J. Noncommut. Geom. 8 (2014), 587–609
DOI 10.4171/JNCG/165

Journal of Noncommutative Geometry

Twisted Calabi–Yau property of Ore extensions

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Abstract. Suppose that $E = A[x;\sigma,\delta]$ is an Ore extension with σ an automorphism. It is proved that if A is twisted Calabi–Yau of dimension d, then E is twisted Calabi–Yau of dimension $d+1$. The relation between their Nakayama automorphisms is also studied. As an application, the Nakayama automorphisms of a class of 5-dimensional Artin–Schelter regular algebras are given explicitly.

Mathematics Subject Classification (2010). Primary 16E40, 16S36; Secondary 16E65.

Keywords. Ore extension, twisted Calabi–Yau algebra, Nakayama automorphism, Artin–Schelter regular algebra.

Introduction

In the last twenty years, a lot of research appears on Artin–Schelter regular graded algebras arising from noncommutative projective algebraic geometry, and on Artin-Schelter regular Hopf algebras/quantum groups. Brown and Zhang proved that a noetherian Artin–Schelter regular Hopf algebra is rigid Gorenstein [BZ08], which is called the twisted Calabi–Yau condition in this paper. Such a class of algebras is called twisted Calabi–Yau algebra (see Definition 1.1). Van den Bergh duality [VdB98] holds for any twisted Calabi–Yau algebra. A noetherian Hopf algebra is Artin–Schelter regular if and only if it is twisted Calabi–Yau. In the noetherian connected graded case, an algebra is Artin–Schelter regular if and only if it is graded twisted Calabi–Yau. Associated to a twisted Calabi–Yau algebra, there is an automorphism, called the Nakayama automorphism in general, which is unique up to an inner automorphism. A twisted Calabi–Yau algebra is Calabi–Yau in the sense of Ginzburg [Gin06] if and only if its Nakayama automorphism is inner. Calabi–Yau algebra is an algebraic structure arising from the geometry of Calabi–Yau manifolds and mirror symmetry. It has attracted much interest in recent years.

For any finite-dimensional Lie algebra $\mathfrak{g}$, Yekutieli constructed the rigid dualizing complex of $U(\mathfrak{g})$ [Yek00]. In the terminology now used, in fact he proved that $U(\mathfrak{g})$ is Calabi–Yau if and only if $\operatorname{tr}(\operatorname{ad} x) = 0$ for all $x \in \mathfrak{g}$. This result is generalized to a more general situation – the PBW deformations of Koszul Calabi–Yau algebras [WZ13]. The quantized enveloping algebra of a complex semisimple

Lie algebra is always Calabi–Yau [Che04]. In [BZ08], Brown and Zhang also described the Nakayama automorphism explicitly by using the homological integral for any noetherian Artin–Schelter regular Hopf algebra. Recently, some people are interested in quantum homogeneous spaces, which are right coideal subalgebras of Hopf algebras satisfying some additional conditions. One question is to study when quantum homogeneous spaces are Artin–Schelter regular or twisted Calabi–Yau. The first-named and the third-named authors studied the twisted Calabi–Yau property of the right coideal subalgebras of a quantized enveloping algebra [LW14]. A class of right coideal subalgebras of a quantized enveloping algebra can be obtained by iterated Ore extensions. This motivates us to study the Nakayama automorphism and the twisted Calabi–Yau property of Ore extensions in this paper. Ore extensions are noncommutative analogues of polynomial extensions. If $E = A[x; \sigma, \delta]$ is a graded Ore extension with σ an automorphism and A is Artin–Schelter regular, then so is E. This means the twisted Calabi–Yau property of a connected graded algebra is preserved by (graded) Ore extensions. It is natural to ask whether Ore extensions preserve the twisted Calabi–Yau property in general situations? The answer is positive when σ is an automorphism.

Let $E = A[x; \sigma, \delta]$ be an Ore extension with σ an automorphism. There is a short exact sequence of E^{e}-modules (see Lemma 2.1)

$$0 \to E \otimes_A {}^{\sigma^{-1}}E \xrightarrow{\rho} E \otimes_A E \xrightarrow{\mu} E \to 0.$$

Then an E^{e}-projective resolution of E can be constructed by using an A^{e}-projective resolution of A. In particular, taking the bar complex of A, the construction is nothing but the construction given by Guccione–Guccione [GG97]. Using this construction, we compute the Hochschild cohomology $H^*(E, E \otimes E)$ and obtain a family of short exact sequences (Theorem 2.7).

Theorem 1. *Let A be a projective $\Bbbk$-algebra and $E = A[x; \sigma, \delta]$ be an Ore extension with σ an automorphism. Suppose that A admits a finitely generated projective resolution as an A^{e}-module. Then for any $n \in \mathbb{N}$,*

$$0 \to H^n(A, E \otimes E) \to H^n(A, E \otimes E^{\sigma^{-1}}) \to H^{n+1}(E, E \otimes E) \to 0$$

is an exact sequence of E^{e}-modules.

We prove that Ore extensions preserve the twisted Calabi–Yau property and describe the relation between the Nakayama automorphisms of A and E (Theorem 3.3).

Theorem 2. *Let A be a projective $\Bbbk$-algebra and $E = A[x; \sigma, \delta]$ be an Ore extension with σ an automorphism. Suppose that A is ν-twisted Calabi–Yau of dimension d. Then E is twisted Calabi–Yau of dimension $d + 1$, and the Nakayama automorphism ν' of E satisfies that $\nu'|_A = \sigma^{-1}\nu$ and $\nu'(x) = ux + b$ for some $u, b \in A$ with u invertible.*

As an application, we focus on a class of Artin–Schelter regular algebras of dimension 5 which were investigated in detail by the second-named and the third-named authors [WW12]. These algebras may be constructed by iterated Ore extensions and their Nakayama automorphisms are given explicitly.

The paper is organized as follows. In Section 1, we recall the definitions of twisted Calabi–Yau algebras and Ore extensions, and fix some notation. In Section 2, following [GG97], we study Hochschild cohomology on Ore extensions instead of Hochschild homology. Some exact sequences are obtained and Theorem 1 is proved. In Section 3, we prove the main result Theorem 2, that is, Ore extensions preserve the twisted Calabi–Yau property if σ is an automorphism. The relation between their Nakayama automorphisms is also described. In Section 4, the main result is applied to multi-parameter quantum affine spaces and a class of Artin–Schelter regular algebras of dimension 5 which can be constructed by iterated Ore extensions.

1. Preliminaries

1.1. Twisted Calabi–Yau algebras. Throughout, $\Bbbk$ is a unital commutative ring and all algebras are $\Bbbk$-algebras. Unadorned $\otimes$ means $\otimes_{\Bbbk}$ and Hom means $\mathrm{Hom}_{\Bbbk}$. Suppose that A is an algebra. Let A^{op} be the opposite algebra of A and $A^{\mathrm{e}} = A \otimes A^{\mathrm{op}}$ be the enveloping algebra of A. The term A^{e}-modules is used for A-A-bimodules.

For any two $\Bbbk$-modules M, N, let $\tau_{M,N} : M \otimes N \to N \otimes M$ be the flip map. The subscript is often omitted if there is no confusion. For any A^{e}-module M and any endomorphisms ν, σ of A, denote by ${}^{\nu}M^{\sigma}$ the A^{e}-module whose ground $\Bbbk$-module is M and the action is given by $a \cdot m \cdot b = \nu(a)m\sigma(b)$ for all $a, b \in A$ and $m \in M$. If one of ν and σ is the identity map, then it is usually omitted.

Suppose that M and N are both A^{e}-modules. It is easy to see that there are two A^{e}-module structures on $M \otimes N$, one is called the outer structure defined by $(a \otimes b) \rightharpoonup (m \otimes n) = am \otimes nb$, and the other is called the inner structure defined by $(m \otimes n) \leftharpoonup (a \otimes b) = ma \otimes bn$, for any $a, b \in A$, $m \in M$, $n \in N$. Since A^{e} is identified with $A \otimes A$ as a $\Bbbk$-module, $A \otimes A$ endowed with the outer (resp. inner) structure is nothing but the left (resp. right) regular A^{e}-module A^{e}. Hence we often say A^{e} has the outer and inner A^{e}-module structures. In the following definition, the outer structure on A^{e} is used when computing the homology $\mathrm{Ext}^*_{A^{\mathrm{e}}}(A, A^{\mathrm{e}})$. Thus $\mathrm{Ext}^*_{A^{\mathrm{e}}}(A, A^{\mathrm{e}})$ admits an A^{e}-module structure induced by the inner one on A^{e}.

Definition 1.1. An algebra A is called *ν-twisted Calabi–Yau* of dimension d for some automorphism ν of A and for some integer $d \geq 0$ if

(1) A is homologically smooth, that is, as an A^{e}-module, A has a finitely generated projective resolution of finite length;

(2) $\mathrm{Ext}^i_{A^{\mathrm{e}}}(A, A^{\mathrm{e}}) \cong \begin{cases} 0, & i \neq d, \\ A^{\nu}, & i = d, \end{cases}$ as A^{e}-modules.

Sometimes condition (2) is called the twisted Calabi–Yau condition. In this case, ν is called the *Nakayama automorphism* of A.

The Nakayama automorphism is unique up to an inner automorphism. A ν-twisted Calabi–Yau algebra A is Calabi–Yau in the sense of Ginzburg [Gin06] if and only if ν is an inner automorphism of A.

Graded twisted Calabi–Yau algebras are defined similarly. Condition (1) is equivalent to that A, when viewed as a complex concentrated in degree 0, is a compact object in the derived category $\boldsymbol{D}(A^{\mathrm{e}})$ [Nee01], i.e., the functor $\mathrm{Hom}_{\boldsymbol{D}(A^{\mathrm{e}})}(A, -)$ commutes with arbitrary coproducts.

1.2. Artin–Schelter regular algebras. In this section, $\Bbbk$ is a field.

Definition 1.2. Suppose that A is an algebra with an augmentation map $\varepsilon\colon A \to \Bbbk$. Then A is called left *Artin–Schelter regular* (for short, AS-regular) if

(1) A has finite left global dimension d,

(2) $\dim_{\Bbbk} \mathrm{Ext}^d_A({}_A\Bbbk, {}_AA) = 1$ and $\mathrm{Ext}^i_A({}_A\Bbbk, {}_AA) = 0$ for all $i \neq d$.

