

Asymptotic analysis for Korteweg models

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This paper deals with a sharp interface limit of the isothermal Navier–Stokes–Korteweg system. The sharp interface limit is performed by matched asymptotic expansions of the fields in powers of the interface width ε . These expansions are considered in the interfacial region (inner expansions) and in the bulk (outer expansion) and are matched order by order. Particularly we consider the first orders of the corresponding inner equations obtained by a change of coordinates in an interfacial layer. For a specific scaling we establish solvability criteria for these inner equations and recover the results within the general setting of jump conditions for sharp interface models.

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1. Introduction

Phase transitions in single substance flows are usually described by two kinds of models: the sharp interface model and the diffuse phase field model. The conventional and physically more intuitive approach is the sharp interface model. In the sharp interface approach, interfaces separating the coexisting phases or structural domains are modeled as hypersurfaces at which certain quantities such as the density or the pressure suffer jump conditions. Local quantities at the interface are then determined from the boundary conditions or are calculated from the driving force for interfacial motion. The structure of possible interfacial conditions is quite well-founded from the view point of thermodynamics. However from the numerical point of view the sharp interface approach involves the explicit tracking of the interface and becomes often numerically impractical for complicated microstructures. Because of these disadvantages, the phase field approach has emerged as a powerful method during the last twenty years. A phase field model represents a microstructure, both the compositional domains and the interfaces, as a whole. The interface between different phases is described by a small transition region, where an order parameter, representing the phases, changes

its state smoothly. The microstructural evolution is modeled by a system of partial differential equations. The phase field model contains the corresponding sharp interface description as a particular limit, i.e., if the interfacial thickness tends to zero. An overview about a large class of phase field models and their sharp interface limits can be found in [2]. For phase field models it is of overall interest to validate them, by investigating the sharp interface limit in appropriate scaling regimes.

In this work we consider a sharp interface limit of the isothermal Navier–Stokes–Korteweg phase field model for a particular scaling. The sharp interface limit is performed by matched asymptotic expansions of the fields of these models in powers of the interface width ϵ . These expansions are considered in the interfacial region (inner expansion) and in the bulk (outer expansion), and are matched order by order. This results in partial differential equations for the diffuse field and a series of boundary conditions at the interface from which we achieve jump conditions in the sharp interface limit.

We consider the case where the Mach number is of order $\mathcal{O}(1)$ and viscosity and capillarity are both of order $\mathcal{O}(\epsilon^2)$. Our particular scaling of the viscosity leads to a no-entropy-dissipation kinetic relation in the leading order $\mathcal{O}(1)$. The resulting sharp interface model is the isothermal Euler model. In the first order $\mathcal{O}(\epsilon)$ we obtain a Young–Laplace law and a non-zero entropy dissipation, which is determined by the viscosity. The fact that the surface tension is of order $\mathcal{O}(\epsilon)$ is in agreement with the results obtained in [9, 20, 21] for equilibria. A different kinetic relation was considered in [5], there surface tension and entropy dissipation are of order $\mathcal{O}(1)$ and $\mathcal{O}(\epsilon)$ respectively. There are also other physically meaningful scalings, for instance the case where the viscosity is of order $\mathcal{O}(\epsilon)$, which implies entropy dissipation already in the leading order [3, 16, 23]. Another possibility is to look at a low Mach number limit linked to the sharp interface limit, which can be found in [15], and in [17, 18, 22] for the one phase case.

The paper is organized as follows: We start with describing the Navier–Stokes–Korteweg phase field model in Section 2. The corresponding sharp interface setting is introduced in Section 3. By non-dimensionalization we introduce the smallness parameter ϵ with physical interpretation, see Section 4. In Section 5 we introduce the asymptotic expansions and provide formulas for the geometric properties of the interface in two dimensions. Then we state our main results in Section 6, which are proven in Sections 7 and 8 respectively. In Section 9 we investigate the kinetic relation comprised in the jump conditions derived in Section 6. We show that the state on one side of the phase boundary already determines the interfacial velocity and the state on the other side of the phase boundary. Furthermore this property does not follow from the jump conditions for mass and momentum alone.

2. The Navier–Stokes–Korteweg phase field model

We consider an isothermal, compressible fluid with density $\rho \in (0, b)$ and velocity $\mathbf{u} \in \mathbb{R}^d$, that is capable to undergo phase transitions. We assume that the stress tensor can be additively decomposed according to

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_{NS} + \boldsymbol{\sigma}_K,$$

where $\boldsymbol{\sigma}_{NS}$ denotes the classical Navier–Stokes stress and $\boldsymbol{\sigma}_K$ is called Korteweg stress that takes care of possible phase transitions. The Navier–Stokes stress has the form

$$\boldsymbol{\sigma}_{NS} := \lambda(\operatorname{div}\mathbf{u})I + \mu(\nabla\mathbf{u} + (\nabla\mathbf{u})^T), \quad (2.1)$$

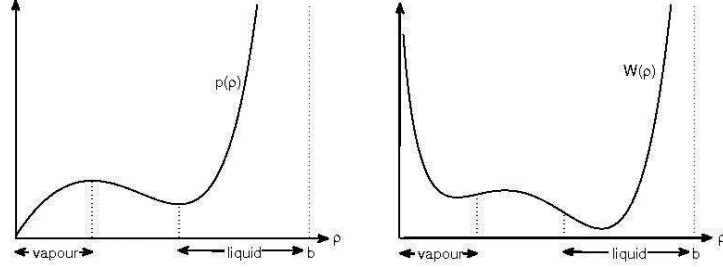


FIG. 1. Van-der-Waals pressure and free energy density function

where λ is the bulk viscosity and μ is the shear viscosity, which satisfy $\mu > 0$, $\lambda + \frac{2\mu}{d} \geq 0$, where d is the space dimension. The Korteweg tensor is given by

$$\sigma_K := \left(\rho\gamma\Delta\rho + \frac{1}{2}\gamma|\nabla\rho|^2 \right) I - (\gamma\nabla\rho \otimes \nabla\rho), \quad (2.2)$$

where γ is a constant which models capillarity effects. The evolution of the model is described by the isothermal Navier–Stokes–Korteweg equations

$$\rho_t + \operatorname{div}(\rho\mathbf{u}) = 0, \quad \text{in } \mathbb{R}^d \times (0, T). \quad (2.3)$$

$$(\rho\mathbf{u})_t + \operatorname{div}(\rho\mathbf{u} \otimes \mathbf{u}) + \nabla p(\rho) = \operatorname{div}\sigma_{NS} + \operatorname{div}\sigma_K, \quad (2.4)$$

We assume that the local part of the pressure $p = p(\rho)$ is a non-monotone function of the density given by a van-der-Waals law. From the density the phase can be directly derived, see Figure 1. The corresponding Helmholtz free energy density function $W(\rho)$ is related to the pressure by

$$p(\rho) = \rho W'(\rho) - W(\rho) \quad (2.5)$$

and has the following properties:

- $W \in C^2((0, b), [0, \infty))$,
- $\exists a_1, a_2 \in (0, b) : W'' > 0$ in $(0, a_1) \cup (a_2, b)$, $W'' < 0$ in (a_1, a_2) ,
- $\lim_{\rho \rightarrow b} W(\rho) = \infty$.

We will denote the Gibbs free energy by g , which is given by $g(\rho) = W(\rho)$.

Smooth solutions of (2.3), (2.4) identically satisfy the entropy inequality, which is given by

$$\begin{aligned} & \left(W(\rho) + \frac{\rho}{2}|\mathbf{u}|^2 + \frac{\gamma}{2}|\nabla\rho|^2 \right)_t + \operatorname{div} \left(\left(W(\rho) + \frac{\rho}{2}|\mathbf{u}|^2 + \frac{\gamma}{2}|\nabla\rho|^2 \right) \mathbf{u} \right) \\ & \quad + \operatorname{div}((p(\rho)I - \sigma_{NS} - \sigma_K)\mathbf{u} + \gamma\rho\nabla\rho(\nabla \cdot \mathbf{u})) \\ & \quad = -\sigma_{NS} : (\nabla\mathbf{u}) \leq 0. \end{aligned} \quad (2.6)$$

An alternative form of the inequality is derived in [1, 9]. The existence of classical solutions of (2.3),(2.4) was studied in [14, 19], while weak solutions were investigated in [6, 8, 11].

3. The Euler–Korteweg sharp interface model

Now we describe the phase transition by means of a sharp interface model. In this paper we treat the case where the capillarity γ and the viscosities λ and μ are of the same small order. The corresponding sharp interface model is described by the Euler–Korteweg system. In this context the conservation laws of mass and momentum read in the bulk phases $\Omega^-(t) := \{\mathbf{x} \in \mathbb{R}^d : \rho(t, \mathbf{x}) \in (0, a_1)\}$ and $\Omega^+(t) := \{\mathbf{x} \in \mathbb{R}^d : \rho(t, \mathbf{x}) \in (a_2, b)\}$, $t \in [0, T)$:

$$\rho_t + \operatorname{div}(\rho \mathbf{u}) = 0, \quad (3.1)$$

$$(\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = 0. \quad (3.2)$$

The interface between two adjacent phases Ω^\pm is described by a moving hypersurface $\Gamma(t)$, i.e., a *sharp interface* of zero thickness. Across $\Gamma(t)$ the bulk quantities may have discontinuities. We are particularly interested in the case that the interface itself is equipped with mass, momentum, energy and entropy. The jump conditions for these quantities rely on the conservation laws for mass, momentum and energy and on the entropy inequality across the interface Γ . To state these jump conditions we will use the following notation for some quantity ψ having a jump at the interface. By ψ^\pm we denote the limit at the interface from Ω^\pm and we abbreviate

$$\llbracket \psi \rrbracket := \psi^+ - \psi^-, \quad \{\psi\} := \frac{\psi^+ + \psi^-}{2}.$$

For more details of the following equations we refer to [10].

The general forms of conservation of mass and momentum read:

$$\llbracket \rho(u_\nu - w_\nu) \rrbracket = -\frac{\partial \rho_\Gamma}{\partial t} - (\operatorname{div}_\Gamma(w_t) - \kappa w_\nu) \rho_\Gamma, \quad (3.3)$$

$$\llbracket \rho(u_\nu - w_\nu)(\mathbf{u} - \mathbf{w}) + p \boldsymbol{\nu} \rrbracket = -\rho_\Gamma \frac{\partial \mathbf{w}}{\partial t} + \operatorname{div}_\Gamma(\boldsymbol{\sigma}_\Gamma). \quad (3.4)$$

The newly introduced quantities are the normal component of the fluid velocity u_ν , the interfacial mass density ρ_Γ , the interfacial velocity \mathbf{w} which may be decomposed into normal- and tangential speed, i.e., $\mathbf{w} = w_t \mathbf{t} + w_\nu \boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\Gamma \in \mathbb{R}^{d \times (d-1)}$ is the surface stress tensor. Each surface point is equipped with a tangential matrix \mathbf{t} and a normal vector $\boldsymbol{\nu}$ and κ denotes the sum of the principal curvatures.

We consider exclusively isothermal processes at the temperature T_0 , and these are guaranteed by corresponding heat fluxes \mathbf{q} , \mathbf{q}_Γ in the bulk and on the interface. The conservation of energy serves to eliminate the heat fluxes in the entropy inequality, which hereafter becomes the relevant inequality for the isothermal Euler–Korteweg model.

The general form of conservation of energy across the interfaces reads:

$$\begin{aligned} \left[\left[\rho(u_\nu - w_\nu) \left(e + \frac{1}{2} |\mathbf{u}|^2 \right) + \mathbf{q} \cdot \boldsymbol{\nu} + u_\nu p \right] \right] = \\ - \frac{\partial \rho_\Gamma \left(e_\Gamma + \frac{1}{2} |\mathbf{w}|^2 \right)}{\partial t} - (\operatorname{div}_\Gamma(w_t) - \kappa w_\nu) \rho_\Gamma \left(e_\Gamma + \frac{1}{2} |\mathbf{w}|^2 \right) - \operatorname{div}_\Gamma(\mathbf{q}_\Gamma - \mathbf{w}^T \boldsymbol{\sigma}_\Gamma). \end{aligned} \quad (3.5)$$

The internal energy densities of bulk and interface are denoted by ρe and $\rho_\Gamma e_\Gamma$, respectively.

