

Stable commutator length in word-hyperbolic groups

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Abstract. In this paper we obtain uniform positive lower bounds on the stable commutator length of elements in word-hyperbolic groups and certain groups acting on hyperbolic spaces (namely the mapping class group acting on the complex of curves, and an amalgamated free product acting on an associated Bass–Serre tree). If G is a word-hyperbolic group that is δ -hyperbolic with respect to a symmetric generating set S , then there is a positive constant C depending only on δ and on $|S|$ such that every element of G either has a power which is conjugate to its inverse, or else the stable commutator length of the element is at least equal to C . By Bavard’s theorem, these lower bounds on stable commutator length imply the existence of quasimorphisms with uniform control on the defects; however, we show how to construct such quasimorphisms directly.

We also prove various separation theorems on families of elements in such groups, constructing homogeneous quasimorphisms (again with uniform estimates) which are positive on some prescribed element while vanishing on some family of independent elements whose translation lengths are uniformly bounded.

Finally, we prove that the first accumulation point for stable commutator length in a torsion-free word-hyperbolic group is contained between $1/12$ and $1/2$. This gives a universal sense of what it means for a conjugacy class in a hyperbolic group to have a small stable commutator length, and can be thought of as a kind of “homological Margulis lemma”.

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1. Introduction

Let G be a group, and let $[G, G]$ denote the commutator subgroup. Given $g \in [G, G]$ the *commutator length* of g , denoted $\text{cl}(g)$, is the least number of commutators in G whose product is equal to g . The *stable commutator length*, denoted $\text{scl}(g)$, is the limit of $\text{cl}(g^n)/n$ as n goes to infinity. If $g^n \in [G, G]$ for some least positive integer n , define $\text{scl}(g) = \text{scl}(g^n)/n$, and define $\text{scl}(g) = \infty$ if no power of g is contained in $[G, G]$; see § 2 for precise definitions. For a general introduction to the theory of stable commutator length, see [6].

Informally, if $G = \pi_1(X)$ for some topological space X , commutator length is the smallest genus of a surface in X whose boundary represents a given loop in X , and stable commutator length “rationally” measures the same quantity. At the homological level, stable commutator length is an L^1 filling norm with \mathbb{Q} coefficients, in the sense of Gersten [18] or Gromov [20].

This paper is concerned with obtaining uniform positive lower bounds on stable commutator length in (word-) hyperbolic groups and groups acting on (δ -) hyperbolic spaces. Morally, (δ -) hyperbolic spaces are those whose geometry can be efficiently probed by maps of triangles and other surfaces into the space, so it should not be surprising that they can be studied effectively with stable commutator length. However, there is a sense in which our results *are* counterintuitive, which we briefly explain.

It is a widely observed fact that in the presence of (coarse) negative curvature, one can obtain upper bounds on the “size” of a surface which is efficient in some sense. Here size is measured with respect to some kind of norm; for instance area, Whitney b -norm, L^1 norm on homology etc. Efficiency might vary from context to context (e.g. harmonic, minimal, normal) and will depend on the way in which we measure the size of the surface. Our main results say that in the presence of (coarse) negative curvature, there is a uniform (positive) *lower* bound on the (homological) size of certain surfaces.

We concentrate on *worst-case* behavior, rather than *typical behavior*, and our results are frequently sharp. Other authors have studied commutator length and its relation with negative curvature, especially Gromov [22], § 6.C₂. In our language, Gromov observes that in a word-hyperbolic group, if $g \in [G, G]$ is not torsion and does not have a power conjugate to its inverse, then $\text{scl}(g) > 0$ (actually, Gromov neglects to mention the second possibility). The key innovation in our paper is that our bounds are *uniform*, and depend only on macroscopic features of the group, specifically δ and number of generators. Note that any hyperbolic group may be made δ -hyperbolic for some universally small δ (say $\delta < 10$) just by increasing the number of generators, so our results in this sense are best possible.

Perhaps our most striking theorem is Theorem B, a kind of “spectral gap” theorem, which says that in a non-elementary torsion-free word-hyperbolic group, the first accumulation point for stable commutator length as a function from conjugacy classes to positive real numbers is between $1/12$ and $1/2$. These bounds imply that there is a *universal* sense of what it means for an element in a word-hyperbolic group to have a small stable commutator length. This should perhaps be compared with Margulis’ Lemma, which says that there is a universal sense of what it means for a closed geodesic in a hyperbolic manifold to be *short*. The difficulty in transporting Margulis’ Lemma to geometric group theory is that geometric notions in a group are typically only defined up to a certain ambiguity (e.g. quasi-isometry) which obscures small scale geometric phenomena. The advantage of working with stable commutator length, and the power of our theorem, is precisely that it captures such small scale phenomena. This comparison is more than superficial – the key to obtaining the sharp lower bound is to use estimates due to Mineyev which reproduce, in a gen-

eral δ -hyperbolic space, geometric phenomena which are strictly analogous to C^1 phenomena in Lie groups.

We are able to give similar uniform lower bounds on the stable commutator length of certain elements in two other important classes of groups: mapping class groups of surfaces (hereafter denoted $\text{MCG}(S)$), and amalgamated free products. In general, neither kind of group is δ -hyperbolic, but each acts naturally on a certain δ -hyperbolic space: the complex of curves, and the Bass–Serre tree respectively. Obtaining lower bounds on stable commutator length in mapping class groups is intimately tied to important problems in 4-dimensional symplectic geometry; for example, estimating the complexity of a symplectic 4-manifold filling a given contact 3-manifold, or controlling the ratios of characteristic numbers (e.g. Euler characteristic and signature). This point of view has been pioneered by D. Kotschick, sometimes in collaboration with H. Endo, in a series of papers which include [25], [12], [26] and [13]. From another point of view, there are relations between symplectic geometry and stable commutator length which are more intimately connected with invariants like Hofer length and subgroup distortion; see work of Polterovich, especially [33].

The Bavard Duality Theorem (see Theorem 2.5) gives a duality between stable commutator length, and certain kinds of functions on a group, called *homogeneous quasimorphisms*. A function $f : G \rightarrow \mathbb{R}$ is a homogeneous quasimorphism if it is homogeneous (i.e. if it satisfies $f(g^n) = nf(g)$ for all $g \in G$ and $n \in \mathbb{Z}$) and if there is a least real number $D(f)$ called the *defect*, for which $|f(gh) - f(g) - f(h)| \leq D(f)$ for all $g, h \in G$. The Duality Theorem says that obtaining lower bounds for stable commutator length is equivalent to constructing homogeneous quasimorphisms with small defects (see § 2). Bavard’s theorem is non-constructive, and uses the Axiom of Choice (in the form of the Hahn–Banach Theorem); however in our paper we are able to construct explicit quasimorphisms with small defects directly. We are also able to prove various *separation theorems*, constructing quasimorphisms with small defects that take big values on prescribed elements and vanish on others. This is a quantitative improvement on the kinds of separation theorems proved or conjectured by various people in the past, and we expect it to have a number of applications.

Let us stress that this paper is concerned more with developing foundations, and understanding what (to us) seems like a fundamental algebraic/geometric inequality, manifested in several important group theoretic contexts, than with deducing topological or other corollaries.

1.1. Statement of results. We now give a brief summary of the contents of the paper. In § 2 we give definitions, standardize notation, and recall some basic elements of the theory of δ -hyperbolic groups and spaces, and bounded cohomology. We spend some time discussing Mineyev’s geodesic flow space, which is a technical tool that we find useful for establishing uniform bounds on stable commutator length directly by means of geometry rather than using quasimorphisms (see § 2 for definitions).

There are other versions of the geodesic flow space of a word-hyperbolic group or space, and that of Gromov [21] is probably best known. However, Mineyev’s space

has some features that make it technically easier to use for our purposes.

In § 3 we give the first version of our first main result, the Gap Theorem:

Theorem A (Gap Theorem, weak version). *Let G be a word-hyperbolic group that is δ -hyperbolic with respect to a symmetric generating set S with $|S|$ generators. Then there is a constant $C(\delta, |S|) > 0$ such that for every $a \in G$, either $\text{scl}(a) \geq C$ or else there is some positive integer n and some $b \in G$ such that $ba^{-n}b^{-1} = a^n$.*

Note that if $ba^{-n}b^{-1} = a^n$ then $\text{scl}(a) = 0$.

It is easy to produce examples of word-hyperbolic groups which contain elements with arbitrarily small (positive) stable commutator length, so the dependence on δ and $|S|$ is necessary.

In § 4 and § 5 we introduce so-called *counting quasimorphisms*, and use them to prove the second (stronger) version of the Gap Theorem. In the following theorem, we use the notation $\tau(a)$ to denote the *translation length* of an element $a \in G$, as measured in the Cayley graph $\Gamma_S(G)$ of G with respect to a fixed generating set S . It is worth keeping in mind that if a is not torsion, there is a positive lower bound on $\tau(a)$ that depends only on δ and $|S|$.

Theorem A' (Gap Theorem, strong version). *Let G be a word-hyperbolic group that is δ -hyperbolic with respect to a symmetric generating set S with $|S|$ generators. Let $a \in G$ be a (non-torsion) element. Assume that there is no $n > 0$ and no $b \in G$ with $ba^{-n}b^{-1} = a^n$. Then there is a homogeneous quasimorphism h on G such that*

- (1) $h(a) = 1$,
- (2) *the defect of h is $\leq C(\delta, |S|)$.*

Moreover, let $a_i \in G$ be a collection of elements for which $T = \sup_i \tau(a_i)$ is finite. Suppose that for all integers $n, m \neq 0$ and all elements $b \in G$ and indices i there is an inequality

$$ba^n b^{-1} \neq a_i^m.$$

Then there is a homogeneous quasimorphism h on G such that

- (1) $h(a) = 1$, and $h(a_i) = 0$ for all i ,
- (2) *the defect of h is $\leq C'(\delta, |S|, T)$.*

The weak version of the Gap Theorem follows from the strong version. Moreover, the first part of the strong version follows from the weak version together with an application of Bavard's theorem, but the implication in this direction uses the Axiom of Choice. Note that the constants C appearing in the two versions are different in either case but related.