Right AS-regular algebras are defined similarly, and A is called AS-regular if A is both left and right AS-regular. A noetherian Hopf algebra is AS-regular if and only if it is twisted Calabi–Yau. One direction is proved in [BZ08], Lemma 5.2 and Proposition 4.5, where they used the term rigid Gorenstein for twisted Calabi–Yau. The other direction follows from next lemma, which we can not locate a reference.

Lemma 1.3. *Suppose that A is an algebra with an augmentation map $\varepsilon\colon A \to \Bbbk$. If A is twisted Calabi–Yau, then A is AS-regular.*

Proof. Since

$$\begin{aligned}
\Bbbk \overset{\mathrm{L}}{\otimes}_A A^{\nu}[-d] &\cong \Bbbk \overset{\mathrm{L}}{\otimes}_A \mathrm{RHom}_{A^{\mathrm{e}}}(A, A^{\mathrm{e}}) \\
&\cong \mathrm{RHom}_{A^{\mathrm{e}}}(A, A \otimes \Bbbk) \\
&\cong \mathrm{RHom}_{A^{\mathrm{e}}}(A, \mathrm{Hom}(\Bbbk, A)) \\
&\cong \mathrm{RHom}_A(A \otimes_A \Bbbk, A) \\
&\cong \mathrm{RHom}_A(\Bbbk, A),
\end{aligned}$$

it follows that $\dim_{\Bbbk} \mathrm{Ext}^d_A({}_A\Bbbk, {}_AA) = 1$ and $\mathrm{Ext}^i_A({}_A\Bbbk, {}_AA) = 0$ for all $i \neq d$. Similarly we can prove that A is also right AS-regular. $\square$

For a connected graded algebra A, A is left AS-regular if and only if it is right AS-regular. By the same argument as in the above lemma, A is AS-regular if A is twisted Calabi–Yau. On the other hand, if A is noetherian AS-regular, then A has a rigid dualizing complex [VdB97], which implies that A is twisted Calabi–Yau.

1.3. Ore extensions. Let A be a $\Bbbk$-algebra, σ be an endomorphism of A and δ be a σ-derivation (i.e., $\delta\colon A \to A$ is a $\Bbbk$-linear map such that $\delta(ab) = \delta(a)b + \sigma(a)\delta(b)$ for all $a, b \in A$). Then σ, δ uniquely determine a ring extension E/A satisfying

(1) E is a free left A-module with basis $\{1, x, x^2, \ldots\}$,

(2) For any $a \in A$, $xa = \sigma(a)x + \delta(a)$.

The algebra E is denoted by $A[x; \sigma, \delta]$ and is called the *Ore extension* of A associated to σ and δ. For graded algebras, *graded Ore extensions* are defined similarly. However, the Koszul sign convention does not apply in this context.

If σ is the identity map, $A[x; \sigma, \delta]$ is often simply written as $A[x; \delta]$; and if $\delta = 0$, as $A[x; \sigma]$. The polynomial extension $A[x]$ is a special Ore extension.

If σ is an automorphism, then $\{1, x, x^2, \ldots\}$ is also a basis for E as a free right A-module. In this case, $Ax^k \subseteq \sum_{i=0}^{k} x^i A$ and $x^l A \subseteq \sum_{j=0}^{l} Ax^j$ for any $k, l \in \mathbb{N}$. Let p_i^n be the $\Bbbk$-linear map which is the sum of all the compositions $\sigma_1\sigma_2 \ldots \sigma_n$ with σ_j being σ or δ, and σ appearing i times in each composition. Then for any $a \in A$ and $n \geq 1$,

$$x^n a = \sum_{i=0}^{n} p_i^n(a)x^i. \tag{1.1}$$

Similarly, let q_i^n be the $\Bbbk$-linear map which is the sum of all the compositions $\sigma_1\sigma_2 \ldots \sigma_n$ with σ_j being σ^{-1} or $-\delta\sigma^{-1}$, and σ^{-1} appearing i times in each composition. Then for any $a \in A$ and $n \geq 1$,

$$ax^n = \sum_{i=0}^{n} x^i q_i^n(a). \tag{1.2}$$

Many ring-theoretic and homological properties are preserved by Ore extensions under certain conditions. We list some of them as follows.

- If A is an integral domain and σ is injective, then E is an integral domain.
- If A is a prime ring and σ is an automorphism, then E is a prime ring.
- If A has finite right global dimension and σ is an automorphism, then E has finite right global dimension, in fact,

$$\operatorname{r.gl.dim} A \leq \operatorname{r.gl.dim} E \leq \operatorname{r.gl.dim} A + 1.$$

- If $\Bbbk$ is a noetherian ring, A is (strongly) right noetherian and σ is an automorphism, then E is (strongly) right noetherian.

For the details and other properties of Ore extensions, we refer to [MR01], [GW04], [ASZ99], etc.

Here are some examples of iterated Ore extensions: multi-parameter quantum affine n-spaces $\mathcal{O}_{\mathrm{q}}(\Bbbk^n)$, Weyl algebras $A_n(\Bbbk)$, enveloping algebras $U(\mathfrak{g})$ of finite-dimensional solvable Lie algebras $\mathfrak{g}$, the Borel part of quantized enveloping algebras $U_q(\mathfrak{g})$ of complex semisimple Lie algebras $\mathfrak{g}$, and some classes of AS-regular algebras.

1.4. Notations. We fix some notation about complexes and graded modules.

Suppose that $(P_., d)$ is a chain complex. The l-shift of $P_.$, denoted by $P[l]_.$, is defined by $P[l]_n = P_{n-l}$ and $d[l]_n = (-1)^l d_{n-l}$. If $(P'_., d')$ is another chain complex and $f : P_. \to P'_.$ is a morphism of complexes, the mapping cone of f, denoted by $\mathrm{cone}(f)$, is defined by $\mathrm{cone}(f)_n = P_{n-1} \oplus P'_n$ and the differential sending $(p, p') \in \mathrm{cone}(f)_n$ to $(-d_{n-1}(p), d'_n(p') - f_{n-1}(p))$. Dually, suppose that $(Q^\cdot, d)$, $(Q'^\cdot, d')$ are cochain complexes and $g : Q^\cdot \to Q'^\cdot$ is a morphism of complexes. The l-shift of $Q^\cdot$, denoted by $Q[l]^\cdot$, is defined by $Q[l]^n = Q^{n+l}$ and $d[l]^n = (-1)^l d^{n+l}$. The mapping cone, $\mathrm{cone}(g)$, is defined by $\mathrm{cone}(g)^n = Q^{n+1} \oplus Q'^n$ and the differential sending $(q, q') \in \mathrm{cone}(g)^n$ to $(-d^{n+1}(q), d'^n(q') + g^{n+1}(q))$. If $f : P_. \to P'_.$ is a morphism of A-module complexes and ${}_AM$ is an A-module, then $\mathrm{cone}(\mathrm{Hom}_A(f, M)) \cong \mathrm{Hom}_A(\mathrm{cone}(f), M)[1]$.

For any graded A-module M, the n-shift $M(n)$ of M is defined by $M(n)_i = M_{n+i}$.

We mainly refer to [Lod98] for Hochschild homology and cohomology.

2. Hochschild cohomology on Ore extensions

We investigate the Hochschild cohomology on Ore extensions in this section. From now on, σ is always required to be an automorphism.

Lemma 2.1. *Let A be an algebra and $E = A[x; \sigma, \delta]$ be an Ore extension. Then the sequence of E^{e}-modules*

$$0 \to E \otimes_A {}^{\sigma^{-1}}E \xrightarrow{\rho} E \otimes_A E \xrightarrow{\mu} E \to 0 \tag{2.1}$$

is exact, where $\rho(e \otimes e') = ex \otimes e' - e \otimes xe'$ and μ is given by multiplication.

Proof. First of all, ρ is well defined since

$$\begin{aligned}
\rho(1 \otimes \sigma^{-1}(a)) &= x \otimes \sigma^{-1}(a) - 1 \otimes x\sigma^{-1}(a) \\
&= x \otimes \sigma^{-1}(a) - 1 \otimes ax - 1 \otimes \delta\sigma^{-1}(a) \\
&= x\sigma^{-1}(a) \otimes 1 - a \otimes x - \delta\sigma^{-1}(a) \otimes 1 \\
&= ax \otimes 1 - a \otimes x \\
&= \rho(a \otimes 1).
\end{aligned}$$

Suppose that $\sum_{i=0}^n x^i \otimes e_i \in \mathrm{Ker}\,\rho$. Then $\sum_{i=0}^n x^{i+1} \otimes e_i - \sum_{i=0}^n x^i \otimes xe_i = 0$. Note that $x^{n+1} \otimes e_n$ is the unique term containing x^{n+1} as the first tensor factor. It follows that $e_n = 0$ and so $\sum_{i=0}^n x^i \otimes e_i = 0$. Thus ρ is injective.

Now suppose that $\sum_{i=0}^{n} x^i \otimes e'_i \in \operatorname{Ker}\mu$ with $e'_n \neq 0$. Then

$$\begin{aligned}\sum_{i=0}^{n} x^i \otimes e'_i &= 1 \otimes e'_0 + \sum_{i=1}^{n} x^i \otimes e'_i \\ &= 1 \otimes e'_0 + \sum_{i=1}^{n} x^i \otimes e'_i - \sum_{i=1}^{n} x^{i-1} \otimes xe'_i + \sum_{i=1}^{n} x^{i-1} \otimes xe'_i \\ &= 1 \otimes e'_0 + \rho(\sum_{i=1}^{n} x^{i-1} \otimes e'_i) + \sum_{i=0}^{n-1} x^i \otimes xe'_{i+1} \\ &= \sum_{i=0}^{n-1} x^i \otimes e''_i \pmod{\operatorname{Im}\rho},\end{aligned}$$

where $e''_i \in E$ and $e''_{n-1} \neq 0$. By induction on n, we obtain $\operatorname{Ker}\mu = \operatorname{Im}\rho$.