Finally we give the entropy inequality across the interface. In [1, 9] it is shown that the sharp interface version of the Navier–Stokes–Korteweg phase field model has entropy fluxes in the bulk and at the interface that are given by \mathbf{q}/T_0 and \mathbf{q}_Γ/T_0 , respectively. In this case the entropy inequality across the interface reads

$$\left[\left[\rho(u_v - w_v)s + \frac{\mathbf{q} \cdot \mathbf{v}}{T_0} \right] \right] \geq -\frac{\partial \rho_\Gamma s_\Gamma}{\partial t} - (\operatorname{div}_\Gamma(w_t) - \kappa w_v)\rho_\Gamma s_\Gamma - \operatorname{div}_\Gamma\left(\frac{\mathbf{q}_\Gamma}{T_0}\right), \quad (3.6)$$

where ρs is the entropy density of the bulk, and the entropy density of the interface is denoted by $\rho_\Gamma s_\Gamma$.

Next we multiply the momentum balance (3.4) by \mathbf{w} and subtract it from the energy law (3.5). The result is simplified by means of the jump condition for the mass, leading to the general form of the jump condition for the internal energy

$$\left[\left[\rho(u_v - w_v) \left(e + \frac{p}{\rho} + \frac{1}{2}|\mathbf{u} - \mathbf{w}|^2 \right) + \mathbf{q} \cdot \mathbf{v} \right] \right] = -\frac{\partial \rho_\Gamma e_\Gamma}{\partial t} - (\operatorname{div}_\Gamma(w_t) - \kappa w_v)\rho_\Gamma e_\Gamma - \operatorname{div}_\Gamma(\mathbf{q}_\Gamma) + \nabla_\Gamma(\mathbf{w}) : \boldsymbol{\sigma}_\Gamma^T. \quad (3.7)$$

We are interested here in a special case where the constitutive law for the surface stress vector $\boldsymbol{\sigma}_\Gamma$ has a simple structure:

$$\boldsymbol{\sigma}_\Gamma = \gamma_\Gamma \frac{\mathbf{t}}{|\mathbf{t}|^2}. \quad (3.8)$$

The quantity γ_Γ is called surface tension. In this case the jump condition for the internal energy assumes the special form

$$\left[\left[\rho(u_v - w_v) \left(e + \frac{p}{\rho} + \frac{1}{2}|\mathbf{u} - \mathbf{w}|^2 \right) + \mathbf{q} \cdot \mathbf{v} \right] \right] = -\frac{\partial \rho_\Gamma e_\Gamma}{\partial t} - (\rho_\Gamma e_\Gamma - \gamma_\Gamma)(\operatorname{div}_\Gamma(w_t) - \kappa w_v). \quad (3.9)$$

Finally we multiply the entropy inequality by T_0 and subtract it from the jump condition for the internal energy. There follows the interfacial inequality for the Helmholtz free energy $W_\Gamma = \rho_\Gamma e_\Gamma - T_0 \rho_\Gamma s_\Gamma$:

$$\begin{aligned} & \frac{\partial W_\Gamma}{\partial t} - (\gamma_\Gamma - W_\Gamma)(\operatorname{div}_\Gamma(w_t) - \kappa w_v) \\ & + \{\rho(u_v - w_v)\} \left[\left[W'(\rho) + \frac{1}{2}|\mathbf{u} - \mathbf{w}|^2 \right] \right] + \llbracket \rho(u_v - w_v) \rrbracket \left\{ W'(\rho) + \frac{1}{2}|\mathbf{u} - \mathbf{w}|^2 \right\} \leq 0. \end{aligned} \quad (3.10)$$

Among the objectives of this study is to deduce the Euler–Korteweg sharp interface model from the Navier–Stokes–Korteweg phase field model. Note that the sharp interface model must fit into the setting given here, i.e., we have to recover (3.3), (3.4) and (3.10).

4. Non-dimensionalization

We introduce the following reference quantities $x_r, t_r, u_r, \rho_r, p_r, \mu_r$ and γ_r such that

$$\mathbf{x} = x_r \mathbf{x}^*, \quad t = t_r t^*, \quad \mathbf{u} = u_r \mathbf{u}^*, \quad \rho = \rho_r \rho^*, \quad p = p_r p^*, \quad \lambda = \lambda^* \mu_r, \quad \mu = \mu^* \mu_r, \quad \gamma = \gamma_r \gamma^*.$$

Then we may rewrite the Navier–Stokes–Korteweg equations (2.3), (2.4) as

$$\begin{aligned} \frac{x_r}{u_r t_r} \rho_{t^*}^* + \operatorname{div}^*(\rho^* \mathbf{u}^*) &= 0, \\ \frac{t_r u_r}{x_r} (\rho^* \mathbf{u}^*)_{t^*} + \frac{t_r^2 u_r^2}{x_r^2} \operatorname{div}^*(\rho^* \mathbf{u}^* \otimes \mathbf{u}^*) + \frac{t_r^2 u_r^2}{x_r^2 M^2} \nabla^* p^* &= \frac{t_r^2 u_r^2}{x_r^2 \operatorname{Re}} \operatorname{div}^* \boldsymbol{\sigma}_{NS}^* + C \gamma^* \operatorname{div}^* \boldsymbol{\sigma}_K^*, \end{aligned}$$

where M denotes the Mach number $M = u_r \sqrt{\frac{\rho_r}{p_r}}$, $\operatorname{Re} = \frac{\rho_r}{\mu_r} u_r x_r$ is the Reynolds number and $C = \frac{t_r^2 \rho_r \gamma_r}{x_r^4}$ the capillarity number.

There are several possibilities to non-dimensionalize the NSK-system. In the sequel we choose $u_r = \frac{x_r}{t_r}$ and consider for a small parameter $\varepsilon > 0$ the following scaling:

$$M = 1, \quad \frac{1}{\operatorname{Re}} = \varepsilon^2, \quad C = \varepsilon^2. \quad (4.1)$$

This leads to

$$\rho_{t^*}^* + \operatorname{div}^*(\rho^* \mathbf{u}^*) = 0, \quad (4.2)$$

$$(\rho^* \mathbf{u}^*)_{t^*} + \operatorname{div}^*(\rho^* \mathbf{u}^* \otimes \mathbf{u}^*) + \nabla^* p^* = \varepsilon^2 \operatorname{div}^* \boldsymbol{\sigma}_{NS}^* + \varepsilon^2 \gamma^* \operatorname{div}^* \boldsymbol{\sigma}_K^*. \quad (4.3)$$

We will call this scaling the capillarity regime. We remark that (4.2),(4.3) is equivalent to

$$\rho_{t^*}^* + \operatorname{div}^*(\rho^* \mathbf{u}^*) = 0, \quad (4.4)$$

$$\mathbf{u}_{t^*}^* + (\mathbf{u}^* \cdot \nabla^*) \mathbf{u}^* + \nabla^* g(\rho^*) = \varepsilon^2 \frac{1}{\rho^*} \operatorname{div}^* \boldsymbol{\sigma}_{NS}^* + \varepsilon^2 \gamma^* \nabla^* \Delta^* \rho^*. \quad (4.5)$$

There are further choices that lead to physically meaningful limits. More details can be found in the introduction.

For simplification we omit the symbol $*$ in the forthcoming considerations. Letting the parameter $\varepsilon \rightarrow 0$ leads to jump conditions of the corresponding sharp interface model, which we will deduce in Section 7. Before that, we need some prerequisites for the asymptotic analysis.

5. Inner and outer expansions and matching conditions

We consider the two dimensional case $d = 2$. The position of the phase boundary in the sharp interface limit is described by a function $\mathbf{r}(t, s)$, where s is some coordinate parameterizing the interface. We can calculate normal- and tangent vectors as well as the velocity of the interface from \mathbf{r} . We want to mention that there is some freedom in the choice of a parameterization, especially we can prescribe any tangential velocity. We will later on see that the tangential velocity is continuous across the interface. Therefore we may choose a particular parameterization such that several terms drop out, cf. Remark 7.2.

The tangent vector pointing in counterclockwise direction is given by

$$\mathbf{t}(t, s) = \left(\frac{\partial r^1}{\partial s}(t, s), \frac{\partial r^2}{\partial s}(t, s) \right)^T,$$

where r^1 and r^2 denote the components of the vector \mathbf{r} in Cartesian coordinates. The inner unit normal to the interface is given by

$$\mathbf{v}(t, s) = \frac{1}{|\mathbf{t}(t, s)|} \left(-\frac{\partial r^2}{\partial s}(t, s), \frac{\partial r^1}{\partial s}(t, s) \right)^T.$$

The mean curvature, which in two space dimensions coincides with the sum of the principal curvatures, is defined by

$$\kappa := \frac{r_s^1 r_{ss}^2 - r_{ss}^1 r_s^2}{((r_s^1)^2 + (r_s^2)^2)^{\frac{3}{2}}}. \quad (5.1)$$

Next we define the interface velocity and its decomposition into tangential and normal components by

$$\mathbf{w} := \frac{\partial \mathbf{r}}{\partial t} = w_t \mathbf{t} + w_v \mathbf{v}. \quad (5.2)$$

We observe that

$$\left(\frac{t^j}{|\mathbf{t}|} \right)_s = \kappa |\mathbf{t}| v^j \quad \text{and} \quad (v^j)_s = -t^j \kappa, \quad (j = 1, 2) \quad (5.3)$$

and

$$(v^j)_\xi v^j = 0, \quad \left(\frac{t^j}{|\mathbf{t}|} \right)_\xi \frac{t^j}{|\mathbf{t}|} = 0, \quad \left(\frac{t^j}{|\mathbf{t}|} \right)_\xi v^j = -\frac{t^j}{|\mathbf{t}|} (v^j)_\xi, \quad (5.4)$$

where the index ξ in the expressions denotes the time or tangential derivative. Note that in equation (5.4) and all subsequent calculations we sum over all indices occurring twice.

For some generic tangent vector ψ the surface divergence in one dimension is defined by

$$\operatorname{div}_\Gamma(\psi) = \frac{1}{|\mathbf{t}|} \frac{\partial(|\mathbf{t}|\psi)}{\partial s}. \quad (5.5)$$

5.1 Outer setting

We assume the existence of expansions in ε for the density and velocity in the bulk phases:

$$\mathbf{u}(t, x^1, x^2; \varepsilon) \sim \sum_{i=0}^{\infty} \varepsilon^i \mathbf{u}_i(t, x^1, x^2), \quad \rho(t, x^1, x^2; \varepsilon) \sim \sum_{i=0}^{\infty} \varepsilon^i \rho_i(t, x^1, x^2), \quad (5.6)$$

where the \sim implicitly means that the infinite sum converges uniformly. Inserting these expansions into the NSK equations leads to the following equations in the first two orders: In the $\mathcal{O}(\varepsilon^0)$ order:

$$\rho_{0,t} + \operatorname{div}(\rho_0 \mathbf{u}_0) = 0, \quad (5.7)$$

$$(\mathbf{u}_0)_t + (\mathbf{u}_0 \cdot \nabla) \mathbf{u}_0 + \nabla g(\rho_0) = 0. \quad (5.8)$$

In the $\mathcal{O}(\varepsilon^1)$ order:

$$\rho_{1,t} + \operatorname{div}(\rho_1 \mathbf{u}_0 + \rho_0 \mathbf{u}_1) = 0, \quad (5.9)$$

$$(\mathbf{u}_1)_t + (\mathbf{u}_0 \cdot \nabla) \mathbf{u}_1 + (\mathbf{u}_1 \cdot \nabla) \mathbf{u}_0 + \nabla(g'(\rho_0)\rho_1) = 0. \quad (5.10)$$

The conditions at the interface are obtained via matching.

