

Local similarities and the Haagerup property

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With an appendix by Daniel S. Farley

Abstract. A new class of groups, the locally finitely determined groups of local similarities on compact ultrametric spaces, is introduced and it is proved that these groups have the Haagerup property (that is, they are a - T -menable in the sense of Gromov). The class includes Thompson's groups, which have already been shown to have the Haagerup property by D. S. Farley, as well as many other groups acting on boundaries of trees. A sufficient condition, used in this article, for the Haagerup property is shown in the appendix by D. S. Farley to be equivalent to the well-known property of having a proper action on a space with walls.

Mathematics Subject Classification (2000). 20F65, 22D10, 54E45.

Keywords. Local similarity, ultrametric space, Haagerup property.

1. Introduction

This article is motivated by D. Farley's theorem [9] that R. Thompson's famous infinite, finitely presented, simple group V has the Haagerup property. Farley's result and method are extended here to a new class of countable, discrete groups, which includes many Thompson-like groups and groups of local similarities on locally rigid, compact ultrametric spaces.

A countable discrete group Γ has the *Haagerup property* if there exists an isometric action $\Gamma \curvearrowright \mathcal{H}$ on some affine Hilbert space \mathcal{H} such that the action is metrically proper, which means for every bounded subset B of \mathcal{H} , the set $\{g \in \Gamma \mid gB \cap B \neq \emptyset\}$ is finite. The Haagerup property is also called *Gromov's a - T -menability property*. We refer to Cherix, Cowling, Jolissaint, Julg, and Valette [6] for a detailed discussion of the Haagerup property.

One reason for interest in the Haagerup property is that Higson and Kasparov [12] proved that the Baum–Connes conjecture with coefficients is true for groups with the Haagerup property.

*Supported in part by NSF Grant DMS-0504176.

The groups for which we verify the Haagerup property come with actions on compact ultrametric spaces. Examples of such spaces are the end spaces, or boundaries, of rooted, locally finite simplicial trees. See Section 2, especially Remark 2.3, for more details.

The actions of the groups on compact ultrametric spaces are by local similarities. There is a finiteness condition on the local restrictions of these local similarities. See Section 3 for the precise definitions.

The following is the main result of this article.

Theorem 1.1. *If Γ is a locally finitely determined group of local similarities on a compact ultrametric space X , then Γ has the Haagerup property.*

Examples of groups satisfying the hypothesis of Theorem 1.1 are given in Section 4. These include Thompson's groups (F , T , and V) as well as other Thompson-like groups. Moreover, if X is a locally rigid, compact ultrametric space, then the full group $\text{LS}(X)$ of all local similarities on X is shown to satisfy the hypothesis. Such spaces include the end spaces of rigid trees in the sense of Bass and Lubotzky [2] with many interesting examples constructed by Bass and Kulkarni [1] and Bass and Tits [3]. See Hughes [14] for more on locally rigid ultrametric spaces.

Theorem 1.1 is proved in Section 6 by showing that the given action of Γ on X induces a zipper action of Γ on some set. Zipper actions are defined in Section 5. This concept is implicit in Farley [9] and is a special case of Valette's characterization of the Haagerup property for countable, discrete groups [6], Proposition 7.5.1.

In the appendix, Farley provides a proof that zipper actions are equivalent to proper actions on spaces with walls, a well-known sufficient condition for the Haagerup property (see Cherix et al. [6], Section 1.2.7). In addition to [9], Farley has a separate proof [8], [10], using this condition, that Thompson's groups have the Haagerup property. See Cherix, Martin, and Valette [7] for a characterization of the Haagerup property for countable, discrete groups in terms of spaces of measured walls. One should also note the similarity of zipper actions with the criterion of Sageev [20] for a group pair to be multi-ended. Example 6.6 shows that zipper actions do not naively lead to spaces with walls.

Acknowledgments. I have benefited from conversations with Dan Farley, Slava Grigorchuk, Mark Sapir, Shmuel Weinberger, and Guoliang Yu.

2. Ultrametric spaces and local similarities

This section contains some background on ultrametric spaces and local similarities.

Definition 2.1. An *ultrametric space* is a metric space (X, d) such that $d(x, y) \leq \max\{d(x, z), d(z, y)\}$ for all $x, y, z \in X$.

Classical examples of ultrametrics arise from p -adic norms, where p is a prime. For example, by $|x|_p := p^{-\max\{n \in \mathbb{N} \cup \{0\} \mid p^n \text{ divides } x\}}$ the p -adic norm $|\cdot|_p$ on the integers \mathbb{Z} is defined. The corresponding metric on \mathbb{Z} is an ultrametric. For more on the relationship between ultrametrics and p -adics, see Robert [18].

For the purposes of this article, the most important examples of ultrametrics arise as end spaces of trees, which are recalled in the following example.

Example 2.2. Let T be a *locally finite simplicial tree*; that is, T is the geometric realization of a locally finite, one-dimensional, simply connected, simplicial complex. There is a natural unique metric d on T such that (T, d) is an \mathbb{R} -tree,¹ every edge is isometric to the closed unit interval $[0, 1]$, and the distance between distinct vertices v_1, v_2 is the minimum number of edges in a sequence of edges e_0, e_1, \dots, e_n with $v_1 \in e_0, v_2 \in e_n$ and $e_i \cap e_{i+1} \neq \emptyset$ for $0 \leq i \leq n-1$. Whenever we refer to a locally finite simplicial tree T , the metric d on T will be understood to be the natural one just described. Choose a *root* (i.e., a base vertex) $v \in T$ and define the *end space* of (T, v) by

$$\text{end}(T, v) = \{x: [0, \infty) \rightarrow T \mid x(0) = v \text{ and } x \text{ is an isometric embedding}\}.$$

For $x, y \in \text{end}(T, v)$, define

$$d_e(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1/e^{t_0} & \text{if } x \neq y \text{ and } t_0 = \sup\{t \geq 0 \mid x(t) = y(t)\}. \end{cases}$$

It follows that $(\text{end}(T, v), d_e)$ is a compact ultrametric space of diameter ≤ 1 .