Therefore, the sequence (2.1) is exact. $\square$

Remark 2.2. The graded version of Lemma 2.1 is also true. If $\deg(x) = l$, the short exact sequence (2.1) should be modified by

$$0 \to E \otimes_A {}^{\sigma^{-1}}E(-l) \xrightarrow{\rho} E \otimes_A E \xrightarrow{\mu} E \to 0.$$

For any A^{e}-projective resolution $P.$ of A with an augmentation map ε, $E \otimes_A P. \otimes_A {}^{\sigma^{-1}}E$ and $E \otimes_A P. \otimes_A E$ are E^{e}-projective resolutions of $E \otimes_A {}^{\sigma^{-1}}E$ and $E \otimes_A E$ respectively. By the Comparison lemma, ρ can be lifted to a morphism of E^{e}-module complexes from $E \otimes_A P. \otimes_A {}^{\sigma^{-1}}E$ to $E \otimes_A P. \otimes_A E$, say ψ. Then $\operatorname{cone}(\psi)$ is an E^{e}-projective resolution of E via $\mu(\mathrm{id}_E \otimes \varepsilon \otimes \mathrm{id}_E)$.

Now we start to look at the Hochschild cohomology. Let $P.$ be the bar complex of A,

$$0 \to A^{\otimes 2} \xleftarrow{b'} A^{\otimes 3} \xleftarrow{b'} \cdots \xleftarrow{b'} A^{\otimes n+1} \xleftarrow{b'} A^{\otimes n+2} \xleftarrow{b'} \cdots$$

where $b' \colon A^{\otimes n+2} \to A^{\otimes n+1}$ is the map

$$b'(a_0 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \cdots \otimes a_{n+1}.$$

A lifting map of ρ is constructed in [GG97] as follows.

The two complexes $E \otimes_A P. \otimes_A {}^{\sigma^{-1}}E$ and $E \otimes_A P. \otimes_A E$ are $(E \otimes A^{\otimes *} \otimes {}^{\sigma^{-1}}E, b'_{1,*})$ and $(E \otimes A^{\otimes *} \otimes E, b'_{0,*})$, respectively, where the differentials are

$$\begin{aligned} b'_{1,n}(a_0 \otimes \cdots \otimes a_{n+1}) &= \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1} \\ &\quad + (-1)^n a_0 \otimes \cdots \otimes a_{n-1} \otimes \sigma^{-1}(a_n) a_{n+1}, \\ b'_{0,n}(a_0 \otimes \cdots \otimes a_{n+1}) &= \sum_{i=0}^{n} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}. \end{aligned}$$

The lifting map $\{\psi'_n\colon E\otimes A^{\otimes n}\otimes {}^{\sigma^{-1}}E\to E\otimes A^{\otimes n}\otimes E\}_{n\in\mathbb{N}}$ is defined by

$$\begin{aligned}
&\psi'_n(1\otimes a_1\otimes\cdots\otimes a_n\otimes 1)\\
&\quad= x\otimes\sigma^{-1}(a_1)\otimes\cdots\otimes\sigma^{-1}(a_n)\otimes 1-1\otimes a_1\otimes\cdots\otimes a_n\otimes x\\
&\qquad-\sum_{j=1}^{n}1\otimes a_1\otimes\cdots\otimes a_{j-1}\otimes\delta\sigma^{-1}(a_j)\otimes\sigma^{-1}(a_{j+1})\otimes\cdots\otimes\sigma^{-1}(a_n)\otimes 1.
\end{aligned}$$

By the above argument, we have

Lemma 2.3 ([GG97], Propositions 1.1 and 1.2). *Let A be an algebra and $E=A[x;\sigma,\delta]$ be an Ore extension. Then*

$$\begin{tikzcd}
E\otimes {}^{\sigma^{-1}}E \arrow[d, "\psi'_0"] & E\otimes A\otimes {}^{\sigma^{-1}}E \arrow[l, "b'_{1,1}"'] \arrow[d, "\psi'_1"] & E\otimes A^{\otimes 2}\otimes {}^{\sigma^{-1}}E \arrow[l, "b'_{1,2}"'] \arrow[d, "\psi'_2"] & \cdots \arrow[l, "b'_{1,3}"'] \\
E\otimes E & E\otimes A\otimes E \arrow[l, "b'_{0,1}"'] & E\otimes A^{\otimes 2}\otimes E \arrow[l, "b'_{0,2}"'] & \cdots \arrow[l, "b'_{0,3}"']
\end{tikzcd} \tag{2.2}$$

is a commutative diagram of E^{e}-modules, and

$$\operatorname{cone}(\psi')\xrightarrow{\mu}E\to 0 \tag{2.3}$$

is an exact sequence. If further, A is flat (resp. projective) over $\Bbbk$, then (2.3) is a flat (resp. projective) resolution of E as an E^{e}-module.

In the following statements, we sometimes write $f(a_1\otimes\cdots\otimes a_n)$ as $f(a_1,\ldots,a_n)$ for convenience.

Let M be an E^{e}-module. Applying $\operatorname{Hom}_{E^{\mathrm{e}}}(-,M)$ to (2.2), we have the commutative diagram

$$\begin{tikzcd}
\operatorname{Hom}(\Bbbk,M^{\sigma^{-1}}) \arrow[r, "b^{1,0}"] & \operatorname{Hom}(A,M^{\sigma^{-1}}) \arrow[r, "b^{1,1}"] & \operatorname{Hom}(A^{\otimes 2},M^{\sigma^{-1}}) \arrow[r, "b^{1,2}"] & \cdots \\
\operatorname{Hom}(\Bbbk,M) \arrow[u, "\theta^0"] \arrow[r, "b^{0,0}"] & \operatorname{Hom}(A,M) \arrow[u, "\theta^1"] \arrow[r, "b^{0,1}"] & \operatorname{Hom}(A^{\otimes 2},M) \arrow[u, "\theta^2"] \arrow[r, "b^{0,2}"] & \cdots,
\end{tikzcd} \tag{2.4}$$

where the maps are given by, for any $f \in \mathrm{Hom}(A^{\otimes n}, M)$, $\tilde{f} \in \mathrm{Hom}(A^{\otimes n}, M^{\sigma^{-1}})$,

$$\begin{aligned}
&b^{0,n}(f)(a_1, \ldots, a_{n+1}) \\
&\quad = a_1 f(a_2, \ldots, a_{n+1}) + \sum_{i=1}^{n} (-1)^i f(a_1, \ldots, a_i a_{i+1}, \ldots, a_{n+1}) \\
&\qquad + (-1)^{n+1} f(a_1, \ldots, a_n) a_{n+1}, \\
&b^{1,n}(\tilde{f})(a_1, \ldots, a_{n+1}) \\
&\quad = a_1 \tilde{f}(a_1, \ldots, a_{n+1}) + \sum_{i=1}^{n} (-1)^i \tilde{f}(a_1, \ldots, a_i a_{i+1}, \ldots, a_{n+1}) \\
&\qquad + (-1)^{n+1} \tilde{f}(a_1, \ldots, a_n) \sigma^{-1}(a_{n+1}), \\
&\theta^n(f)(a_1, \ldots, a_n) \\
&\quad = x f(\sigma^{-1}(a_1), \ldots, \sigma^{-1}(a_n)) - f(a_1, \ldots, a_n) x \\
&\qquad - \sum_{j=1}^{n} f(a_1, \ldots, a_{j-1}, \delta\sigma^{-1}(a_j), \sigma^{-1}(a_{j+1}), \ldots, \sigma^{-1}(a_n)).
\end{aligned}$$

Obviously, when M is viewed as an A^{e}-module, the two rows in the diagram (2.4) are the Hochschild complexes $C^*(A, M^{\sigma^{-1}})$ and $C^*(A, M)$. In general, for any A^{e}-module M, the differentials of $C^*(A, M)$ and $C^*(A, M^{\sigma^{-1}})$ are denoted by b and $\mathsf{b}_{\sigma^{-1}}$, respectively, if there is no confusion. On the other hand, by Lemma 2.3, we can compute $H^n(E, M)$ by using $\mathrm{cone}(\psi')$ or $\mathrm{cone}(\theta)$.

Lemma 2.4. *Let A be a projective $\Bbbk$-algebra and $E = A[x; \sigma, \delta]$ be an Ore extension and let M be an E^{e}-module. For any $n \in \mathbb{N}$, $H^n(E, M) \cong H^{n-1}(\mathrm{cone}(\theta))$.*

Proof. By (2.3) and (2.4),

$$\begin{aligned}
H^n(E, M) &= H^n(\mathrm{Hom}_{E^{\mathrm{e}}}(\mathrm{cone}(\psi'), M)) \\
&\cong H^n(\mathrm{cone}(\theta)[-1]) \\
&= H^{n-1}(\mathrm{cone}(\theta)).
\end{aligned}$$ $\square$

Now let $M = E \otimes E$. By the definition of mapping cones, there is a short exact sequence of E^{e}-module complexes

$$0 \to C^*(A, E \otimes E^{\sigma^{-1}}) \to \mathrm{cone}(\theta) \to C^*(A, E \otimes E)[1] \to 0,$$

where the E^{e}-module structure on each complex is induced by the inner structure on $E \otimes E$. It follows that

$$\begin{aligned}
\cdots \to H^{n-1}(C^*(A, E \otimes E^{\sigma^{-1}})) \to H^{n-1}(\mathrm{cone}(\theta)) \to H^{n-1}(C^*(A, E \otimes E)[1]) \\
\xrightarrow{\partial} H^n(C^*(A, E \otimes E^{\sigma^{-1}})) \to H^n(\mathrm{cone}(\theta)) \to H^n(C^*(A, E \otimes E)[1]) \to \cdots
\end{aligned}$$

is an exact sequence of E^{e}-modules. By Lemma 2.4, the above sequence becomes

$$\begin{aligned} \cdots \to H^{n-1}(A, E \otimes E^{\sigma^{-1}}) \to H^n(E, E \otimes E) \to H^n(A, E \otimes E) \\ \xrightarrow{\partial} H^n(A, E \otimes E^{\sigma^{-1}}) \to H^{n+1}(E, E \otimes E) \to H^{n+1}(A, E \otimes E) \to \cdots, \end{aligned} \tag{2.5}$$

where the connecting homomorphism $\partial = H^n(\theta)$.