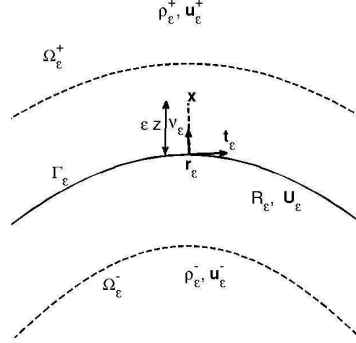


FIG. 2. Boundary layer region

5.2 Inner setting

Throughout the rest of this paper we will assume that the interface Γ_ε defined as

$$\Gamma_\varepsilon(t) = \{(x^1, x^2) \in \mathbb{R}^2 : \rho_\varepsilon(t, x^1, x^2) = \rho_*\} \quad (5.11)$$

is a $C^1([0, T], C^2(\Omega))$ -hypersurface. We like to mention that this has not to be the case in general for all times t . The value ρ_* is some point in the elliptic region, i.e., $p'(\rho_*) < 0$. The interface Γ_ε is explicitly described by \mathbf{r}_ε , the inner unit normal \mathbf{v}_ε and the tangential vector \mathbf{t}_ε respectively. A point in the neighborhood of the interface is represented by

$$\begin{pmatrix} x^1 \\ x^2 \end{pmatrix}(\tau, s, z) = \mathbf{r}_\varepsilon(\tau, s) + \varepsilon z \mathbf{v}_\varepsilon(\tau, s), \quad (5.12)$$

where z denotes the distance from the interface in normal direction. The reader may note that we could have used t instead of τ in (5.12), but it will become clear in the subsequent analysis why we want to rename the time variable in inner coordinates. The small parameter ε is introduced to zoom in the interfacial region. The representation (5.12) can be used to change variables from $(x^1, x^2) \leftrightarrow (s, z)$ without changing the time variable.

We suppose a scalar or a Cartesian component of a vector ψ is defined in inner and outer coordinates, i.e., $\psi(t, x^1, x^2) = \Psi(\tau, s, z)$. In the following we denote quantities in inner coordinates by capital letters. In particular R is the density in inner coordinates. The partial derivatives of ψ transform as follows:

$$\begin{pmatrix} \frac{\partial \psi}{\partial x^1} \\ \frac{\partial \psi}{\partial x^2} \\ \frac{\partial \psi}{\partial t} \end{pmatrix} = \begin{pmatrix} (1 + \varepsilon z \kappa) \frac{1}{|\mathbf{t}_\varepsilon|^2} t_\varepsilon^1 & \varepsilon^{-1} v_\varepsilon^1 & 0 \\ (1 + \varepsilon z \kappa) \frac{1}{|\mathbf{t}_\varepsilon|^2} t_\varepsilon^2 & \varepsilon^{-1} v_\varepsilon^2 & 0 \\ -(1 + \varepsilon z \kappa) ((w_t)_\varepsilon + \varepsilon z \frac{1}{|\mathbf{t}_\varepsilon|^2} t_\varepsilon^i (v_\varepsilon^i)_\tau) & -\varepsilon^{-1} (w_v)_\varepsilon & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial \Psi}{\partial s} \\ \frac{\partial \Psi}{\partial z} \\ \frac{\partial \Psi}{\partial \tau} \end{pmatrix} + \mathcal{O}(\varepsilon^2), \quad (5.13)$$

where $(w_t)_\varepsilon = w_\varepsilon^i \frac{t_\varepsilon^i}{|\mathbf{t}_\varepsilon|^2}$ and $(w_v)_\varepsilon = w_\varepsilon^i v_\varepsilon^i$, for any $(\tau, s, z) \in [0, T] \times I \times \mathbb{R}$, where $I \subset \mathbb{R}$ is some compact interval used to parameterize the interface.

Accordingly to the outer expansion (5.6) we assume the existence of inner expansions in ε . In particular we assume

$$\mathbf{U}(\tau, s, z; \varepsilon) \sim \sum_{i=0}^{\infty} \varepsilon^i \mathbf{U}_i(\tau, s, z), \quad R(\tau, s, z; \varepsilon) \sim \sum_{i=0}^{\infty} \varepsilon^i R_i(\tau, s, z), \quad (5.14)$$

where \mathbf{U}_i, R_i do not depend on ε for all $i \in \mathbb{N}_0$. Furthermore we suppose expansions of the geometric quantity

$$\mathbf{r}_\varepsilon(\tau, s) \sim \sum_{i=0}^{\infty} \varepsilon^i \mathbf{r}_i(\tau, s), \quad (5.15)$$

which induces ε -expansions of $\mathbf{v}_\varepsilon, \mathbf{t}_\varepsilon$ and \mathbf{w}_ε .

Particularly we get the following expansions, up to terms of order $\mathcal{O}(\varepsilon^2)$:

$$(w_v)_\varepsilon = \underbrace{v_0^i w_0^i}_{=:w_{v0}} + \varepsilon \underbrace{(v_1^i w_0^i + v_0^i w_1^i)}_{=:w_{v1}} + \mathcal{O}(\varepsilon^2), \quad (5.16)$$

$$(U_v)_\varepsilon = \underbrace{v_0^i U_0^i}_{=:U_{v0}} + \varepsilon \underbrace{(v_1^i U_0^i + v_0^i U_1^i)}_{=:U_{v1}} + \mathcal{O}(\varepsilon^2), \quad (5.17)$$

$$(U_t)_\varepsilon = \underbrace{\frac{t_0^i U_0^i}{|\mathbf{t}_0|^2}}_{=:U_{t0}} + \varepsilon \underbrace{\left(\frac{t_1^i U_0^i}{|\mathbf{t}_0|^2} - 2 \frac{t_0^i U_0^i t_0^j t_1^j}{|\mathbf{t}_0|^4} + \frac{t_0^i U_1^i}{|\mathbf{t}_0|^2} \right)}_{=:U_{t1}} + \mathcal{O}(\varepsilon^2), \quad (5.18)$$

$$(w_t)_\varepsilon = \underbrace{\frac{t_0^i w_0^i}{|\mathbf{t}_0|^2}}_{=:w_{t0}} + \varepsilon \underbrace{\left(\frac{t_1^i w_0^i}{|\mathbf{t}_0|^2} - 2 \frac{t_0^i w_0^i t_0^j t_1^j}{|\mathbf{t}_0|^4} + \frac{t_0^i w_1^i}{|\mathbf{t}_0|^2} \right)}_{=:w_{t1}} + \mathcal{O}(\varepsilon^2), \quad (5.19)$$

$$J_\varepsilon := R_\varepsilon((U_v)_\varepsilon - (w_v)_\varepsilon) \quad (5.20)$$

$$= \underbrace{R_0(U_{v0} - w_{v0})}_{=:J_0} + \varepsilon \underbrace{(R_1(U_{v0} - w_{v0}) + R_0(U_{v1} - w_{v1}))}_{=:J_1} + \mathcal{O}(\varepsilon^2). \quad (5.21)$$

With respect to the expansions of the geometric quantities we observe

$$1 = |\mathbf{v}_\varepsilon|^2 = v_0^i v_0^i + 2\varepsilon v_0^i v_1^i + \mathcal{O}(\varepsilon^2),$$

which implies

$$v_0^i v_0^i = 1, \quad v_0^i v_1^i = 0. \quad (5.22)$$

Furthermore we have

$$0 = t_\varepsilon^i v_\varepsilon^i = t_0^i v_0^i + \varepsilon(t_1^i v_0^i + t_0^i v_1^i) + \mathcal{O}(\varepsilon^2),$$

which implies

$$t_0^i v_0^i = 0, \quad v_0^i t_1^i + v_1^i t_0^i = 0. \quad (5.23)$$

Additionally an easy calculation shows

$$(v_0^i)_s = -\kappa_0 t_0^i, \quad \left(\frac{t_0^i}{|\mathbf{t}_0|} \right)_s = \kappa_0 |\mathbf{t}_0| v_0^i. \quad (5.24)$$

Using (5.12), (5.14) and (5.15) we get the inner equations by comparing coefficients of different powers of ε .

5.3 Matching conditions

Inner- and outer quantities are matched by the usual procedure, see [7] for details. For convenience of the reader we sketch the arguments. Let ψ and Ψ be any function possessing expansions like (5.6) and (5.14). Near the interfacial layer we formally equate the two expansions

$$\Psi(\tau, s, z; \varepsilon) \sim \sum_{i=0}^{\infty} \varepsilon^i \Psi_i(\tau, s, z) = \sum_{i=0}^{\infty} \varepsilon^i \psi_i(t, \mathbf{r}_\varepsilon(\tau, s) + \varepsilon z \mathbf{v}_\varepsilon(\tau, s)) \sim \psi(t, x^1, x^2; \varepsilon). \quad (5.25)$$

Then we expand the right-hand side in a Taylor series yielding

$$\sum_{i=0}^{\infty} \varepsilon^i \Psi_i(\tau, s, z) = \sum_{i=0}^N \varepsilon^i P_i^\pm(\tau, s, z) + \varepsilon^{N+1} R_{N+1}^\pm, \quad (5.26)$$

where

$$P_i^\pm = \frac{1}{i!} \frac{d^i}{d\varepsilon^i} \psi(\tau, \mathbf{r}_\varepsilon(\tau, s) + \varepsilon z \mathbf{v}_\varepsilon(\tau, s)) \Big|_{\varepsilon=0} \quad (5.27)$$

and R_{N+1}^\pm is a Lagrange remainder. We observe that P_i^\pm is a polynomial of order i in z with coefficients depending on \mathbf{r} and its derivatives and on (x^1, x^2) -derivatives of ψ . We will assume that all the (x^1, x^2) -derivatives of ψ up to order $N + 1$ are bounded in a neighborhood of the interface. The polynomials P_i^+ , P_i^- are valid for $z > 0$ and $z < 0$ respectively. Now let $z \rightarrow \pm\infty$ with ε coupled to z such that $\varepsilon z^{N+1} \rightarrow 0$ but otherwise arbitrary. Then the remainder term in (5.26) has higher order than all the preceding terms. This fact leads to the identification of the inner quantities with the Taylor polynomials for $z \rightarrow \infty$, i.e.,

$$\Psi_i(\tau, s, z) = P_i^\pm(\tau, s, z) + o(1) \text{ for } z \rightarrow \pm\infty. \quad (5.28)$$

Applying this to the first two orders of the asymptotic expansions shows

$$\Psi_0(\tau, s, z) \rightarrow \psi_0^\pm(\tau, \mathbf{r}_0(\tau, s)) \quad z \rightarrow \pm\infty, \quad (5.29)$$

$$\Psi_1(\tau, s, z) - \psi_1^\pm(\tau, \mathbf{r}_0(\tau, s)) - \left(\frac{\partial \psi_0}{\partial x^j} \right)^\pm(\tau, \mathbf{r}_0(\tau, s)) r_1^j(\tau, s) = v_0^j(\tau, s) z + o(1) \quad z \rightarrow \pm\infty. \quad (5.30)$$

By differentiating both sides in (5.25) with respect to s , z or τ we can derive further matching conditions:

$$\Psi_{0,\tau} \rightarrow \left(\frac{\partial \psi_0}{\partial x^j} \right)^\pm(\tau, \mathbf{r}_0(\tau, s)) w_0^j(\tau, s) + \left(\frac{\partial \psi_0}{\partial t} \right)^\pm(\tau, \mathbf{r}_0(\tau, s)) \quad z \rightarrow \pm\infty, \quad (5.31)$$

$$\Psi_{0,s} \rightarrow \left(\frac{\partial \psi_0}{\partial x^j} \right)^\pm(\tau, \mathbf{r}_0(\tau, s)) t_0^j(\tau, s) \quad z \rightarrow \pm\infty, \quad (5.32)$$

$$\Psi_{0,z} \rightarrow 0 \quad z \rightarrow \pm\infty, \quad (5.33)$$

$$\Psi_{1,z} \rightarrow \left(\frac{\partial \psi_0}{\partial x^j} \right)^\pm(\tau, \mathbf{r}_0(\tau, s)) v_0^j(\tau, s) \quad z \rightarrow \pm\infty. \quad (5.34)$$

We will assume that all these limits are attained superlinearly fast. When we redefine

$$\psi_0^+(s, \tau) := \lim_{z>0, z \rightarrow 0} \psi_0(\tau, \mathbf{r}_0(s, \tau) + z\mathbf{v}_0(s, \tau)), \quad (5.35)$$

$$\psi_0^-(s, \tau) := \lim_{z<0, z \rightarrow 0} \psi_0(\tau, \mathbf{r}_0(s, \tau) + z\mathbf{v}_0(s, \tau)), \quad (5.36)$$

we can rewrite (5.31) and (5.32) as

$$\Psi_{0,\tau} \rightarrow (\psi_0^\pm)_\tau \quad z \rightarrow \pm\infty, \quad (5.37)$$

$$\Psi_{0,s} \rightarrow (\psi_0^\pm)_s \quad z \rightarrow \pm\infty. \quad (5.38)$$

6. Main results

In our considerations we assume that for all relevant quantities there exists an asymptotic expansion in ε . In particular we consider solutions of the structure given below:

DEFINITION 6.1 Let $(\rho_\varepsilon, \mathbf{u}_\varepsilon)$ be a classical solution of (4.4), (4.5) existing in the time interval $[0, T)$ with inner and outer expansions as in (5.6) and (5.14).

