

Graphs and two-step nilpotent Lie algebras

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Abstract. We consider a method popular in the literature of associating a two-step nilpotent Lie algebra with a finite simple graph. We prove that the two-step nilpotent Lie algebras associated with two graphs are Lie isomorphic if and only if the graphs from which they arise are isomorphic.

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1. Introduction

Recently in connection with the study of some very interesting geometrical and dynamical properties of nilmanifolds, the simply connected nilpotent Lie groups or equivalently their Lie algebras have received the attention of many researchers. Among these nilpotent Lie algebras, the two-step ones are the simplest (after the abelian ones) and most widely studied [1, 2, 3, 4, 5, 6, 10, 11, 12, 13, 14, 15, 17, 18, 19, 20].

Let k be a field with characteristic not equal to 2. We recall that a non-abelian finite dimensional Lie algebra \mathfrak{n} over k is said to be a *two-step nilpotent Lie algebra* if $[\mathfrak{n}, [\mathfrak{n}, \mathfrak{n}]] = \{0\}$. Every two-step nilpotent Lie algebra \mathfrak{n} over k can be realized as a vector space direct sum $V \oplus (\wedge^2 V)/W$, where V is a finite dimensional k -vector space and W is a subspace of the exterior power $\wedge^2 V$. The Lie bracket structure on \mathfrak{n} is given by

- (1) $[v_1, v_2] = v_1 \wedge v_2 \bmod W$ for $v_1, v_2 \in V$, and
- (2) $[x, y] = 0$ for $x \in \mathfrak{n}$ and $y \in (\wedge^2 V)/W$

A combinatorial approach for construction of two-step nilpotent Lie algebras was described in [1]. Subsequently this construction has been used by many authors [12, 13, 20, 18]. We recall the construction. Let (S, E) be a finite simple

graph, where S is the set of vertices and E is the set of edges, where it is assumed that there are no loops and no multiple edges connecting the same pair of vertices. We associate with (S, E) a two-step nilpotent Lie algebra $\mathfrak{n} = \mathfrak{n}(S, E)$ over k in the following way. The underlying vector space of \mathfrak{n} is $V \oplus (\wedge^2 V)/W$, where V is the k -vector space consisting of formal k -linear combinations of elements of S (so that S is a basis of V), and W is the subspace of the exterior power $\wedge^2 V$ spanned by the vectors $\alpha \wedge \beta$, where $\alpha\beta \notin E$. The Lie bracket structure on \mathfrak{n} is defined as above. We note that the space $(\wedge^2 V)/W$ has dimension $|E|$. In fact it is spanned by $\{\alpha \wedge \beta \bmod W \mid \alpha\beta \in E\}$. Consequently $\mathfrak{n}(S, E)$ has dimension $|S| + |E|$.

Note that the construction of $\mathfrak{n}(S, E)$ from the graph (S, E) is functorial in the sense that if $f: (S, E) \rightarrow (S', E')$ is an isomorphism of graphs, then we obtain an isomorphism $f_*: \mathfrak{n}(S, E) \rightarrow \mathfrak{n}(S', E')$ in the natural way: $f_*(\alpha) = f(\alpha)$ for each $\alpha \in S$ and for each edge $\alpha\beta \in E$, $f_*(\alpha \wedge \beta) = f(\alpha) \wedge f(\beta)$. It is easy to see that f_* extends linearly to a Lie algebra isomorphism from $\mathfrak{n}(S, E)$ to $\mathfrak{n}(S', E')$.

In this note, we consider the converse question and prove the following:

Theorem 1.1. *Let (S, E) and (S', E') be finite simple graphs. If the two Lie algebras $\mathfrak{n}(S, E)$ and $\mathfrak{n}(S', E')$ are isomorphic then the graphs (S, E) and (S', E') are also isomorphic.*

This result has already found use in some recent investigations [18, 20]. In [18], the authors provide a method to construct Einstein solvmanifolds by using the Lie algebras $\mathfrak{n}(S, E)$. Using our main theorem, these solvmanifolds are isometric if and only if the graphs are isomorphic, which gives us examples of nonisometric Einstein solvmanifolds.