Since, as A^{e}-modules, $E \otimes E \cong A \otimes A \otimes \Bbbk[x]^{\otimes 2}$, $ax^l \otimes x^k b \mapsto a \otimes b \otimes x^l \otimes x^k$, and similarly

$$E \otimes E^{\sigma^{-1}} \cong A \otimes A^{\sigma^{-1}} \otimes \Bbbk[x]^{\otimes 2} \cong A \otimes {}^{\sigma}A \otimes \Bbbk[x]^{\otimes 2},$$

there exist two canonical morphisms of $\Bbbk$-module complexes

$$\begin{gathered} C^*(A, A \otimes A^{\sigma^{-1}}) \otimes \Bbbk[x]^{\otimes 2} \to C^*(A, E \otimes E^{\sigma^{-1}}), \\ C^*(A, A \otimes A) \otimes \Bbbk[x]^{\otimes 2} \to C^*(A, E \otimes E). \end{gathered}$$

where the differentials of the left two complexes are $\mathsf{b}_{\sigma^{-1}} \otimes \mathrm{id}^{\otimes 2}$, $\mathsf{b} \otimes \mathrm{id}^{\otimes 2}$, respectively.

We hope to equip the left two complexes with suitable E^{e}-module structures such that the above are morphisms of E^{e}-module complexes. To this end, for any $\tilde{f} \in C^n(A, A \otimes A^{\sigma^{-1}})$, define

$$\begin{aligned} x \cdot (\tilde{f} \otimes x^l \otimes x^k) &= \tilde{f} \otimes x^l \otimes x^{k+1} \quad \text{for all } k, l \in \mathbb{N}, \\ a \cdot (\tilde{f} \otimes x^l \otimes x^k) &= \sum_{i=0}^{k} q_i^k(a) \cdot \tilde{f} \otimes x^l \otimes x^i \quad \text{for all } a \in A, \\ (\tilde{f} \otimes x^l \otimes x^k) \cdot x &= \tilde{f} \otimes x^{l+1} \otimes x^k, \\ (\tilde{f} \otimes x^l \otimes x^k) \cdot a &= \sum_{i=0}^{l} \tilde{f} \cdot p_i^l(a) \otimes x^i \otimes x^k, \end{aligned}$$

where p_i^l and q_i^k are defined in (1.1) and (1.2) respectively, the actions $q_i^k(a) \cdot \tilde{f}$ and $\tilde{f} \cdot p_i^l(a)$ are induced from the inner structure on $A \otimes A^{\sigma^{-1}}$.

This makes $C^*(A, A \otimes A^{\sigma^{-1}}) \otimes \Bbbk[x]^{\otimes 2}$ be a complex of E^{e}-modules and similarly for $C^*(A, A \otimes A) \otimes \Bbbk[x]^{\otimes 2}$.

Lemma 2.5. *Suppose that A is a flat $\Bbbk$-algebra and $E = A[x; \sigma, \delta]$ is an Ore extension. Then there exists a morphism of E^{e}-module complexes*

$$\eta\colon C^*(A, A \otimes A) \otimes \Bbbk[x]^{\otimes 2} \to C^*(A, A \otimes A^{\sigma^{-1}}) \otimes \Bbbk[x]^{\otimes 2}$$

such that the diagram

$$\begin{tikzcd} C^*(A, A \otimes A^{\sigma^{-1}}) \otimes \Bbbk[x]^{\otimes 2} \arrow[r] & C^*(A, E \otimes E^{\sigma^{-1}}) \\ C^*(A, A \otimes A) \otimes \Bbbk[x]^{\otimes 2} \arrow[r] \arrow[u, "\eta"] & C^*(A, E \otimes E) \arrow[u, "\theta"'] \end{tikzcd} \tag{2.6}$$

is commutative.

Proof. For any $y \in A \otimes A^{\sigma^{-1}}$, we use Sweedler's notation $y = \sum y' \otimes y''$. For any $f \otimes x^l \otimes x^k \in C^n(A, A \otimes A) \otimes \Bbbk[x]^{\otimes 2}$, let

$$[f, x^l \otimes x^k] \colon (a_1, \dots, a_n) \mapsto \sum f(a_1, \dots, a_n)' x^l \otimes x^k f(a_1, \dots, a_n)''$$

be the corresponding element in $C^n(A, E \otimes E)$. Then

$$\begin{aligned}
&\theta^n([f, x^l \otimes x^k])(a_1, \dots, a_n) \\
&\quad = x([f, x^l \otimes x^k](\sigma^{-1}(a_1), \dots, \sigma^{-1}(a_n))) - ([f, x^l \otimes x^k](a_1, \dots, a_n))x \\
&\qquad - \sum_{j=1}^{n} [f, x^l \otimes x^k](a_1, \dots, a_{j-1}, \delta\sigma^{-1}(a_j), \sigma^{-1}(a_{j+1}), \dots, \sigma^{-1}(a_n)) \\
&\quad = \sum x f(\sigma^{-1}(a_1), \dots, \sigma^{-1}(a_n))' x^l \otimes x^k f(\sigma^{-1}(a_1), \dots, \sigma^{-1}(a_n))'' \\
&\qquad - \sum f(a_1, \dots, a_n)' x^l \otimes x^k f(a_1, \dots, a_n)'' x \\
&\qquad - \sum_{j=1}^{n} [f, x^l \otimes x^k](a_1, \dots, a_{j-1}, \delta\sigma^{-1}(a_j), \sigma^{-1}(a_{j+1}), \dots, \sigma^{-1}(a_n)) \\
&\quad = \sum \sigma(f(\sigma^{-1}(a_1), \dots, \sigma^{-1}(a_n))') x^{l+1} \otimes x^k f(\sigma^{-1}(a_1), \dots, \sigma^{-1}(a_n))'' \\
&\qquad + \sum \delta(f(\sigma^{-1}(a_1), \dots, \sigma^{-1}(a_n))') x^l \otimes x^k f(\sigma^{-1}(a_1), \dots, \sigma^{-1}(a_n))'' \\
&\qquad - \sum f(a_1, \dots, a_n)' x^l \otimes x^{k+1} \sigma^{-1}(f(a_1, \dots, a_n)'') \\
&\qquad + \sum f(a_1, \dots, a_n)' x^l \otimes x^k \delta\sigma^{-1}(f(a_1, \dots, a_n)'') \\
&\qquad - \textstyle\sum_{j=1}^{n} [f, x^l \otimes x^k](a_1, \dots, a_{j-1}, \delta\sigma^{-1}(a_j), \sigma^{-1}(a_{j+1}), \dots, \sigma^{-1}(a_n)) \\
&\quad = [(\sigma \otimes \mathrm{id}) f (\sigma^{-1})^{\otimes n}, x^{l+1} \otimes x^k](a_1, \dots, a_n) \\
&\qquad - [(\mathrm{id} \otimes \sigma^{-1}) f, x^l \otimes x^{k+1}](a_1, \dots, a_n) \\
&\qquad + [(\delta \otimes \mathrm{id}) f (\sigma^{-1})^{\otimes n}, x^l \otimes x^k](a_1, \dots, a_n) \\
&\qquad + [(\mathrm{id} \otimes \delta\sigma^{-1}) f, x^l \otimes x^k](a_1, \dots, a_n) \\
&\qquad - \sum_{j=1}^{n} [f(\mathrm{id}^{\otimes j-1} \otimes \delta\sigma^{-1} \otimes (\sigma^{-1})^{\otimes n-j}), x^l \otimes x^k](a_1, \dots, a_n).
\end{aligned}$$

Thus η can be defined as follows, so that the diagram (2.6) is commutative. For any $n \in \mathbb{N}$ and $f \in C^n(A, A \otimes A)$,

$$\eta^n(f \otimes x^l \otimes x^k) = f_1 \otimes x^{l+1} \otimes x^k - f_2 \otimes x^l \otimes x^{k+1} + f_3 \otimes x^l \otimes x^k \tag{2.7}$$

with

$$f_1 := (\sigma \otimes \mathrm{id}) f (\sigma^{-1})^{\otimes n} \tag{2.8}$$

$$f_2 := (\mathrm{id} \otimes \sigma^{-1}) f \tag{2.9}$$

$$f_3 := (\delta \otimes \mathrm{id}) f (\sigma^{-1})^{\otimes n} + (\mathrm{id} \otimes \delta\sigma^{-1}) f - \sum_{j=1}^{n} f(\mathrm{id}^{\otimes j-1} \otimes \delta\sigma^{-1} \otimes (\sigma^{-1})^{\otimes n-j}). \tag{2.10}$$

It remains to check that η^n is E^{e}-linear for all n. In fact, it is obvious that $\eta^n(x \cdot (f \otimes x^l \otimes x^k) \cdot x) = x \cdot \eta^n(f \otimes x^l \otimes x^k) \cdot x$. Thus it suffices to show

$$\eta^n(a \cdot (f \otimes 1 \otimes 1)) = a \cdot \eta^n(f \otimes 1 \otimes 1), \tag{2.11}$$

$$\eta^n((f \otimes 1 \otimes 1) \cdot a) = \eta^n(f \otimes 1 \otimes 1) \cdot a. \tag{2.12}$$