In addition let

$$\begin{aligned} \Omega_\varepsilon^-(t) &:= \{(x^1, x^2) \in \mathbb{R}^2 : \rho_\varepsilon(t, x^1, x^2) < \rho_*\}, \\ \Omega_\varepsilon^+(t) &:= \{(x^1, x^2) \in \mathbb{R}^2 : \rho_\varepsilon(t, x^1, x^2) > \rho_*\}, \\ \Gamma_\varepsilon(t) &:= \{(x^1, x^2) \in \mathbb{R}^2 : \rho_\varepsilon(t, x^1, x^2) = \rho_*\}, \end{aligned}$$

for $t \in [0, T)$ and let Γ_ε be a $C^1([0, T), C^2(\mathbb{R}^2))$ -hypersurface with an asymptotic ε -expansion as in (5.15).

We call $(\rho_0, \mathbf{u}_0, \rho_1, \mathbf{u}_1)$ an *outer solution* if

$$\rho_0, \rho_1 \in C^1([0, T), C^0(\bar{\Omega}_0^\pm(t))) \cap C^0([0, T), C^1(\bar{\Omega}_0^\pm(t))), \quad (6.1)$$

$$\mathbf{u}_0, \mathbf{u}_1 \in C^1([0, T), C^0(\bar{\Omega}_0^\pm(t), \mathbb{R}^2)) \cap C^0([0, T), C^1(\bar{\Omega}_0^\pm(t), \mathbb{R}^2)) \quad (6.2)$$

and $(\rho_0, \mathbf{u}_0, \rho_1, \mathbf{u}_1)$ satisfies

$$\rho_{0,t} + \operatorname{div}(\rho_0 \mathbf{u}_0) = 0, \quad (6.3)$$

$$(\mathbf{u}_0)_t + (\mathbf{u}_0 \cdot \nabla) \mathbf{u}_0 + \nabla g(\rho_0) = 0, \quad (6.4)$$

$$\rho_{1,t} + \operatorname{div}(\rho_1 \mathbf{u}_0 + \rho_0 \mathbf{u}_1) = 0, \quad (6.5)$$

$$(\mathbf{u}_1)_t + (\mathbf{u}_0 \cdot \nabla) \mathbf{u}_1 + (\mathbf{u}_1 \cdot \nabla) \mathbf{u}_0 + \nabla(g'(\rho_0)\rho_1) = 0. \quad (6.6)$$

We call $(R_0, \mathbf{U}_0, R_1, \mathbf{U}_1)$ an *inner solution* if

$$R_0, R_1 \in C^1([0, T) \times I, C^3(\mathbb{R})), \quad (6.7)$$

$$\mathbf{U}_0, \mathbf{U}_1 \in C^1([0, T) \times I, C^2(\mathbb{R}, \mathbb{R}^2)), \quad (6.8)$$

where I is the compact interval used to parameterize the interface and $(R_0, \mathbf{U}_0, R_1, \mathbf{U}_1)$ satisfies

$$-w_{v_0} R_{0,z} + (U_{v_0} R_0)_z = 0, \quad (6.9)$$

$$-w_{v0}(U_0^j)_z + U_{v0}(U_0^j)_z + v_0^j g(R_0)_z - v_0^j \gamma R_{0,zzz} = 0, \quad (6.10)$$

$$-w_{v1}R_{0,z} - w_{v0}R_{1,z} - w_{t0}R_{0,s} + R_{0,\tau} + \frac{t_0^i}{|\mathbf{t}_0|^2}(R_0U_0^i)_s + (R_1U_{v0} + R_0U_{v1})_z = 0, \quad (6.11)$$

$$\begin{aligned} & -w_{v1}(U_0^j)_z - w_{v0}(U_1^j)_z + (U_0^j)_\tau - w_{t0}(U_0^j)_s + U_{v1}(U_0^j)_z + U_{v0}(U_1^j)_z \\ & + U_{t0}(U_0^j)_s + g(R_0)_z v_1^j + (g'(R_0)R_1)_z v_0^j + g(R_0)_s \frac{t_0^j}{|\mathbf{t}_0|^2} \\ & - (\lambda + 2\mu) v_0^j \frac{1}{R_0} (U_{v0})_{zz} \\ & = \mu t_0^j \frac{1}{R_0} (U_{t0})_{zz} + \gamma v_0^j R_{1,zzz} + \gamma v_1^j R_{0,zzz} - \gamma \kappa_0 v_0^j R_{0,zz} + \gamma \frac{t_0^j}{|\mathbf{t}_0|^2} R_{0,zzz}. \end{aligned} \quad (6.12)$$

We call $(\rho_0, \mathbf{u}_0, \rho_1, \mathbf{u}_1, R_0, \mathbf{U}_0, R_1, \mathbf{U}_1)$ a *matching solution*, when $(\rho_0, \mathbf{u}_0, \rho_1, \mathbf{u}_1)$ is an outer solution, $(R_0, \mathbf{U}_0, R_1, \mathbf{U}_1)$ is an inner solution and both are linked by the matching conditions (5.29)–(5.34). We call a matching solution *admissible*, if the mass flux $\rho_0(u_{v0} - w_{v0}) \neq 0$.

REMARK 6.2 Equations (6.3)–(6.6) were motivated in Section 5.1. To derive (6.9)–(6.12) we perform the coordinate change (5.12) in (4.4) and (4.5), which changes the partial derivatives according to (5.13). Then we insert the expansions (5.14) and compare coefficients of different powers of ε .

REMARK 6.3 Real phase transitions take only place if the mass flux is non-zero, i.e., $\rho_0(u_{v0} - w_{v0}) \neq 0$. Therefore, we have chosen this condition as an admissibility condition. We like to point out that the subsequent analysis does not cover the case $\rho_0(u_{v0} - w_{v0}) = 0$.

For every quantity ψ in outer variables having an asymptotic expansion in ε we will use the following notation for the jump of its leading orders at the interface:

$$\llbracket \psi \rrbracket_{0,1} := \llbracket \psi_0 \rrbracket + \varepsilon \llbracket \psi_1 \rrbracket \quad \text{and} \quad \{\psi\}_{0,1} := \{\psi_0\} + \varepsilon \{\psi_1\}.$$

Now we are well-prepared to state our main results.

THEOREM 6.4 If $(\rho_0, \mathbf{u}_0, \rho_1, \mathbf{u}_1, R_0, \mathbf{U}_0, R_1, \mathbf{U}_1)$ is an admissible matching solution defined as in Definition 6.1, then the following jump conditions are fulfilled:

$$\llbracket \rho_\varepsilon((u_v)_\varepsilon - (w_v)_\varepsilon) \rrbracket_{0,1} = -\varepsilon \frac{\partial \rho_\Gamma}{\partial \tau} - \varepsilon (\operatorname{div}_\Gamma(w_{t0}) - \kappa_0 w_{v0}) \rho_\Gamma, \quad (6.13)$$

$$\llbracket \rho_\varepsilon((u_v)_\varepsilon - (w_v)_\varepsilon)(\mathbf{u}_\varepsilon - \mathbf{w}_\varepsilon) + p(\rho_\varepsilon) \mathbf{v}_\varepsilon \rrbracket_{0,1} = -\varepsilon \rho_\Gamma \frac{\partial \mathbf{w}_0}{\partial \tau} + \varepsilon \operatorname{div}_\Gamma(\boldsymbol{\sigma}_\Gamma), \quad (6.14)$$

$$\begin{aligned} \left[\left[\frac{1}{2} \left(\frac{j_\varepsilon}{\rho_\varepsilon} \right)^2 + g(\rho_\varepsilon) \right] \right]_{0,1} &= -\varepsilon \int_0^\infty v_0^j (U_0^j - (u_0^j)^+)_\tau dz \\ & - \varepsilon \int_{-\infty}^0 v_0^j (U_0^j - (u_0^j)^-)_\tau dz \\ & - \varepsilon (\lambda + 2\mu) j_0 \int_{-\infty}^\infty \left(\left(\frac{1}{R_0} \right)_z \right)^2 dz, \end{aligned} \quad (6.15)$$

where

$$j_\varepsilon := \rho_\varepsilon((u_v)_\varepsilon - (w_v)_\varepsilon) \quad (6.16)$$

is the massflux across the interface and

$$\rho_\Gamma := \int_0^\infty (R_0 - \rho_0^+) dz - \int_{-\infty}^0 (R_0 - \rho_0^-) dz, \quad (6.17)$$

$$\sigma_\Gamma := \sigma_\Gamma^j = \left(\int_0^\infty \left(\frac{j_0^2}{R_0} - \frac{j_0^2}{\rho_0^+} + \gamma R_{0,z}^2 \right) dz + \int_{-\infty}^0 \dots dz \right) \frac{t_0^j}{|\mathbf{t}_0|^2}. \quad (6.18)$$

REMARK 6.5 We want to emphasize that (6.13) and (6.14) recover (3.3) and (3.4) respectively and we will show in Theorem 6.7 that (6.13)–(6.15) are in agreement with (3.10).

REMARK 6.6 The zeroth orders of (6.13)–(6.15) imply that the states ρ_0^\pm are the Maxwell points of the free energy density function

$$W_{j_0}(\rho) := W(\rho) - \frac{j_0^2}{2\rho}.$$

The Maxwell points α and β are the uniquely determined points satisfying

$$W'_{j_0}(\alpha) = W'_{j_0}(\beta) = \frac{W_{j_0}(\beta) - W_{j_0}(\alpha)}{\beta - \alpha}.$$

THEOREM 6.7 If $(\rho_0, \mathbf{u}_0, \rho_1, \mathbf{u}_1, R_0, \mathbf{U}_0, R_1, \mathbf{U}_1)$ is an admissible matching solution defined as in Definition 6.1, then for ε sufficiently small the jump conditions derived in Theorem 6.4 imply the inequality

$$\begin{aligned} 0 &\geq -\varepsilon(\lambda + 2\mu)j_0^2 \int_{-\infty}^\infty \left(\left(\frac{1}{R_0} \right)_z \right)^2 dz \quad (6.19) \\ &= \varepsilon \frac{\partial W_\Gamma}{\partial \tau} - \varepsilon(\gamma_\Gamma - W_\Gamma) (\operatorname{div}_\Gamma((w_t)_\varepsilon) - \kappa_\varepsilon(w_v)_\varepsilon) + \llbracket j_\varepsilon \rrbracket_{0,1} \left\{ W'(\rho_\varepsilon) + \frac{|\mathbf{u}_\varepsilon - \mathbf{w}_\varepsilon|^2}{2} \right\}_{0,1} \\ &\quad + \{j_\varepsilon\}_{0,1} \left[\left[W'(\rho_\varepsilon) + \frac{|\mathbf{u}_\varepsilon - \mathbf{w}_\varepsilon|^2}{2} \right] \right]_{0,1}. \end{aligned}$$

Recall that the surface tension γ_Γ is related to the surface stress tensor by (3.8). The interface mass density ρ_Γ and the surface stress tensor are as in (6.17), (6.18) and the Helmholtz surface energy density W_Γ is given by

$$\begin{aligned} W_\Gamma &= \int_0^\infty \left(W(R_0) - W(\rho_0^+) + \frac{1}{2} \frac{j_0^2}{R_0} - \frac{1}{2} \frac{j_0^2}{\rho_0^+} + \frac{\gamma}{2} R_{0,z}^2 \right) dz \\ &\quad + \int_{-\infty}^0 \left(W(R_0) - W(\rho_0^-) + \frac{1}{2} \frac{j_0^2}{R_0} - \frac{1}{2} \frac{j_0^2}{\rho_0^-} + \frac{\gamma}{2} R_{0,z}^2 \right) dz. \quad (6.20) \end{aligned}$$

The inequality (6.19) is identical to the entropy inequality (3.10). Note that in (6.19) the interface terms contribute in the first order of ε . Thus the jump conditions derived in Theorem 6.4 are compatible to the second law of thermodynamics.