Remark 2.3. There is a well-known relationship between trees and ultrametrics. For example, if (X, d) is a compact ultrametric space, then there exists a rooted, locally finite simplicial tree (T, v) and a homeomorphism $h: [0, \infty) \rightarrow [0, \infty)$ such that (X, hd) is isometric to $\text{end}(T, v)$. Moreover, every compact ultrametric space (X, d) of diameter ≤ 1 is isometric to the endspace of a rooted, proper \mathbb{R} -tree (T, v) , but not necessarily one whose edges have length less than 1. See Hughes [13] and [14] for more details and further references.

Let (X, d) be a metric space, $x \in X$, and $\varepsilon > 0$. Then we denote by $B(x, \varepsilon) = \{y \in X \mid d(x, y) < \varepsilon\}$ the *open ball about x of radius ε* , and by $\bar{B}(x, \varepsilon) = \{y \in X \mid d(x, y) \leq \varepsilon\}$ the *closed ball about x of radius ε* .

In an ultrametric space, if two balls intersect, then one must contain the other. Moreover, closed balls are open sets and open balls are closed sets. In the compact case, there is the following result, the proof of which is elementary and is left to the reader.

¹An \mathbb{R} -tree is a metric space (T, d) that is uniquely arcwise connected, and for any two points $x, y \in T$ the unique arc from x to y is isometric to the subinterval $[0, d(x, y)]$ of \mathbb{R} .

Lemma 2.4. *If X is a compact ultrametric space and $Y \subseteq X$, then the following are equivalent:*

- (1) Y is open and closed.
- (2) Y is a finite union of open balls in X .
- (3) Y is a finite union of closed balls in X .

We conclude this section with the basic definitions concerning local similarities.

Definition 2.5. If $\lambda > 0$, then a map $g: X \rightarrow Y$ between metric spaces (X, d_X) and (Y, d_Y) is a λ -similarity provided that $d_Y(gx, gy) = \lambda d_X(x, y)$ for all $x, y \in X$.

Definition 2.6. A homeomorphism $g: X \rightarrow X$ between metric spaces is a *local similarity* if for every $x \in X$ there exist $r, \lambda > 0$ such that g restricts to a surjective λ -similarity $g|: B(x, r) \rightarrow B(gx, \lambda r)$.

Definition 2.7. For a metric space X , we denote by $\text{LS}(X)$ the *group of all local similarities from X onto X* .

We will be concerned with the group $\text{LS}(X)$ only when X is a compact ultrametric space. It has a natural topology (the compact-open topology), but in this article we always endow subgroups of $\text{LS}(X)$ with the discrete topology.

3. Locally finitely determined groups of local similarities

In this section we introduce the groups that are the object of study in this article and establish some of their elementary properties. Throughout this section, let X be a compact ultrametric space with ultrametric d . The groups are defined in terms of an extra structure on X , which we now define.

Definition 3.1. A *finite similarity structure for X* is a function, denoted by Sim , that assigns to each ordered pair B_1, B_2 of closed balls in X a (possibly empty) finite set $\text{Sim}(B_1, B_2)$ of surjective similarities $B_1 \rightarrow B_2$ such that whenever B_1, B_2, B_3 are closed balls in X , the following properties hold:

- (1) (Identities) $\text{id}_{B_1} \in \text{Sim}(B_1, B_1)$.
- (2) (Inverses) If $h \in \text{Sim}(B_1, B_2)$, then $h^{-1} \in \text{Sim}(B_2, B_1)$.
- (3) (Compositions) If $h_1 \in \text{Sim}(B_1, B_2)$ and $h_2 \in \text{Sim}(B_2, B_3)$, then $h_2 h_1 \in \text{Sim}(B_1, B_3)$.
- (4) (Restrictions) If $h \in \text{Sim}(B_1, B_2)$ and $B_3 \subseteq B_1$, then $h|_{B_3} \in \text{Sim}(B_3, h(B_3))$.

When it is necessary to indicate the dependence of Sim on X , the notation Sim_X is used.

The word *finite* is used here to describe the similarity structure, not to imply that there only finitely many similarities involved (in general, there are infinitely many), rather to emphasize that given any two closed balls only finitely many similarities between them are chosen by Sim .

Example 3.2. The *trivial* finite similarity structure on X is given by

$$\text{Sim}(B_1, B_2) = \begin{cases} \{\text{id}_{B_1}\} & \text{if } B_1 = B_2, \\ \emptyset & \text{otherwise,} \end{cases}$$

for each pair of closed balls B_1, B_2 in X .

More examples are given in the next section.

Definition 3.3. Let B be a closed ball in X . A function $g: B \rightarrow X$ is a *local similarity embedding* if for each $x \in B$ there exist $r, \lambda > 0$ such that $\overline{B}(x, r) \subseteq B$ and $g|: \overline{B}(x, r) \rightarrow \overline{B}(gx, \lambda r)$ is a surjective λ -similarity. If the choices can be made so that $g| \in \text{Sim}(\overline{B}(x, r), \overline{B}(gx, \lambda r))$, then g is *locally determined* by Sim .

Definition 3.4. A subgroup Γ of $\text{LS}(X)$ is *locally determined* by the finite similarity structure Sim for X if every $g \in \Gamma$ is locally determined by Sim . In this case, the group Γ is said to be a *locally finitely determined group of local similarities on X* .

Proposition 3.5. *If Sim_X is a finite similarity structure for X , then*

$$\Gamma := \{g \in \text{LS}(X) \mid g \text{ is locally determined by } \text{Sim}_X\}$$

is the unique largest subgroup of $\text{LS}(X)$ that is locally determined by Sim_X .