In [20], the authors construct symplectic two-step nilpotent Lie algebras associated with graphs. In this case our main result implies that *there are exactly five non-isomorphic two-step nilpotent Lie algebras of dimension six associated with graphs in the above manner*. Indeed, there are five non-isomorphic graphs (S, E) such that $|S| + |E| = 6$. Now, $\dim \mathfrak{n}(S, E) = |S| + |E|$. Hence using our Theorem 1.1, there are exactly five non-isomorphic two-step nilpotent Lie algebras of dimension six associated with graphs. This fact was used in [20, Remark 1].

The group of Lie automorphisms of $\mathfrak{n}(S, E)$ was determined in terms of the graph (S, E) in [1]. Also, Anosov and ergodic automorphisms on corresponding nilmanifolds were studied. In [18], explicit examples and non-examples of Einstein solvmanifolds were constructed using the Lie algebras $\mathfrak{n}(S, E)$ (see also [12]). A combinatorial construction of the first and second cohomology groups of $\mathfrak{n}(S, E)$ was given in [20], and was used to construct symplectic and contact nilmanifolds.

Remark 1.2. Suppose the underlying field $k = \mathbb{R}$. Let $N(S, E)$ denote the unique simply connected nilpotent Lie group corresponding to the Lie algebra $\mathfrak{n}(S, E)$. Then the Lie group exponential map $\exp: \mathfrak{n}(S, E) \rightarrow N(S, E)$ is a diffeomorphism (see [21], p. 6). We note that the Baker-Campbell-Hausdorff formula ([23], p. 119)) gives the multiplication law in $N(S, E)$ as follows:

$$\exp(x) \exp(y) = \exp\left(x + y + \frac{1}{2}[x, y]\right)$$

for all $x, y \in \mathfrak{n}(S, E)$.

More precisely, we can realize $N(S, E)$ as $\mathfrak{n}(S, E)$ (via the exponential map) with the multiplication defined by

$$(v_1, x_1).(v_2, x_2) = \left(v_1 + v_2, x_1 + x_2 + \frac{1}{2}[v_1, v_2]\right)$$

for all $v_1, v_2 \in V$ and $x_1, x_2 \in (\wedge^2 V)/W$.

Then our Theorem 1.1 implies that the simply connected Lie groups $N(S, E)$ and $N(S', E')$ are Lie isomorphic if and only if the graphs (S, E) and (S', E') are isomorphic. This can be seen by using the fact that the simply connected Lie groups are Lie isomorphic if and only if their Lie algebras are Lie isomorphic (see [24], p. 101).

Remark 1.3. There have been some other constructions of algebraic structures associated with a simple graph. Some, though not all of these are related with the construction considered here. In [16], the authors consider the K -algebra associated with a simple graph (S, E) over a field K generated by the set of vertices S and with relations $\alpha\beta = \beta\alpha$ if and only if $\alpha\beta \notin E$. They proved that these K -algebras are isomorphic if and only if the corresponding graphs are isomorphic. This result was further used to prove an analogous result for graph groups in [7]. In [7], the author considers the group associated with graph (S, E) which is defined as the group generated by the set S and with relations $\alpha\beta = \beta\alpha$ if and only if $\alpha\beta \in E$. Here we remark that the simply connected nilpotent Lie group $N(S, E)$ (as in Remark 1.2) is not finitely generated.

Later in [8], the authors introduce the free partially commutative Lie algebra $l(S, E)$ associated with (S, E) which is the quotient of the free Lie algebra on the set S modulo the Lie ideal generated by $\{[\alpha, \beta] \mid \alpha\beta \notin E\}$. In fact, the authors study these structures in more generality but we will not go into the details here. Following [7, 9, 16], it can be shown that these free partially commutative Lie algebras are Lie isomorphic if and only if the corresponding graphs are isomorphic (see Theorem 1 in [7] for example). Here we note that the quotient of $l(S, E)$

by the Lie ideal \mathfrak{l}^3 generated by the set $\{[x, [y, z]] \mid x, y, z \in \mathfrak{l}(S, E)\}$, is Lie isomorphic to our two-step nilpotent Lie algebra $\mathfrak{n}(S, E)$. The isomorphism result for the free partially commutative Lie algebras can be recaptured by using our Theorem 1.1 (if the characteristic of the field K is not equal to 2) as follows. Let $\mathfrak{l} = \mathfrak{l}(S, E)$ and $\mathfrak{l}' = \mathfrak{l}(S', E')$ be Lie isomorphic free partially commutative Lie algebras associated with graphs (S, E) and (S', E') respectively. Then the quotient Lie algebras $\mathfrak{l}/\mathfrak{l}^3$ and $\mathfrak{l}'/\mathfrak{l}'^3$ are Lie isomorphic. That means the two-step nilpotent Lie algebras $\mathfrak{n}(S, E)$ and $\mathfrak{n}(S', E')$ are isomorphic and our Theorem 1.1 implies that the graphs (S, E) and (S', E') are isomorphic.