The second part of this strong version of the Gap Theorem can be thought of as a kind of *separation theorem*, which can be summarized in words as follows: given an element a and a finite set of elements $a_i \in G$ whose nontrivial powers are never equal

to a conjugate of a nontrivial power of a , one can (explicitly) construct a homogeneous quasimorphism that vanishes on all the a_i and is positive on a ; moreover, the defect of this quasimorphism can be *uniformly* controlled just in terms of the *translation lengths* of the a_i , and in terms of δ and $|S|$.

Such separation theorems have been pursued by several authors. For example, Polterovich and Rudnick [34] proved such a separation theorem for the group $SL(2, \mathbb{Z})$, and asked if one can generalize it to hyperbolic groups; our work gives a quantitative positive answer to their question.

In § 6 we state and prove the Accumulation Theorem:

Theorem B (Accumulation Theorem). *Let G be a torsion free non-elementary word-hyperbolic group. Then the first accumulation point δ_∞ for stable commutator length satisfies*

$$\frac{1}{12} \leq \delta_\infty \leq \frac{1}{2}.$$

Here δ_∞ is the first accumulation point for the values (with multiplicity) of the function scl on the set of conjugacy classes in G , thought of as an ordered subset of \mathbb{R} . As mentioned above, we think of this theorem as a kind of “homological Margulis Lemma”. One can obtain similar theorems for groups with torsion, but the statement is not so clean or universal.

In § 7 and § 8 we discuss two important examples of groups acting on δ -hyperbolic spaces: the action of the mapping class group on the complex of curves, and the action of an amalgamated free product on its associated tree.

Our main theorem about the mapping class group is:

Theorem C (Mapping Class Theorem). *Let S be a compact orientable surface of hyperbolic type and $MCG(S)$ its mapping class group. Then there is a positive integer P depending on S such that for any pseudo-Anosov element a , either there is an $0 < n \leq P$ and an element $b \in MCG(S)$ with $ba^{-n}b^{-1} = a^n$, or else there exists a homogeneous quasimorphism h on $MCG(S)$ such that $h(a) = 1$ and the defect of h is $\leq C(S)$, where $C(S)$ depends only on S .*

Moreover, let $a_i \in MCG(S)$ be a collection of elements for which $T = \sup_i \tau(a_i)$ is finite. Suppose that for all integers $n, m \neq 0$ and all elements $b \in MCG(S)$ and indices i there is an inequality

$$ba^n b^{-1} \neq a_i^m.$$

Then there is a homogeneous quasimorphism h on $MCG(S)$ such that

- (1) $h(a) = 1$, and $h(a_i) = 0$ for all i ,
- (2) the defect of h is $\leq C'(S, T)$.

Note that if b is not pseudo-Anosov, then $\tau(b) = 0$. Moreover, for any surface S , there are infinitely many conjugacy classes of elements b with $\tau(b) \leq O(1/g \log(g))$ where $g = \text{genus}(S)$ (see [15], especially Theorem 1.5).

In the case of the mapping class group, one knows that not all the quasimorphisms are of the kind promised by this theorem. Along completely different lines, Endo and Kotschick [12] managed to obtain lower bounds on stable commutator length for elements $a \in S$ which are products of *positive Dehn twists in disjoint curves*. Such elements are reducible, and therefore fix a simplex in the complex of curves $\mathcal{C}(S)$; by contrast, our methods do not give any information about elements with fixed points.

Note that our arguments make use of a key *acylindricity* property proved by Bowditch, which is something like a bicombing in the context of $\mathcal{C}(S)$. Our lower bounds *do* depend on the surface S . There is some evidence that this dependence is necessary: a forthcoming paper by Kotschick [27] shows that stable commutator length goes to zero under stabilizing genus by adding handles.

Note that as in Theorem A', Theorem C includes a *separation theorem*. Endo and Kotschick gave examples that set some limits on how far such a theorem might be generalized. Firstly, they show that there are elements $a \in \text{MCG}(S)$ which are not pseudo-Anosov, but nevertheless for which there are homogeneous quasimorphisms which do not vanish on a . Secondly and conversely, they show that there are elements $b \in \text{MCG}(S)$ of infinite order, for which no power of b is conjugate to its inverse, but for which $h(b) = 0$ for any homogeneous quasimorphism h on $\text{MCG}(S)$. These examples indicate the subtlety of the function scl on $\text{MCG}(S)$, and suggest that a multiplicity of approaches are necessary to appreciate its full richness.

See [12] and [13] respectively for details.

Our main theorem about amalgamated free products is:

Theorem D (Amalgamation Theorem). *Let $G = A *_C B$. Let w be a word on A, B which is reduced and cyclically reduced such that $|w| > 1$, and let \bar{w} denote the corresponding element in G . Suppose that the double coset $C\bar{w}C$ does not contain the element corresponding to any cyclic conjugate of w^{-1} . Then there exists a homogeneous quasimorphism h on G such that $h(\bar{w}) = 1$, and the defect of h is ≤ 312 .*

One can make a similar statement about HNN extensions.

Here a word w is said to be *cyclically reduced* if w is not equal to $v_1 v_2 v_1^{-1}$ ($v_1 \neq \emptyset$) as a word. The double coset condition is at first glance somewhat odd. We examine a particular example (of essential slopes in knot complements in 3-manifolds) in some detail and show why it is natural and unavoidable.

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2. Background Material

2.1. δ -hyperbolic groups and spaces. We assume that the reader is familiar with basic elements of the theory of δ -hyperbolic groups and spaces: quasi-isometries, quasigeodesics, word-hyperbolicity, δ -thin triangles, translation length, etc. For a reference, see [21].

Let G be a δ -hyperbolic group with a fixed generating set S . We denote the translation length of an element $g \in G$ on the Cayley graph $\Gamma_S(G)$ by $\tau(g)$. This length is defined by

$$\tau(g) = \lim_{n \rightarrow \infty} \frac{\text{dist}(\text{id}, g^n)}{n},$$

where distance is measured in the usual way in $\Gamma_S(G)$. Observe that τ is a class function.

If g is torsion, $\tau(g) = 0$. Otherwise, recall that there is a constant $N(|S|, \delta)$ and an integer $n < N$ such that a power g^n of g leaves invariant a bi-infinite geodesic *axis* and acts on this axis as translation through a distance $n\tau(g)$. Since this distance is necessarily a positive integer, this implies that there is a positive constant $B(|S|, \delta)$ such that $\tau(g) \geq B$ for every non-torsion element g . See e.g. Theorem 5.1 of [14].

2.2. Stable commutator length

Definition 2.1. Let G be a group, and $a \in G$. The *commutator length* of a , denoted $\text{cl}(a)$, is the minimum number of commutators whose product is equal to a . If a is not in $[G, G]$, we set $\text{cl}(a) = \infty$. The *stable commutator length*, denoted $\text{scl}(a)$, is the \liminf

$$\text{scl}(a) = \liminf_{n \rightarrow \infty} \frac{\text{cl}(a^n)}{n}.$$

Notice that $\text{scl}(a) = \infty$ if and only if the order of $[a]$ in $H_1(G; \mathbb{Z})$ is infinite.

The functions cl and scl , like τ , are class functions. Observe that cl and scl are *monotone* under homomorphisms. That is, if $\phi: G \rightarrow H$ is a homomorphism, then

$$\text{scl}(\phi(a)) \leq \text{scl}(a)$$

for all $a \in G$, and similarly for cl .

Example 2.2 (Mirror Condition). Let G be a group and suppose that there are elements $a, b \in G$ and integers $n \neq m$ such that $ba^n b^{-1} = a^m$. Then

$$n \cdot \text{scl}(a) = \text{scl}(a^n) = \text{scl}(ba^n b^{-1}) = \text{scl}(a^m) = m \cdot \text{scl}(a)$$

and therefore $\text{scl}(a) = 0$. (Note that the defining property of a means that $a^{n-m} \in [G, G]$, so $\text{scl}(a) \neq \infty$.)

Such elements can be found in *Baumslag–Solitar groups*. We compute

$$|n| \cdot \tau(a) = \tau(a^n) = \tau(ba^n b^{-1}) = \tau(a^m) = |m| \cdot \tau(a),$$

so if $\tau(a)$ is not zero (which can only happen for a torsion element when G is hyperbolic), this is only possible if $n = \pm m$. The group

$$\langle a, b \mid ba^{-1}b^{-1} = a, b^2 = \text{id} \rangle$$

may be thought of as the (orbifold) fundamental group of the *interval with mirrored endpoints*, which is sometimes just called a *mirror interval*. In a hyperbolic manifold, an element of π_1 conjugate to its inverse is represented by a geodesic which “double covers” a geodesic segment with both endpoints on an orbifold stratum of order 2 (i.e., it double covers a mirror interval). Thus we will sometimes say that an element $a \in G$ for which there is no $b \in G$ and no integer $n \neq 0$ for which $ba^n b^{-1} = a^{-n}$ satisfies the *no mirror condition*.

Analogues of this condition will occur in the hypotheses of all our main theorems throughout this paper.

Example 2.3. Let K be a knot complement in S^3 with genus g , and let $M_{p,q}$ be the result of (p, q) Dehn filling on K . If $a \in \pi_1(M_{p,q})$ represents the core of the added solid torus, then $\text{scl}(a) \leq g/p$ (in fact, by taking covers and tubing boundary components together, one can actually obtain $\text{scl}(a) \leq (g - 1/2)/p$; see [5]). If K is not a satellite knot or a torus knot, for all but finitely many pairs (p, q) the resulting manifold $M_{p,q}$ is hyperbolic, and its fundamental group is word-hyperbolic. (See e.g. [35]).

2.3. Quasimorphisms

Definition 2.4. Let G be a group. A *quasimorphism* is a function

$$\phi: G \rightarrow \mathbb{R}$$

for which there is a least constant $D(\phi) \geq 0$, called the *defect*, such that

$$|\phi(a) + \phi(b) - \phi(ab)| \leq D(\phi)$$

for all $a, b \in G$. In words, a quasimorphism fails to be linear by a bounded amount.

A quasimorphism is *homogeneous* if $\phi(a^n) = n\phi(a)$ for all integers n and all $a \in G$.

If ϕ is a quasimorphism on G , then one can obtain a homogeneous quasimorphism $\bar{\phi}$ by the formula

$$\bar{\phi}(a) = \lim_{n \rightarrow \infty} \frac{\phi(a^n)}{n}.$$

Note that the defining property of a quasimorphism (that it is “almost linear”) implies that the limit exists. A homogeneous quasimorphism is a class function.