Proof. The Compositions and Restrictions Properties of Sim_X show that Γ is closed under multiplication. Likewise, the Inverses Property shows that Γ is closed under inverses. Hence, Γ is a subgroup of $\text{LS}(X)$ locally determined by Sim_X . Clearly, any other locally determined group of local similarities on X is a subgroup of Γ . \square

Throughout the remainder of this section, let $\Gamma \leq \text{LS}(X)$ be a group locally determined by the finite similarity structure Sim . Recall that Γ is given the discrete topology.

Definition 3.6. A *region* for $g \in \Gamma$ is a closed ball B in X such that $g(B)$ is a ball and $g|B \in \text{Sim}(B, g(B))$. A region B for $g \in \Gamma$ is a *maximum region* for g if it is not properly contained in any region for g .

Lemma 3.7. *For each $g \in \Gamma$ and for each $x \in X$ there exists a unique maximum region B for g such that $x \in B$.*

Proof. By definition, x is contained in some region R for g . Compactness of X implies R is contained in only finitely many closed balls of X . Thus, there is a largest (with respect to set inclusion) such ball B that is a region for g , and it must be a maximum region. It is the unique maximum region for g containing x because any two intersecting balls of X have the property that one contains the other. \square

It follows that for each $g \in \Gamma$, the maximum regions of g form a partition of X (that is, the maximum regions of g cover X and are mutually disjoint), and any closed ball in X contains, or is contained in, a maximum region of g .

Definition 3.8. If $g \in \Gamma$, then the *maximum partition for g* is the partition of X into the maximum regions of g .

Thus, any partition of X into regions for an element $g \in \Gamma$ refines the maximum partition for g .

The following lemma follows immediately from the definitions and the Inverses Property.

Lemma 3.9. *If $g \in \Gamma$ and R is a region for g , then $g(R)$ is a region for g^{-1} . In addition, if R is a maximum region for g , then $g(R)$ is a maximum region for g^{-1} .*

Lemma 3.10. *Let \mathcal{P}_+ and \mathcal{P}_- be two partitions of X into closed balls. The set*

$$\Gamma(\mathcal{P}_{\pm}) = \{g \in \Gamma \mid \mathcal{P}_+ \text{ is the maximum partition for } g \text{ and} \\ \mathcal{P}_- \text{ is the maximum partition for } g^{-1}\}$$

is finite.

Proof. Say $\mathcal{P}_+ = \{B_1, \dots, B_n\}$ where $n = |\mathcal{P}_+|$ is the cardinality of \mathcal{P}_+ . Let $\text{Bi}(\mathcal{P}_+, \mathcal{P}_-)$ denote the finite set of bijections from \mathcal{P}_+ to \mathcal{P}_- . For $h \in \text{Bi}(\mathcal{P}_+, \mathcal{P}_-)$, let $S_h := \prod_{i=1}^n \text{Sim}(B_i, h(B_i))$ and note that S_h is finite. Define the finite set F to be the disjoint union $F := \bigsqcup_{h \in \text{Bi}(\mathcal{P}_+, \mathcal{P}_-)} S_h$, which we prefer to write as $F = \bigcup_{h \in \text{Bi}(\mathcal{P}_+, \mathcal{P}_-)} (h, S_h)$. If $g \in \Gamma(\mathcal{P}_{\pm})$, then $g_* \in \text{Bi}(\mathcal{P}_+, \mathcal{P}_-)$ is defined by $g_*(B) = g(B)$ for all $B \in \mathcal{P}_+$. Clearly, there is an injection $\Gamma(\mathcal{P}_{\pm}) \rightarrow F$ given by $g \mapsto (g_*, (g|_{B_1}, \dots, g|_{B_n}))$. \square

Recall that if \mathcal{P} and \mathcal{Q} are two collections, then \mathcal{P} *refines* \mathcal{Q} means for every $P \in \mathcal{P}$ there exists $Q \in \mathcal{Q}$ such that $P \subseteq Q$.

Lemma 3.11. *Let \mathcal{P}_+ and \mathcal{P}_- be two partitions of X into closed balls. The set*

$$\Gamma_{\text{ref}}(\mathcal{P}_{\pm}) = \{g \in \Gamma \mid \mathcal{P}_+ \text{ refines the maximum partition for } g \text{ and} \\ \mathcal{P}_- \text{ refines the maximum partition for } g^{-1}\}$$

is finite.

Proof. Given any closed ball B in X , there exist only finitely many distinct closed balls of X containing B . Hence, any partition of X into closed balls refines only finitely many other partitions of X into closed balls. Thus, there exist only finitely many pairs, say $(\mathcal{P}_+^i, \mathcal{P}_-^i)$ for $i = 1, \dots, n$, of partitions of X into closed balls such that \mathcal{P}_+ refines \mathcal{P}_+^i and \mathcal{P}_- refines \mathcal{P}_-^i for all $i = 1, \dots, n$. Clearly, $\Gamma_{\text{ref}}(\mathcal{P}_{\pm}) = \bigcup_{i=1}^n \Gamma(\mathcal{P}_{\pm}^i)$ and the result follows from Lemma 3.10. \square

Lemma 3.12. Γ *is countable.*

Proof. The space X has only countably many closed balls; hence, X has only countably many partitions into closed balls and only countably many pairs, say $(\mathcal{P}_+^i, \mathcal{P}_-^i)$ for $i = 1, 2, 3, \dots$, of partitions of X into closed balls. Clearly, $\Gamma = \bigcup_{i=1}^{\infty} \Gamma(\mathcal{P}_{\pm}^i)$ and the result follows from Lemma 3.10. \square

4. Examples

In this section we give examples of locally finitely determined groups of local similarities on compact ultrametric spaces. The examples include Thompson’s groups so that Farley’s result [9] is recovered from Theorem 1.1. The examples also include many other Thompson-like groups, as well as the full local similarity groups of end spaces of certain trees constructed by Bass and Kulkarni [1] and Bass and Tits [3].