By the definition of η,

$$\begin{aligned}\eta^n(a \cdot (f \otimes 1 \otimes 1)) &= \eta^n(a \cdot f \otimes 1 \otimes 1) \\ &= (a \cdot f)_1 \otimes x \otimes 1 - (a \cdot f)_2 \otimes 1 \otimes x + (a \cdot f)_3 \otimes 1 \otimes 1,\end{aligned}$$

and

$$\begin{aligned}a \cdot \eta^n(f \otimes 1 \otimes 1) &= a \cdot (f_1 \otimes x \otimes 1 - f_2 \otimes 1 \otimes x + f_3 \otimes 1 \otimes 1) \\ &= a \cdot f_1 \otimes x \otimes 1 - \sigma^{-1}(a) \cdot f_2 \otimes 1 \otimes x \\ &\quad + \delta\sigma^{-1}(a) \cdot f_2 \otimes 1 \otimes 1 + a \cdot f_3 \otimes 1 \otimes 1 \\ &= a \cdot f_1 \otimes x \otimes 1 - \sigma^{-1}(a) \cdot f_2 \otimes 1 \otimes x \\ &\quad + (\delta\sigma^{-1}(a) \cdot f_2 + a \cdot f_3) \otimes 1 \otimes 1.\end{aligned}$$

It is easy to verify that $(a \cdot f)_1 = a \cdot f_1$, $(a \cdot f)_2 = \sigma^{-1}(a) \cdot f_2$ and

$$\begin{aligned}&(a \cdot f)_3(a_1, \ldots, a_n) \\ &\quad = (\delta \otimes \mathrm{id})(a \cdot f)(\sigma^{-1}(a_1), \ldots, \sigma^{-1}(a_n)) + (\mathrm{id} \otimes \delta\sigma^{-1})(a \cdot f)(a_1, \ldots, a_n) \\ &\qquad - \sum_{j=1}^{n} (a \cdot f)(a_1, \ldots, a_{j-1}, \delta\sigma^{-1}(a_j), \sigma^{-1}(a_{j+1}), \ldots, \sigma^{-1}(a_n)) \\ &\quad = \sum \delta(f(\sigma^{-1}(a_1), \ldots, \sigma^{-1}(a_n))') \otimes a f(\sigma^{-1}(a_1), \ldots, \sigma^{-1}(a_n))'' \\ &\qquad + \sum f(a_1, \ldots, a_n)' \otimes \delta\sigma^{-1}(a f(a_1, \ldots, a_n)'') \\ &\qquad - \textstyle\sum_{j=1}^{n} (a \cdot f)(a_1, \ldots, a_{j-1}, \delta\sigma^{-1}(a_j), \sigma^{-1}(a_{j+1}), \ldots, \sigma^{-1}(a_n)) \\ &\quad = \sum \delta(f(\sigma^{-1}(a_1), \ldots, \sigma^{-1}(a_n))') \otimes a f(\sigma^{-1}(a_1), \ldots, \sigma^{-1}(a_n))'' \\ &\qquad + \sum f(a_1, \ldots, a_n)' \otimes \delta\sigma^{-1}(a)\sigma^{-1}(f(a_1, \ldots, a_n)'') \\ &\qquad + \sum f(a_1, \ldots, a_n)' \otimes a\delta\sigma^{-1}(f(a_1, \ldots, a_n)'') \\ &\qquad - \textstyle\sum_{j=1}^{n} (a \cdot f)(a_1, \ldots, a_{j-1}, \delta\sigma^{-1}(a_j), \sigma^{-1}(a_{j+1}), \ldots, \sigma^{-1}(a_n)) \\ &\quad = a \cdot (\delta \otimes \mathrm{id}) f(\sigma^{-1}(a_1), \ldots, \sigma^{-1}(a_n))\end{aligned}$$

$$
\begin{aligned}
&+\delta\sigma^{-1}(a)\cdot(\mathrm{id}\otimes\sigma^{-1})f(a_1,\dots,a_n)\\
&+a\cdot(\mathrm{id}\otimes\delta\sigma^{-1})f(a_1,\dots,a_n)\\
&-\sum_{j=1}^{n}(a\cdot f)(a_1,\dots,a_{j-1},\delta\sigma^{-1}(a_j),\sigma^{-1}(a_{j+1}),\dots,\sigma^{-1}(a_n))\\
=\ &(\delta\sigma^{-1}(a)\cdot f_2+a\cdot f_3)(a_1\dots,a_n).
\end{aligned}
$$

Thus (2.11) holds and (2.12) can be checked in a similar way. Therefore, η is constructed as desired. □

Lemma 2.6. *Suppose that A is a flat $\Bbbk$-algebra and $E = A[x;\sigma,\delta]$ is an Ore extension. Let $f \in C^n(A, A\otimes A)$ ($n\in\mathbb{N}$) and f_1, f_2, f_3 be given by* (2.8), (2.9), (2.10). *The following are equivalent:*

(1) *f is a cocycle (resp. coboundary) in $C^n(A, A\otimes A)$,*

(2) *f_1 is a cocycle (resp. coboundary) in $C^n(A, A\otimes A^{\sigma^{-1}})$,*

(3) *f_2 is a cocycle (resp. coboundary) in $C^n(A, A\otimes A^{\sigma^{-1}})$.*

If the above conditions are satisfied, f_3 is also a cocycle (resp. coboundary) in $C^n(A, A\otimes A^{\sigma^{-1}})$.

Proof. Take $l = k = 1$ in (2.7), then

$$
\begin{aligned}
\eta^{n+1}(\mathsf{b}f\otimes 1\otimes 1) &= (\mathsf{b}f)_1\otimes x\otimes 1-(\mathsf{b}f)_2\otimes 1\otimes x+(\mathsf{b}f)_3\otimes 1\otimes 1\\
&=\mathsf{b}_{\sigma^{-1}}f_1\otimes x\otimes 1-\mathsf{b}_{\sigma^{-1}}f_2\otimes 1\otimes x+\mathsf{b}_{\sigma^{-1}}f_3\otimes 1\otimes 1.
\end{aligned}
$$

It follows that

$$
(\mathsf{b}f)_1=(\sigma\otimes\mathrm{id})(\mathsf{b}f)(\sigma^{-1})^{\otimes n+1}=\mathsf{b}_{\sigma^{-1}}f_1,\ (\mathsf{b}f)_2=(\mathrm{id}\otimes\sigma^{-1})(\mathsf{b}f)=\mathsf{b}_{\sigma^{-1}}f_2
$$

and $(\mathsf{b}f)_3=\mathsf{b}_{\sigma^{-1}}f_3$.

So f is a cocycle if and only if f_1 is a cocycle, if and only if f_2 is a cocycle. If any one of f, f_1 and f_2 is a cocycle, then f_3 is also a cocycle.

If f is a coboundary, say $f=\mathsf{b}g$, then $f_1=(\sigma\otimes\mathrm{id})(\mathsf{b}g)(\sigma^{-1})^{\otimes n}=\mathsf{b}_{\sigma^{-1}}g_1$, $f_2=\mathsf{b}_{\sigma^{-1}}g_2$ and $f_3=\mathsf{b}_{\sigma^{-1}}g_3$. Thus f_1, f_2 and f_3 are all coboundaries.

If either $f_1=(\sigma\otimes\mathrm{id})f(\sigma^{-1})^{\otimes n}$ or $f_2=(\mathrm{id}\otimes\sigma^{-1})f$ is a coboundary, then f is a coboundary. □

Theorem 2.7. *Let A be a projective $\Bbbk$-algebra and $E = A[x;\sigma,\delta]$ be an Ore extension. Suppose that A admits a finitely generated projective resolution as an A^{e}-module. Then for any $n\in\mathbb{N}$,*

$$
0\to H^n(A, E\otimes E)\xrightarrow{\partial} H^n(A, E\otimes E^{\sigma^{-1}})\to H^{n+1}(E, E\otimes E)\to 0
$$

is an exact sequence of E^{e}-modules.

Proof. Since A admits a finitely generated projective resolution as an A^{e}-module, the two horizontal arrows in (2.6) are quasi-isomorphisms of E^{e}-module complexes. Thus the sequence (2.5) becomes

$$\begin{aligned}\cdots \to H^n(A, A\otimes A)\otimes \Bbbk[x]^{\otimes 2} \xrightarrow{\tilde{\partial}} H^n(A, A\otimes A^{\sigma^{-1}})\otimes \Bbbk[x]^{\otimes 2} \\ \to H^{n+1}(E, E\otimes E) \to \cdots,\end{aligned}$$

where $\tilde{\partial}$ is induced by ∂ and $\tilde{\partial} = H^n(\eta)$.

It is sufficient to show $\tilde{\partial}$ is injective.

Suppose that $\sum_{(l,k)} f^{l,k}\otimes x^l\otimes x^k$ is a cocycle in $C^n(A, A\otimes A)\otimes \Bbbk[x]^{\otimes 2}$ such that $\tilde{\partial}(\sum_{(l,k)} f^{l,k}\otimes x^l\otimes x^k + \mathrm{Im}(\mathsf{b}^{n-1}\otimes \mathrm{id}^{\otimes 2})) = 0$. Then

$$\begin{aligned}\eta^n(\sum_{(l,k)} f^{l,k}\otimes x^l\otimes x^k) &= \sum_{(l,k)} f_1^{l,k}\otimes x^{l+1}\otimes x^k - \sum_{(l,k)} f_2^{l,k}\otimes x^l\otimes x^{k+1} \\ &\quad + \sum_{(l,k)} f_3^{l,k}\otimes x^l\otimes x^k \qquad (2.13)\\ &\in \mathrm{Im}(\mathsf{b}^{n-1}_{\sigma^{-1}}\otimes \mathrm{id}^{\otimes 2}).\end{aligned}$$

Endow $\mathbb{N}^2$ with the lexicographical order from right to left, that is, $(a,b) > (c,d)$ if $b > d$ or $(b = d, a > c)$. So the set consisting of all pairs (l,k) such that $f^{l,k} \neq 0$ is a totally ordered set with respect to the order. Pick the greatest index (l_0, k_0) and observe that $f_2^{l_0,k_0}\otimes x^{l_0}\otimes x^{k_0+1}$ is the unique term in (2.13) containing $x^{l_0}\otimes x^{k_0+1}$ as its tensor factor. Therefore, $f_2^{l_0,k_0}$ is a coboundary and so is f^{l_0,k_0}. It follows that $\tilde{\partial}$ is injective. □

3. Ore extensions preserve the twisted Calabi–Yau property

In this section, we will show that the twisted Calabi–Yau property is preserved by Ore extensions. First of all, recall the short exact sequence (2.1). If A admits a finitely generated A^{e}-projective resolution of finite length, say $P.$, and

$$\psi : E\otimes_A P.\otimes_A {}^{\sigma^{-1}}E \to E\otimes_A P.\otimes_A E$$

is a morphism lifting ρ, then $\mathrm{cone}(\psi)$ is a bounded complex of finitely generated E^{e}-projective modules. Thus the following proposition is concluded immediately.