The defect $D(\bar{\phi})$ is related to $D(\phi)$ by

$$D(\bar{\phi}) \leq 2 \cdot D(\phi);$$

see e.g. [6], Corollary 2.59. Homogeneous quasimorphisms are often easier to work with than ordinary quasimorphisms, but ordinary quasimorphisms are easier to construct. We use this averaging procedure to move back and forth between the two concepts.

We denote the vector space of all homogeneous quasimorphisms on G by $Q(G)$. Observe that for any commutator $[a, b] \in G$ and any $\phi \in Q(G)$ we have

$$|\phi([a, b])| \leq D(\phi).$$

It turns out that there is an equality

$$\sup_{a,b} |\phi([a, b])| = D(\phi);$$

see [1], Lemma 3.6.

Quasimorphisms and stable commutator length are related by Bavard’s Duality Theorem (c.f. [1]):

Theorem 2.5 (Bavard’s Duality Theorem). *Let G be a group. Then for any $a \in [G, G]$ we have an equality*

$$\text{scl}(a) = \frac{1}{2} \sup_{\phi \in Q(G)} \frac{|\phi(a)|}{D(\phi)}.$$

Note that one must take the supremum over $\phi \in Q(G)$ with $\phi(a) \neq 0$ (and therefore $D(\phi) > 0$) for this to make sense; also, if $Q(G) = H^1(G)$ then $\text{scl}(a) = 0$ for $a \in [G, G]$. Note further that the theorem makes sense and is true for a satisfying $a^n \in [G, G]$ for some positive n .

Bavard’s theorem depends on the Hahn–Banach theorem and L^1 - L^∞ duality. Note that the Hahn–Banach theorem is equivalent to the Axiom of Choice. In particular, the quasimorphisms promised by Bavard’s theorem are typically *not constructible*. Therefore we take explicit note in the sequel of when our arguments make use of Bavard’s theorem and when they do not.

Example 2.6. For non-elementary hyperbolic groups G , the space $Q(G)$ has an uncountable dimension ([14]). But for certain groups, one can completely understand $Q(G)$. For example, let \tilde{G} denote the universal central extension of the group of all orientation-preserving homeomorphisms of S^1 . Then $Q(\tilde{G}) = \mathbb{R}$, generated by Poincaré’s rotation number. In particular, every non-negative real number is equal to scl of some conjugacy class in this group. Similarly, let \tilde{T} denote the universal central

extension of Thompson’s group T of dyadic piecewise linear homeomorphisms of S^1 . Then \hat{T} is a finitely presented group which realizes every non-negative rational number as scl. See [19] for more about the group T and its bounded cohomology.

2.4. Mineyev’s geodesic flow space. To understand stable commutator length in word-hyperbolic groups, one needs to control the geometry of maps of surfaces into δ -hyperbolic spaces. Naively, following the usual practice in hyperbolic manifolds, one triangulates a surface and “straightens” the simplices, and then appeals to the Gauss–Bonnet theorem to control area and therefore diameter in terms of injectivity radius. In a δ -hyperbolic space, the straightening must be done in a careful way. There are at least three technical approaches to this straightening:

- (1) Gromov’s geodesic flow space ([21], chapter 8),
- (2) Mineyev–Monod–Shalom’s homological ideal bicombing ([30]),
- (3) Mineyev’s geodesic flow space ([29]).

The approach in [21] is not entirely fleshed out, and the “geodesic flow” is really a quasigeodesic flow; therefore for our applications, Mineyev’s flow space is best suited.

Mineyev constructs from a hyperbolic complex X (for instance, a Cayley graph for a word-hyperbolic group) a geodesic flow space $\mathcal{F}(X)$, which consists of a union of parameterized lines joining ordered pairs of distinct points in the ideal boundary ∂X . The space $\mathcal{F}(X)$ admits a number of metrics and pseudo-metrics, of which the pseudo-metric d^\times is most important to us.

The following summarizes some of the main properties of $\mathcal{F}(X)$ which we use.

Theorem 2.7 (Mineyev). *Let X, d_X be a δ -hyperbolic complex with valence $\leq n$. Then there exists a pseudo-metric space $\mathcal{F}(X), d^\times$ called the flow space of X with the following properties:*

- (1) $\mathcal{F}(X)$ is homeomorphic to $(\partial X \times \partial X - \Delta) \times \mathbb{R}$. The factors (p, q, \cdot) under this homeomorphism are called the flowlines.
- (2) There is an \mathbb{R} -action on $\mathcal{F}(X)$ (the geodesic flow) which acts as an isometric translation on each flowline (p, q, \cdot) .
- (3) There is a $\mathbb{Z}/2\mathbb{Z}$ action $x \rightarrow x^*$ which anti-commutes with the \mathbb{R} action, which satisfies $d^\times(x, x^*) = 0$, and which interchanges the flowlines (p, q, \cdot) and (q, p, \cdot) .
- (4) There is a natural action of $\text{Isom}(X)$ on $\mathcal{F}(X)$ by isometries. If $g \in \text{Isom}(X)$ is hyperbolic with fixed points p^\pm in ∂X then g fixes the flowline (p^-, p^+, \cdot) of $\mathcal{F}(X)$ and acts on it as a translation by a distance which we denote $\tau(g)$. This action of $\text{Isom}(X)$ commutes with the \mathbb{R} and $\mathbb{Z}/2\mathbb{Z}$ actions.
- (5) There are constants $M \geq 0$ and $0 \leq \lambda < 1$ such that for all triples $a, b, c \in \partial X$, there is a natural isometric parameterization of the flowlines $(a, c, \cdot), (b, c, \cdot)$

for which there is exponential convergence

$$d^\times((a, c, t), (b, c, t)) \leq M\lambda^t.$$

- (6) If X admits a cocompact isometric action, and $G \subset \text{Isom}(X)$ is torsion free, then there is a G -equivariant (K, ϵ) quasi-isometry between $\mathcal{F}(X)$, d^\times and X , d_X .

Moreover, all constants as above depend only on δ and n .

Note that although $\mathcal{F}(X)$ is homeomorphic to $(\partial X \times \partial X - \Delta) \times \mathbb{R}$, this topology is not induced by the pseudo-metric d^\times , since this pseudo-metric fails to separate pairs of points interchanged by the $\mathbb{Z}/2\mathbb{Z}$ action.

This is a conflation of several results in [29]. The pseudo-metric d^\times is defined in Section 3.2 and Section 8.6 on a slightly larger space which Mineyev calls the *symmetric join*. The flow space, defined in Section 13, is a natural subset of this. The basic properties of the \mathbb{R} , $\mathbb{Z}/2\mathbb{Z}$ and $\text{Isom}(X)$ action are proved in Section 2. The remaining properties are subsets of Theorem 44 (p. 459) and Theorem 57 (p. 468). There is another natural metric d_* on $\mathcal{F}(X)$ which induces the topology on $\mathcal{F}(X)$, and for which the \mathbb{R} action is by bi-Lipschitz homeomorphisms.

For our applications, the key points are that the action of hyperbolic isometries on the flowlines is by translations, and the exponential convergence of flowlines with common endpoints at infinity.

Remark 2.8. Since the flowlines (p, q, \cdot) and (q, p, \cdot) are distance 0 apart in d^\times , we may think of them as different parameterizations of the same lines in a suitable quotient on which d^\times is a metric (and not just a pseudo-metric).

Remark 2.9. In bullet (5), the synchronous parameterizations of (a, c, t) and (b, c, t) for which exponential convergence holds are precisely those for which $(a, c, 0)$ is the point on the flowline (a, c, \cdot) closest to b , and $(b, c, 0)$ is the point on the flowline (b, c, \cdot) closest to a (interpreted in terms of suitable horofunctions). In particular, for every triple a, b, c of distinct points in $\mathcal{F}(X)$, if Δ is a triangle obtained from the union of three flowlines joining these points in pairs, the edges of Δ are exponentially close to each other away from a compact subset of Δ of uniformly bounded diameter (i.e., the diameter of the “thick part” of Δ is bounded independently of the choice of a, b, c).

3. The Gap Theorem, first version

The Gap Theorem, to be proved below in its first version, says that in a word-hyperbolic group G , if a is an element which satisfies the no mirror condition (from

Example 2.2) and has a sufficiently long translation length, then the stable commutator length of a can be *uniformly* bounded from below. Example 2.2 and Example 2.3 together point to the necessity of both hypotheses.

Theorem A (Gap Theorem, weak version). *Let G be a word-hyperbolic group that is δ -hyperbolic with respect to a symmetric generating set S with $|S|$ generators. Then there is a constant $C(\delta, |S|) > 0$ such that for every $a \in G$, either $\text{scl}(a) \geq C$ or else there is some positive integer n and some $b \in G$ such that $ba^{-n}b^{-1} = a^n$.*

Remark 3.1. By Bavard's theorem, the uniform lower bound on $\text{scl}(a)$ is equivalent to the existence of a homogeneous quasimorphism ϕ with $\phi(a) = 1$ for which there is a uniform upper bound on the defect $D(\phi)$ depending only on δ and $|S|$.

Remark 3.2. If $ba^{-n}b^{-1} = a^n$ for some b and positive integer n , then b^2 and a^n commute, and since G is hyperbolic, suitable powers of a and b generate an infinite dihedral group.

It follows that if G is torsion free, no such element b can exist. Conversely, note that every torsion element a satisfies $a^n = a^{-n} = \text{id}$ for some positive n . It follows that the no mirrors condition is vacuously satisfied in a torsion-free hyperbolic group.

The following proof uses properties of Mineyev's geodesic flow space. We prove a stronger theorem in § 5, by directly constructing sufficiently many quasimorphisms. The second proof is logically superior to the first, since the construction is direct, and moreover the existence of these quasimorphisms does not depend on Bavard's Theorem and the Axiom of Choice. However, the first proof is more "geometric".

Proof. Let $\Gamma_S(G)$ denote the Cayley graph of G with respect to the generating set S . For any element $a \in G$ recall that $\tau(a)$ denotes the translation length of a . As observed in § 2.1, every element $a \in G$ is either torsion, or has a power a^n where $n \leq C_2(|S|, \delta)$ that fixes an axis l_a , and therefore has $\tau(a) \geq C_1(|S|, \delta)$. For the remainder of the proof we replace a by a suitable power a^n , and assume that a fixes an axis l_a .

