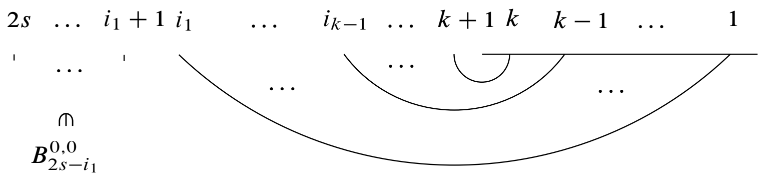


Figure 15. A generic element in $a(k, k+1)$.

We thus get the formula

$$\vartheta\Big(\bigcup_{a(k,k+1)\in B_{2s-k}^{0,k}} S_{a(k,k+1)}\Big) = \sum_{i=0}^{s-k}(i+1)C_i\Big[\Big(\frac{1-\sqrt{1-4x}}{2x}\Big)^{k-1}\Big]_i,$$

where $\vartheta(X)$ denotes the number of cells in X, C_i denotes the n-th Catalan number, and $[f(x)]_i$ is the coefficient of x^i in the Maclaurin expansion of $f(x)$. Recall that $\frac{1-\sqrt{1-4x}}{2x}$ is the generating function of $\{C_i\}$. The factor $(i+1)C_i$ corresponds to the number of cells outside the bottommost type II arc and $[(\frac{1-\sqrt{1-4x}}{2x})^{k-1}]_i$ is equal to the total number of crossingless matchings inside the bottommost type II arc. Note that the arcs inside the bottommost type II arc contribute nothing to the cell structure, see Figure 13. After simplifying the above formula we get

$$\vartheta\Big(\bigcup_{a(k,k+1)\in B^{0,k}_{2s-k}} S_{a(k,k+1)}\Big) = \Big[\frac{1}{\sqrt{1-4x}}\Big(\frac{1-\sqrt{1-4x}}{2x}\Big)^{k-1}\Big]_{s-k}.$$

By induction on s and k it is easy to prove that

$$\Big[\frac{1}{\sqrt{1-4x}}\Big(\frac{1-\sqrt{1-4x}}{2x}\Big)^{k-1}\Big]_{s-k} = \binom{2s-k}{s-k} - \binom{2s-k-1}{s-k-1}$$

and the proposition follows. □

Proposition 12. [2] *The cohomology ring of $\mathcal{B}_{\frac{n+m}{2},\frac{n-m}{2}}$ has dimension $\binom{n}{\frac{n-m}{2}}$ and is isomorphic to the quotient ring of $R = \mathbb{Z}[X_1,\dots,X_n]$ by the ideal R_1 generated by $e_k(I)$ for all $k+|I| = n+1$, X_I for all $|I| = \frac{n-m}{2}+1$, and X_i^2 for $i \in [1,n]$, where*

$$I \subset \{1,2,\dots,n\}, \quad X_I = \prod_{i\in I} X_i, \quad e_k(I) = \sum_{|J|=k, J\subset I} X_J.$$

We now prove the main theorem of this section.

Proof of Theorem 3. It follows from Theorem 4 and Lemma 3 that to prove Theorem 3 it suffices to show that $H(\widetilde{S}^{0,m}_n) \cong H(\mathcal{B}_{\frac{n+m}{2},\frac{n-m}{2}})$. Denote by X a generator of $H^2(S)$. We have the following maps

$$\widetilde{S}^{0,m}_n \xrightarrow{\ \iota\ } S^{\times n} \xrightarrow{\ \psi_i\ } S',$$

where ι is the inclusion and ψ_i is the projection onto the i-th component. Define $X_i \in \widetilde{S}^{0,m}_n$ to be the pull back of X under the map $\psi_i \circ \iota$

$$X_i \stackrel{\text{def}}{=} (-1)^i \iota^* \circ \psi_i^*(X).$$

It is obvious that those $\{X_i\}$ generate $H(\widetilde{S}^{0,m}_n)$. It follows from Proposition 12 and Proposition 11 that to prove the theorem we only need to verify the following relations:

$$X_i^2 = 0, \quad i \in [1,n]; \tag{9}$$

$$X_I = 0, \quad |I| = \frac{n-m}{2}+1; \tag{10}$$

$$e_k(I) = 0, \quad k+|I| = n+1. \tag{11}$$

The first two relations are obvious. Consider the map $i_a^*\colon H(\widetilde{S}^{0,m}_n) \to H(S_a)$ induced by the inclusion $i_a\colon S_a \hookrightarrow \widetilde{S}^{0,m}_n$. Since $\sum_a i_a^*(H(\widetilde{S}^{0,m}_n)) \to \oplus_a H(S_a)$ is an inclusion, (11) will follow once we verify that

$$\sum_{|J|=k, J\subset I} i_a^*(X_J) = 0, \qquad k+|I| = n+1 \tag{12}$$

for all $a \in B_n^{0,m}$.

