J. Noncommut. Geom. 5 (2011), 295–298
DOI 10.4171/JNCG/76
Journal of Noncommutative Geometry

Appendix to V. Mathai and J. Rosenberg's paper "A noncommutative sigma-model"

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Abstract. Two problems posed in the paper [6] by V. Mathai and J. Rosenberg are resolved.

Mathematics Subject Classification (2010). Primary 58B34; Secondary 46L87.
Keywords. Noncommutative tori, energy.

This short note is an appendix to [6].

Let $\theta \in \mathbb{R}$. Denote by A_θ the rotation C*-algebra generated by unitaries U and V subject to $UV = e^{2\pi i\theta}VU$, and by A_θ^∞ its canonical smooth subalgebra. Denote by Tr the canonical faithful tracial state on A_θ determined by $\mathrm{Tr}(U^mV^n) = \delta_{m,0}\delta_{n,0}$ for all $m, n \in \mathbb{Z}$. Denote by δ_1 and δ_2 the unbounded closed $*$-derivations of A_θ defined on some dense subalgebras of A_θ and determined by $\delta_1(U) = 2\pi i U$, $\delta_1(V) = 0$, and $\delta_2(U) = 0$, $\delta_2(V) = 2\pi i V$. The *energy* [9], $E(u)$, of a unitary u in A_θ is defined as

$$E(u) = \frac{1}{2}\mathrm{Tr}(\delta_1(u)^*\delta_1(u) + \delta_2(u)^*\delta_2(u)) \tag{1}$$

when u belongs to the domains of δ_1 and δ_2, and ∞ otherwise.

Rosenberg has the following conjecture [9], Conjecture 5.4, p. 108.

Conjecture 1. *For any $m, n \in \mathbb{Z}$, in the connected component of U^mV^n in the unitary group of A_θ^∞, the functional E takes its minimal value exactly at the scalar multiples of U^mV^n.*

For a $*$-endomorphism φ of A_θ^∞, its *energy* [6], $\mathcal{L}(\varphi)$, is defined as $2E(\varphi(U)) + 2E(\varphi(V))$. Mathai and Rosenberg's Conjecture 3.1 in [6] about the minimal value of $\mathcal{L}(\varphi)$ follows directly from Conjecture 1.

Denote by H the Hilbert space associated to the GNS representation of A_θ for Tr, and denote by $\|\cdot\|_2$ its norm. We shall identify A_θ as a subspace of H as usual. Then (1) can be rewritten as

$$E(u) = \frac{1}{2}(\|\delta_1(u)\|_2^2 + \|\delta_2(u)\|_2^2).$$

*Partially supported by NSF Grant DMS-0701414.

Now we prove Conjecture 1, and hence also prove Conjecture 3.1 of [6].

Theorem 2. *Let $\theta \in \mathbb{R}$ and $m, n \in \mathbb{Z}$. Let $u \in A_\theta$ be a unitary whose class in $K_1(A_\theta)$ is the same as that of $U^m V^n$. Then $E(u) \geq E(U^m V^n)$, and "$=$" holds if and only if u is a scalar multiple of $U^m V^n$.*

Proof. We may assume that u belongs to the domains of δ_1 and δ_2. Set $a_j = u^* \delta_j(u)$ for $j = 1, 2$. For any closed $*$-derivation δ defined on a dense subset of a unital C*-algebra A and any tracial state τ of A vanishing on the range of δ, if unitaries v_1 and v_2 in the domain of δ have the same class in $K_1(A)$, then $\tau(v_1^* \delta(v_1)) = \tau(v_2^* \delta(v_2))$ [7], p. 281. Thus

$$\operatorname{Tr}(a_j) = \operatorname{Tr}((U^m V^n)^* \delta_j(U^m V^n)) = \begin{cases} 2\pi i m & \text{if } j = 1, \\ 2\pi i n & \text{if } j = 2. \end{cases}$$

We have

$$\begin{aligned} \|\delta_j(u)\|_2^2 &= \|a_j\|_2^2 \\ &= \|\operatorname{Tr}(a_j)\|_2^2 + \|a_j - \operatorname{Tr}(a_j)\|_2^2 \\ &\geq \|\operatorname{Tr}(a_j)\|_2^2 \\ &= |\operatorname{Tr}(a_j)|^2 = \begin{cases} 4\pi^2 m^2 & \text{if } j = 1, \\ 4\pi^2 n^2 & \text{if } j = 2, \end{cases} \end{aligned}$$

and "$=$" holds if and only if $a_j = \operatorname{Tr}(a_j)$. It follows that $E(u) \geq 2\pi^2(m^2 + n^2)$, and "$=$" holds if and only if $\delta_1(u) = 2\pi i m u$ and $\delta_2(u) = 2\pi i n u$. Now the theorem follows from the fact that the elements a in A_θ satisfying $\delta_1(a) = 2\pi i m a$ and $\delta_2(a) = 2\pi i n a$ are exactly the scalar multiples of $U^m V^n$. □