For convenience, we let K be a 2-complex coming from a finite presentation for G with generating set S . Observe that $\Gamma_S(G)$ is the 1-skeleton of the universal cover \tilde{K} . Suppose that $\text{scl}(a)$ is very small so that there is an expression

$$a^n = \prod_{i=1}^m [b_i, c_i]$$

for some $b_i, c_i \in G$ such that m/n is small. Let Σ be a surface of genus m , and $\sigma: \Sigma \rightarrow K$ a simplicial map which takes the boundary to the immersed circle $\gamma = l_a/a^n$. We would like to choose a representative surface whose area and geometry can be controlled by $\chi(\Sigma)$ and $\text{length}(\gamma)$. Since we need to control constants, this must be done carefully.

We appeal to Mineyev’s Theorem 2.7. Using this theorem, one proceeds as follows. One picks a 1-vertex triangulation of Σ , and spins the vertex along $\partial\Sigma$ thereby producing an ideal triangulation whose edges can be realized as a subset in the quotient space $\mathcal{F}(\Gamma_S(G))/G$. This is covered by a $\pi_1(\Sigma)$ equivariant map from the 1-skeleton of $\tilde{\Sigma}$ to $\mathcal{F}(\Gamma_S(G))$, where $\pi_1(\Sigma)$ acts on $\mathcal{F}(\Gamma_S(G))$ by its image in G . Each ideal triangle in $\tilde{\Sigma}$ corresponds to a union of three flowlines in $\mathcal{F}(\Gamma_S(G))$.

In \mathcal{F} , as pointed out in Remark 2.9, each triangle consists of a thick *core* of diameter at most C_3 (depending only on δ and $|S|$) together with thin regions consisting of pairs of geodesic rays whose distance converges exponentially fast to zero. The image of Σ consists of $2|\chi(\Sigma)| = 4m - 2$ triangles. Because of the exponential convergence of geodesics in \mathcal{F} , we can give Σ a hyperbolic metric of constant curvature $-\kappa$ in such a way that the triangles are totally geodesic, and the map to $\mathcal{F}(\Gamma_S(G))$ is distance decreasing on the 1-skeleton, where $\kappa > 0$ depends only on δ and $|S|$. It follows from the Gauss–Bonnet theorem that away from a thick part consisting of at most $(4m - 2)$ regions whose diameters sum to at most $(4m - 2)C_3$ for some constant C_3 , the thickness of Σ is bounded by a constant C_4 , where C_3 and C_4 depend only on $|S|$ and δ . In fact, by choosing C_3 sufficiently large, we may assume that C_4 is as small as we like, a fact which we will not use.

After composing with a quasi-isometry

$$\phi: \mathcal{F}(X) \rightarrow \tilde{K}$$

and filling in the map on triangles, we get an induced map

$$\tilde{\sigma}_1: \tilde{\Sigma} \rightarrow \tilde{K}.$$

It does not really matter how the map is filled in on triangles, since we are ultimately only interested in the distances between points contained in the image of $\partial\tilde{\Sigma}$. Again, by Theorem 2.7, we can assume that the quasi-isometry constants of ϕ depend only on δ and $|S|$.

Under this quasi-isometry, the constants C_3 and C_4 for the diameter of the core and the thickness of the complementary region must be replaced by analogous constants C'_3 and C'_4 which still depend only on δ and $|S|$. The image of every boundary component is a quasigeodesic which is within a bounded distance of some translate of the axis l_a ; it follows that for a suitable choice of ϕ , without changing the constants involved, we can assume that $\tilde{\Sigma}_1$ takes $\partial\tilde{\Sigma}$ in a $\pi_1(\Sigma)$ -equivariant way to a union of translates of l_a .

By abuse of notation, we denote the image of this map as $\tilde{\Sigma}_1$. Let α be a fundamental domain for the action of a^n on l_a . By our estimates, away from a subset of α of length at most $(4m - 2)C'_3$, every point $p \in \alpha$ can be joined by an arc $\tilde{\beta}_p$ in the image of $\tilde{\Sigma}_1$, such that the endpoints of $\tilde{\beta}_p$ are a distance at most C'_4 apart in \tilde{K} , and lie on distinct components l_a and l_a^i of $\tilde{\Sigma}_1(\partial\tilde{\Sigma}_1)$. Pulling back by $\tilde{\Sigma}_1$ and projecting to Σ_1 , we obtain a homotopically essential proper arc $\beta_p \in \Sigma_1$. If p_1, p_2 are at least distance $2C'_4$ apart in α , we see that the pulled-back arcs β_{p_1}, β_{p_2} can be isotoped to

be disjoint in Σ_1 . A surface of genus m with one boundary component contains at most $6m - 3$ disjoint nonparallel proper essential arcs, and each arc has 2 endpoints. It follows that there are at most $12m - 6$ components l_a^i which are joined by arcs β_p with nearby endpoints to points in α .

For a fixed component l_a^i , the set of points where l_a^i is close to l_a is (coarsely) connected, by convexity of quasigeodesics in δ -hyperbolic spaces. Consequently there is a connected arc $\sigma' \subset \alpha$ satisfying

$$\text{length}(\sigma') \geq \frac{\text{length}(\alpha) - (4m - 2)(C'_3 + C'_4)}{12m - 6},$$

which cobounds a strip R' of $\tilde{\Sigma}_1$ of width $\leq C'_4$ with some fixed l_a^i .

By convexity of quasigeodesics, there is a connected subinterval $\sigma \subset \sigma'$ satisfying

$$\text{length}(\sigma) \geq \text{length}(\sigma') - 2C'_4$$

such that σ cobounds a strip R of $\tilde{\Sigma}_1$ of width C_5 depending only on δ with some fixed l_a^i . The strip R is much longer than it is wide, and it makes sense to say that a choice of orientations on the l_a and l_a^i agree or disagree (see Figure 1):



Figure 1. The orientations on l_a and l_a^i might agree or disagree along R .

Now, since Σ_1 is oriented, the induced orientations on l_a and l_a^i disagree. Therefore l_a^i is a nontrivial translate of the axis l_a . That is, there is some c with $l_a^i = c(l_a)$. Then $e = ca^{-1}c^{-1}$ stabilizes l_a^i , and moves points in (roughly) the same direction as a .

Let p be the midpoint of σ . Suppose that $\text{length}(\gamma) = \tau(a)$ is big compared to the constant C_5 (e.g., $\text{length}(\gamma) \geq 2C_5$ will do). We consider the translates $e^{-w}a^w(p)$ for

$$|w| \leq \frac{\text{length}(\sigma)}{3 \cdot \text{length}(\gamma)} \leq \frac{n}{36m} + 1,$$

where the last inequality follows from our assumptions about $\text{length}(\gamma)$.

Then by our estimate on the width of the strip R , we have

$$d(p, e^{-w}a^w(p)) \leq 4C_5.$$

There are at most $|S|^{4C_5}$ elements in the group which move the point p a distance $\leq 4C_5$, and therefore if

$$\frac{n}{36m} + 1 > |S|^{4C_5}$$

then by the pigeonhole principle there are $w_1 \neq w_2$ such that

$$e^{-w_1}a^{w_1} = e^{-w_2}a^{w_2}$$

and therefore

$$ca^{w_1-w_2}c^{-1} = e^{w_2-w_1} = a^{w_2-w_1}.$$

In words: if $\text{scl}(a)$ is too small, a nontrivial power of a is conjugate to its inverse.

It follows contrapositively that when $\tau(a) \geq 2C_5$ and no power of a is conjugate to its inverse, we have $\text{scl}(a) \geq \frac{1}{36}|S|^{-4C_5}$. If a is not torsion, $\tau(a) \geq C_1$. So for any a with no power conjugate to its inverse, we have an estimate

$$\text{scl}(a^{2C_5/C_1}) \geq \frac{1}{36}|S|^{-4C_5}.$$

Therefore the constant

$$C(\delta, |S|) = \frac{1}{36}|S|^{-4C_5} \frac{C_1}{2C_5}$$

satisfies the claim of the theorem. \square

Remark 3.3. It follows from the proof that if a does not satisfy the no mirror condition, so that there is a positive n with $a^n = ca^{-n}c^{-1}$ for some c , then n may be chosen to be less than some constant depending only on δ and $|S|$.

It is interesting to ask the following:

Question 3.4. Let G be a group, and let H be the set of elements in G which satisfy the no mirror condition, i.e., for which $a^n \neq ba^{-n}b^{-1}$ whenever $b \in G$ and $n \neq 0$. Under what circumstances is $\inf_{a \in H} \text{scl}(a)$ positive?

We say that a group G has the *stable commutator gap property* if it has the property described in Question 3.4. Theorem A shows that hyperbolic groups have the stable commutator gap property.

Example 3.5. In [7], Comerford and Edmunds show that in a free group F , every nonzero element a satisfies $\text{scl}(a) \geq 1/2$ (this lower bound essentially follows from an earlier result of Duncan and Howie [10] and improves an earlier estimate $\text{scl}(a) \geq 1/6$ obtained by Culler in [8]). Let G be residually free; i.e., suppose that for all nonzero $a \in G$ there is a homomorphism $\phi_a: G \rightarrow F$ to a free group, for which $\phi_a(a)$ is nonzero. Since scl is nonincreasing under homomorphisms, G satisfies the stable commutator gap property.

4. Counting functions

Here we review so-called *counting functions*, which generalize a construction introduced by Brooks [4] to construct quasimorphisms on free groups. See [14], [16] or [2] for more details.

Suppose that G is a group with a fixed symmetric generating set S , and $\Gamma := \Gamma_S(G)$ is its Cayley graph. Let w be a word in the generating set. Let α be a (directed) path in Γ , and $|\alpha|$ its length. Define $|\alpha|_w$ to be the maximal number of times that w can be seen as an (oriented) subword of α without overlapping.

Example 4.1. $|xyxyx|_{xy} = 2$. $|xyxyx|_{xyx} = 1$. $|xxyxy|_{yx} = 1$.

Let $0 < W < |w|$ be a constant. For $x, y \in \Gamma$, define

$$c_{w,W}(x, y) := d(x, y) - \inf_{\alpha} (|\alpha| - W|\alpha|_w),$$

where α ranges over all the paths from x to y . If the infimum is attained by α , we say that α is a *realizing path* for $c_{w,W}$ from x to y . If γ is a geodesic from x to y , then define $c_{w,W}(\gamma) = c_{w,W}(x, y)$.