REMARK 6.8 A straightforward computation shows

$$W_\Gamma - \gamma_\Gamma = \rho_\Gamma \left(g(\rho_0^\pm) + \frac{1}{2} \left(\frac{j_0}{\rho_0^\pm} \right)^2 \right)$$

which is a special case of the Gibbs adsorption law, see [12].

REMARK 6.9 The inequality (6.19) can be obtained by straightforward but cumbersome calculations using asymptotic analysis to the entropy inequality in the bulk (see (2.6)), which is derived in [9].

7. Proof of Theorem 6.4

7.1 $\mathcal{O}(\varepsilon^{-1})$ -terms

In the following lemma we will establish the zeroth order of (6.13), (6.14) and (6.15).

LEMMA 7.1 If $(\rho_0, \mathbf{u}_0, \rho_1, \mathbf{u}_1, R_0, \mathbf{U}_0, R_1, \mathbf{U}_1)$ is an admissible matching solution defined as in Definition 6.1, then the following jump conditions of zeroth order are satisfied:

$$\llbracket \rho_\varepsilon((u_v)_\varepsilon - (w_v)_\varepsilon) \rrbracket_0 = 0, \quad (7.1)$$

$$\llbracket \rho_\varepsilon((u_v)_\varepsilon - (w_v)_\varepsilon)(\mathbf{u}_\varepsilon - \mathbf{w}_\varepsilon) + p(\rho_\varepsilon)\mathbf{v}_\varepsilon \rrbracket_0 = 0, \quad (7.2)$$

$$\llbracket (u_t)_\varepsilon \rrbracket_0 = 0, \quad (7.3)$$

$$\left[\left[\frac{1}{2} \left(\frac{j_\varepsilon}{\rho_\varepsilon} \right)^2 + g(\rho_\varepsilon) \right] \right]_0 = 0. \quad (7.4)$$

Proof. We recall that the mass flux across the interface is given by $j = \rho(u_v - w_v)$. Hence equation (6.9) implies $J_{0,z} = 0$ and therefore we get the jump condition (7.1), i.e.,

$$\llbracket j \rrbracket_0 = 0. \quad (7.5)$$

Multiplying (6.10) with the zeroth order of the tangent vector \mathbf{t}_0 we obtain

$$\frac{J_0}{R_0} (U_{t_0})_z = 0. \quad (7.6)$$

Hence, the leading order of the tangent velocity U_{t_0} is constant across the interface. Furthermore to solve (6.9) and (6.10) it is sufficient to substitute $U_{v_0} = \frac{J_0}{R_0} - w_{v_0}$ in (6.10) and to solve (6.10) times \mathbf{v}_0 which gives

$$\frac{J_0}{R_0} \left(\frac{J_0}{R_0} \right)_z + g(R_0)_z = \gamma R_{0,zzz}. \quad (7.7)$$

By Proposition 1.2 in the work of Benzoni-Gavage et. al. [4] there exist values ρ_0^\pm satisfying certain jump conditions which we will state in (7.9) and (7.12) below. Given these states ρ_0^\pm there exists a solution R_0 of (7.7) attaining ρ_0^\pm as boundary values.

In the sequel we will show that (7.9) and (7.12) are necessary conditions for boundary values of solutions of (7.7). From (7.7) we infer

$$\frac{1}{2} \left(\frac{J_0}{R_0} \right)^2 + g(R_0) = \gamma R_{0,zz} + c_1 \quad (7.8)$$

for some $c_1 \in \mathbb{R}$. Using the matching conditions (5.29) and (5.33) this implies

$$c_1 = \frac{1}{2} \left(\frac{j_0^-}{\rho_0^-} \right)^2 + g(\rho_0^-) = \frac{1}{2} \left(\frac{j_0^+}{\rho_0^+} \right)^2 + g(\rho_0^+). \quad (7.9)$$

Multiplying (7.7) by R_0 we get

$$J_0 \left(\frac{J_0}{R_0} \right)_z + p(R_0)_z = \gamma R_0 R_{0,zzz}, \quad (7.10)$$

which implies

$$\frac{J_0^2}{R_0} + p(R_0) = \gamma \left(R_0 R_{0,zz} - \frac{1}{2} (R_{0,z})^2 \right) + c_2 \quad (7.11)$$

for some c_2 in \mathbb{R} . Applying (5.29) and (5.33) to (7.11) we find

$$c_2 = \frac{j_0^2}{\rho_0} + p(\rho_0^-) = \frac{j_0^2}{\rho_0^+} + p(\rho_0^+). \quad (7.12)$$

□

REMARK 7.2 In the proof of Lemma 7.1 we have seen that the tangential velocity is constant across the interface and up to now we have not chosen a specific parameterization of the interface. So we are free to impose

$$U_0^i = w_{t0} t_0^i + U_{v0} v_0^i, \quad \text{i.e., } U_{t0} = w_{t0}. \quad (7.13)$$

We want to point out that the admissibility condition from Definition 6.1 is crucial to establish the equality of the tangent velocities of the fluid and the interface.

7.2 Prerequisites for $\mathcal{O}(1)$ -terms

The following technical lemmata are used to derive the first order jump conditions. Lemma 7.4 shows solvability criteria for the inner equations on an abstract level and is exploited in Lemma 7.5.

LEMMA 7.3 Let $Q \in C^2(\mathbb{R})$ be given with

$$Q_z(z), Q_{zz}(z) \rightarrow 0, \quad \text{for } |z| \rightarrow \infty. \quad (7.14)$$

Further, define

$$\varphi(z) = \int_0^z \frac{1}{(Q_z(\tilde{z}))^2} d\tilde{z}.$$

Then

$$|\varphi(z) Q_z(z)| \rightarrow \infty, \quad \text{for } |z| \rightarrow \infty. \quad (7.15)$$

Proof. Due to (7.14) there exists some $K > 0$ such that

$$|Q_z(z)| < \varepsilon, \quad |Q_{zz}(z)| < \varepsilon, \quad \text{for } |z| > K.$$

We consider some pair (z, \bar{z}) with $|\bar{z}| > K + 1$, $|z| > K$ and $z\bar{z} > 0$. Then we have

$$|Q_z(z) - Q_z(\bar{z})| = \left| \int_{\bar{z}}^z Q_{zz}(\tilde{z}) d\tilde{z} \right| \leq |z - \bar{z}|\varepsilon. \quad (7.16)$$

Therefore

$$|Q_z(z)| \leq 2|Q_z(\bar{z})|, \quad \text{for all } |z - \bar{z}| \leq \frac{|Q_z(\bar{z})|}{\varepsilon} (< 1).$$

Next we define $z^* := \bar{z} - \operatorname{sgn}(\bar{z})\frac{|Q_z(\bar{z})|}{\varepsilon}$ and find

$$|\varphi(\bar{z})| \geq \left| \int_{z^*}^{\bar{z}} \left(\frac{1}{Q_z(\bar{z})} \right)^2 d\tilde{z} \right| \geq |\bar{z} - z^*| \frac{1}{4} \left| \frac{1}{Q_z(\bar{z})} \right|^2, \quad (7.17)$$

which implies

$$|\varphi(\bar{z})Q_z(\bar{z})| \geq \frac{1}{4} |\bar{z} - z^*| \left| \frac{1}{Q_z(\bar{z})} \right| = \frac{1}{4\varepsilon}. \quad (7.18)$$

Hence for every $\varepsilon > 0$ we find some $K > 0$ such that

$$|\varphi(\bar{z})Q_z(\bar{z})| \geq \frac{1}{4\varepsilon} \quad \text{for all } |\bar{z}| > K + 1, \quad (7.19)$$

which proves (7.15). \square

LEMMA 7.4 Let $(\rho_0, \mathbf{u}_0, \rho_1, \mathbf{u}_1, R_0, \mathbf{U}_0, R_1, \mathbf{U}_1)$ be an admissible matching solution and let the operator

$$L : W^{3,1}(\mathbb{R}) \times W^{1,1}(\mathbb{R}) \times W^{1,1}(\mathbb{R}) \rightarrow L^1(\mathbb{R}) \times L^1(\mathbb{R}) \times L^1(\mathbb{R})$$

be given by

$$L \begin{pmatrix} A \\ B \\ C \end{pmatrix} := \begin{pmatrix} (A(U_{v0} - w_{v0}) + R_0 B)_z \\ ((U_{v0} - w_{v0})B + g'(R_0)A - \gamma A_{zz})_z \\ (U_{v0} - w_{v0})C_z \end{pmatrix}. \quad (7.20)$$

Then

$$L \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}, \quad f_1, f_2, f_3 \in L^1(\mathbb{R}), \quad (7.21)$$

has a solution if and only if

$$\int_{-\infty}^{\infty} f_1 dz = 0, \quad (7.22)$$

$$\int_{-\infty}^{\infty} f_2 dz = 0, \quad (7.23)$$

$$\int_{-\infty}^{\infty} R_0 f_3 dz = 0, \quad (7.24)$$

$$\int_{-\infty}^{\infty} \frac{j_0}{R_0} f_1 + R_0 f_2 dz = 0. \quad (7.25)$$

Proof. By the Fredholm alternative theorem the system (7.21) is solvable if and only if the right hand side satisfies

$$\int_{-\infty}^{\infty} f_1 \cdot \tilde{A} + f_2 \cdot \tilde{B} + f_3 \cdot \tilde{C} dz = 0$$

for every solution $(\tilde{A}, \tilde{B}, \tilde{C})^T$ of the homogeneous problem for the adjoint operator. The adjoint operator

$$L^* : L^\infty(\mathbb{R}) \times L^\infty(\mathbb{R}) \times L^\infty(\mathbb{R}) \rightarrow W^{-3,1}(\mathbb{R}) \times W^{-1,1}(\mathbb{R}) \times W^{-1,1}(\mathbb{R})$$

is given by

$$L^* \begin{pmatrix} \tilde{A} \\ \tilde{B} \\ \tilde{C} \end{pmatrix} := \begin{pmatrix} -(U_{v0} - w_{v0})\tilde{A}_z - g'(R_0)\tilde{B}_z + \gamma\tilde{B}_{zzz} \\ -R_0\tilde{A}_z - (U_{v0} - w_{v0})\tilde{B}_z \\ -((U_{v0} - w_{v0})\tilde{C})_z \end{pmatrix}. \quad (7.26)$$

In order to determine solvability criteria for (7.21) we have to find all $(\tilde{A}, \tilde{B}, \tilde{C})^T \in (L^\infty(\mathbb{R}))^3$ satisfying

$$L^*(\tilde{A}, \tilde{B}, \tilde{C})^T = 0. \quad (7.27)$$

Considering the homogeneous problem (7.27) it is well known (cf. [13] for example) that all distributional solutions are already classical solutions. There are five linearly independent solutions of (7.27) in $C^3(\mathbb{R}) \times C^1(\mathbb{R}) \times C^1(\mathbb{R})$. We have to determine whether they are elements of $(L^\infty(\mathbb{R}))^3$.