When $\theta \in \mathbb{R}$ is irrational, the C*-algebra A_θ is simple [10], Theorem 3.7, has real rank zero [1], Theorem 1.5, and is an $A\mathbb{T}$-algebra [5], Theorem 4. It is a result of Elliott that for any pair of $A\mathbb{T}$-algebras with real rank zero, every homomorphism between their graded K-groups preserving the graded dimension range is induced by a $*$-homomorphism between them [4], Theorem 7.3. The graded dimension range of a unital simple $A\mathbb{T}$-algebra A is the subset $\{(g_0, g_1) \in K_0(A) \oplus K_1(A) : 0 \lneqq g_0 \leq [1_A]_0\} \cup (0, 0)$ of the graded K-group $K_0(A) \oplus K_1(A)$ [8], p. 51. It follows that, when θ is irrational, for any group endomorphism ψ of $K_1(A_\theta)$, there is a unital $*$-endomorphism φ of A_θ inducing ψ on $K_1(A_\theta)$. It is an open question when one can choose φ to be smooth in the sense of preserving A_θ^∞, though it was shown in [2], [3] that if θ is irrational and φ restricts to a $*$-automorphism of A_θ^∞, then ψ must be an automorphism of the rank-two free abelian group $K_1(A_\theta)$ with determinant 1. When ψ is the zero endomorphism, from Theorem 2 one might guess that $\mathcal{L}(\varphi)$ could be arbitrarily small. It is somehow surprising, as we show now, that in fact there is a common positive lower bound for $\mathcal{L}(\varphi)$ for all $0 < \theta < 1$. This answers a question Rosenberg raised at the Noncommutative Geometry workshop at Oberwolfach in September 2009.

Theorem 3. *Suppose that* $0 < \theta < 1$. *For any unital* $*$*-endomorphism* φ *of* A_θ, *one has* $\mathcal{L}(\varphi) \geq 4(3-\sqrt{5})\pi^2$.

Theorem 3 is a direct consequence of the following lemma.

Lemma 4. *Let* $\theta \in \mathbb{R}$ *and let* u, v *be unitaries in* A_θ *with* $uv = \lambda vu$ *for some* $\lambda \in \mathbb{C} \setminus \{1\}$. *Then* $E(u) + E(v) \geq 2(3-\sqrt{5})\pi^2$.

Proof. We have

$$\operatorname{Tr}(uv) = \operatorname{Tr}(\lambda vu) = \lambda \operatorname{Tr}(uv),$$

and hence $\operatorname{Tr}(uv) = 0$. Thus

$$\begin{aligned} -\operatorname{Tr}(u)\operatorname{Tr}(v) &= \operatorname{Tr}(uv - \operatorname{Tr}(u)\operatorname{Tr}(v)) \\ &= \operatorname{Tr}((u-\operatorname{Tr}(u))v) + \operatorname{Tr}(\operatorname{Tr}(u)(v-\operatorname{Tr}(v))) \\ &= \operatorname{Tr}((u-\operatorname{Tr}(u))v). \end{aligned}$$

We may assume that both u and v belong to the domains of δ_1 and δ_2. For any $m, n \in \mathbb{Z}$, denote by $a_{m,n}$ the Fourier coefficient $\langle u, U^m V^n \rangle$ of u. Then $a_{0,0} = \operatorname{Tr}(u)$ and

$$\begin{aligned} (2\pi)^2 \|u - \operatorname{Tr}(u)\|_2^2 &= \sum_{\substack{m,n\in\mathbb{Z},\\ m^2+n^2>0}} |2\pi a_{m,n}|^2 \\ &\leq \sum_{\substack{m,n\in\mathbb{Z},\\ m^2+n^2>0}} |2\pi a_{m,n}|^2 (m^2+n^2) \\ &= \|\delta_1(u)\|_2^2 + \|\delta_2(u)\|_2^2 = 2E(u). \end{aligned}$$

Thus

$$|\operatorname{Tr}(u)|^2 = \|\operatorname{Tr}(u)\|_2^2 = \|u\|_2^2 - \|u - \operatorname{Tr}(u)\|_2^2 \geq 1 - \frac{1}{2\pi^2} E(u)$$

and

$$|\operatorname{Tr}((u - \operatorname{Tr}(u))v)| \leq \|(u - \operatorname{Tr}(u))v\|_2 = \|u - \operatorname{Tr}(u)\|_2 \leq \left(\frac{1}{2\pi^2} E(u)\right)^{1/2}.$$

Similarly, $|\operatorname{Tr}(v)|^2 \geq 1 - \frac{1}{2\pi^2} E(v)$.

Write $\frac{1}{2\pi^2} E(u)$ and $\frac{1}{2\pi^2} E(v)$ as t and s, respectively. We just need to show that $t + s \geq 3 - \sqrt{5}$. If $t \geq 1$ or $s \geq 1$, then this is trivial. Thus we may assume that $1-t, 1-s > 0$. Then

$$(1-t)(1-s) \leq |\operatorname{Tr}(u)\operatorname{Tr}(v)|^2 \leq t.$$

Equivalently, $t(1-s) \geq 1-(t+s)$. Without of loss generality, we may assume $s \geq t$. Write $t+s$ as w. Then

$$t(1-w/2) \geq t(1-s) \geq 1-(t+s) = 1-w,$$

and hence

$$w = t+s \geq \frac{1-w}{1-w/2} + \frac{w}{2}.$$

It follows that $w^2 - 6w + 4 \leq 0$. Thus $w \geq 3-\sqrt{5}$. □

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Received October 5, 2009

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