Fix a point $x \in \Gamma$. Define, for $a \in G$,

$$c_{w,W}(a) := c_{w,W}(x, ax).$$

$c_{w,W}$ is called the *counting function* for the pair (w, W) . Let w^{-1} denote the inverse word of w . We define

$$h_{w,W} := c_{w,W} - c_{w^{-1},W}.$$

In [14], the normalization $W = 1$ is used. This is an appropriate choice of constant when $w^* := \dots w w w \dots$ is a bi-infinite geodesic, which is the case in our applications throughout this paper.

More generally, when G acts on a graph Γ (not necessarily properly), we modify the definition of counting functions as follows. Let w be a path in Γ and call aw for $a \in G$ a *copy of w* or (interchangeably) a *translate of w* . For a path α in Γ , define $|\alpha|_w$ to be the maximal number of disjoint oriented copies of w which can be obtained as subpaths of α . All other definitions are as above. These modified counting functions will be important in the sequel when we discuss the action of the mapping class group $\text{MCG}(S)$ on the complex of curves $\mathcal{C}(S)$, for some surface S .

Proposition 4.2 (Lemma 3.3 [16]). *If α is a realizing path for $c_{w,W}$, then it is a (K, ϵ) -quasigeodesic, where*

$$K = \frac{|w|}{|w| - W}, \quad \epsilon = \frac{2W|w|}{|w| - W}.$$

It is known that in a δ -hyperbolic space any two (K, ϵ) -quasigeodesics which have same end points stay in the $L(K, \epsilon, \delta)$ -neighborhood of each other for some universal constant L (see [21]).

Let $L = L(|w|/(|w| - W), 2W|w|/(|w| - W), \delta)$. Let α be a geodesic from x to y . From Proposition 4.2 we deduce that a realizing path for α must be contained in the L -neighborhood of α . Consequently, if the L -neighborhood of α does not contain a copy of w , then $c_{w,W}(\alpha) = 0$.

Remark 4.3. We will use this fact later in our argument to avoid “reverse counting”. Roughly speaking, let w be a word such that w^n is a geodesic. Then, for $n > 0$,

$$c_{w,W}(w^n) \geq Wn$$

because $|w^n|_w = n$.

Suppose that the L -neighborhood of w^n does not contain a copy of w^{-1} . Here we are thinking of the L -neighborhood of w^n , for large n , like a long narrow tube whose core has a definite orientation, agreeing with the orientation on w . By “a copy of w^{-1} ” here we mean a copy of w whose orientation *disagrees* with that of the core of the tube (compare with Figure 1 and the accompanying discussion). Then it follows that $c_{w^{-1},W}(w^n) = 0$ because for a realizing path α for $c_{w^{-1},W}$ at w^n we must have $|\alpha|_{w^{-1}} = 0$. We thus obtain for all $n > 0$ an inequality $h_{w,W}(w^n) \geq nW$.

Let $D(h)$ be the defect of h . Then we have the following inequality:

Proposition 4.4 (Proposition 3.10 [16]).

$$D(h_{w,W}) \leq 12L + 6W + 48\delta.$$

Remark 4.5. Note that the defect only depends on $|w|$, W , δ . If we take $W = 1$, then L depends only on δ if $|w| \geq 2$. In particular, the upper bound in Proposition 4.4 depends *only* on δ . We will take $W = 1$ in all the applications in this paper.

5. The Gap Theorem, improved version

We are now in a position to state and prove the improved version of the Gap Theorem, using counting quasimorphisms. The following theorem improves Theorem A in at least two ways: it does not use Bavard’s theorem or Mineyev’s flow space, and the constants in questions can be effectively estimated.

Theorem A’ (Gap Theorem, strong version). *Let G be a word-hyperbolic group that is δ -hyperbolic with respect to a symmetric generating set S with $|S|$ generators. Let $a \in G$ be a (non-torsion) element. Assume that there is no $n > 0$ and no $b \in G$ with $ba^{-n}b^{-1} = a^n$. Then there is a homogeneous quasimorphism h on G such that*

- (1) $h(a) = 1$,
- (2) *the defect of h is $\leq C(\delta, |S|)$.*

Moreover, let $a_i \in G$ be a collection of elements for which $T = \sup_i \tau(a_i)$ is finite. Suppose that for all integers $n, m \neq 0$ and all elements $b \in G$ and indices i there is an inequality

$$ba^n b^{-1} \neq a_i^m.$$

Then there is a homogeneous quasimorphism h on G such that

- (1) $h(a) = 1$, and $h(a_i) = 0$ for all i ,
- (2) the defect of h is $\leq C'(\delta, |S|, T)$.

Remark 5.1. Note that in the second part of the theorem, we must state a generalization of the no mirror condition, involving both a and the elements a_i .

Remark 5.2. Hamenstädt ([24]) used dynamical methods to directly construct quasi-morphisms on word-hyperbolic groups. Similar ideas were developed by Picaut ([32]) in the special case of surface groups. It is not clear to us how to use these methods to obtain uniform estimates on defects.

Proof. We follow [14], [16]. We assume in what follows that a is not torsion. Let N be a constant such that a^N stabilizes a bi-infinite geodesic l_a . Note that one may find an N whose size can be bounded in terms of δ and $|S|$. We also choose N so that $N\tau(a) \gg L(\delta)$. Note that since $\tau(a)$ can be bounded from below by a positive constant depending only on δ and $|S|$, the constant N still depends only on δ and $|S|$ (see the proof of Theorem A).

After replacing a by a conjugate if necessary, we may assume that $1 \in l_a$. Set $b = a^N$. l_a is an axis for b . We denote the subpath of l_a from 1 to b by $l_a|b$ and the subpath from 1 to b^{-1} by $l_a|b^{-1}$. Note that we think of these as *oriented* segments.

Now let c_b be the counting function on G for the pair $(l_a|b, 1)$ (note that we are setting $W = 1$ in the notation of § 4). Then for any $n > 0$, we obtain an estimate $c_b(b^n) \geq n$, since the subpath of l_a from 1 to b^n can be tiled by n disjoint translates of $l_a|b$. At the cost of possibly replacing N by $2N$ if necessary, we may assume that $|l_a|b| \geq 2$, so that L (as in § 4) depends only on δ .

Claim 1: There is a constant $C_1(\delta, |S|)$ such that if $N \geq C_1$ then there is no translate of $l_a|b^{-1}$ in the L -neighborhood of l_a whose orientation agrees with that of l_a .

Note that $l_a|b^{-1}$ is a copy of $l_a|b$ with the opposite orientation, so we could just as well state Claim 1 as saying that there is no translate of $l_a|b$ in the L -neighborhood of l_a whose orientation *disagrees* with that of l_a . We will prove the claim later; we call the conclusion of the claim “no reverse counting”, and prove our theorem under this hypothesis.

Take N to satisfy $N \geq C_1$ as well in the following. Note that N can still be chosen satisfying these criteria with size bounded from above in terms of δ and $|S|$.

Let $c_{b^{-1}}$ be the counting function for the pair $(l_a|b^{-1}, 1)$. It follows from Claim 1 that

$$c_{b^{-1}}(b^n) = 0$$

for all $n > 0$. We define $h_b = c_b - c_{b^{-1}}$ and then obtain

$$h_b(b^n) \geq n$$

for any $n > 0$.

By Proposition 4.4 and Remark 4.5, there is a constant $K(\delta)$, which depends only on δ such that

$$D(h_b) \leq K(\delta).$$

Since $b = a^N$, we get $h_b(a^n) \geq n/N$ for all $n > 0$. By averaging, we may replace h_b by a homogeneous quasimorphism $h' = \bar{h}_b$ (see § 2.3). Then $D(h') \leq 4K(\delta)$ and $h'(a) \geq 1/N$. Define $h = mh'$ for some constant m for which $h(a) = 1$. Then h is a homogeneous quasimorphism, and satisfies $D(h) \leq 4NK(\delta)$.

This proves the first part of the theorem, modulo Claim 1.

The second part follows by almost the same argument. First of all, we may assume that a_i has infinite order (equivalently, $\tau(a_i) > 0$) since every homogeneous quasimorphism will already vanish on torsion. Since we are looking for a homogeneous quasimorphism which vanishes on a_i , without loss of generality, we may replace each a_i by a conjugate of a non-trivial power ($\leq N(\delta, |S|)$). Therefore, we may assume that a_i has a geodesic axis l_{a_i} , with $1 \in l_{a_i}$.

Let v be an oriented geodesic path in $\Gamma_S(G)$ from 1 to a^N , and let v^{-1} be v with the opposite orientation. In other words, $v = l_a|_{a^N}$.

A relative version of Claim 1 proves the second part of the theorem. We give two different proofs, which give different constants. Recall our notation $b = a^N$.

Claim 2: Suppose that N satisfies

$$N \geq C_1 |S|^{\tau(a)} \frac{\tau(a)}{\tau(a_i) + 1}$$

for suitable C_1 depending only on δ and $|S|$. Then there is no copy of $l_a|b$ or $l_a|b^{-1}$ in the L -neighborhood of l_{a_i} .

Claim 2': Suppose that N satisfies

$$N \geq C_1 |S|^T$$

for suitable C_1 depending only on δ and $|S|$. Then there is no copy of $l_a|b$ or $l_a|b^{-1}$ in the L -neighborhood of l_{a_i} .

It follows from Claim 2 or 2' that h as constructed above satisfies $h(a_i) = 0$ for all i .

Proof of Claim 1. In fact, the proof of the claim follows by the same argument as the end of the proof of Theorem A. A copy of $l_a|b^{-1}$ in the L -neighborhood of l_a is contained in an axis of some element e which is conjugate to a^{-1} ; i.e. $e = ca^{-1}c^{-1}$ for some c .

Let v denote such a copy of $l_a|b^{-1}$, and let v_0 represent its initial point. The geodesic l_a is invariant under a^N but not necessarily under a itself. Nevertheless, there is a constant C_2 depending only on δ such that $a^i(v_0)$ is within distance C_2 of l_a , for any integer i . Then for all $i < N$ the element $e^{-i}a^i$ satisfies

$$d(v_0, e^{-i}a^i(v_0)) \leq 2L + 2C_2.$$

So if $N > |S|^{2L+2C_2}$ we must have $e^{-i_1}a^{i_1} = e^{-i_2}a^{i_2}$ for two distinct indices i_1, i_2 , and therefore the $(i_2 - i_1)$ -th power of a is conjugate by c to its inverse. Set $C_1 = |S|^{2L+2C_2}$.