2. Some Preliminaries

Before we prove Theorem 1.1, we need a few facts regarding the structure of the automorphism group of $\mathfrak{n} = \mathfrak{n}(S, E)$. We denote by $\text{Aut}(\mathfrak{n})$ the group of Lie automorphisms of \mathfrak{n} and T the set of automorphisms $\tau \in \text{Aut}(\mathfrak{n})$ such that $\tau(V) = V$, where we recall that V is the k -vector space with basis S , and $\mathfrak{n} = V \oplus (\wedge^2 V)/W$. We denote by G the subgroup of $\text{GL}(V)$ consisting of the linear automorphisms $\tau|_V$, where $\tau \in T$.

Next we see that the group G consists of $g \in \text{GL}(V)$ such that the subspace $W \subseteq \wedge^2 V$ is invariant under the induced natural action $\wedge^2 g$ of g on $\wedge^2 V$. Indeed, if $g \in G$ (i.e. if $g = \tau|_V$ for some $\tau \in T$) and if $\alpha\beta \notin E$, then

$$\begin{aligned} g(\alpha) \wedge g(\beta) \bmod W &= [\tau(\alpha), \tau(\beta)] \\ &= \tau[\alpha, \beta] \\ &= 0, \end{aligned}$$

since τ is an automorphism of \mathfrak{n} and $[\alpha, \beta] = 0$ in \mathfrak{n} . Conversely if $g \in \text{GL}(V)$ such that $g(\alpha) \wedge g(\beta) \in W$ for all $\alpha\beta \notin E$, then $\tau = g \oplus \wedge^2 g$ defines a Lie automorphism of \mathfrak{n} such that $\tau(V) = V$ and hence $g \in G$. Since the condition that an element of G stabilizes W under the natural action on $\wedge^2 V$ is represented by polynomial equations, we have therefore the following:

Lemma 2.1. *G is an algebraic group.*

In fact, for any two-step nilpotent Lie algebra $\mathfrak{m} = V \oplus (\wedge^2 V)/W$ one can analogously define the subgroup G of $\text{GL}(V)$ consisting of the restrictions of automorphisms of \mathfrak{m} fixing V . A similar argument shows that G is an algebraic group. However our next lemma holds only for two-step nilpotent Lie algebras arising from graphs. We prove that all linear automorphisms of V which can be

represented as diagonal matrices with respect to the basis S can be extended to Lie automorphisms of $\mathfrak{n}(S, E)$.

Lemma 2.2. *Let D_S denote the subgroup of $\text{GL}(V)$ consisting of all elements which can be represented as diagonal matrices with respect to the basis S of V . Then $D_S \subseteq G$.*

Proof. We recall that G is a subgroup of $\text{GL}(V)$ consisting of those linear automorphisms of V whose induced action on $\bigwedge^2 V$ leaves the subspace W invariant, where $W \subseteq \bigwedge^2 V$ spanned by elements $\alpha \wedge \beta$ such that $\alpha\beta \notin E$. Let $d \in D_S$, say $d(\alpha) = d_\alpha\alpha$ for some nonzero d_α 's in k and for all $\alpha \in S$. Then if $\alpha\beta \notin E$, we have $d(\alpha) \wedge d(\beta) = d_\alpha d_\beta (\alpha \wedge \beta) \in W$. Hence $d \in G$ and $D_S \subseteq G$. \square

3. Proof of Theorem 1.1

Let \bar{k} denote the algebraic closure of k . First we note that if F is an isomorphism from the Lie algebra $\mathfrak{n} = \mathfrak{n}(S, E)$ to $\mathfrak{n}' = \mathfrak{n}(S', E')$, then $\bar{F} = F \otimes_k \text{id}_{\bar{k}}$ is an isomorphism of \bar{k} -Lie algebras $\bar{\mathfrak{n}} = \mathfrak{n} \otimes_k \bar{k}$ and $\bar{\mathfrak{n}}' = \mathfrak{n}' \otimes_k \bar{k}$. To see this, recall that the Lie bracket in $\bar{\mathfrak{n}}$ is defined by

$$[x \otimes a, y \otimes b] = [x, y] \otimes ab \quad \text{for all } x, y \in \mathfrak{n} \text{ and } a, b \in \bar{k}.$$