Fix any $a \in B_n^{0,m}$ and $I \subset \{1, 2, \ldots, n\}$. Since $n - |I| = k - 1$ there exists at most $k - 1$ type I arcs where I intersects with each one at only one point. Therefore, for each $J \subset I$ such that $|J| = k$, J must either contain an end point of a type II arc or contain an end point of a type I arc (p_1, p_2) such that $\{p_1, p_2\} \in I$. If J contains an end point of a type II arc, then $i_a^*(X_J) = 0$. For a type I arc (p_1, p_2), because of the term $(-1)^i$ in the definition of X_i and the fact that $p_1 + p_2$ is odd, we have $i_a^*(X_{p_1} X_{p_2}) = 0$ and $i_a^*(X_{p_1} + X_{p_2}) = 0$. Therefore

$$\sum_{J \subset I, |J|=k, \{p_1, p_2\} \cap J \neq \emptyset} i_a^*(X_J) = 0.$$

For the remaining terms in the summation in (12), pick another type I arc and repeat the above process. After finitely many reductions we can get the relation (12). □

4. Categorification of level two representations of quantum sl_N

4.1. Level two representations of quantum sl_N. Let $V, \wedge^2 V, \ldots, \wedge^{N-1} V$ be the irreducible representations of $U_q(sl_N)$ with highest weights $\omega_1, \omega_2, \ldots, \omega_{N-1}$ respectively, where $\omega_i = L_1 + \cdots + L_i$ and the L_j's are the fundamental weights. A level two representation W of $U_q(sl_N)$ is an irreducible representation with the highest weight $\lambda = \omega_s + \omega_{s+k}$ for some $k \geq 0$. Fix W for the rest of this paper. W decomposes into weight spaces $W = \bigoplus_\mu W_\mu$. Following [3], we call μ *admissible* if μ appears in the weight space decomposition of W. A weight μ is admissible if and only if it can be written as the sum $\mu_1 L_1 + \mu_2 L_2 + \cdots + \mu_N L_N$ such that

- $0 \leq \mu_i \leq 2$, for all $1 \leq i \leq N$,
- $\sum_{i=1}^N \mu_i = 2s + k$, and
- $\mu_1 + \cdots + \mu_i \leq \lambda_1 + \cdots + \lambda_i$, for all $1 \leq i \leq N$,

where λ_i is the coefficient of L_i in the decomposition

$$\lambda = \omega_s + \omega_{s+k} = (L_1 + \cdots + L_s) + (L_1 + \cdots + L_{s+k}).$$

For each admissible weight μ let $m(\mu)$ be the number of 1's in the sequence $(\mu_1, \ldots, \mu_N)$. The dimension of W_μ is then determined by $m(\mu)$.

Recall that $U_q(sl_N)$ is defined to be the algebra generated by E_i, F_i, K_i, and K_i^{-1} for $1 \leq i \leq N-1$ with relations

$$\begin{aligned}
&K_i K_i^{-1} = 1 = K_i^{-1} K_i,\\
&K_i K_j = K_j K_i,\\
&K_i E_j = q^{c_{i,j}} E_j K_i,\\
&K_i F_j = q^{-c_{i,j}} F_j K_i,\\
&E_i F_j - F_j E_i = \delta_{i,j} \tfrac{K_i - K_i^{-1}}{q - q^{-1}},\\
&E_i E_j = E_j E_i \quad \text{if } |i-j| > 1,\\
&F_i F_j = F_j F_i \quad \text{if } |i-j| > 1,\\
&E_i^2 E_{i\pm1} - (q+q^{-1}) E_i E_{i\pm1} E_i + E_{i\pm1} E_i^2 = 0,\\
&F_i^2 F_{i\pm1} - (q+q^{-1}) F_i F_{i\pm1} F_i + F_{i\pm1} F_i^2 = 0.
\end{aligned} \tag{13}$$

E_i acts on W by sending weight space W_μ to $W_{\mu+\varepsilon_i}$ and F_i maps W_μ to $W_{\mu-\varepsilon_i}$, where $\varepsilon_i = (\underbrace{0,\ldots,0}_{i-1}, 1, -1, 0, \ldots, 0)$.