Proof of Claim 2. We only sketch the proof of Claim 2 since the details are almost identical to those of the proof of Claim 1. If the L -neighborhood of l_{a_i} contains a copy of a arbitrarily long subpath of l_a , we can find a point v_0 for which

$$d(v_0, e^{k_j}a_i^j(v_0)) \leq 2L + 2C_2 + \tau(a)$$

for some conjugate e of a or a^{-1} , and for any $0 \leq j \leq J$ for an arbitrarily big (fixed) J . It follows that if $N > |S|^{2L+2C_2+\tau(a)}$ then some non-trivial power of a_i is conjugate to some (possibly trivial) power of a , say a^k . Moreover, if

$$N > |S|^{2L+2C_2+\tau(a)} \frac{\tau(a)}{\tau(a_i) + 1}$$

then $k \neq 0$, contrary to hypothesis.

Proof of Claim 2'. This argument interchanges the roles of a and a_i in the proof of Claim 2. Suppose that the L -neighborhood of l_{a_i} contains a copy, v , of $l_a|b$ or $l_a|b^{-1}$. The segment v is a part of an axis of $e = cac^{-1}$ or $ca^{-1}c^{-1}$. Let v_0 be the starting point of v . Then for each $0 \leq j \leq N$, there exists k_j such that

$$d(v_0, a_i^{k_j}e^j(v_0)) \leq 2L + 2C_2 + \tau(a_i) \leq 2L + 2C_2 + T$$

(we use the same constant C_2 for a_i). Therefore, if $N > |S|^{2L+2C_2+T}$, then some non-trivial power of a is conjugate to some (possibly trivial) power of a_i , which is impossible.

This completes the proof of the theorem. □

Remark 5.3. The second statement can be used to give a lower bound of the stable commutator length “relative to $\{a_i\}$ ”. That is, if we can write a as a product

$$a = [b_1, c_1] \dots [b_n, c_n] a_{i_1}^{m_1} \dots a_{i_k}^{m_k}$$

then we say the *relative length* of a is $\leq n + k/2$. The infimum of this number is the *relative commutator length*, and the liminf of the relative commutator length of a^n divided by n as $n \rightarrow \infty$ is the *relative stable commutator length*. Using our theorem, the relative commutator length has a lower bound of $1/2D(h)$, where h is a homogeneous quasimorphism obtained in the second part for a and the a_i .

Remark 5.4. One may reinterpret the second part of Theorem A' as follows. In any group G , let $B_1(G)$ denote the real vector space of group 1-boundaries (i.e., the group

1-cycles that are boundaries of group 2-chains), and for any chain $\sum t_i g_i \in B_1(G)$ define scl by the formula

$$\text{scl}(\sum t_i g_i) = \sup_{\phi} \frac{|\sum t_i \phi(g_i)|}{2D(\phi)}$$

(compare with Theorem 2.5). This function is a pseudo-norm, and agrees with scl on ordinary elements (see [6], § 2.6). It evidently vanishes on the subspace H spanned by cycles of the form $g - hgh^{-1}$ and $g^n - ng$, and descends to a quotient pseudo-norm on $B_1^H(G) := B_1(G)/H$. Then the second part of Theorem A' implies that whenever G is hyperbolic, scl is a genuine *norm* on $B_1^H(G)$.

We refer to the homogeneous quasimorphisms constructed in the proof of Theorem A' as *counting quasimorphisms*, by contrast with the abstract quasimorphisms promised by Bavard's theorem.

It is interesting to speculate that one could use Theorem A or A' as a starting point to invert the word-hyperbolic Dehn surgery theory developed by Groves and Manning and independently by Osin (cf. [23], [31]).

Corollary 5.5. *Let K be a knot in S^3 of genus g . Then for any $\delta > 0$ and every integer $n > 0$ there is a constant $C(\delta, n) > 0$ such that if $M_{p/q}$ is the result of p/q surgery on K and $|p| \geq C \cdot g$, then every homomorphism from $\pi_1(M_{p/q})$ to a torsion-free n -generator δ -hyperbolic group is trivial.*

Proof. By construction, if a represents the core geodesic of $M_{p/q}$, we can estimate

$$\text{scl}(a) \leq g/|p|.$$

Moreover, since K is a knot in S^3 , $\pi_1(M_{p/q})$ is normally generated by a .

Let G be δ -hyperbolic and torsion free. Then no element is conjugate to its inverse, and therefore there is a uniform lower bound on the stable commutator length of any nontrivial element in G , depending only on δ and the size of a generating set for G . If $\rho: \pi_1(M_{p/q}) \rightarrow G$ is any homomorphism, then $\text{scl}(\rho(a)) \leq \text{scl}(a)$, and the image is normally generated by $\rho(a)$. The claim follows. \square

Remark 5.6. One knows that under a degree 1 map between hyperbolic manifolds of the same dimension, volume must go down. It is therefore significant in this corollary that the volumes of the manifolds $M_{p/q}$ go up as $|p| \rightarrow \infty$.

6. First accumulation point

In a torsion free word-hyperbolic group G , stable commutator length defines a function scl from conjugacy classes to \mathbb{R} . By Theorem A or A', the first accumulation point δ_∞ for the image of this function is positive. In this section, we obtain *universal* estimates for δ_∞ which are *independent* of G .

Theorem B (Accumulation Theorem). *Let G be a torsion free non-elementary word-hyperbolic group. Then the first accumulation point δ_∞ for stable commutator length satisfies*

$$\frac{1}{12} \leq \delta_\infty \leq \frac{1}{2}.$$

Proof. To obtain the upper bound, observe that G contains a quasi-isometrically embedded copy of the free group on 2 generators. A nonabelian free group contains infinitely many conjugacy classes with $\text{scl} \leq 1/2$ (see e.g. [1]). Since the embedding is quasi-isometric, the image of infinitely many of these conjugacy classes stay non-conjugate in G . Under any homomorphism, scl cannot go up, so the upper bound is proved.

An elementary argument gives a lower bound of $1/24$. We follow the argument in the proof of Theorem A, and we adopt notation and the setup from that theorem. In any hyperbolic group, there are only finitely many conjugacy classes of elements whose translation length is bounded above by any constant. Therefore, we may assume that the translation length of an element a is as long as we like. In particular, we can assume that there is an axis l_a which is geodesic and invariant under a . As before, let α denote a fundamental domain for a^n . Then $\text{length}(\alpha) = n\tau(a)$.

We suppose, as in the proof of Theorem A, that there is a segment $\sigma \subset l_a$ satisfying

$$\text{length}(\sigma) \geq \frac{\text{length}(\alpha) - (4m-2)(C'_3 + C'_4)}{12m-6} - 2C'_4,$$

which cobounds a strip R of $\tilde{\Sigma}_1$ of width $\leq C_5$ with a translate $c(l_a)$. As before, there is $e = ca^{-1}c^{-1}$ which stabilizes $c(l_a)$ and moves points in almost the same direction as a .

If the translation length of a is long enough, then under the assumption $\text{scl}(a) < 1/24$, the estimate above gives us that $\text{length}(\sigma)$ is more than twice as big as a fundamental domain for a . For, if $\text{scl}(a) < 1/24$ then we can choose n, m as above such that $n/m > 24$. Now, $\text{length}(\alpha) = n\tau(a)$, so

$$\text{length}(\sigma) \geq \frac{n}{12m-6}\tau(a) - O(1) \geq 2\tau(a),$$

providing that $\tau(a)$ is sufficiently big.

In fact, we can assume that

$$\frac{\text{length}(\sigma)}{2\tau(a)} \geq 1 + \epsilon,$$

where ϵ is any number smaller than $1/24 - \text{scl}(a)$. In particular, we can assume that

$$\text{length}(\sigma) - 2\tau(a) = 2C_6,$$

where C_6 is as big as we like. We parameterize σ as $\sigma(i)$ where

$$i \in \left[\frac{-|\text{length}(\sigma)|}{2}, \frac{|\text{length}(\sigma)|}{2} \right].$$

Observe that for all $i \in [-C_6, C_6]$ we have estimates

$$d(e^{-1}a(\sigma(i)), \sigma(i)) \leq 4C_5, \quad d(ae^{-1}(\sigma(i)), \sigma(i)) \leq 4C_5.$$

For all $-C_6 \leq i \leq C_6$ we let $\mu_i \in G$ be such that

$$\mu_i(\sigma(0)) = \sigma(i).$$

It follows that for any $|i| \leq C_6$ we have

$$d(\mu_i^{-1}e^{-1}a\mu_i(\sigma(0)), \sigma(0)) \leq 4C_5, \quad d(\mu_i^{-1}ae^{-1}\mu_i(\sigma(0)), \sigma(0)) \leq 4C_5.$$

If C_6 is very big compared to $(|S|^{4C_5})^2$ then by the pigeonhole principle there are i_1, \dots, i_n where $n \geq C_6/(|S|^{4C_5})^2$ for which $\mu_{i_j}\mu_{i_k}^{-1}$ commutes with both $e^{-1}a$ and ae^{-1} whenever $1 \leq j, k \leq n$. Up to this point, our argument makes no use of the fact that G is torsion free.

In a torsion free word-hyperbolic group, two nontrivial elements which commute are proportional. Since $\mu_{i_j}\mu_{i_k}^{-1}$ commutes with both $e^{-1}a$ and ae^{-1} , it follows that $e^{-1}a$ and ae^{-1} themselves are proportional. Since they are conjugate, they have the same translation length, and are either equal or inverse, since G is torsion free. In the first case, a and e^{-1} commute; since they have the same (positive) translation length, they are either equal or inverse. Since their axes are almost oppositely aligned along σ , they must be inverse, so $a = e$. But $e = ca^{-1}c^{-1}$ which is absurd in a torsion free group. In the second case, $e^{-1}a = ea^{-1}$, so $e^2 = a^2$, and therefore $e = a$ since G is torsion free, and we get a contradiction in any case. This proves the estimate $\delta_\infty \geq 1/24$.

To get the estimate $\delta_\infty \geq 1/12$ we use Theorem 2.7. We argue as above that if $\text{scl}(a) < 1/12$ and a has sufficiently long translation length, then

$$|\text{length}(\sigma) - \tau(a)| = 2C_6$$

is as big as we like. Let p be a point at a distance C_6 from one of the endpoints of σ .