Also \bar{F} is defined by

$$\bar{F}\left(\sum_i x_i \otimes a_i\right) = \sum_i F(x_i) \otimes a_i \quad \text{for all } x_i \in \mathfrak{n} \text{ and } a_i \in \bar{k}.$$

Then \bar{F} is a \bar{k} -vector space isomorphism which follows from the fact that F is a k -vector space isomorphism. Furthermore, for $x, y \in \mathfrak{n}$ and $a, b \in \bar{k}$ we have

$$\begin{aligned} \bar{F}[x \otimes a, y \otimes b] &= \bar{F}([x, y] \otimes ab) \\ &= (F[x, y] \otimes ab) \\ &= [F(x), F(y)] \otimes ab \text{ since } F \text{ is a Lie algebra isomorphism.} \\ &= [F(x) \otimes a, F(y) \otimes b] = [\bar{F}(x \otimes a), \bar{F}(y \otimes b)]. \end{aligned}$$

Now without loss of generality we can assume that the field k is algebraically closed. Indeed if k is not already algebraically closed, and F is an isomorphism from the Lie algebra $\mathfrak{n} = \mathfrak{n}(S, E)$ to $\mathfrak{n}' = \mathfrak{n}(S', E')$, we can replace \mathfrak{n} by $\bar{\mathfrak{n}}$, \mathfrak{n}' by $\bar{\mathfrak{n}}'$ and F by \bar{F} . Then the new F is an isomorphism of \bar{k} -Lie algebras \mathfrak{n} and \mathfrak{n}' as

discussed above. We need to show that the graphs (S, E) and (S', E') defining the Lie algebras \mathfrak{n} and \mathfrak{n}' are isomorphic.

We recall that

$$\mathfrak{n} = V \oplus (\wedge^2 V)/W,$$

where V is the k -vector space with basis S and $W \subset \wedge^2 V$ is as defined in the previous section. Similarly

$$\mathfrak{n}' = V' \oplus (\wedge^2 V')/W',$$

where V' is the k -vector space with basis S' .

The proof will be in three steps. First we construct a new graph (S'', E'') isomorphic to the graph (S, E) . The set of vertices S'' will be a basis of the vector space V' . Then in the next step we look a group $D_{S''}$ analogous to the group D_S considered in Lemma 2.2. The theorem will follow from the relation between $D_{S''}$ and $D_{S'}$ to be explained in the third step.

We begin by constructing a new graph (S'', E'') isomorphic to the graph (S, E) in the following way. Recall that V' denote the k -vector space with S' as basis and let W' denote the subspace of $\wedge^2 V'$ spanned by the wedge products $\alpha \wedge \beta$, where α and β are vertices in S' not connected by an edge in E' . Let $\pi: \mathfrak{n}' \rightarrow V'$ denote the canonical linear projection with respect to the decomposition $\mathfrak{n}' = V' \oplus (\wedge^2 V')/W'$. We define S'' denote the subset of V' given by $\{\pi(F(\alpha)) \mid \alpha \in S\}$ and E'' denote the set of unordered pairs $\pi(F(\alpha))\pi(F(\beta))$ such that $\alpha\beta \in E$, where, recall that F is the isomorphism given between \mathfrak{n} and \mathfrak{n}' .

It follows that S'' is a basis of V' . Indeed, if $\sum_{i=1}^n a_i \pi(F(\alpha_i)) = 0$ for a_i 's $\in \mathbb{R}$ and α_i 's $\in S$, then $F(\sum_{i=1}^n a_i \alpha_i) \in [\mathfrak{n}', \mathfrak{n}']$. Since F is a Lie algebra isomorphism, we can see that $\sum_{i=1}^n a_i \alpha_i \in [\mathfrak{n}, \mathfrak{n}]$. On the other hand each α_i is in S so $\sum_{i=1}^n a_i \alpha_i \in V \cap [\mathfrak{n}, \mathfrak{n}]$, so that $\sum_{i=1}^n a_i \alpha_i = 0$. By the linear independence of S , each $a_i = 0$. Now the derived algebras $[\mathfrak{n}, \mathfrak{n}]$ and $[\mathfrak{n}', \mathfrak{n}']$ have the same dimension since \mathfrak{n} and \mathfrak{n}' are isomorphic. Hence $|E| = |E'|$ since $|E| = \dim[\mathfrak{n}, \mathfrak{n}]$ and $|E'| = \dim[\mathfrak{n}', \mathfrak{n}']$. Hence $|S| = |S'|$ because \mathfrak{n} is of dimension $|S| + |E|$ and \mathfrak{n}' is of dimension $|S'| + |E'|$, and \mathfrak{n} and \mathfrak{n}' are isomorphic. Thus the sets S' and S'' have exactly the same number of elements and hence S'' is a basis of V' since S' is a basis of V' .