Proposition 3.1. *Let A be an algebra and $E = A[x;\sigma,\delta]$ be an Ore extension. If A is homologically smooth, then so is E.*

Next, we consider the cohomology $H^*(E, E\otimes E)$.

Proposition 3.2. *Let A be a projective $\Bbbk$-algebra and $E = A[x;\sigma,\delta]$ be an Ore extension. Suppose that*

(1) *A admits a finitely generated projective resolution as an A^{e}-module,*

(2) *$H^i(A, A \otimes A) = 0$ unless $i = d$ for some $d \in \mathbb{N}$.*

Then $H^i(E, E \otimes E) = 0$ unless $i = d + 1$.

Let ω, ω' and Ω be the cohomology groups $H^d(A, A \otimes A)$, $H^d(A, A \otimes A^{\sigma^{-1}})$ and $H^{d+1}(E, E \otimes E)$, respectively. Then $\Omega \cong \omega' \otimes \Bbbk[x]$ and the E^{e}-module structure on $\omega' \otimes \Bbbk[x]$ is given as follows, for any $a \in A$, $[\tilde{f}] \in \omega'$, $k \in \mathbb{N}$,

$$a \triangleright ([\tilde{f}] \otimes x^k) = \sum_{i=0}^{k} q_i^k(a)[\tilde{f}] \otimes x^i, \tag{3.1}$$

$$x \triangleright ([\tilde{f}] \otimes x^k) = [\tilde{f}] \otimes x^{k+1}, \tag{3.2}$$

$$([\tilde{f}] \otimes x^k) \triangleleft a = [\tilde{f}]a \otimes x^k, \tag{3.3}$$

$$([\tilde{f}] \otimes x^k) \triangleleft x = [f_2] \otimes x^{k+1} - [f_3] \otimes x^k, \tag{3.4}$$

where $f = (\sigma^{-1} \otimes \mathrm{id})\tilde{f}(\sigma^{\otimes d})$, f_2 and f_3 are given by (2.9) and (2.10).

Proof. Since $H^i(A, A \otimes A^{\sigma^{-1}}) \cong H^i(A, A \otimes {}^{\sigma}A)$, by Theorem 2.7, $H^i(E, E \otimes E) = 0$ for all $i \neq d + 1$. And as E^{e}-modules,

$$H^d(A, E \otimes E) \cong \omega \otimes \Bbbk[x]^{\otimes 2}, \quad H^d(A, E \otimes E^{\sigma^{-1}}) \cong \omega' \otimes \Bbbk[x]^{\otimes 2},$$

where the E^{e}-module structure on $\omega' \otimes \Bbbk[x]^{\otimes 2}$ is given by

$$x \cdot ([\tilde{f}] \otimes x^l \otimes x^k) = [\tilde{f}] \otimes x^l \otimes x^{k+1} \quad \text{for all } [\tilde{f}] \in \omega',\ k, l \in \mathbb{N}, \tag{3.5}$$

$$a \cdot ([\tilde{f}] \otimes x^l \otimes x^k) = \sum_{i=0}^{k} q_i^k(a)[\tilde{f}] \otimes x^l \otimes x^i \quad \text{for all } a \in A, \tag{3.6}$$

$$([\tilde{f}] \otimes x^l \otimes x^k) \cdot x = [\tilde{f}] \otimes x^{l+1} \otimes x^k, \tag{3.7}$$

$$([\tilde{f}] \otimes x^l \otimes x^k) \cdot a = \sum_{i=0}^{l} [\tilde{f}] p_i^l(a) \otimes x^i \otimes x^k, \tag{3.8}$$

and the E^{e}-module structure on $\omega \otimes \Bbbk[x]^{\otimes 2}$ is given similarly. By the proof of Theorem 2.7,

$$0 \to \omega \otimes \Bbbk[x]^{\otimes 2} \xrightarrow{\tilde{\partial}} \omega' \otimes \Bbbk[x]^{\otimes 2} \to \Omega \to 0$$

is exact.

To show $\Omega \cong \omega' \otimes \Bbbk[x]$, it suffices to show that $\omega' \otimes \Bbbk[x]$ is the cokernel of $\tilde{\partial}$.

Now, for any cocycle $\tilde{f} \in C^d(A, A \otimes A^{\sigma^{-1}})$, let $f = (\sigma^{-1} \otimes \mathrm{id})\tilde{f}(\sigma^{\otimes d})$. Then, by the definition of η, $f_1 = \tilde{f}$ and

$$\tilde{f} \otimes x^{l+1} \otimes x^k = f_2 \otimes x^l \otimes x^{k+1} - f_3 \otimes x^l \otimes x^k \pmod{\operatorname{Im} \eta^d}. \tag{3.9}$$

By Lemma 2.6, f_2, f_3 are also cocycles. If, in particular, $\tilde{f}$ is a coboundary, then so are f_2, f_3, and vice versa. It follows that for any $l, k \in \mathbb{N}$,

$$\tilde{f} \otimes x^l \otimes x^k = \sum_{j=0}^{l} g_j \otimes 1 \otimes x^{j+k} \pmod{\operatorname{Im} \eta^d} \tag{3.10}$$

for some cocycles g_j in $C^d(A, A \otimes A^{\sigma^{-1}})$, and $\tilde{f}$ is a coboundary if and only if all of the g_j's are coboundaries.

Obviously, $f = 0$ if and only if $f_1 = 0$. It follows from (2.13) that $\sum_j g_j \otimes 1 \otimes x^j \in \operatorname{Im} \eta^d$ if and only if $g_j = 0$ for all j. This implies that the cocycles g_j in (3.10) are unique. Hence there exists a bijection

$$\begin{aligned} \Phi_1 \colon (\omega' \otimes \Bbbk[x]^{\otimes 2}) \big/ \operatorname{Im} \tilde{\partial} &\to \omega' \otimes \Bbbk[x] \\ [\tilde{f}] \otimes x^l \otimes x^k + \operatorname{Im} \tilde{\partial} &\longmapsto \sum_{j=0}^{l} [g_j] \otimes x^{j+k}. \end{aligned}$$

Therefore, $\Omega \cong \omega' \otimes \Bbbk[x]$. It follows from (3.5), (3.6), (3.8) that the induced E^{e}-module structure on $\omega' \otimes \Bbbk[x]$ satisfies (3.1), (3.2), (3.3). By (3.9),

$$\Phi_1([f_1] \otimes x \otimes 1 + \operatorname{Im} \tilde{\partial}) = [f_2] \otimes x - [f_3] \otimes 1. \tag{3.11}$$

Then it follows from (3.2) and (3.7) that $([\tilde{f}] \otimes x^k) \triangleleft x = [f_2] \otimes x^{k+1} - [f_3] \otimes x^k$, i.e., (3.4) holds. $\square$

Theorem 3.3. *Let A be a projective $\Bbbk$-algebra and $E = A[x; \sigma, \delta]$ be an Ore extension. Suppose that A is ν-twisted Calabi–Yau of dimension d. Then E is twisted Calabi–Yau of dimension $d + 1$ and the Nakayama automorphism ν' of E satisfies that $\nu'|_A = \sigma^{-1}\nu$ and $\nu'(x) = ux + b$ with $u, b \in A$ and u invertible.*

Proof. We still use ω, ω' and Ω as above. As $\omega' \cong {}^{\sigma}\omega$ and $\omega \cong A^{\nu}$, we may fix a bimodule isomorphism $\varphi \colon \omega' \to {}^{\sigma}A^{\nu}$.

It follows from Proposition 3.2 that $\Omega \cong \omega' \otimes \Bbbk[x] \cong {}^{\sigma}A^{\nu} \otimes \Bbbk[x]$. The $E \otimes A^{\mathrm{op}}$-module structure on ${}^{\sigma}A^{\nu} \otimes \Bbbk[x]$ is induced from (3.1), (3.2) and (3.3). Let us prove ${}^{\sigma}A^{\nu} \otimes \Bbbk[x] \cong E^{\sigma^{-1}\nu}$ as $E \otimes A^{\mathrm{op}}$-modules.

In fact, the composition

$${}^{\sigma}A^{\nu} \otimes \Bbbk[x] \xrightarrow{\sigma^{-1} \otimes \mathrm{id}} A^{\sigma^{-1}\nu} \otimes \Bbbk[x] \xrightarrow{\tau} \Bbbk[x] \otimes A^{\sigma^{-1}\nu} \xrightarrow{\mu} E^{\sigma^{-1}\nu},$$

denoted by Φ_3, is an isomorphism of $E \otimes A^{\mathrm{op}}$-modules.

Clearly, Φ_3 is bijective. For any $a', a \in A$ and $k \in \mathbb{N}$,

$$\begin{aligned} \Phi_3((a' \otimes x^k) \triangleleft a) &= \Phi_3(a'\nu(a) \otimes x^k) \\ &= x^k \sigma^{-1}(a') \sigma^{-1}\nu(a) \\ &= \Phi_3(a' \otimes x^k) \cdot a, \end{aligned}$$

$$\begin{aligned}\Phi_3(x \triangleright (a' \otimes x^k)) &= \Phi_3(a' \otimes x^{k+1}) = x^{k+1}\sigma^{-1}(a') \\ &= x x^k \sigma^{-1}(a') \\ &= x \cdot \Phi_3(a' \otimes x^k).\end{aligned}$$

Recall the maps $q_i^k : A \to A$ in (1.2) such that $ax^k = \sum_{i=0}^k x^i q_i^k(a)$,

$$\begin{aligned}\Phi_3(a \triangleright (a' \otimes x^k)) &= \Phi_3(\sum_{i=0}^k \sigma(q_i^k(a))a' \otimes x^i) \\ &= \sum_{i=0}^k x^i q_i^k(a)\sigma^{-1}(a') = ax^k\sigma^{-1}(a') \\ &= a \cdot \Phi_3(a' \otimes x^k).\end{aligned}$$

So $\Omega \cong E^{\sigma^{-1}\nu}$ as $E \otimes A^{\mathrm{op}}$-modules. There exists an endomorphism ν' of E such that $\Omega \cong E^{\nu'}$ as E^{e}-modules and $\nu'|_A = \sigma^{-1}\nu$. In such a way, Φ_3 is indeed an isomorphism of E^{e}-modules.