We begin by recalling standard alphabet language and notation. An excellent reference is Nekrashevych [16]. An *alphabet* is a non-empty finite set A and finite (perhaps empty), ordered subsets of A are *words*. The set of all words is denoted A^* and the set of *infinite words* is denoted A^ω ; that is,

$$A^* = \prod_{n=0}^{\infty} A^n \quad \text{and} \quad A^\omega = \prod_1^{\infty} A.$$

Let T_A be the tree associated to A . The vertices of T_A are words in A ; two words v, w are connected by an edge if and only if there exists $x \in A$ such that $v = wx$ or $vx = w$. The root of T_A is \emptyset . Thus, $A^\omega = \text{end}(T_A, \emptyset)$ and so comes with a natural ultrametric as described in Example 2.2 making A^ω compact. We may assume that A is totally ordered. There is then an induced total order on A^ω , namely the lexicographic order.

Example 4.1 (The Higman–Thompson groups $G_{d,1}$). Let $\Gamma = \text{LS}_{\text{top}}(A^\omega)$ be the subgroup of $\text{LS}(A^\omega)$ consisting of locally order preserving local similarities on A^ω , where a map $h: A^\omega \rightarrow A^\omega$ is *locally order preserving* if for each $x \in A^\omega$ there exists $\varepsilon > 0$ such that $h|: B(x, \varepsilon) \rightarrow A^\omega$ is order preserving. We denote $\text{id}_{A^\omega} = e$; it is the unique order preserving isometry $A^\omega \rightarrow A^\omega$. Any closed ball in A^ω has a unique order preserving similarity onto A^ω ; hence, if B_1 and B_2 are two closed balls in A^ω , then there is a unique order preserving similarity of B_1 onto B_2 . Let $\text{Sim}(B_1, B_2)$ consist solely of that unique order preserving similarity. This can be described using alphabet language quite easily as follows. A closed ball in A^ω is given by vA^ω , where $v \in A^*$. For $v, w \in A^*$, $\text{Sim}(vA^\omega, wA^\omega)$ consists of the unique order preserving similarity $vA^\omega \rightarrow wA^\omega$; $vx \mapsto wx$. Clearly, this defines a finite similarity structure Sim_{A^ω} and Γ is locally determined by Sim_{A^ω} .

When the alphabet is $A = \{0, 1\}$, we get Thompson’s group $V = \text{LS}_{\text{top}}(A^\omega)$. The subgroups $F \leq T \leq V$ are also locally determined by the same finite similarity structure Sim_{A^ω} (elements of T are further required to be cyclicly order preserving; those of F , to be order preserving). In general, $\text{LS}_{\text{top}}(A^\omega)$ is the Higman–Thompson group $G_{d,1}$, where $d = |A|$. For background on these groups, see Cannon, Floyd, and Parry [4] and for other references, see Hughes [14], Section 12.3.

Example 4.2 (Generalized Higman–Thompson groups $\text{LS}_{\text{top}}(X)$). The previous example can easily be extended to end spaces of rooted, ordered, proper \mathbb{R} -trees (T, ν) so that the groups $\text{LS}_{\text{top}}(X)$, where $X = \text{end}(T, \nu)$, defined in Hughes [14], Section 12.3, become locally finitely determined groups of local similarities on X . In particular, it is easy to see that the Higman–Thompson groups $G_{d,n}$, $n \geq 1$, fit into this framework.

Example 4.3 (Subgroups). A subgroup H of a group Γ of local similarities locally determined by the finite similarity structure Sim is also locally determined by Sim . This is clear because Definition 3.4 is a condition on elements of Γ , which therefore holds for each element of H .

Example 4.4 (Nekrashevych–Röver groups $V_d(H)$, H finite). Suppose that H is a finite, self-similar group over the alphabet A , with $d = |A|$ (see Nekrashevych [16]). Nekrashevych [15] defines a group $V_d(H) \leq \text{LS}(A^\omega)$ generalizing a construction of Röver [19]. To describe these groups note that there is a natural similarity from A^ω onto any closed ball of A^ω ; thus, any surjective similarity $h: B_1 \rightarrow B_2$ between closed balls gives rise to an isometry h_* of A^ω :

$$h_*: A^\omega \rightarrow B_1 \xrightarrow{h} B_2 \rightarrow A^\omega.$$

Then an element $g \in \text{LS}(A^\omega)$ is in $V_d(H)$ if and only if for each $x \in A^\omega$ there exists $\varepsilon, \lambda > 0$ such that $g|: B(x, \varepsilon) \rightarrow B(gx, \lambda\varepsilon)$ is a λ -similarity and $(g|)_* \in H$. For his

general construction, Nekrashevych does not require H to be finite, but we require it in order to define the following finite similarity structure on A^ω : if B_1, B_2 are closed balls of A^ω , then $\text{Sim}(B_1, B_2)$ consists of all surjective similarities $h: B_1 \rightarrow B_2$ such that $h_* \in H$. The Restrictions Property follows from the self-similarity property of H . Clearly, $V_d(H)$ is finitely determined by Sim_{A^ω} .

For example, note that in the special case $H = \{1\}$, $V_d(H) = G_{d,1}$.

For a nontrivial example, let Σ_d be the symmetric group on A . The action of Σ_d on A^* given by $\sigma(a_1 \dots a_n) = \sigma(a_1) \dots \sigma(a_n)$ induces an action of Σ_d on the tree T_A and we let $H \cong \Sigma_d$ be the corresponding self-similar subgroup of $\text{Aut}(T_A)$. Note that $G_{d,1} \leq V_d(\Sigma_d)$ and that $\Gamma := V_d(\Sigma_d) \cap \text{Aut}(T_A)$ is a contracting self-similar group with nucleus Σ_d (see Nekrashevych [16], Section 2.11, for the definitions).

Generalizing this last observation, let Γ be any contracting self-similar subgroup of $\text{Aut}(T_A)$ whose nucleus \mathcal{N} is a finite group (in general, contracting self-similar groups have nuclei that are finite sets – the condition that the nucleus be a group is quite restrictive). It follows that \mathcal{N} is a finite self-similar group and we can form the locally finitely determined group $V_d(\mathcal{N})$. For each pair B_1, B_2 of closed balls in A^ω , $\text{Sim}(B_1, B_2)$ is naturally identified with \mathcal{N} . Note that $\Gamma \leq V_d(\mathcal{N}) \cap \text{Aut}(T_A)$.