The homogeneous problem (7.27) decouples into one problem for \tilde{A} and \tilde{B} and one for \tilde{C} . The latter immediately implies that \tilde{C} has to be some multiple of R_0 , because J_0 is constant in z . The equations for \tilde{A} and \tilde{B} are more involved. They amount to

$$\frac{j_0}{R_0}\tilde{A}_z + g'(R_0)\tilde{B}_z - \gamma\tilde{B}_{zzz} = 0, \quad (7.28)$$

$$R_0\tilde{A}_z + (U_{v0} - w_{v0})\tilde{B}_z = 0. \quad (7.29)$$

We can solve (7.29) for \tilde{A}_z , which yields

$$\tilde{A}_z = -\frac{j_0}{R_0^2}\tilde{B}_z. \quad (7.30)$$

Inserting (7.30) in (7.28) we obtain

$$-\frac{j_0^2}{R_0^3}\tilde{B}_z + g'(R_0)\tilde{B}_z - \gamma\tilde{B}_{zzz} = 0. \quad (7.31)$$

For simplification of notation we introduce a new energy density function and a corresponding Gibbs free energy function

$$\tilde{W}(R_0) := W(R_0) - \frac{1}{2}\frac{j_0^2}{R_0}, \quad \tilde{g}(R_0) := \tilde{W}'(R_0) = g(R_0) + \frac{1}{2}\frac{j_0^2}{R_0^2}. \quad (7.32)$$

Then (7.31) becomes

$$\tilde{g}'(R_0)\tilde{B}_z - \gamma\tilde{B}_{zzz} = 0. \quad (7.33)$$

We observe that every constant function and $\tilde{B} = R_0$, owing to (7.7), are solutions of (7.33). Hence we only have to find one more linearly independent solution of (7.33). When we define $\tilde{D} := \tilde{B}_z$ and $\tilde{E} = \tilde{D}_z$, we get from (7.33)

$$\begin{pmatrix} \tilde{D} \\ \tilde{E} \end{pmatrix}_z = \begin{pmatrix} 0 & 1 \\ \frac{1}{\gamma} \tilde{g}'(R_0) & 0 \end{pmatrix} \begin{pmatrix} \tilde{D} \\ \tilde{E} \end{pmatrix}. \quad (7.34)$$

As $\tilde{B} = R_0$ is a solution of (7.33), $(R_{0,z}, R_{0,zz})^T$ is a solution of (7.34). Due to the d'Alembert reduction principle we make the following ansatz:

$$\tilde{D}(z) = \varphi(z) R_{0,z}(z), \quad (7.35)$$

$$\tilde{E}(z) = \varphi(z) R_{0,zz}(z) + \psi(z), \quad (7.36)$$

where the functions φ and ψ are to be determined. Inserting (7.35), (7.36) in (7.34) we get

$$\varphi_z(z) R_{0,z}(z) + \varphi(z) R_{0,zz}(z) = \varphi(z) R_{0,zz} + \psi(z), \quad (7.37)$$

$$\varphi_z(z) R_{0,zz}(z) + \varphi(z) R_{0,zzz}(z) + \psi_z(z) = \frac{1}{\gamma} \tilde{g}'(R_0) \varphi(z) R_{0,z}(z). \quad (7.38)$$

We can solve (7.37) with respect to φ_z , which gives

$$\varphi_z = \frac{\psi}{R_{0,z}}. \quad (7.39)$$

Using (7.7) and (7.32) in (7.38) we obtain

$$\psi_z(z) = -\varphi_z(z) R_{0,zz}(z) \stackrel{(7.39)}{=} -\psi(z) \frac{R_{0,zz}(z)}{R_{0,z}(z)}. \quad (7.40)$$

From this we find

$$\psi = \frac{k}{R_{0,z}}$$

for $k \in \mathbb{R}$ and hence by (7.39)

$$\varphi_z = \frac{k}{(R_{0,z})^2}. \quad (7.41)$$

We only need one solution F of (7.33) which is linearly independent of the constant and R_0 . Thus we consider the special case

$$\varphi(z) = \int_0^z \frac{1}{(R_{0,z}(\tilde{z}))^2} d\tilde{z}.$$

Then F is given as a primitive function of $\varphi R_{0,z}$. From Lemma 7.3 we conclude that $F \notin L^\infty(\mathbb{R})$ as $|\varphi(z) R_{0,z}(z)| \rightarrow \infty$ for $|z| \rightarrow \infty$. Hence there are only two linearly independent solutions of (7.33) in $L^\infty(\mathbb{R})$.

Using (7.30) the general solution of the homogeneous problem (7.27) in $(L^\infty(\mathbb{R}))^3$ has the form

$$\begin{aligned} \tilde{A} &= k_1 + k_2 \frac{j_0}{R_0}, \\ \tilde{B} &= k_3 + k_2 R_0, \\ \tilde{C} &= k_4 R_0, \end{aligned} \quad (7.42)$$

for arbitrary coefficients $k_1, k_2, k_3, k_4 \in \mathbb{R}$. This gives rise to the four solvability criteria

$$\begin{aligned} \int_{-\infty}^{\infty} f_1 dz &= 0, & \int_{-\infty}^{\infty} f_2 dz &= 0, \\ \int_{-\infty}^{\infty} R_0 f_3 dz &= 0, & \int_{-\infty}^{\infty} \frac{j_0}{R_0} f_1 + R_0 f_2 dz &= 0. \end{aligned}$$

□

LEMMA 7.5 Let $(\rho_0, \mathbf{u}_0, \rho_1, \mathbf{u}_1, R_0, \mathbf{U}_0, R_1, \mathbf{U}_1)$ be an admissible matching solution and let the functions $\Phi, \Psi_v, \Psi_t \in C^\infty(\mathbb{R})$ satisfy

$$\Phi(z) = \rho_1^\pm + \left(\frac{\partial \rho_0}{\partial x^k} \right)^\pm (r_1^k + z v_0^k), \quad (7.43)$$

$$\Psi_v(z) = (u_0^j)^\pm v_1^j + (u_1^j)^\pm v_0^j + \left(\frac{\partial u_0^j}{\partial x^k} \right)^\pm v_0^j (r_1^k + z v_0^k), \quad (7.44)$$

$$\Psi_t(z) = (u_1^j)^\pm \frac{t_0^j}{|\mathbf{t}_0|^2} + \left(\frac{\partial u_0^j}{\partial x^k} \right)^\pm \frac{t_0^j}{|\mathbf{t}_0|^2} (r_1^k + z v_0^k) + (u_0^j)^\pm \left(\frac{t_1^j}{|\mathbf{t}_0|^2} - 2 \frac{t_0^j t_0^i t_1^i}{|\mathbf{t}_0|^4} \right), \quad (7.45)$$

for $z > 1$ and $z < -1$ respectively. Further, let $\tilde{R}_1 = R_1 - \Phi$, $\tilde{U}_{v1} = U_{v1} - \Psi_v$ and $\tilde{U}_{t1} = U_{t1} - \Psi_t$. Then equations (6.11), (6.12) can be equivalently written as

$$L \begin{pmatrix} \tilde{R}_1 \\ \tilde{U}_{v1} \\ \tilde{U}_{t1} \end{pmatrix} = \begin{pmatrix} f_1[R_0, \mathbf{U}_0, \Phi, \Psi] \\ f_2[R_0, \mathbf{U}_0, \Phi, \Psi] \\ f_3[R_0, \mathbf{U}_0, \Phi, \Psi] \end{pmatrix}, \quad (7.46)$$

where L is the operator from Lemma 7.4 and

$$\begin{aligned} f_1[R_0, \mathbf{U}_0, \Phi, \Psi] &:= -\frac{1}{|\mathbf{t}_0|} (w_{t0} |\mathbf{t}_0| R_0)_s - R_{0,\tau} + \kappa_0 R_0 U_{v0} \\ &\quad - (\Phi(U_{v0} - w_{v0}) + R_0(\Psi_v - w_{v1}))_z, \end{aligned} \quad (7.47)$$

$$\begin{aligned} f_2[R_0, \mathbf{U}_0, \Phi, \Psi] &:= -v_0^j (U_0^j)_\tau + (\lambda + 2\mu) \frac{1}{R_0} (U_{v0})_{zz} - \gamma \kappa_0 R_{0,zz} \\ &\quad - ((U_{v0} - w_{v0})(\Psi_v - w_{v1}) + g'(R_0)\Phi - \gamma \Phi_{zz})_z, \end{aligned} \quad (7.48)$$

$$f_3[R_0, \mathbf{U}_0, \Phi, \Psi] := -\left(\frac{t_0^j}{|\mathbf{t}_0|^2} \right) (U_0^j)_\tau - \frac{1}{|\mathbf{t}_0|^2} (g(R_0) - \gamma R_{0,zz})_s - (U_{v0} - w_{v0})(\Psi_t)_z. \quad (7.49)$$

Proof. We will not consider (6.12) directly but separate it into a normal and a tangential part. The normal part is given by (6.12) times v_0^j plus (6.9) times v_1^j , i.e.,

$$\begin{aligned} &-w_{v1}(U_{v0})_z - w_{v0}(U_{v1})_z + v_0^j (U_0^j)_\tau + U_{v1}(U_{v0})_z + U_{v0}(U_{v1})_z + (g'(R_0)R_1)_z \\ &= \frac{1}{R_0} (\lambda + 2\mu)(U_{v0})_{zz} + \gamma R_{1,zzz} - \kappa_0 \gamma R_{0,zz}. \end{aligned} \quad (7.50)$$

Rearranging the terms in (7.50) gives

$$\begin{aligned} & ((U_{v0} - w_{v0})(U_{v1} - w_{v1}) + g'(R_0)R_1)_z \\ &= -v_0^j (U_0^j)_\tau + \frac{1}{R_0}(\lambda + 2\mu)(U_{v0})_{zz} + \gamma R_{1,zzz} - \kappa_0 \gamma R_{0,zz}. \end{aligned} \quad (7.51)$$

The tangential part of (6.12) is given by (6.12) times $\frac{t_0^j}{|\mathbf{t}_0|^2}$ plus (6.9) times $\frac{t_1^j}{|\mathbf{t}_0|^2} - 2\frac{t_0^j t_0^i t_1^i}{|\mathbf{t}_0|^4}$, i.e.,

$$\begin{aligned} & -w_{v1}(U_{t0})_z - w_{v0} \left(\frac{t_0^j U_1^j}{|\mathbf{t}_0|^2} \right)_z + \frac{t_0^j}{|\mathbf{t}_0|^2} (U_0^j)_\tau + U_{v1}(U_{t0})_z U_{v0} \left(\frac{t_0^j U_1^j}{|\mathbf{t}_0|^2} \right)_z \\ &+ \frac{t_0^j v_1^j}{|\mathbf{t}_0|^2} g(R_0)_z + \frac{1}{|\mathbf{t}_0|^2} g(R_0)_s + (U_{v0} - w_{v0}) \left(\frac{t_1^j U_0^j}{|\mathbf{t}_0|^2} - 2\frac{t_0^j U_0^j t_0^i t_1^i}{|\mathbf{t}_0|^4} \right)_z + \frac{t_1^j v_0^j}{|\mathbf{t}_0|^2} g(R_0)_z \\ &= \mu \frac{1}{R_0} (U_{t0})_{zz} + \gamma \frac{v_1^j t_0^j}{|\mathbf{t}_0|^2} R_{0,zzz} + \gamma \frac{1}{|\mathbf{t}_0|^2} R_{0,zzz} + \gamma \frac{t_1^j v_0^j}{|\mathbf{t}_0|^2} R_{0,zzz}. \end{aligned} \quad (7.52)$$