4.2. Semi-standard tableaux and arc rings. We give in this section an explicit bijection between semi-standard tableaux and crossingless matchings with one platform. First recall the definition of semi-standard tableaux. For any $\lambda = (\lambda_1, \ldots, \lambda_{N-1}, 0)$ in the weight lattice of $U_q(sl_N)$, there exists an irreducible representation W_λ with the highest weight λ. Weight $\mu = (\mu_1, \ldots, \mu_N)$ appears in the weight space decomposition of W if and only if μ is admissible. The dimension of the weight space $W_\lambda(\mu)$ equals to the number of ways one can fill the Young diagram corresponding to λ with μ_1 1's, μ_2 2's, …, and μ_N N's in such a way that each column is strictly increasing and each row is non-decreasing. Each such filling is called a *semi-standard tableau* (see Figure 16).

$\lambda = (2, 2, 1, 0)$ $\qquad \begin{array}{|c|c|}\hline & \\ \hline & \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$

$\mu = (1, 1, 2, 1)$ $\qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 3 \\ \hline 4 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$

Figure 16. Semi-standard tableaux.

The Young diagram Y_λ corresponding to the level two representation W with the highest weight $\lambda = \omega_s + \omega_{s+k}$ has two columns of length $s+k$ and s respectively.

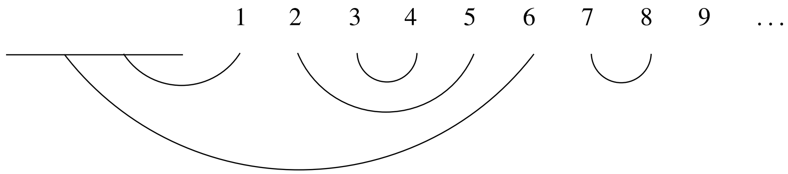
Fix an admissible weight μ. Let M_μ be the set of i's such that $\mu_i = 1$

$$M_\mu \stackrel{\text{def}}{=} \{1 \leq i \leq N \mid \mu_i = 1\},$$

and N_μ be the set of i's such that $\mu_i = 2$

$$N_\mu \stackrel{\text{def}}{=} \{1 \leq i \leq N \mid \mu_i = 2\}.$$

Note that $|M_\mu| = m(\mu)$. Let T_μ be the set of semi-standard tableaux of Y_λ corresponding to μ. For each semi-standard tableau $T^i_\mu \in T_\mu$, let $T^i_\mu(r)$ and $T^i_\mu(l)$ be the set of numbers on the right and left column of T^i_μ respectively. Write M_μ as an ordered sequence $\{a_1, a_2, \dots, a_{m(\mu)}\}$. Assume that $\{a_{i_1}, a_{i_2}, \dots, a_{i_t}\} = M_\mu \bigcap T^i_\mu(r)$. Consider all integer points $\{1, 2, 3, \dots\}$ lying on the x-axis. Put a platform on the x-axis to the left of all points (see Figure 17). First draw an arc in the lower half plane connecting a_{i_1} with the first point in M_μ to its left which is not connected to any point. That point always exists and lies in $T^i_\mu(l)$ since T^i_μ is semi-standard. Repeat the above step for $a_2, a_3, \dots$ in order until each point in $M_\mu \bigcap T^i_\mu(r)$ is connected to some point. Finally, connect the remaining free points in $M_\mu \bigcap T^i_\mu(l)$ to the platform by arcs in the unique way that no two arcs intersect.



$M_\mu \bigcap T^i_\mu(r) = \{4, 5, 8\}$ $\qquad$ $M_\mu \bigcap T^i_\mu(l) = \{1, 2, 3, 6, 7\}$

Figure 17. An element in $B^{2,0}_8(M_\mu)$.

The resulting graph is a crossingless matching among the points in M_μ with one platform. Denote by $B^{k,0}_{m(\mu)}(M_\mu)$ the set of all such elements. Note that $B^{k,0}_{m(\mu)}(M_\mu) \cong B^{k,0}_{m(\mu)}$. The map from T_μ to $B^{k,0}_{m(\mu)}(M_\mu)$ is denoted by φ_μ.