Let p, q be the ideal points stabilized by a , and r, s the ideal points stabilized by b . Then for suitable parameterizations of the flowlines $(p, q, \cdot), (r, s, \cdot)$ the points (p, q, t) and (r, s, t) are within distance $Me^{\lambda C_6}$ for $t \in [C_6, \text{length}(\sigma) - C_6]$, where M and $\lambda < 1$ are universal constants (depending only on δ and $|S|$), but C_6 is as big as we like.

This requires some explanation: by Theorem 2.7 (5), for any three ideal points x, y, z , there are parameterizations of (x, z, \cdot) and (y, z, \cdot) for which

$$d^\times((x, z, t), (y, z, t)) \leq M\lambda^t$$

for suitable M and $0 \leq \lambda < 1$. In fact, by Remark 2.9, these parameterizations are exactly those for which $(x, z, 0)$ is the closest point on (x, z, \cdot) to y , and $(y, z, 0)$ is the closest point on (y, z, \cdot) to x . Now, if (p, q, \cdot) and (r, s, \cdot) are flowlines

which have long sub-segments which are distance $\leq C_5$ apart, then if m is the point on (p, q, \cdot) which is closest to r , and n is the point on (p, q, \cdot) which is closest to s , then m, n are uniformly close to the endpoints of σ . Similarly, if m', n' are the points on (r, s, \cdot) closest to p, q respectively, then m', m are close, and so are n, n' . Now consider the flowline (p, s, \cdot) . This flowline converges exponentially fast to (p, q, \cdot) along the ray from n to p . Similarly, it converges exponentially fast to (r, s, \cdot) along the ray from m' to s . Hence away from a bounded neighborhood of the endpoints of σ , the flowlines (p, q, \cdot) and (r, s, \cdot) are themselves exponentially close. This is the kind of convexity argument which is very standard in strictly negatively curved spaces; Mineyev's technology allows us to transplant it to \mathcal{F} .

It follows that

$$d^\times(e^{-1}a \cdot (p, q, C_6), (p, q, C_6)) \leq 2Me^{\lambda C_6}.$$

By the triangle inequality, for any n we estimate

$$d^\times((e^{-1}a)^n \cdot (p, q, C_6), (p, q, C_6)) \leq 2nMe^{\lambda C_6}$$

and therefore

$$\tau(e^{-1}a) \leq \frac{2nKMe^{\lambda C_6} + \epsilon}{n},$$

where K, ϵ are as in the last bullet of Theorem 2.7. The constants K, M, λ, ϵ all depend only on δ and $|S|$, whereas we may choose C_6 as big as we like, and n as big as we like. In particular, $\tau(e^{-1}a)$ may be made arbitrarily small by choosing C_6 very big.

On the other hand, if a is not equal to e , then $\tau(e^{-1}a) \geq C_1 > 0$ for C_1 depending only on δ and $|S|$. This gives a contradiction for sufficiently big C_6 (chosen depending on C_1).

This contradiction proves the theorem. \square

Example 6.1. The upper bound $1/2$ is *sharp*, and is realized in a nonabelian free group, or closed hyperbolic surface group, by [7].

Example 6.2. Suppose that $H = \langle h_1, h_2 \rangle$ is a non-free 2-generator subgroup of (any group) G . Then one can show $\text{scl}(\langle h_1, h_2 \rangle) < 1/2$. However, Delzant showed ([9]) that in any word-hyperbolic group there are only finitely many conjugacy classes of non-free 2-generator subgroups.

If G is allowed to have torsion, things become slightly more complicated.

Example 6.3. Let S be a (surface) orbifold containing two orbifold points p_2, p_3 of order 2 and 3 respectively. If α is any embedded arc in S from p_2 to p_3 , then the boundary of a regular neighborhood of α has scl at most $1/12$. For typical S there are infinitely many distinct isotopy classes of such arcs α .

It is straightforward to see that in any word-hyperbolic group G , there is a positive first accumulation point for scl on conjugacy classes satisfying the no mirror condition (in fact this follows directly from Theorem A and Theorem A'), but we have not been able to show that this lower bound is independent of G , and therefore we pose the following

Question 6.4. Is there a universal positive constant C such that the first accumulation point for scl on conjugacy classes satisfying the no mirror condition in a hyperbolic group is at least C ?

Finally, the explicitness and universality of the constants in Theorem B, together with Example 6.1 motivates the following

Question 6.5. Can the lower bound in Theorem B be improved to $1/4$?

7. Mapping class groups

Our theorems may be generalized to groups which are not themselves hyperbolic but which act suitably on δ -hyperbolic spaces. In this section and the next, we concentrate on two important examples.

We show how to adapt our Gap Theorem to the action of the mapping class group on the complex of curves. For an introduction to this complex and its properties, see [28] or [3]. The proof follows much the same outline as the proof of Theorem A'. A significant difference is that the action of $\text{MCG}(S)$ on $\mathcal{C}(S)$ is not *proper*; nevertheless it is *weakly proper* in a suitable sense [2], and this weak properness is enough. The technical tool we use is the *acylindricity* of the action of $\text{MCG}(S)$ on $\mathcal{C}(S)$ as observed by Bowditch [3].

We denote the translation length of an element $a \in \text{MCG}(S)$ on $\mathcal{C}(S)$ by $\tau(a)$.

Theorem C (Mapping Class Theorem). *Let S be a compact orientable surface of hyperbolic type and $\text{MCG}(S)$ its mapping class group. Then there is a positive integer P depending on S such that for any pseudo-Anosov element a , either there is an $0 < n \leq P$ and an element $b \in \text{MCG}(S)$ with $ba^{-n}b^{-1} = a^n$, or else there exists a homogeneous quasimorphism h on $\text{MCG}(S)$ such that $h(a) = 1$ and $D(h) \leq C(S)$, where $C(S)$ depends only on S .*

Moreover, let $a_i \in \text{MCG}(S)$ be a collection of elements for which $T = \sup_i \tau(a_i)$ is finite. Suppose that for all integers $n, m \neq 0$ and all elements $b \in \text{MCG}(S)$ and indices i there is an inequality

$$ba^n b^{-1} \neq a_i^m.$$

Then there is a homogeneous quasimorphism h on $\text{MCG}(S)$ such that

- (1) $h(a) = 1$, and $h(a_i) = 0$ for all i ,
- (2) the defect of h is $\leq C'(S, T)$.

Remark 7.1. By Thurston's classification of surface automorphisms (see e.g. [36]) every element of infinite order in $\text{MCG}(S)$ is either reducible or pseudo-Anosov. An element $a \in \text{MCG}(S)$ has $\tau(a) = 0$ on $\mathcal{C}(S)$ if and only if it has finite order or it is reducible.

Remark 7.2. Note the reappearance of the no mirror condition. Also note the separation theorem; compare with the statement of Theorem A'.

Remark 7.3. The dependence of $C(S)$ on S is somewhat subtle and indirect, and does not seem to be easy to estimate.

Proof. The basic structure of the proof should be reasonably clear at this point.

Let $\mathcal{C}(S)$ be the curve graph of S . $\mathcal{C}(S)$ is δ -hyperbolic [28]. Any pseudo-Anosov element acts as an axial isometry. Moreover, by Bowditch [3], there exists $B(S)$ such that for any pseudo-Anosov element a , a^B has a geodesic axis in $\mathcal{C}(S)$. So, in particular, $\tau(a) \geq 1/B$.

If there exists $n > 0$ and $b \in \text{MCG}(S)$ with $ba^n b^{-1} = a^{-n}$, then one may assume that $n \leq P(S)$, where $P(S)$ depends only on S . This follows because the action of $\text{MCG}(S)$ on $\mathcal{C}(S)$ is *acylindrical* in the sense of Bowditch [3]. Here is the precise statement of acylindricity: for any $A > 0$, there exists E, M such that for any two points $x, y \in \mathcal{C}(S)$ with $d(x, y) \geq E$ then there are at most M elements $b \in \text{MCG}(S)$ such that $d(x, bx) \leq A, d(y, by) \leq A$. We consider the case $A = 10\delta$. So, in the following we assume that for all $0 < n$ and $b \in \text{MCG}(S)$, we have $ba^n b^{-1} \neq a^{-n}$.

From above, one finds that there is $P' > 0$ such that if $n \geq P'$, then (a copy of the geodesic segment) a^{-n} does not appear in the L -neighborhood of a geodesic axis of a^B , where L is the constant from Section 4, which depends only on δ in this setting. This is because, otherwise, one finds that a^m is conjugate to a^{-m} for some $0 < m$, which is a contradiction.

So there exists $R(S)$ such that for any a as in the theorem there exists $N \leq R$ such that a^N has a geodesic axis, and no reverse counting happens for a^N .

Let α be a geodesic axis for a^N . Let $x \in \alpha$, and denote the subpath from x to $a^N x$ as a^N . Let $h = c - c_-$ be the quasimorphism constructed using the counting functions for the pairs $(a^N, 1)$ and $(a^{-N}, 1)$. Then $c(a^{Nn}) \geq n$ and $c_-(a^{Nn}) = 0$ for any $n > 0$, since there is no reverse counting. So, $h(a^{Nn}) \geq n$ for all $n > 0$. We know that $D(h) \leq K(\delta)$, where $K(\delta)$ depends only on S by Proposition 4.4. Therefore, given a , we can construct (by averaging h) a homogeneous quasimorphism f such that $f(a) = 1$ and $D(f) \leq 4KR$, where the constant KR depends only on S .

The argument to prove the second part is very similar to the proof of the second part of Theorem A'. Given a collection of elements a_i and a uniform upper bound T on their translation lengths, if there is a translate of the axis of a which stays close to an axis of a_i on a sufficiently long interval, then by acylindricity and the pigeonhole principle, we can conclude that some power of a is conjugate to a power of a_i .