Using (7.13) and (5.23) this is equivalent to

$$(U_{v0} - w_{v0})(U_{t1})_z + \frac{t_0^j}{|\mathbf{t}_0|^2} (U_0^j)_\tau + \frac{1}{|\mathbf{t}_0|^2} g(R_0)_s = \gamma \frac{1}{|\mathbf{t}_0|^2} R_{0,zzz}. \quad (7.53)$$

By the matching conditions, R_1, U_{v1}, U_{t1} must have the following asymptotic behavior for $z \rightarrow \pm\infty$:

$$R_1(z) = \rho_1^\pm + \left(\frac{\partial \rho_0}{\partial x^k} \right)^\pm (r_1^k + z v_0^k) + o(1), \quad (7.54)$$

$$U_{v1}(z) = (u_1^j)^\pm v_1^j + (u_1^j)^\pm v_0^j + \left(\frac{\partial u_0^j}{\partial x^k} \right)^\pm v_0^j (r_1^k + z v_0^k) + o(1), \quad (7.55)$$

$$U_{t1}(z) = (u_1^j)^\pm \frac{t_0^j}{|\mathbf{t}_0|^2} + \left(\frac{\partial u_0^j}{\partial x^k} \right)^\pm \frac{t_0^j}{|\mathbf{t}_0|^2} (r_1^k + z v_0^k) + (u_0^j)^\pm \left(\frac{t_1^j}{|\mathbf{t}_0|^2} - 2\frac{t_0^j t_0^i t_1^i}{|\mathbf{t}_0|^4} \right) + o(1). \quad (7.56)$$

Hence, solving (6.11),(6.12) is equivalent to solving

$$\begin{aligned} & (\tilde{R}_1(U_{v0} - w_{v0}) + R_0 \tilde{U}_{v1})_z = -\frac{1}{|\mathbf{t}_0|} (w_{t0} |\mathbf{t}_0| R_0)_s - R_{0,\tau} + \kappa_0 R_0 U_{v0} \\ & \quad - (\Phi(U_{v0} - w_{v0}) + R_0(\Psi_v - w_{v1}))_z, \\ & ((U_{v0} - w_{v0}) \tilde{U}_{v1} + g'(R_0) \tilde{R}_1 - \gamma \tilde{R}_{1,zz})_z = -v_0^j (U_0^j)_\tau + (\lambda + 2\mu) \frac{1}{R_0} (U_{v0})_{zz} - \gamma \kappa R_{0,zz} \\ & \quad - ((U_{v0} - w_{v0})(\Psi_v - w_{v1}) + g'(R_0)\Phi - \gamma \Phi_{zz})_z \end{aligned}$$

and

$$(U_{v0} - w_{v0})(\tilde{U}_{t1})_z = - \left(\frac{t_0^j}{|\mathbf{t}_0|^2} \right) (U_0^j)_\tau - \frac{1}{|\mathbf{t}_0|^2} (g(R_0) - \gamma R_{0,zz})_s - (U_{v0} - w_{v0})(\Psi_t)_z,$$

with the new boundary conditions

$$\tilde{R}_1(z), \tilde{U}_{v1}(z), \tilde{U}_{t1}(z) \rightarrow 0 \text{ for } z \rightarrow \infty. \quad (7.57)$$

□

In the following we deduce the first order jump conditions in Theorem 6.4 by exploiting the solvability conditions from Lemma 7.4.

7.3 $\mathcal{O}(1)$ -order terms

7.3.1 *Mass balance.* Relying on Lemma 7.4 and Lemma 7.5 we can determine the first order of the jump of the mass flux across the interface.

LEMMA 7.6 Let $(\rho_0, \mathbf{u}_0, \rho_1, \mathbf{u}_1, R_0, \mathbf{U}_0, R_1, \mathbf{U}_1)$ be an admissible matching solution then

$$\llbracket \rho_\varepsilon((u_v)_\varepsilon - (w_v)_\varepsilon) \rrbracket_1 = -\frac{\partial \rho_\Gamma}{\partial \tau} - (\operatorname{div}_\Gamma(w_{t0}) - \kappa_0 w_{v0}) \rho_\Gamma, \quad (7.58)$$

where

$$\rho_\Gamma = \int_0^\infty (R_0 - \rho_0^+) dz - \int_{-\infty}^0 (R_0 - \rho_0^-) dz + \mathcal{O}(\varepsilon). \quad (7.59)$$

Proof. From Lemma 7.4 and Lemma 7.5 we know

$$\int_{-\infty}^\infty f_1[R_0, \mathbf{U}_0, \Phi, \Psi] dz = 0,$$

where

$$\begin{aligned} f_1[R_0, \mathbf{U}_0, \Phi, \Psi] = & -\frac{1}{|\mathbf{t}_0|} (w_{t0} |\mathbf{t}_0| R_0)_s - R_{0,\tau} + \kappa_0 R_0 U_{v0} \\ & - (\Phi(U_{v0} - w_{v0}) + R_0(\Psi_v - w_{v1}))_z, \end{aligned}$$

when $f_1 \in L^1(\mathbb{R})$. We start with showing that $f_1 \in L^1(\mathbb{R})$.

We observe that

$$(\rho_{0,t})^\pm + \frac{t_0^i}{|\mathbf{t}_0|^2} \left(\frac{\partial(\rho_0 u_0^i)}{\partial x^j} \right)^\pm t_0^j + v_0^i \left(\frac{\partial(\rho_0 u_0^i)}{\partial x^j} \right)^\pm v_0^j = 0, \quad (7.60)$$

because ρ_0, \mathbf{u}_0 satisfy the mass conservation equation in the bulk. Utilizing $w_{v0} v_0^i + w_{t0} t_0^i = w_0^i$ equation (7.60) is equivalent to

$$\begin{aligned} & -w_{v0} \left(\frac{\partial \rho_0}{\partial x^i} \right)^\pm v_0^i - w_{t0} \left(\frac{\partial \rho_0}{\partial x^i} \right)^\pm t_0^i + (\rho_{0,t})^\pm \\ & + \left(\frac{\partial \rho_0}{\partial x^i} \right)^\pm w_0^i + \frac{t_0^i}{|\mathbf{t}_0|^2} \left(\frac{\partial(\rho_0 u_0^i)}{\partial x^j} \right)^\pm t_0^j + v_0^i \left(\frac{\partial(\rho_0 u_0^i)}{\partial x^j} \right)^\pm v_0^j = 0. \quad (7.61) \end{aligned}$$

Using the different forms of the matching conditions implies

$$-w_{v_0} \left(\frac{\partial \rho_0}{\partial x^i} \right)^\pm v_0^i - w_{t_0} (\rho_0^\pm)_s + (\rho_0^\pm)_\tau + \frac{t_0^i}{|\mathbf{t}_0|} (\rho_0^\pm (u_0^i)^\pm)_s + v_0^i \left(\frac{\partial (\rho_0 u_0^i)}{\partial x^j} \right)^\pm v_0^j = 0. \quad (7.62)$$

Due to (5.24) and (7.13) we obtain

$$(u_{v_0}^\pm - w_{v_0}) \left(\frac{\partial \rho_0}{\partial x^i} \right)^\pm v_0^i + \rho_0^\pm \left(\frac{\partial u_0^i}{\partial x^j} \right)^\pm v_0^i v_0^j + \frac{1}{|\mathbf{t}_0|} (w_{t_0} |\mathbf{t}_0|)_s \rho_0^\pm + (\rho_0^\pm)_\tau - \kappa_0 \rho_0^\pm u_{v_0}^\pm = 0. \quad (7.63)$$

Keeping in mind the definition of $f_1[R_0, \mathbf{U}_0, \Phi, \Psi]$ (see (7.47)), equation (7.63) implies due to (5.31), (5.32) and (5.34)

$$f_1[R_0(z), \mathbf{U}_0(z), \Phi(z), \Psi(z)] \rightarrow 0, \quad \text{for } |z| \rightarrow \infty \quad (7.64)$$

superlinearly and hence $f_1[R_0, \mathbf{U}_0, \Phi, \Psi] \in L^1(\mathbb{R})$.

Thus we can write (7.22) as

$$\begin{aligned} 0 = & \lim_{a \rightarrow \infty} \left[\left(\frac{\partial \rho_0}{\partial x^j} \right)^+ (v_0^j a + r_1^j) (u_{v_0}^+ - w_{v_0}) - \left(\frac{\partial \rho_0}{\partial x^j} \right)^- (r_1^j - v_0^j a) (u_{v_0}^- - w_{v_0}) \right. \\ & + \rho_1^+ (u_{v_0}^+ - w_{v_0}) - \rho_1^- (u_{v_0}^- - w_{v_0}) + \rho_0^+ \left(\frac{\partial u_0^i}{\partial x^j} \right)^+ v_0^i (v_0^j a + r_1^j) - \rho_0^- \left(\frac{\partial u_0^i}{\partial x^j} \right)^- v_0^i (r_1^j - v_0^j a) \\ & - w_{v_1} \rho_0^+ + w_{v_1} \rho_0^- + \rho_0^+ (u_0^j)^+ v_1^j + \rho_0^+ (u_1^j)^+ v_0^j - \rho_0^- (u_0^j)^- v_1^j - \rho_0^- (u_1^j)^- v_0^j \\ & \left. - \int_{-a}^a \left(-\frac{1}{|\mathbf{t}_0|} (w_{t_0} |\mathbf{t}_0|)_s R_0 - R_{0,\tau} + \kappa_0 R_0 U_{v_0} \right) dz \right]. \end{aligned}$$

Now we would like to decompose the limit into several parts, which have to converge. This can be achieved by subtracting a times (7.63) $^\pm$ before letting $a \rightarrow \infty$, i.e.,

$$\begin{aligned} 0 = & \lim_{a \rightarrow \infty} \left[\left(\frac{\partial \rho_0}{\partial x^j} \right)^+ r_1^j (u_{v_0}^+ - w_{v_0}) - \left(\frac{\partial \rho_0}{\partial x^j} \right)^- r_1^j (u_{v_0}^- - w_{v_0}) \right. \\ & + \rho_1^+ (u_{v_0}^+ - w_{v_0}) - \rho_1^- (u_{v_0}^- - w_{v_0}) + \rho_0^+ \left(\frac{\partial u_0^i}{\partial x^j} \right)^+ v_0^i r_1^j - \rho_0^- \left(\frac{\partial u_0^i}{\partial x^j} \right)^- v_0^i r_1^j \\ & - w_{v_1} \rho_0^+ + w_{v_1} \rho_0^- + \rho_0^+ (u_0^j)^+ v_1^j + \rho_0^+ (u_1^j)^+ v_0^j - \rho_0^- (u_0^j)^- v_1^j - \rho_0^- (u_1^j)^- v_0^j \\ & \left. - \int_{-a}^a -\frac{1}{|\mathbf{t}_0|} (w_{t_0} |\mathbf{t}_0|)_s (R_0 - \rho_0^+) - (R_0 - \rho_0^+)_\tau + \kappa_0 (R_0 U_{v_0} - \rho_0^+ u_{v_0}^+) dz - \int_{-a}^0 \dots dz \right]. \quad (7.65) \end{aligned}$$

Inserting (A.3) in (7.65) and using

$$R_0 U_{v_0} - \rho_0^+ u_{v_0}^+ = R_0 w_{v_0} - \rho_0^+ w_{v_0},$$

we obtain

$$\begin{aligned} & \left[\left[\rho_\varepsilon((u_\nu)_\varepsilon - (w_\nu)_\varepsilon) \right] \right]_1 \\ &= \int_0^\infty (-R_{0,\tau} + (\rho_0^+)_\tau) dz + \int_{-\infty}^0 \dots \\ &+ \int_0^\infty \left(-\frac{1}{|\mathbf{t}_0|} (|\mathbf{t}_0|w_{t0})_s R_0 + \frac{1}{|\mathbf{t}_0|} (|\mathbf{t}_0|w_{t0})_s \rho_0^+ + \kappa_0 R_0 w_{v0} - \kappa_0 \rho_0^+ w_{v0} \right) dz + \int_{-\infty}^0 \dots \end{aligned} \quad (7.66)$$

In view of (3.3) we can identify the surface mass density as

$$\rho_\Gamma := \int_0^\infty (R_0 - \rho_0^+) dz + \int_{-\infty}^0 (R_0 - \rho_0^-) dz + \mathcal{O}(\varepsilon^2), \quad (7.67)$$

which finishes the proof. \square

7.3.2 Gibbs free energy. We deduce the first order of the jump of the Gibbs free energy across the interface. The proof is based on Lemma 7.4 and Lemma 7.5.