Example 4.5 (Groups acting on trees with finite vertex stabilizers). Let (T, v) be a geodesically complete, rooted, locally finite simplicial tree, where *geodesically complete* means no vertex, except possibly the root, has valency 1. Let Γ be a subgroup of the isometry group $\text{Isom}(T)$ such that Γ has finite vertex stabilizers (that is, for each vertex $w \in T$, the isotropy group Γ_w is finite). There is a well-known homomorphism $\epsilon: \text{Isom}(T) \rightarrow \text{LS}(X)$, where $X = \text{end}(T, v)$, explicitly described in Hughes [14], Section 12.1. We will show that $\epsilon(\Gamma)$ is locally finitely determined. If B is a closed ball in X , then there exists a vertex $w_B \in T$ such that $B = \{x \in X \mid x(d(v, w_B)) = w_B\}$ and $T_B = \{x(t) \mid x \in B \text{ and } t \geq d(v, w_B)\}$ is a subtree of T with B similar to $\text{end}(T_B, w_B)$. Define a finite similarity structure Sim as follows. If B_1, B_2 are closed balls in X , let

$$\text{Sim}(B_1, B_2) = \{\epsilon(g) \mid B_1 \rightarrow B_2 \mid g \in \Gamma, g(w_{B_1}) = w_{B_2}, \text{ and } g(T_{B_1}) = T_{B_2}\}.$$

The finite vertex stabilizers assumption implies that $\text{Sim}(B_1, B_2)$ is finite. The other properties of a similarity structure are easy to verify. Moreover, $\epsilon(\Gamma)$ is locally determined by Sim . Note that ϵ is an injection except when T is isometric to \mathbb{R} . In particular, finitely generated free groups are locally finitely determined. Of course, it is well known that discrete groups acting on trees with finite vertex stabilizers have the Haagerup property (see Cherix et al. [6], Section 1.2.3).

Example 4.6 (Local similarity groups of locally rigid, compact ultrametric spaces). Let X be a locally rigid, compact ultrametric space as defined in Hughes [14]. It is proved there that a compact ultrametric space X is locally rigid if and only if the

isometry group $\text{Isom}(X)$ is finite. In particular, the isometry group of any closed ball in X is also finite. From this it follows easily that for any two closed balls B_1, B_2 in X , the set of all surjective similarities from B_1 to B_2 is finite. We can therefore define a finite similarity structure Sim by letting $\text{Sim}(B_1, B_2)$ be the set of all similarities from B_1 onto B_2 . Then the group $\Gamma = \text{LS}(X)$ of all local similarities of X onto itself is locally determined by Sim .

Example 4.7 (Local similarity groups of end spaces of rigid trees). Let T be a locally finite simplicial tree that is *rigid*; that is, the group of automorphisms $\text{Aut}(T)$ is discrete; see Bass and Lubotzky [2]. Let $X = \text{end}(T, v)$, where v is a chosen vertex of T . Assuming that (T, v) is geodesically complete, the rigidity of T is equivalent to local rigidity of X ; see Hughes [14], Section 12.2. Hence, $\Gamma := \text{LS}(X)$ is locally finitely determined as described in the preceding example. An interesting source of examples of rigid trees come from π -*rigid* graphs of Bass and Kulkarni [1] and Bass and Tits [3]. These are finite, connected, simplicial graphs G with the property that if \tilde{G} is the universal covering tree of G , then $\text{Aut}(\tilde{G}) = \pi_1(G)$. In particular, \tilde{G} is rigid and $\text{LS}(\text{end}(\tilde{G}, v))$ is finitely determined.

5. Zipper actions

In this section we discuss a sufficient condition, called a zipper action, for a discrete group to have the Haagerup property. This condition is implicit in Farley [9] and is a special case of the necessary and sufficient condition due to Valette [6], Proposition 7.5.1. Moreover, in the appendix to this article, Farley shows that zipper actions are equivalent to proper actions on spaces of walls. Apart from the terminology, there is nothing original in this section.

Definition 5.1. A discrete group Γ has a *zipper action* if there is a left action $\Gamma \curvearrowright \mathcal{E}$ of Γ on a set \mathcal{E} and a subset $Z \subseteq \mathcal{E}$ such that

- (1) for every $g \in \Gamma$, the symmetric difference $gZ \Delta Z$ is finite, and
- (2) for every $r > 0$, $\{g \in \Gamma \mid |gZ \Delta Z| \leq r\}$ is finite.

Note that if Γ is an infinite group then condition (2) implies Z must also be infinite. Also, the action $\Gamma \curvearrowright \mathcal{E}$ is not assumed to be effective; however, condition (2) implies that the kernel of the action is finite.

The terminology arises as follows. We think of Z as being an infinite zipper in \mathcal{E} that is unzipped by the action of Γ . Only a finite portion is unzipped by any finite subset of Γ , but as one takes larger finite subsets of Γ , more of Z is unzipped.

Example 5.2. We show that the group \mathbb{Z} has a zipper action. Let

$$\mathcal{E} = \mathbb{Z} \quad \text{and} \quad Z = \{n \in \mathcal{E} \mid n \leq 0\}.$$

An action $\mathbb{Z} \curvearrowright \mathcal{E}$ is defined by $g \cdot n = g + n$. If $g \in \mathbb{Z}$ and $g \geq 0$, then $Z \subseteq gZ$ and $gZ \Delta Z = \{n \in \mathcal{E} \mid 0 < n \leq g\}$. If $g \in \mathbb{Z}$ and $g \leq 0$, then $gZ \subseteq Z$ and $gZ \Delta Z = \{n \in \mathcal{E} \mid g < n \leq 0\}$. One may say “ Z is taken further off itself as $g \rightarrow +\infty$ in \mathbb{Z} ” and “ Z is taken deeper into itself as $g \rightarrow -\infty$ in \mathbb{Z} ”. Thus, $|gZ \Delta Z| = |g|$ for all $g \in \mathbb{Z}$. If $r \geq 0$, then $\{g \in \mathbb{Z} \mid |gZ \Delta Z| \leq r\} = \{g \in \mathbb{Z} \mid |g| \leq r\}$, which is finite.