Now we try to decide $\nu'(x)$. Let $\Phi_2 = \varphi \otimes \mathrm{id} : \omega' \otimes \Bbbk[x] \to {}^{\sigma}A^{\nu} \otimes \Bbbk[x]$. Since $\omega' \cong {}^{\sigma}A^{\nu}$ via φ, there exists a cocycle $\tilde{f} \in C^d(A, A \otimes A^{\sigma^{-1}})$ such that

$$\Phi_2\Phi_1([\tilde{f}] \otimes 1 \otimes 1 + \operatorname{Im}\tilde{\partial}) = 1_A \otimes 1.$$

Define $f, h \in C^d(A, A \otimes A)$ by $f = (\sigma^{-1} \otimes \mathrm{id})\tilde{f}(\sigma^{\otimes d})$ and $h = (\mathrm{id} \otimes \sigma)\tilde{f}$, respectively. Clearly, $\tilde{f} = f_1 = h_2$. Thus f and h are both cocycles. Then

$$\begin{aligned}\nu'(x) &= 1_E \cdot x \\ &= \Phi_3\Phi_2\Phi_1([\tilde{f}] \otimes 1 \otimes 1 + \operatorname{Im}\tilde{\partial}) \cdot x \\ &= \Phi_3\Phi_2\Phi_1([\tilde{f}] \otimes x \otimes 1 + \operatorname{Im}\tilde{\partial}) \quad \text{by (3.7)} \\ &= \Phi_3\Phi_2\Phi_1([f_1] \otimes x \otimes 1 + \operatorname{Im}\tilde{\partial}) \\ &= \Phi_3\Phi_2([f_2] \otimes x) - \Phi_3\Phi_2([f_3] \otimes 1) \quad \text{by (3.11)} \\ &= \Phi_3(\varphi([f_2]) \otimes x) - \Phi_3(\varphi([f_3]) \otimes 1) \\ &= x\sigma^{-1}\varphi([f_2]) - \sigma^{-1}\varphi([f_3]) \\ &= \varphi([f_2])x + \delta\sigma^{-1}\varphi([f_2]) - \sigma^{-1}\varphi([f_3]).\end{aligned}$$

Let $u = \varphi([f_2])$ and $b = \delta\sigma^{-1}\varphi([f_2]) - \sigma^{-1}\varphi([f_3])$. Then $\nu'(x) = ux + b$.

On the other hand,

$$\begin{aligned}x &= x \cdot \Phi_3\Phi_2\Phi_1([\tilde{f}] \otimes 1 \otimes 1 + \operatorname{Im}\tilde{\partial}) = \Phi_3\Phi_2\Phi_1([h_2] \otimes 1 \otimes x + \operatorname{Im}\tilde{\partial}) \\ &= \Phi_3\Phi_2\Phi_1([h_1] \otimes x \otimes 1 + \operatorname{Im}\tilde{\partial}) + \Phi_3\Phi_2\Phi_1([h_3] \otimes 1 \otimes 1 + \operatorname{Im}\tilde{\partial}) \\ &= \Phi_3\Phi_2\Phi_1([h_1] \otimes 1 \otimes 1 + \operatorname{Im}\tilde{\partial}) \cdot x + \Phi_3\Phi_2\Phi_1([h_3] \otimes 1 \otimes 1 + \operatorname{Im}\tilde{\partial}) \\ &= \sigma^{-1}\varphi([h_1]) \cdot x + \sigma^{-1}\varphi([h_3]).\end{aligned}$$

Let $v = \sigma^{-1}\varphi([h_1])$, $c = \sigma^{-1}\varphi([h_3])$. Then

$$x = v \cdot x + c = v(ux + b) + c = vux + vb + c,$$

which implies $vu = 1_A$ and $vb + c = 0$.

Since $\nu'|_A$ is an automorphism and u is left invertible, $x \in \operatorname{Im} \nu'$, namely, ν' is surjective. Suppose that $\nu'(\sum_{i=0}^{n} x^i a_i) = 0$. Then develop $\nu'(\sum_{i=0}^{n} x^i a_i) = \sum_{i=0}^{n}(ux + b)^i \sigma^{-1}\nu(a_i)$ to the form $\sum_{i=0}^{n} x^i a_i'$. It is easy to show that the leading term is $x^n \sigma^{-n}(u) \dots \sigma^{-2}(u)\sigma^{-1}(u)\sigma^{-1}\nu(a_n)$. So the coefficient is zero. Since u is left invertible and σ, ν are automorphisms, $a_n = 0$. Consequently, ν' is injective.

Finally, we prove that u is also right invertible. In fact, for any $a \in A$, $xa = \sigma(a)x + \delta(a)$. Under the action of ν',

$$\begin{aligned}(ux + b)\sigma^{-1}\nu(a) &= u(\nu(a)x + \delta\sigma^{-1}\nu(a)) + b\sigma^{-1}\nu(a)\\ &= \sigma^{-1}\nu\sigma(a)(ux + b) + \sigma^{-1}\nu\delta(a).\end{aligned}$$

Comparing the coefficients of x, we have $\sigma^{-1}\nu\sigma(a)u = u\nu(a)$ for any $a \in A$. In particular, let $a = \sigma^{-1}\nu^{-1}\sigma(v)$ and so u is also right invertible.

Therefore, by Propositions 3.1, 3.2, E is twisted Calabi–Yau of dimension $d + 1$ and the Nakayama automorphism ν' satisfies the required conditions. □

Remark 3.4. By the definition of η in (2.6), $f_1 = f_2$ if $\sigma = \mathrm{id}$, and $f_3 = 0$ if $\delta = 0$. Thus $\nu'(x) = x + b$ if $\sigma = \mathrm{id}$, and $\nu'(x) = ux$ if $\delta = 0$.

4. Applications

One motivation of studying the twisted Calabi–Yau property of Ore extensions is studying the right coideal subalgebras of the positive Borel part of a quantized enveloping algebra and computing their Nakayama automorphisms [LW14] by the first-named and the third-named authors. Such algebras can be obtained by iterated Ore extensions. In [LW14], a class of right coideal subalgebras (quantum homogeneous spaces) $C \subseteq U_q(\mathfrak{g})$ is proved to be twisted Calabi–Yau, and the Nakayama automorphisms are given explicitly in some cases.

In this section, the base ring $\Bbbk$ is assumed to be a field.

4.1. Quantum affine spaces. As stated in Section 1, multi-parameter quantum affine n-spaces $\mathcal{O}_{\mathrm{q}}(\Bbbk^n)$ can be obtained by iterated Ore extensions. Their Nakayama automorphisms can be computed by using Theorem 3.3. Of course, all the results in this section are known and can be deduced in some other way.

Let $n \geq 1$ and q be a matrix $(q_{ij})_{n\times n}$ whose entries are in $\Bbbk$ satisfying $q_{ii} = 1$ and $q_{ij}q_{ji} = 1$ for all $1 \leq i, j \leq n$. The quantum affine n-space $\mathcal{O}_{\mathrm{q}}(\Bbbk^n)$ is defined to be a $\Bbbk$-algebra generated by $x_1, \dots, x_n$ with the relations $x_j x_i = q_{ij} x_i x_j$ for all $1 \leq i, j \leq n$.

Proposition 4.1. *The quantum affine n-space $\mathcal{O}_{\mathrm{q}}(\Bbbk^n)$ is twisted Calabi–Yau of dimension n, whose Nakayama automorphism ν sends x_i to $\prod_{j=1}^{n} q_{ji} x_i$.*

Proof. If $n = 1$, $\mathcal{O}_{\mathrm{q}}(\Bbbk) = \Bbbk[x_1]$. The conclusion is true.

If $n > 1$, we assume the conclusion holds for $n-1$. Let $\mathbf{q}'$ be an $(n-1)\times(n-1)$ matrix obtained by deleting the n^{th} row and the n^{th} column of q, and $\mathbf{q}''$ by deleting the first row and the first column of q. Now consider the following two quantum $(n-1)$-spaces

$$\mathcal{O}_{\mathbf{q}'}(\Bbbk^{n-1}) = \Bbbk\langle x_1, \dots, x_{n-1} \mid x_j x_i = q_{ij} x_i x_j,\ 1 \le i, j \le n-1\rangle,$$
$$\mathcal{O}_{\mathbf{q}''}(\Bbbk^{n-1}) = \Bbbk\langle x_2, \dots, x_n \mid x_j x_i = q_{ij} x_i x_j,\ 2 \le i, j \le n\rangle.$$

Clearly, $\mathcal{O}_{\mathrm{q}}(\Bbbk^n) = \mathcal{O}_{\mathbf{q}'}(\Bbbk^{n-1})[x_n; \sigma']$ where $\sigma'(x_i) = q_{in} x_i$ for $1 \le i \le n-1$, and $\mathcal{O}_{\mathrm{q}}(\Bbbk^n) = \mathcal{O}_{\mathbf{q}''}(\Bbbk^{n-1})[x_1; \sigma'']$ where $\sigma''(x_i) = q_{i1} x_i$ for $2 \le i \le n$.

By the inductive hypothesis, $\mathcal{O}_{\mathbf{q}'}(\Bbbk^{n-1})$ and $\mathcal{O}_{\mathbf{q}''}(\Bbbk^{n-1})$ are both twisted Calabi–Yau of dimension $n-1$ and their Nakayama automorphisms ν', ν'' are given by

$$\nu'(x_i) = \prod_{j=1}^{n-1} q_{ji} x_i, \quad 1 \le i \le n-1,$$
$$\nu''(x_i) = \prod_{j=2}^{n} q_{ji} x_i, \quad 2 \le i \le n,$$

respectively.