Since the graph (S'', E'') as constructed above is isomorphic to the graph (S, E) , to prove the theorem, it is enough to show that there exists an isomorphism of graphs $f: (S', E') \rightarrow (S'', E'')$.

Now as in Lemma 2.2, $D_{S'}$ denote the subgroup of $\text{GL}(V')$ consisting of all elements which can be represented as diagonal matrices with respect to the basis S' of V' . By Lemma 2.2, we have $D_{S'} \subset G'$. Similarly let $D_{S''}$ denote the subgroup of $\text{GL}(V')$ consisting of all elements which can be represented as diagonal matrices with respect to the basis $S'' = \{\pi(F(\alpha)) \mid \alpha \in S\}$ of V' .

We claim that $D_{S''} \subset G'$ as well. To see this, let $d \in D_{S''}$ and $d(\pi(F(\alpha))) = d_\alpha \pi(F(\alpha))$ for all $\alpha \in S$ and for some nonzero d_α 's in k . It suffices to show that if γ and δ are vertices in S' not connected by an edge in E' , then $d(\gamma) \wedge d(\delta) \in W'$ (i.e. $[d(\gamma), d(\delta)] = 0$ in \mathfrak{n}' .) Now since $\gamma, \delta \in V'$ and S'' is a basis for V' , we represent

$$\gamma = \sum_{\alpha \in S} a_\alpha \pi(F(\alpha)) \quad \text{and} \quad \delta = \sum_{\beta \in S} b_\beta \pi(F(\beta)),$$

where each a_α, b_β is in k . Since $[\gamma, \delta] = 0$, we have

$$\left[\pi\left(F\left(\sum_{\alpha \in S} a_\alpha \alpha\right)\right), \pi\left(F\left(\sum_{\beta \in S} b_\beta \beta\right)\right) \right] = 0.$$

This means that

$$\left[F\left(\sum_{\alpha \in S} a_\alpha \alpha\right), F\left(\sum_{\beta \in S} b_\beta \beta\right) \right] = 0,$$

because in \mathfrak{n}' , $[v + w, v' + w'] = 0$ if and only if $[v, v'] = 0$ for $v, v' \in V'$ and $w, w' \in (\wedge^2 V')/W'$. Hence

$$\left[\sum_{\alpha \in S} a_\alpha \alpha, \sum_{\beta \in S} b_\beta \beta \right] = 0$$

in \mathfrak{n} since F is an isomorphism. Now we define σ , an element of $\text{GL}(V)$ by $\sigma(\alpha) = d_\alpha \alpha$ for each $\alpha \in S$. Then $\sigma \in D_S$, where D_S is the subgroup of $\text{GL}(V)$ consisting of diagonal automorphisms of V with respect to S and hence $\sigma \in G$ by Lemma 2.2. So

$$\left[\sum_{\alpha \in S} a_\alpha \sigma(\alpha), \sum_{\beta \in S} b_\beta \sigma(\beta) \right] = 0$$

in \mathfrak{n} which means that

$$\left[\sum_{\alpha \in S} a_\alpha d_\alpha \alpha, \sum_{\beta \in S} b_\beta d_\beta \beta \right] = 0.$$

Since F is an isomorphism,

$$\left[\sum_{\alpha \in S} a_\alpha d_\alpha F(\alpha), \sum_{\beta \in S} b_\beta d_\beta F(\beta) \right] = 0$$

in \mathfrak{n}' and hence

$$\left[\sum_{\alpha \in S} a_{\alpha} d_{\alpha} \pi(F(\alpha)), \sum_{\beta \in S} b_{\beta} d_{\beta} \pi(F(\beta)) \right] = 0.$$

Thus $[d(\gamma), d(\delta)] = 0$ in \mathfrak{n}' as we wanted to show.