Conversely, for any $a \in B^{k,0}_{m(\mu)}(M_\mu)$ with t type I arcs $c_1, c_2, \dots, c_t$, let R_a be the set of right end points of all c_i. A semi-standard tableau of Y_λ is constructed by putting $R_a \bigcup N_\mu$ into the right column and $(M_\mu \backslash R_a) \bigcup N_\mu$ into the left column (both in increasing order). The map from $B^{k,0}_{m(\mu)}$ to T_μ is denoted by ψ_μ. It is easy to verify that ψ_μ is indeed the inverse of φ_μ. Thus we have a bijection between T_μ and $B^{k,0}_{m(\mu)}$

$$T_\mu \underset{\psi_\mu}{\overset{\varphi_\mu}{\rightleftarrows}} B^{k,0}_{m(\mu)}. \tag{14}$$

See Figure 18 for an example of this bijection where W is a level two representation of $U_q(sl_5)$.

4.3. Category $\mathcal{C}$ and exact functors. Starting with $B^{k,0}_{m(\mu)}(M_\mu)$, we repeat the definition of $A^{k,l}_n$ and get $A^{k,0}_{m(\mu)}(M_\mu)$, or simply A_μ. Note that $A_\mu \cong \mathbb{Z}$ when $m(\mu) = 0$. For an admissible weight μ define $\mathcal{C}_\mu$ to be the category of finitely generated graded left A_μ-modules. By taking direct sum over all admissible μ we collect those $\mathcal{C}_\mu$ into a single category $\mathcal{C}$

$$\mathcal{C} \overset{\text{def}}{=} \bigoplus_\mu \mathcal{C}_\mu.$$

Note that when $k = 0$ our category $\mathcal{C}_\mu$ is the same as $\mathcal{C}(\mu)$ in [3].

The functors $\mathcal{E}_i$, $\mathcal{F}_i$, and $\mathcal{K}_i$ defined by Khovanov and Huerfano naturally extend to our category $\mathcal{C}$. Recall that $\mathcal{E}_i : \mathcal{C} \to \mathcal{C}$ is defined to be the sum over all admissible μ of the functors $\mathcal{E}^\mu_i : \mathcal{C}_\mu \to \mathcal{C}_{\mu+\varepsilon_i}$. If $\mu+\varepsilon_i$ is not admissible $\mathcal{E}^\mu_i$ is the zero functor. Otherwise, define $\mathcal{E}^\mu_i$ to be tensoring with the $(A_{\mu+\varepsilon_i}, A_\mu)$-bimodule $\mathcal{F}(T^\mu_i)$ where T^μ_i is the simplest flat tangle with bottom end points corresponding to μ and top end points corresponding to $\mu + \varepsilon_i$. Figure 19 shows an example of the functor $\mathcal{E}^\mu_i$. The definition of $\mathcal{F}_i$ is similar. See [3] for details. Define $\mathcal{K}_i$ to be the functor which shifts the grading of $M \in \mathcal{C}_\mu$ up by $\mu_i - \mu_{i+1}$

$$\mathcal{K}_i(M) \overset{\text{def}}{=} M\{\mu_i - \mu_{i+1}\}.$$

Proposition 13. *There are functor isomorphisms*

$$\begin{aligned}
&\mathcal{K}_i\mathcal{K}_i^{-1} \cong \mathrm{Id} \cong \mathcal{K}_i^{-1}\mathcal{K}_i,\\
&\mathcal{K}_i\mathcal{K}_j \cong \mathcal{K}_j\mathcal{K}_i,\\
&\mathcal{K}_i\mathcal{E}_j \cong \mathcal{E}_j\mathcal{K}_i\{c_{i,j}\},\\
&\mathcal{K}_i\mathcal{F}_j \cong \mathcal{F}_j\mathcal{K}_i\{-c_{i,j}\},\\
&\mathcal{E}_i\mathcal{F}_j \cong \mathcal{F}_j\mathcal{E}_i \quad \text{if } i \neq j,\\
&\mathcal{E}_i\mathcal{E}_j \cong \mathcal{E}_j\mathcal{E}_i \quad \text{if } |i-j| > 1,\\
&\mathcal{F}_i\mathcal{F}_j \cong \mathcal{F}_j\mathcal{F}_i \quad \text{if } |i-j| > 1,\\
&\mathcal{E}_i^2\mathcal{E}_j \oplus \mathcal{E}_j\mathcal{E}_i^2 \cong \mathcal{E}_i\mathcal{E}_j\mathcal{E}_i\{1\} \oplus \mathcal{E}_i\mathcal{E}_j\mathcal{E}_i\{-1\} \quad \text{if } j = i \pm 1,\\
&\mathcal{F}_i^2\mathcal{F}_j \oplus \mathcal{F}_j\mathcal{F}_i^2 \cong \mathcal{F}_i\mathcal{F}_j\mathcal{F}_i\{1\} \oplus \mathcal{F}_i\mathcal{F}_j\mathcal{F}_i\{-1\} \quad \text{if } j = i \pm 1,
\end{aligned} \tag{15}$$