In slightly more detail: let l_{a_i} be an axis for a_i , and let v be an axis for a conjugate $e = cac^{-1}$ of a which stays close to l_{a_i} on a sufficiently long segment β (we say how long in a moment). Let integers n_j, m_j be chosen for which

$$|n_j \tau(a_i) - m_j \tau(a)| < C_1$$

for some fixed constant C_1 . Given an upper bound on T and $\tau(a)$, we can find at least C_2 many such pairs (n_j, m_j) whose absolute values are bounded by C_3 where C_2 is as big as we like, and where C_3 depends on $T, \tau(a), S, C_1$ (and *not* on the specific element a_i). Let β be longer than $C_3 T, C_3 \tau(a)$. Then (after possibly replacing n_j by $-n_j$ for some j) we observe that

$$d(a_i^{n_j} e^{m_j}(p), p) \leq C'_1$$

for some C'_1 which depends only on C_1 and S . By acylindricity and the pigeonhole principle, there are distinct indices (which we denote by n_1, m_1, n_2, m_2 , respectively) for which

$$a_i^{n_1} e^{m_1} = a_i^{n_2} e^{m_2}$$

and therefore some nonzero power of a_i is equal to some nonzero power of e , which is itself conjugate to a , contrary to hypothesis.

This shows that the length β can be bounded above in terms of $S, T, \tau(a)$.

Note that one constructs by this argument a *single* quasimorphism h whose value grows linearly on powers of a , and which vanishes on all powers of a_i simultaneously. The fact that there are (typically) infinitely many a_i on which h vanishes is immaterial.

We give another proof of the second claim which makes more explicit use of acylindricity and gives the slightly better constants claimed in the statement of the theorem.

The argument is similar to the one to show for sufficiently large N , there is no reverse counting for a^N , since otherwise, for some $0 < n \leq N$, a^n is conjugate to a^{-n} by the acylindricity of the action, which is impossible. Also, see the proof of Claim 2' in the proof of Theorem A'.

We want to show that $c_{a^{\pm 2N}}(a_i) = 0$ for all i if N is bigger than a constant depending only on S and T . Suppose that $c_{a^{2N}}(a_i) > 0$ for some (fixed) i (the argument is precisely analogous for $c_{a^{-2N}}$). Let l_{a_i} be an axis for a_i , and let l_e be an axis for a conjugate $e = cac^{-1}$ of a which stays close to, namely in the $L(\delta)$ -neighborhood of, l_{a_i} on a segment v such that one endpoint is v_0 and the other endpoint is $e^{2N}(v_0)$.

For simplicity, we assume that both l_{a_i}, l_e are geodesics (replace a, a_i by a^B, a_i^B if necessary). Then v is 2δ -close to l_{a_i} , where δ is the hyperbolicity constant for $\mathcal{C}(S)$ (here we assume that v is much longer than δ , which follows if we take N bigger than a constant depending only on S).

We observe that for all $0 \leq j \leq N$, there exists n_j such that

$$d(a_i^{n_j} e^j(v_0), v_0), d(a_i^{n_j} e^{j+N}(v_0), e^N(v_0)) \leq \tau(a_i) + 4\delta \leq T + 4\delta.$$

We set $A = T + 4\delta$ and obtain corresponding constants $D(A)$, $M(A)$ for the acylindricity of the action. Now assume that N is such that $d(e^N(v_0), v_0) \geq D(A)$. In other words, $N\tau(a) \geq D(A)$ (we know $\tau(a) \geq 1/B$, so take $N \geq BD$). Then, by the acylindricity, there are at most M distinct elements in $a_i^{n_j} e^j$ ($0 \leq j \leq N$). It follows that if $N > M$, then some non-trivial power of a is conjugated by c to a (possibly trivial) power of a_i , which is impossible. We thereby obtain an upper bound for N by A , $D(A)$, $M(A)$, B , which depend only on S , T , to have $c_{a^{2N}}(a_i) > 0$. We obtain the same upper bound from $c_{a^{-2N}}(a_i) > 0$ as well. \square

8. Amalgamations

In this section we adapt our theorem to the special case of an amalgamated free product acting on its associated tree, and construct many quasimorphisms with uniform lower bounds. One must be slightly careful: the group $\mathrm{SL}(2, \mathbb{Q}_p)$ is an amalgam of two copies of $\mathrm{SL}(2, \mathbb{Z}_p)$. Nevertheless, $\mathrm{SL}(2, \mathbb{Q}_p)$ is uniformly perfect, and therefore admits no nonzero homogeneous quasimorphisms at all. As in the case of the mapping class group acting on $\mathcal{C}(S)$, one must ensure (by fiat) that the action of the amalgam on its associated tree is *weakly proper*; this is guaranteed by a suitable malnormality condition.

Theorem D (Amalgamation Theorem). *Let $G = A *_C B$. Let w be a word on A, B which is reduced and cyclically reduced such that $|w| > 1$, and let \bar{w} denote the corresponding element in G . Suppose that the double coset $C\bar{w}C$ does not contain the element corresponding to any cyclic conjugate of w^{-1} . Then there exists a homogeneous quasimorphism h on G such that $h(\bar{w}) = 1$, and the defect of h is ≤ 312 .*

Proof. We use [17]. Here is a review. Let Γ be the Cayley graph of G for the generating set $A \cup B$.

Then a geodesic between 1 and a is exactly a word for a reduced as a word in $A *_C B$ (Lemma 3.1 [17]). Let w be a reduced word for a . Assume that w is cyclically reduced. Then $w^* = \dots w w w \dots$ is an infinite reduced word, which is a geodesic in Γ since w is cyclically reduced. For such w , let c, c_{-1} be the counting function for $(w, 1)$, $(w^{-1}, 1)$. Then $c(a^n) \geq n$ for all $n > 0$.

By the double coset condition, $c_{-1}(a^n) = 0$ for all $n > 0$, namely, no reverse counting. The argument is essentially in [17]. Suppose that $c_{-1}(a^n) > 0$. Then there is a realizing path α from 1 to a^n which contains w^{-1} as a subword. It is shown in Lemma 3.2 [17] that one can always take a realizing path to be a geodesic. Let $\beta = w^n$, which is a geodesic from 1 to a^n . Since both reduced words α and β represent the same element a^n , by Britton's lemma, there must be $c, c' \in C$ such that $c\bar{w}c' = \bar{v}$, where v is some cyclic conjugate of w^{-1} (see Section 4 of [17] for details). But this is prohibited.

It follows that $h(a^n) \geq n$, where $h = c - c_{-1}$. It is shown in Proposition 3.1 [17] that $D(h) \leq 78$. When we make a quasimorphism homogeneous, an upper bound for the defect becomes $312 = 78 \times 4$. \square

Remark 8.1. There is no implied suggestion that the constant 312 is optimal! The main point is that it is a *universal* constant, which does not depend in any way on the group G .

Remark 8.2. If C is trivial, so that G is a free product, Bavard ([1], Proposition 3.7.2) obtains the (optimal) lower bound $\text{scl} \geq 1/2$.

Remark 8.3. Note that the double coset condition is the precise analogue of the no mirror condition in the context of amalgamated free products.

Question 8.4. What is the optimal constant in Theorem D?

The following example clarifies the geometry of the double coset condition in the context of 3-manifold topology.

Example 8.5. Let M be a knot complement in a rational homology sphere. After choosing an orientation for the knot and for M , there is a natural choice of meridian and longitude m, l on ∂M . The longitude is defined by the property that it is virtually trivial in homology. So there is some surface S of least Euler characteristic in M whose boundary is a multiple of the longitude.

Suppose that $[\partial S] = n \cdot [l]$ in homology. Define $\chi_{\mathbb{Q}} = \frac{\chi(S)}{n}$. Let $M_{p/q}$ be the result of Dehn filling M along the slope p/q . Let $\gamma \subset M_{p/q}$ denote the core of the added solid torus. As in Example 2.3, we have an estimate for stable commutator length

$$\text{scl}(\gamma) \leq \frac{-\chi_{\mathbb{Q}}}{2p}.$$

As before, for K a knot in S^3 , this estimate becomes

$$\text{scl}(\gamma) \leq \frac{g - 1/2}{p},$$

where g is the genus of a Seifert surface for K .

A slope p/q on ∂M is said to be a *boundary slope* if there is an essential, oriented, proper surface $\Sigma \subset M$ such that $\partial \Sigma$ is a nonempty union of curves isotopic to the p/q curve, with either orientation.

After filling M to $M_{p/q}$, the manifold $M_{p/q}$ splits along a surface Σ' obtained by filling in the boundary components of Σ , into two submanifolds which by abuse of notation we denote M_A, M_B with fundamental groups A, B . For brevity, we denote $\pi_1(\Sigma) = C$. The core geodesic γ intersects the two submanifolds M_A, M_B efficiently, in a collection of proper arcs which represent elements of the double coset

spaces CAC and CBC . If the volume of M (and therefore that of $M_{p/q}$) is small, and the area of Σ' is large, most of M_A , M_B must be very thin, and have the structure of an I -bundle over some subsurfaces Σ'_A , Σ'_B of $\Sigma' = \partial M_A = \partial M_B$. These I -bundles are known as the *characteristic* submanifolds of M_A , M_B and we denote them by I_A , I_B , respectively.

If Σ' is connected, the boundary components of I_A and I_B are contained in a single surface. Suppose that ∂I_A and ∂I_B are connected. The geodesic γ is decomposed into a sequence of geodesic segments γ_j with $0 \leq j \leq i$ which are the connected components of $\gamma \cap (M_{p/q} - \Sigma')$. Each γ_j is properly embedded in M_A or M_B , and is contained in the corresponding characteristic submanifold I_A or I_B . Since we are assuming ∂I_A , ∂I_B are connected, each oriented geodesic segment γ_j can be dragged around I_A (say) into itself in such a way that the orientation is reversed at the end. Composing these proper isotopies, we can drag all the γ_j simultaneously in such a way that the common endpoints of γ_j and γ_{j-1} agree during the dragging, for each $j \leq i$. The initial point of γ_0 and the final point of γ_i will not necessarily follow homotopic paths under this dragging, and their difference is an element of $\pi_1(\Sigma')$. Translating this into algebraic information, we have exhibited a conjugate of γ as an element of the double coset space $\pi_1(\Sigma')\gamma^{-1}\pi_1(\Sigma')$.

This example actually occurs: Nathan Dunfield [11] has produced examples of Montesinos knots with genus 1 for which $p/1$ is an essential slope where $p \sim 20,000$. The corresponding core geodesics γ satisfy $\text{scl}(\gamma) \leq \frac{1}{40,000}$ and therefore γ is conjugate into $\pi_1(\Sigma')\gamma^{-1}\pi_1(\Sigma')$ by Theorem D.

The double coset condition shows us how to think about the geometry of the resulting manifolds, and the way in which γ sits inside them.

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