By Lemma 2.1, G' is an algebraic group. Moreover, $D_{S'}$ and $D_{S''}$ are maximal tori in the connected component in Zariski topology of identity of G' . But maximal tori in a connected algebraic group over an algebraically closed field (such as our k) are conjugate (see [22], p. 108), and hence there exists $g \in G'$ such that

$$D_{S'} = g(D_{S''})g^{-1}.$$

To define an isomorphism of graphs $f: S' \rightarrow S''$, we construct a special element of $D_{S'}$. For each $\alpha \in S'$, choose a nonzero $d'_{\alpha} \in k$ such that unless the set $\{\alpha, \beta\}$ equals the set $\{\gamma, \delta\}$ we have $d'_{\alpha}d'_{\beta} \neq d'_{\gamma}d'_{\delta}$. Since by hypothesis the field k is algebraically closed, it is in particular infinite, and such d'_{α} 's may be chosen. Define $d' \in D_{S'}$ be the element whose matrix representation with respect to S' given by $d'(\alpha) = d'_{\alpha}\alpha$. As note above, there exists $d'' \in D_{S''}$ such that

$$d' = gd''g^{-1}.$$

Since d' and d'' are similar, they have the same eigenvalues. Further since d' is diagonal with respect to S' and d'' is diagonal with respect to S'' , it follows that these diagonal entries are the same up to a permutation. Hence there exists a bijection f of S' with S'' such that

$$d''(f(\alpha)) = d'_{\alpha}f(\alpha) \quad \text{for all } \alpha \in S'.$$

We claim that f is in fact an isomorphism of the graph (S', E') with the graph (S'', E'') . Establishing this claim will conclude the proof of our theorem.

It suffices now to show that $\alpha\beta \in E'$ if and only if $f(\alpha)f(\beta) \in E''$. Now $\alpha\beta \in E'$ if and only if $[\alpha, \beta] \neq 0$ in \mathfrak{n}' . Also $f(\alpha)f(\beta) \in E''$ if and only if $[f(\alpha), f(\beta)] \neq 0$ in \mathfrak{n}' . This follows from the definition of E'' and using that F is an isomorphism. Indeed $f(\alpha) = \pi(F(\alpha'))$ and $f(\beta) = \pi(F(\beta'))$ for some $\alpha', \beta' \in S$. Hence $f(\alpha)f(\beta) \in E''$ if and only if $\alpha'\beta' \in E$. Since F is an isomorphism, $\alpha'\beta' \in E$ if and only if $[F(\alpha'), F(\beta')] \neq 0$ in \mathfrak{n}' i.e. $[\pi(F(\alpha')), \pi(F(\beta'))] \neq 0$ in \mathfrak{n}' .

We will show that for $\alpha, \beta \in S'$, $[\alpha, \beta] \neq 0$ if and only if $[f(\alpha), f(\beta)] \neq 0$. Since the diagonal entries of the chosen element $d' \in D_{S'}$ (defined as above) are such that the pairwise products of the diagonal entries are distinct, $d'_{\alpha}d'_{\beta}$ is an eigenvalue of an extended automorphism d' of \mathfrak{n}' if and only if $[\alpha, \beta] \neq 0$.

Similarly $d'_\alpha d'_\beta$ is an eigenvalue of an extended automorphism d'' of \mathfrak{n}' if and only if $[f(\alpha), f(\beta)] \neq 0$. Now we note that

$$[g^{-1}(d'(\alpha)), g^{-1}(d'(\beta))] = d'_\alpha d'_\beta [g^{-1}(\alpha), g^{-1}(\beta)].$$

On the other hand, we have

$$[g^{-1}(d'(\alpha)), g^{-1}(d'(\beta))] = d''[g^{-1}(\alpha), g^{-1}(\beta)], \quad \text{since } g^{-1}d' = d''g^{-1}.$$

Since $g \in G'$, $[g^{-1}(\alpha), g^{-1}(\beta)] \neq 0$ if and only if $[\alpha, \beta] \neq 0$. Hence $d'_\alpha d'_\beta$ is an eigenvalue of d'' if and only if $d'_\alpha d'_\beta$ is an eigenvalue of d'' . Thus $\alpha\beta \in E'$ if and only if $f(\alpha)f(\beta) \in E''$. Hence $f: (S', E') \rightarrow (S'', E'')$ is an isomorphism of graphs, this complete the proof of Theorem 1.1.

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