where

$$c_{i,j} = \begin{cases} 2 & \text{if } j = i,\\ -1 & \text{if } j = i \pm 1,\\ 0 & \text{if } |j-i| > 1. \end{cases}$$

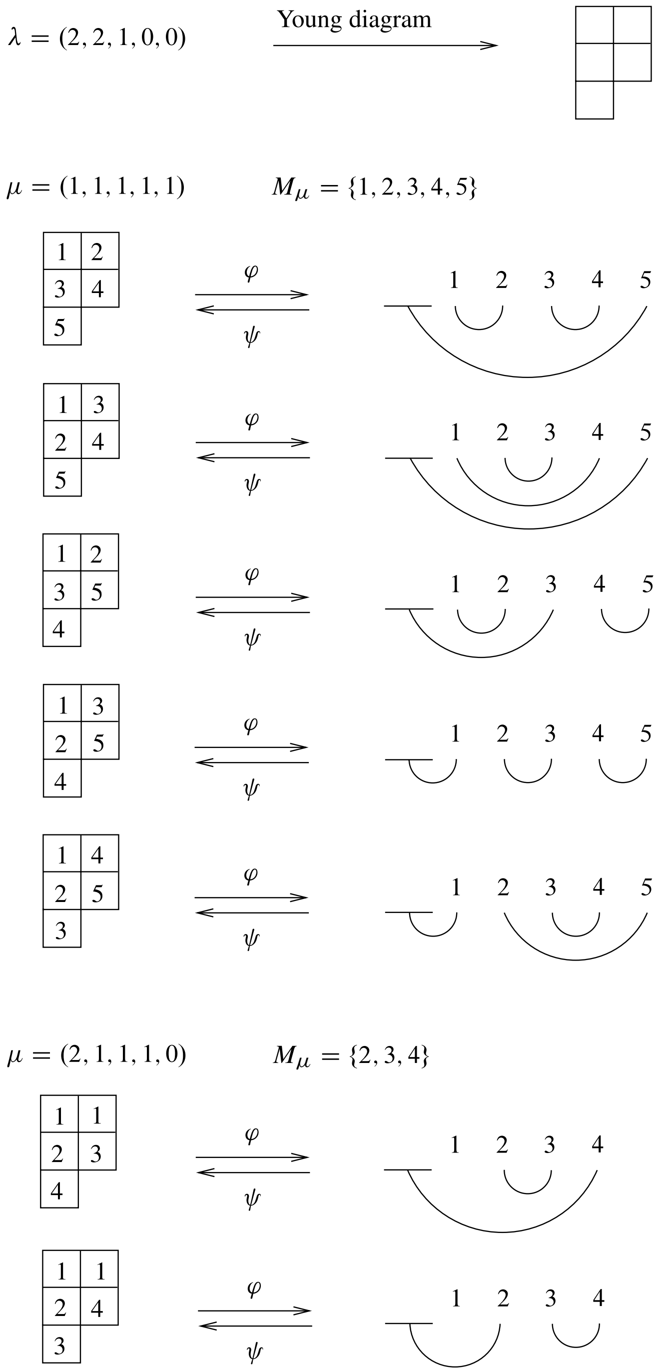
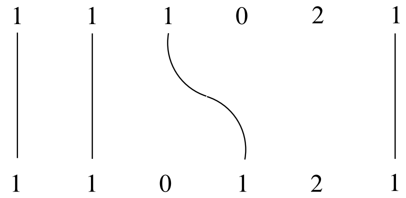


Figure 18. Bijection between semi-standard tableaux and crossingless matchings with one platform.