The proof of the following theorem, which is a special case of Valette [6], Proposition 7.5.1, is implicit in Farley [9], but is included for completeness.

Theorem 5.3. *If the discrete group Γ has a zipper action, then Γ has the Haagerup property.*

Proof. Define $\pi : \Gamma \rightarrow \ell^\infty(\mathcal{E})$ by $\pi(g) = \chi_{gZ} - \chi_Z$ (where χ_Y denotes the characteristic function of $Y \subseteq \mathcal{E}$). Note the following:

- (1) The support of $\pi(g)$ is $gZ \Delta Z$; hence, $\pi(g)$ is finitely supported and $\pi(g)$ is in the Hilbert space $\ell^2(\mathcal{E})$ for all $g \in \Gamma$.
- (2) The square of the ℓ^2 -norm $\|\pi(g)\|_2^2 = |gZ \Delta Z|$ for all $g \in \Gamma$.
- (3) For every $r > 0$, $\{g \in \Gamma \mid \|\pi(g)\|_2 \leq r\}$ is finite.

The action of Γ on \mathcal{E} induces a unitary left action of Γ on $\ell^2(\mathcal{E})$, $\rho : \Gamma \rightarrow \mathcal{B}(\ell^2(\mathcal{E}))$, where $\mathcal{B}(\ell^2(\mathcal{E}))$ is the space of bounded linear operators on $\ell^2(\mathcal{E})$. Namely, $\rho(g)(f)(e) = f(g^{-1}e)$ for $g \in \Gamma$, $f : \mathcal{E} \rightarrow \mathbb{C}$ in $\ell^2(\mathcal{E})$, and $e \in \mathcal{E}$.

One checks that π is a 1-cocycle for ρ ; that is, $\pi(g_1g_2) = \rho(g_1)\pi(g_2) + \pi(g_1)$ for all $g_1, g_2 \in \Gamma$. For this, it is useful to observe firstly that the action ρ extends to an action $\hat{\rho} : \Gamma \rightarrow \mathcal{B}(\ell^\infty(\mathcal{E}))$ defined by the same formula ($\hat{\rho}(g)(f)(e) = f(g^{-1}e)$). Then observe secondly that $\hat{\rho}(g)\chi_Y = \chi_{gY}$ in $\ell^\infty(\mathcal{E})$ for any $Y \subseteq \mathcal{E}$.

It follows that $A : \Gamma \rightarrow \text{Isom}(\ell^2(\mathcal{E}))$ defined by $A(g)(f) = \rho(g)(f) + \pi(g)$ is an affine isometric action of Γ on $\ell^2(\mathcal{E})$. Moreover, property (3) above guarantees that A is metrically proper. □

Remark 5.4. The existence of a zipper action is preserved by direct sums of groups. For let Γ_i ($i = 1, 2$) be discrete groups having left actions $\Gamma_i \curvearrowright \mathcal{E}_i$ and subsets $Z_i \subset \mathcal{E}_i$ as in Definition 5.1. Let $\Gamma := \Gamma_1 \oplus \Gamma_2$, $\mathcal{E} := \mathcal{E}_1 \amalg \mathcal{E}_2$, $Z := Z_1 \amalg Z_2$, and define a left action $\Gamma \curvearrowright \mathcal{E}$ in the obvious way: $(g_1, g_2) \cdot e_i = g_i e_i$ where $e_i \in \mathcal{E}_i$ and $i \in \{1, 2\}$. The conditions are readily checked.

6. The main construction

In this section we prove the following theorem.

Theorem 6.1. *If Γ is a locally finitely determined group of local similarities on a compact ultrametric space X , then Γ has a zipper action.*

Clearly, Theorem 1.1 follows from Theorems 5.3 and 6.1.

Throughout this section, X will denote a compact ultrametric space and $\Gamma \leq \text{LS}(X)$ will be a group locally determined by a finite similarity structure Sim on X .

Before defining a set \mathcal{E} with a zipper action $\Gamma \curvearrowright \mathcal{E}$, note that it follows from Lemma 2.4 that the image of a local similarity embedding $f: B \rightarrow X$, where B is a closed ball in X , is a finite union of mutually disjoint closed balls in X .

Now let \mathcal{E} be the set of equivalence classes of pairs (f, B) where B is a closed ball in X and $f: B \rightarrow X$ is a local similarity embedding locally determined by Sim . Two such (f_1, B_1) and (f_2, B_2) are *equivalent* provided there exists $h \in \text{Sim}(B_1, B_2)$ such that $f_2 h = f_1$ (in particular, $f_1(B_1) = f_2(B_2)$). The verification that this is an equivalence relation requires the Identities, Compositions, and Inverses Properties of the similarity structure. Equivalence classes are denoted by $[f, B]$.

Let $Z = \{[f, B] \in \mathcal{E} \mid f(B) \text{ is a closed ball in } X \text{ and } f \in \text{Sim}(B, f(B))\}$.

Note that an element $[f, B] \in Z$ is uniquely determined by the closed ball $f(B)$. In fact, $[f, B] = [\text{incl}_{f(B)}, f(B)]$, where $\text{incl}_Y: Y \rightarrow X$ denotes the inclusion map. Thus,

$$Z = \{[\text{incl}_B, B] \in \mathcal{E} \mid B \text{ is a closed ball in } X\}.$$

In particular, Z can be identified with the collection of all closed balls in X .

There is a left action $\Gamma \curvearrowright \mathcal{E}$ defined by $g[f, B] = [gf, B]$. The fact that $[gf, B] \in \mathcal{E}$ follows from the Compositions and Restrictions Properties of the similarity structure.