Since the invertible elements in $\mathcal{O}_{\mathrm{q}}(\Bbbk^n)$ are those nonzero scalars in $\Bbbk$, the identity map is the only inner automorphism of $\mathcal{O}_{\mathrm{q}}(\Bbbk^n)$. By Theorem 3.3, $\mathcal{O}_{\mathrm{q}}(\Bbbk^n)$ is twisted Calabi–Yau of dimension n whose Nakayama automorphism ν satisfies

$$\nu(x_i) = \sigma'^{-1}\Big(\prod_{j=1}^{n-1} q_{ji} x_i\Big) = \prod_{j=1}^{n} q_{ji} x_i, \quad 1 \le i \le n-1,$$
$$\nu(x_i) = \sigma''^{-1}\Big(\textstyle\prod_{j=2}^{n} q_{ji} x_i\Big) = \displaystyle\prod_{j=1}^{n} q_{ji} x_i, \quad 2 \le i \le n.$$

So $\nu(x_i) = \prod_{j=1}^{n} q_{ji} x_i$ for $1 \le i \le n$.

Therefore, the proposition holds for all $n \ge 1$. □

Remark 4.2. The same method can be applied to Weyl algebras $A_n(\Bbbk)$, $n \ge 1$. As a consequence, Weyl algebra $A_n(\Bbbk)$ is Calabi–Yau of dimension $2n$ [Ber09], Theorem 6.5.

4.2. A 3-dimensional AS-regular algebra. Let A be generated by x, y, z with three relations

$$yx - xy - x^2, \quad zx - xz, \quad zy - yz - 2xz.$$

Then A is a 3-dimensional AS-regular algebra.

Let $B = \Bbbk\langle x, y\rangle/(yx - xy - x^2)$ be the Jordan plane, which is an AS-regular algebra of dimension 2. Obviously, $B = \Bbbk[x][y; \delta_1]$ with $\delta_1(x) = x^2$. It follows that B is twisted Calabi–Yau, but not Calabi–Yau, with the Nakayama automorphism given by $\nu(x) = x$ and $\nu(y) = 2x + y$.

On one hand, $A = B[z; \nu]$ is an Ore extension of Jordan plane. Then A is twisted Calabi–Yau with the Nakayama automorphism ν' such that $\nu'(x) = x$ and $\nu'(y) = y$.

On the other hand, $A = \Bbbk[x, z][y; \delta]$ where δ is given by $\delta(x) = x^2$ and $\delta(z) = -2xz$. So, $\nu'(z) = z$.

It follows that A is Calabi–Yau, which was proved by Berger and Pichereau [BP14].

4.3. A class of AS-regular algebras of dimension 5. Classifying quantum projective spaces $\mathbb{P}^n$ – noncommutative analogues of projective n-spaces – is one of the most important questions in noncommutative projective algebraic geometry. An algebraic approach to construct a quantum $\mathbb{P}^n$ is to form the noncommutative projective scheme Proj A [AZ94], where A is a noetherian connected graded AS-regular algebra of global dimension $n + 1$. So the question turns out to be the classification of AS-regular algebras.

Recently, the second-named and the third-named authors tried to classify quantum $\mathbb{P}^4$s. In [WW12], AS-regular algebras of dimension 5, generated by two generators of degree 1 with three generating relations of degree 4, are classified under some generic condition. There are nine types of such AS-regular algebras in the classification list. Among them, algebras $\boldsymbol{D}$ and $\boldsymbol{G}$ can be realized by iterated Ore extensions ([WW12], Proposition 5.7 and Theorem 5.8).

In this section, we compute the Nakayama automorphisms of these two types of algebras. Assume $\Bbbk$ is a field of characteristic zero. The algebras $\boldsymbol{D}$ and $\boldsymbol{G}$ are of the form $\Bbbk\langle x, y\rangle/(r_1, r_2, r_3)$.

For the algebra $\boldsymbol{D}$,

$$\begin{aligned}
r_1 &= x^3y + px^2yx + qxyx^2 - p(2p^2 + q)yx^3,\\
r_2 &= x^2y^2 - p(p^2 + q)yxyx - q^2y^2x^2 + (q - p^2)xy^2x + (q - p^2)yx^2y,\\
r_3 &= xy^3 + pyxy^2 + qy^2xy - p(2p^2 + q)y^3x,
\end{aligned}$$

where $p, q \in \Bbbk \setminus \{0\}$ and $2p^4 - p^2q + q^2 = 0$.

For the algebra $\boldsymbol{G}$,

$$\begin{aligned}
r_1 &= x^3y + px^2yx + qxyx^2 + syx^3,\\
r_2 &= x^2y^2 + l_2xyxy + l_3yxyx + l_4y^2x^2 + l_5xy^2x + l_5yx^2y,\\
r_3 &= xy^3 + pyxy^2 + qy^2xy + sy^3x,
\end{aligned}$$

where

$$l_2 = -\frac{s^2(qs - g)}{g(qs + g)}, \quad l_3 = s - \frac{pg(ps - q^2)}{q(qs + g)}, \quad l_4 = -\frac{g^2}{s^2}, \quad l_5 = \frac{ps^2 + qg}{qs + g},$$

with $p, q, s, g \in \Bbbk \setminus \{0\}$, $ps^3g + qsg^2 + s^5 + g^3 = 0$, $p^3s = q^3$, $ps \neq q^2$, $q^2s^2 \neq g^2$ and $s^5 + g^3 \neq 0$.

It is proved that the algebras $\boldsymbol{D}$ and $\boldsymbol{G}$ can be obtained as an iterated Ore extension by a unified process [WW12], Section 5.2. We give a sketch of the process here.

Let $A = \Bbbk[y]$ with $\deg y = 1$. Let $a, b \in \Bbbk$ satisfy $ab(a+b)(a^2+b^2)(a^3-b^3) \neq 0$.

Define $A_1 = A[z_1; \sigma_1]$ to be the graded Ore extension of A with $\deg z_1 = 3$, where

$$\sigma_1(y) = ay.$$

Define $A_2 = A_1[z_2; \sigma_2, \delta_2]$ to be the graded Ore extension of A_1 with $\deg z_2 = 2$, where

$$\begin{aligned} \sigma_2(y) &= by, & \sigma_2(z_1) &= az_1, \\ \delta_2(y) &= z_1, & \delta_2(z_1) &= 0. \end{aligned}$$

Define $A_3 = A_2[z_3; \sigma_3, \delta_3]$ to be the graded Ore extension of A_2 with $\deg z_3 = 3$, where

$$\begin{aligned} \sigma_3(y) &= a^{-1}b^3y, & \sigma_3(z_1) &= b^3z_1, & \sigma_3(z_2) &= az_2, \\ \delta_3(y) &= z_2^2, & \delta_3(z_1) &= (a-b)z_2^3, & \delta_3(z_2) &= 0. \end{aligned}$$

Define $A_4 = A_3[x; \sigma_4, \delta_4]$ to be the graded Ore extension of A_3 with $\deg x = 1$, where

$$\begin{aligned} \sigma_4(y) &= a^{-1}b^2y, & \sigma_4(z_1) &= a^{-1}b^3z_1, & \sigma_4(z_2) &= bz_2, & \sigma_4(z_3) &= az_3, \\ \delta_4(y) &= z_2, & \delta_4(z_1) &= \frac{a^3-b^3}{a(a+b)}z_2^2, & \delta_4(z_2) &= \frac{a^3-b^3}{a(a+b)}z_3, & \delta_4(z_3) &= 0. \end{aligned}$$

Let $a = p^{-3}q^2, b = -p^{-1}q$. Then $A_4 \cong \boldsymbol{D}$. Let $a = s^2g^{-1}, b = -p^{-1}q$. Then $A_4 \cong \boldsymbol{G}$. Both isomorphisms send the indeterminants x, y in A_4 to the generators x, y of $\boldsymbol{D}$ and $\boldsymbol{G}$, respectively.

Now let us compute the graded Nakayama automorphism ν of A_4.

By Theorem 3.3, $\nu(y) = \sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}(y) = ab^{-6}y$.

Observe that A_4 can be also obtained as an iterated Ore extension along the opposite direction, that is, adding z_3, z_2, z_1, y to $\Bbbk[x]$ successively. The corresponding automorphisms and derivations are determined by σ_i and δ_i $(1 \leq i \leq 4)$. We do not give their concrete expressions but only the result $\nu(x) = a^{-1}b^6x$.

Return to the algebras $\boldsymbol{D}$ and $\boldsymbol{G}$. For $\boldsymbol{D}$, $a^{-1}b^6 = p^3q^{-2}p^{-6}q^6 = p^{-3}q^4$, and the Nakayama automorphism ν is given by

$$\nu(x) = p^{-3}q^4x, \ \nu(y) = p^3q^{-4}y.$$

For $\boldsymbol{G}$, $a^{-1}b^6 = s^{-2}gp^{-6}q^6 = g$, and the Nakayama automorphism ν is given by

$$\nu(x) = gx, \quad \nu(y) = g^{-1}y.$$

Thus we have

Theorem 4.3. (1) *The algebra* $\boldsymbol{D}$ *is twisted Calabi–Yau with the Nakayama automorphism* ν *given by*

$$\nu(x) = p^{-3}q^4x,\ \nu(y) = p^3q^{-4}y.$$

And $\boldsymbol{D}$ *is Calabi–Yau if and only if that* p, q *satisfy the system of equations*

$$\begin{cases} p^3 = q^4, \\ 2p^4 - p^2q + q^2 = 0. \end{cases}$$

(2) *The algebra* $\boldsymbol{G}$ *is twisted Calabi–Yau with the Nakayama automorphism* ν *given by*

$$\nu(x) = gx,\ \ \nu(y) = g^{-1}y.$$

And $\boldsymbol{G}$ *is Calabi–Yau if and only if* $g = 1$.

Acknowledgments. The authors thank the referee for the valuable comments. This research is supported by the NSFC (key project 10731070), and STCSM (Science and Technology Committee, Shanghai Municipality, project 11XD1400500), and a training program for innovative talents of key disciplines, Fudan University.

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Received November 28, 2011; revised July 7, 2012

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