$\varepsilon_3 = (0, 0, 1, -1, 0, 0)$

$\mu_1 = (1, 1, 0, 1, 2, 1)$

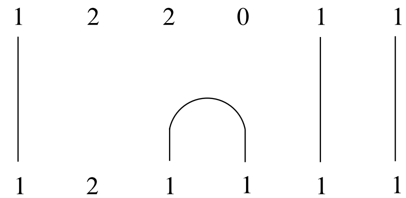
$\mu_1 + \varepsilon_3 = (1, 1, 1, 0, 2, 1)$ $\quad \mathcal{E}_3^{\mu_1} = \mathcal{F}\Big($



$\Big)$

$\mu_2 = (1, 2, 1, 1, 1, 1)$

$\mu_2 + \varepsilon_3 = (1, 2, 2, 0, 1, 1)$ $\quad \mathcal{E}_3^{\mu_2} = \mathcal{F}\Big($



$\Big)$

Figure 19. Examples of the functor $\mathcal{E}_i^{\mu}$.

Proposition 14. *For any admissible μ there is an isomorphism of functors in the category $\mathcal{C}_\mu$*

$$\begin{aligned}
&\mathcal{E}_i\mathcal{F}_i \cong \mathcal{F}_i\mathcal{E}_i \oplus \mathrm{Id}\{1\} \oplus \mathrm{Id}\{-1\} \quad \text{if } (\mu_i, \mu_{i+1}) = (2, 0),\\
&\mathcal{E}_i\mathcal{F}_i \cong \mathcal{F}_i\mathcal{E}_i \oplus \mathrm{Id} \quad \text{if } \mu_i - \mu_{i+1} = 1,\\
&\mathcal{E}_i\mathcal{F}_i \cong \mathcal{F}_i\mathcal{E}_i \quad \text{if } \mu_i = \mu_{i+1},\\
&\mathcal{E}_i\mathcal{F}_i \oplus \mathrm{Id} \cong \mathcal{F}_i\mathcal{E}_i \quad \text{if } \mu_i - \mu_{i+1} = -1,\\
&\mathcal{E}_i\mathcal{F}_i \oplus \mathrm{Id}\{1\} \oplus \mathrm{Id}\{-1\} \cong \mathcal{F}_i\mathcal{E}_i \quad \text{if } (\mu_i, \mu_{i+1}) = (0, 2).
\end{aligned} \tag{16}$$

Proposition 15. *The functor $\mathcal{E}_i$ is left adjoint to $\mathcal{F}_i\mathcal{K}_i^{-1}\{1\}$, the functor $\mathcal{F}_i$ is left adjoint to $\mathcal{E}_i\mathcal{K}_i\{1\}$, and $\mathcal{K}_i$ is left adjoint to $\mathcal{K}_i^{-1}$.*

The above three propositions are from [3]. They work in our case without any modifications since the actions happen away from the platform.

The Grothendieck group of $\mathcal{C}$ is a $\mathbb{Z}[q, q^{-1}]$-module where grading shifts correspond to multiplication by q. The functors $\mathcal{E}_i$, $\mathcal{F}_i$, and $\mathcal{K}_i$ are exact and commute with grading shift action $\{1\}$. Exactness follows from left and right projectivity of bimodule $\mathcal{F}(T)$ for flat tangle T in Section 2. On the Grothendieck group level $\mathcal{E}_i$, $\mathcal{F}_i$, and $\mathcal{K}_i$ descend to $\mathbb{Z}[q, q^{-1}]$-linear endomorphisms $[\mathcal{E}_i]$, $[\mathcal{F}_i]$, and $[\mathcal{K}_i]$ respectively. Functor isomorphisms in Proposition 13 and Proposition 14 correspond to the quantum group relation (13) in $K(\mathcal{C})$. So we can view $K(\mathcal{C})$ as an $U_q(sl_N)$ module. It follows from the bijection (14) that $K(\mathcal{C})$ is isomorphic to W as an $U_q(sl_N)$ module.

Proposition 16. *The Grothendieck group of $\mathcal{C}$ is isomorphic to the irreducible representation of $U_q(sl_N)$ with the highest weight $\omega_k + \omega_{k+s}$*

$$K(\mathcal{C}) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C} \cong W.$$

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