It follows from the description of Z above that

$$gZ = \{[g|_B, B] \in \mathcal{E} \mid B \text{ is a closed ball of } X\}$$

for all $g \in \Gamma$.

The next part of this section is devoted to establishing, in Corollary 6.4 and Lemma 6.5 below, the two properties required of a zipper action.

Lemma 6.2. *Let B be a closed ball in X and $g \in \Gamma$. Then $[\text{incl}_B, B] \in Z \setminus gZ$ if and only if B properly contains a maximum region of g^{-1} .*

Proof. Suppose first that $[\text{incl}_B, B] \in Z \setminus gZ$ and, by way of contradiction, there exists a maximum region R for g^{-1} containing B . Then $g^{-1}R$ is a ball and $g^{-1}|_R \in \text{Sim}(R, g^{-1}R)$. The Restrictions Property implies $g^{-1}|_B \in \text{Sim}(B, g^{-1}B)$ and $[g^{-1}|_B, B] \in Z$. Clearly, $[\text{incl}_B, B] = g[g^{-1}|_B, B] \in gZ$, which is a contradiction.

Conversely, let R be a maximum region of g^{-1} properly contained in B . If $[\text{incl}_B, B] \in gZ$, then there is $[\text{incl}_{B_1}, B_1] \in Z$ with $[g|_{B_1}, B_1] = g[\text{incl}_{B_1}, B_1] = [\text{incl}_B, B]$, which is to say $g(B_1) = B$. Moreover, $[g|_{B_1}, B_1] = [\text{incl}_B, B]$ implies that $g|: B_1 \rightarrow B$ is in $\text{Sim}(B_1, B)$. The Inverses Property implies $g^{-1}|: B \rightarrow B_1$ is in $\text{Sim}(B, B_1)$. In particular, B is a region for g^{-1} , contradicting the maximality of R . Thus, $[\text{incl}_B, B] \notin gZ$. \square

Lemma 6.3. *For each $g \in \Gamma$, the function $[\text{incl}_B, B] \mapsto B$ is a bijection from $Z \setminus gZ$ to the set of closed balls of X properly containing maximum regions of g^{-1} . Moreover, the function $g[\text{incl}_B, B] \mapsto B$ is a bijection from $gZ \setminus Z$ to the set of closed balls of X properly containing maximum regions of g .*

Proof. The first statement follows immediately from the preceding lemma. The second follows from the first together with the observation that $g[\text{incl}_B, B] \mapsto [\text{incl}_B, B]$ is a bijection from $gZ \setminus Z$ to $Z \setminus g^{-1}Z$. \square

Corollary 6.4. *For each $g \in \Gamma$, the symmetric difference $gZ \Delta Z$ is finite.*

Proof. This follows immediately from the preceding lemma because there are only a finite number of closed balls of X containing a maximum region of g or g^{-1} . \square

Lemma 6.5. *For each $r > 0$, $\{g \in \Gamma \mid |gZ \Delta Z| \leq r\}$ is finite.*

Proof. Let $\Gamma_r = \{g \in \Gamma \mid |gZ \Delta Z| \leq r\}$. Since $|gZ \Delta Z| = |g^{-1}Z \Delta Z|$, it follows that $g \in \Gamma_r$ if and only if $g^{-1} \in \Gamma_r$. For each $x \in X$, let

$$M_{r,x} = \{R \mid R \text{ is a maximum region for some } g \in \Gamma_r \text{ and } x \in R\}.$$

By Lemma 6.3 if $g \in \Gamma_r$, then the number of closed balls of X properly containing a maximum region of g is less than or equal to r . In particular, if $R \in M_{r,x}$, there are at most r closed balls of X properly containing R . Since $M_{r,x}$ is totally ordered by inclusion, it follows that $M_{r,x}$ is finite and there exists $R_{r,x} \in M_{r,x}$ such that $R_{r,x} \subseteq R$ for all $R \in M_{r,x}$. The set $\mathcal{P}_r := \{R_{r,x} \mid x \in X\}$ is a partition of X and each $R_{r,x}$ is a region for g and for g^{-1} , for all $g \in \Gamma_r$. That is to say \mathcal{P}_r refines the maximum partitions of both g and g^{-1} for all $g \in \Gamma_r$. Setting $\mathcal{P}_+ = \mathcal{P}_r = \mathcal{P}_-$, this means that $\Gamma_r \subset \Gamma_{\text{ref}}(\mathcal{P}_\pm)$ and the result follows from Lemma 3.11. \square

The proof of Theorem 6.1 is now complete.

Even though it will be shown in the appendix that a zipper action gives rise to a proper action on a space with walls, the next example is included to show that this does not happen in the most naive way. See the appendix for definitions and references.

Example 6.6. Consider the alphabet $A = \{0, 1\}$ and Thompson's group $V \leq \text{LS}(A^\omega)$ as in Example 4.1. The construction above gives sets $Z \subseteq \mathcal{E}$ and a zipper action $V \curvearrowright \mathcal{E}$. It might be expected that $\mathcal{W} := \{(gZ, \mathcal{E} \setminus gZ) \mid g \in V\}$ is a set of walls for \mathcal{E} . However, this is not the case. Specifically, we show there exists $[f_1, B_1], [f_2, B_2] \in \mathcal{E}$ and an infinite subset $G \subseteq V$ such that $[f_1, B_1] \in gZ$ and $[f_2, B_2] \notin gZ$ for every $g \in G$ (that is, there are two elements of \mathcal{E} separated by infinitely many walls). Let

$B_1 = 0A^\omega$ and $B_2 = 1A^\omega$. Let $f_1: B_1 \rightarrow A^\omega$ be the inclusion and let $f_2: B_2 \rightarrow A^\omega$ be defined by

$$\begin{cases} f_2(10w) = 10w, \\ f_2(11w) = 111w, \end{cases}$$

for all $w \in A^\omega$. Let $G = \{g \in V \mid g \text{ is a local isometry and } g|_{B_1} = f_1\}$. The required conditions are readily checked. (Note also that $[f_2, B_2] \in VZ$ so that a set of walls will not result by reducing the size of \mathcal{E} .)

Appendix by Daniel S. Farley: Zipper actions, spaces with walls, and CAT(0) cubical complexes

The purpose of this appendix is to show that the property of having a zipper action is equivalent to having a proper action on a space with walls. Spaces with walls were introduced by Haglund and Paulin [11], who wanted a common language for describing a range of combinatorial structures, among them CAT(0) cubical complexes. Sageev [20] had in effect shown that CAT(0) cubical complexes are spaces with walls, so a group acting properly on a CAT(0) cubical complex also acts properly on a space with walls (see Cherix et al. [6], Section 1.2.7, for a discussion about the relevance of this to the Haagerup property and for more references). Chatterji and Niblo [5] and Nica [17] proved the converse, namely, that a group that acts properly on a space with walls also acts properly on a CAT(0) cube complex. Thus, having a proper action of a group Γ on a CAT(0) cube complex is equivalent to having a zipper action of Γ . In fact, a zipper is closely related to Sageev's notion of an almost invariant set in [20], and the discussion in this appendix is implicit in [20]. At any rate, the experts will find this result familiar.

Theorem A.1. *A discrete group Γ has a zipper action if and only if Γ acts properly on a space with walls.*

Definition A.2. Let S be a set and let Γ be a group.

- (1) A *wall* in S is a pair $W = \{H_1, H_2\}$ such that W is a partition of S ; i.e., $S = H_1 \cup H_2$, $H_1 \cap H_2 = \emptyset$, and $H_1 \neq \emptyset \neq H_2$.
- (2) If $W = \{H_1, H_2\}$ is a wall in S , then H_1 and H_2 are called *half-spaces* in S .
- (3) Elements $x, y \in S$ are *separated* by the wall W if $x \in H_1$ and $y \in H_2$.
- (4) The set S is a *space with walls* if there is given a set of walls in S such that for every $x, y \in S$ there are at most finitely many walls separating x and y . In this case, define $d(x, y)$ to be the number of walls separating x and y and note that d is a pseudometric (that is, it is symmetric and satisfies the triangle inequality).

- (5) The group Γ acts on the space S with walls if there is an action of Γ on S such that Γ permutes the walls. In this case the action of Γ is by isometries of the pseudometric.
- (6) The group Γ is said to act properly on the space S with walls if for all $r \in \mathbb{R}$ and for all $p \in S$, $\{g \in \Gamma \mid d(gp, p) < r\}$ is finite.

Note that a finite group Γ has a zipper action (let $\mathcal{E} = Z = \Gamma$) and also acts properly on a space with walls (let S be the set of all partitions of Γ into two nonempty, disjoint subsets). Therefore, in the proof of Theorem A.1, it is assumed that Γ is infinite.

Proof of Theorem A.1. Suppose first that there is a zipper action $\Gamma \curvearrowright \mathcal{E}$ with zipper $Z \subseteq \mathcal{E}$. Define

$$A = \bigcap_{g \in \Gamma} gZ \quad \text{and} \quad B = \bigcap_{g \in \Gamma} (gZ)^c.$$

Note that $\mathcal{E} \setminus (A \cup B) \neq \emptyset$, for otherwise $Z \Delta gZ = \emptyset$ for all $g \in \Gamma$, and so Z would fail to be a zipper. Let $S = \{gZ \subseteq \mathcal{E} \mid g \in \Gamma\}$. If $x \in \mathcal{E}$, define $H_x^+ = \{gZ \in S \mid x \in gZ\}$ and $H_x^- = \{gZ \in S \mid x \notin gZ\}$. Then $\{\{H_x^+, H_x^-\} \mid x \in \mathcal{E} \setminus (A \cup B)\}$ is a set of walls for S . The condition that $x \notin A \cup B$ ensures that $H_x^+ \neq \emptyset \neq H_x^-$. To see that for $g_1, g_2 \in \Gamma$, there are only finitely many walls separating g_1Z and g_2Z , observe that $\{H_x^+, H_x^-\}$ separates g_1Z and g_2Z if and only if $x \in g_1Z \Delta g_2Z$, and $x \in g_1Z \Delta g_2Z \subseteq (g_1Z \Delta Z) \cup (g_2Z \Delta Z)$, which is finite. The action $\Gamma \curvearrowright S$ is given by $h \cdot gZ = hgZ$. Since $hH_x^\pm = H_{hx}^\pm$ for all $h \in \Gamma$ and $x \in \mathcal{E} \setminus (A \cup B)$ (and $x \in A \cup B$ if and only if $hx \in A \cup B$), it follows that Γ permutes the walls. Since, by the second property of a zipper action (Definition 5.1), there are, for a given $r > 0$, at most finitely many $g \in \Gamma$ such that $d(gZ, Z) = |gZ \Delta Z| < r$, the action of Γ is proper.

Conversely, suppose that there is a proper action $\Gamma \curvearrowright S$ of Γ on a space S with walls. Let \mathcal{E} be the set of all half-spaces of S . For each $x \in S$, define $Z_x = \{H \in \mathcal{E} \mid x \in H\}$. Note that $g \cdot Z_x = Z_{gx}$ for all $g \in \Gamma$ and $x \in S$. Fix a base point $p \in S$ and let $Z = Z_p$ be the zipper for the action. For each $g \in \Gamma$, it follows that

$$\begin{aligned} Z_{gp} \Delta Z_p &= \{H \in \mathcal{E} \mid gp \in H \text{ and } p \notin H\} \cup \{H \in \mathcal{E} \mid gp \notin H \text{ and } p \in H\} \\ &= \{H \in \mathcal{E} \mid H \text{ is a half-space of a wall separating } p \text{ and } gp\}. \end{aligned}$$

Thus, $|Z_{gp} \Delta Z_p| = 2d(p, gp)$ and the properties required of a zipper action follow. \square

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Received January 2, 2008; revised June 29, 2008

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