LEMMA 7.7 Let $(\rho_0, \mathbf{u}_0, \rho_1, \mathbf{u}_1, R_0, \mathbf{U}_0, R_1, \mathbf{U}_1)$ be an admissible matching solution. Then

$$\left[\left[\frac{1}{2} \left(\frac{j_\varepsilon}{\rho_\varepsilon} \right)^2 + g(\rho_\varepsilon) \right] \right]_1 = - \int_0^\infty v_0^j (U_0^j - (u_0^j)^+)_\tau dz \quad (7.68)$$

$$\begin{aligned} & - \int_{-\infty}^0 v_0^j (U_0^j - (u_0^j)^-)_\tau dz \\ & - (\lambda + 2\mu) j_0 \int_{-\infty}^\infty \left(\left(\frac{1}{R_0} \right)_z \right)^2 dz. \end{aligned} \quad (7.69)$$

Proof. From Lemma 7.4 and Lemma 7.5 we know

$$\int_{-\infty}^\infty f_2[R_0, \mathbf{U}_0, \Phi, \Psi] dz = 0,$$

where

$$\begin{aligned} f_2[R_0, \mathbf{U}_0, \Phi, \Psi] &= -v_0^j (U_0^j)_\tau + (\lambda + 2\mu) \frac{1}{R_0} (U_{v0})_{zz} - \gamma \kappa_0 R_{0,zz} \\ &- ((U_{v0} - w_{v0})(\Psi_v - w_{v1}) + g'(R_0)\Phi - \gamma \Phi_{zz})_z, \end{aligned}$$

when $f_2 \in L^1(\mathbb{R})$. We start with showing that $f_2 \in L^1(\mathbb{R})$.

As the outer quantities ρ_0, \mathbf{u}_0 satisfy (4.5), we have

$$v_0^j (u_{0,t}^j)^\pm + (u_0^k)^\pm \left(\frac{\partial u_0^j}{\partial x^k} \right)^\pm v_0^j + g'(\rho_0^\pm) \left(\frac{\partial \rho_0}{\partial x^j} \right)^\pm v_0^j = 0. \quad (7.70)$$

Due to the matching conditions (5.33), (5.34), (5.37) and the assumptions (7.43), (7.44) and (7.45) on Φ , Ψ_v and Ψ_t we get

$$\begin{aligned} f_2[R_0(z), \mathbf{U}_0(z), \Phi(z), \Psi(z)] &\rightarrow -(u_{v_0}^\pm - w_{v_0}) \left(\frac{\partial u_0^j}{\partial x^k} \right)^\pm v_0^j v_0^k - g'(\rho_0^\pm) \left(\frac{\partial \rho_0}{\partial x^k} \right)^\pm v_0^k \\ &\quad - v_0^j \left(\frac{\partial u_0^j}{\partial x^k} \right)^\pm w_0^k - v_0^j (u_{0,t}^j)^\pm. \end{aligned} \quad (7.71)$$

Inserting (7.70) in (7.71) we get

$$\begin{aligned} f_2[R_0(z), \mathbf{U}_0(z), \Phi(z), \Psi(z)] &\rightarrow \left((u_0^k)^\pm - u_{v_0} v_0^k + w_{v_0} v_0^k - w_0^k \right) \left(\frac{\partial u_0^j}{\partial x^k} \right)^\pm v_0^k = 0 \\ &\quad \text{for } |z| \rightarrow \infty, \end{aligned} \quad (7.72)$$

where the convergence is obtained superlinear, whence $f_2[R_0, \mathbf{U}_0, \Phi, \Psi] \in L^1(\mathbb{R})$.

Due to the matching conditions and (7.43), (7.44) the solvability condition (7.23) is equivalent to

$$\begin{aligned} 0 &= \lim_{a \rightarrow \infty} \left[(u_{v_0}^+ - w_{v_0}) w_{v_1} - (u_{v_0}^- - w_{v_0}) w_{v_1} \right. \\ &\quad - (u_{v_0}^+ - w_{v_0}) \left((u_0^j)^+ v_1^j + (u_1^j)^+ v_0^j + \left(\frac{\partial u_0^j}{\partial x^k} \right)^+ v_0^j (r_1^k + v_0^k a) \right) \\ &\quad + (u_{v_0}^- - w_{v_0}) \left((u_0^j)^- v_1^j + (u_1^j)^- v_0^j + \left(\frac{\partial u_0^j}{\partial x^k} \right)^- v_0^j (r_1^k - v_0^k a) \right) \\ &\quad - g'(\rho_0^+) \rho_1^+ - g'(\rho_0^+) \left(\frac{\partial \rho_0}{\partial x^k} \right)^+ (r_1^k + v_0^k a) + g'(\rho_0^-) \rho_1^- \\ &\quad \left. + g'(\rho_0^-) \left(\frac{\partial \rho_0}{\partial x^k} \right)^- (r_1^k - v_0^k a) - \int_{-a}^a v_0^j (U_0^j)_\tau dz + (\lambda + 2\mu) \int_{-a}^a \frac{1}{R_0} (U_{v_0})_{zz} dz \right]. \end{aligned} \quad (7.73)$$

Now we subtract a times (7.70)⁺ and a times (7.70)⁻ before letting $a \rightarrow \infty$ and obtain

$$\begin{aligned} 0 &= \lim_{a \rightarrow \infty} \left[(u_{v_0}^+ - w_{v_0}) w_{v_1} - (u_{v_0}^- - w_{v_0}) w_{v_1} \right. \\ &\quad - (u_{v_0}^+ - w_{v_0}) \llbracket \psi_0 \rrbracket \left((u_0^j)^+ v_1^j + (u_1^j)^+ v_0^j + \left(\frac{\partial u_0^j}{\partial x^k} \right)^+ v_0^j r_1^k \right) \\ &\quad + (u_{v_0}^- - w_{v_0}) \left((u_0^j)^- v_1^j + (u_1^j)^- v_0^j + \left(\frac{\partial u_0^j}{\partial x^k} \right)^- v_0^j r_1^k \right) \\ &\quad - g'(\rho_0^+) \rho_1^+ - g'(\rho_0^+) \left(\frac{\partial \rho_0}{\partial x^k} \right)^+ r_1^k + g'(\rho_0^-) \rho_1^- + g'(\rho_0^-) \left(\frac{\partial \rho_0}{\partial x^k} \right)^- r_1^k \\ &\quad \left. - \int_0^a v_0^j (U_0^j - (u_0^j)^+)_\tau dz - \int_{-a}^0 \dots dz + (\lambda + 2\mu) \int_{-a}^a \frac{1}{R_0} (U_{v_0})_{zz} dz \right]. \end{aligned} \quad (7.74)$$

Using (A.5) equation (7.74) means

$$\begin{aligned} \left[\left[\frac{1}{2}((u_v)_\varepsilon - (w_v)_\varepsilon)^2 + g(\rho_\varepsilon) \right] \right]_1 &= - \int_0^\infty v_0^j \left((U_0^j)_\tau(z) - ((u_0^j)^+)_\tau \right) dz \\ &\quad - \int_{-\infty}^0 v_0^j \left((U_0^j)_\tau(z) - ((u_0^j)^-)_\tau \right) dz \\ &\quad + \int_{-\infty}^\infty (\lambda + 2\mu) \frac{1}{R_0(z)} (U_{v0})_{zz}(z) dz. \end{aligned} \quad (7.75)$$

Inserting $U_{v0} = \frac{J_0}{R_0} + w_v$ in (7.75) yields

$$\begin{aligned} \left[\left[\frac{1}{2}((u_v)_\varepsilon - (w_v)_\varepsilon)^2 + g(\rho_\varepsilon) \right] \right]_1 &= - \int_0^\infty v_0^j \left(U_0^j(z) - (u_0^j)^+ \right)_\tau dz \\ &\quad - \int_{-\infty}^0 v_0^j \left(U_0^j(z) - (u_0^j)^- \right)_\tau dz \\ &\quad - \int_{-\infty}^\infty (\lambda + 2\mu) J_0 \left(\left(\frac{1}{R_0} \right)_z \right)^2 dz. \end{aligned} \quad (7.76)$$

□

7.3.3 Momentum balance. Using the last two solvability conditions from Lemma 7.4 together with Lemma 7.5, we will determine the first order of the pressure jump across the interface.

LEMMA 7.8 Let $(\rho_0, \mathbf{u}_0, \rho_1, \mathbf{u}_1, R_0, \mathbf{U}_0, R_1, \mathbf{U}_1)$ be an admissible matching solution. Then

$$\left[\left[\rho_\varepsilon((u_v)_\varepsilon - (w_v)_\varepsilon)(\mathbf{u}_\varepsilon - \mathbf{w}_\varepsilon) + p(\rho_\varepsilon)\mathbf{v}_\varepsilon \right] \right]_1 = -\rho_\Gamma \frac{\partial \mathbf{w}_0}{\partial \tau} + \operatorname{div}_\Gamma(\boldsymbol{\sigma}_\Gamma), \quad (7.77)$$

where

$$\rho_\Gamma = \int_0^\infty (R_0 - \rho_0^+) dz - \int_{-\infty}^0 (R_0 - \rho_0^-) dz + \mathcal{O}(\varepsilon), \quad (7.78)$$

$$\sigma_\Gamma^j = \left(\int_0^\infty \left(p(\rho_0^+) - p(R_0) - \frac{\gamma}{2}(R_{0,z})^2 \right) dz + \int_{-\infty}^0 \dots \right) \frac{t_0^j}{|\mathbf{t}_0|^2} + \mathcal{O}(\varepsilon). \quad (7.79)$$

Proof. From Lemma 7.4 and Lemma 7.5 we have the solvability criteria

$$\int_{-\infty}^\infty R_0 f_3 dz = 0, \quad \int_{-\infty}^\infty \frac{j_0}{R_0} f_1 + R_0 f_2 dz = 0,$$

where

$$\begin{aligned}
f_1[R_0, \mathbf{U}_0, \Phi, \Psi] &= -\frac{1}{|\mathbf{t}_0|} (w_{t_0} |\mathbf{t}_0| R_0)_s - R_{0,\tau} + \kappa_0 R_0 U_{v_0} \\
&\quad - (\Phi(U_{v_0} - w_{v_0}) + R_0(\Psi_v - w_{v_1}))_z, \\
f_2[R_0, \mathbf{U}_0, \Phi, \Psi] &= -v_0^j (U_0^j)_\tau + (\lambda + 2\mu) \frac{1}{R_0} (U_{v_0})_{zz} - \gamma \kappa_0 R_{0,zz} \\
&\quad - ((U_{v_0} - w_{v_0})(\Psi_v - w_{v_1}) + g'(R_0)\Phi - \gamma\Phi_{zz})_z, \\
f_3[R_0, \mathbf{U}_0, \Phi, \Psi] &= -\left(\frac{t_0^j}{|\mathbf{t}_0|^2}\right) (U_0^j)_\tau - \frac{1}{|\mathbf{t}_0|^2} (g(R_0) - \gamma R_{0,zz})_s - (U_{v_0} - w_{v_0})(\Psi_t)_z,
\end{aligned}$$

when $f_1, f_2, f_3 \in L^1(\mathbb{R})$. From the proofs of Lemma 7.6 and Lemma 7.7 we already know that $f_1, f_2 \in L^1(\mathbb{R})$. It remains to show that $f_3 \in L^1(\mathbb{R})$.

By the matching conditions and (7.45) we get

$$\begin{aligned}
f_3 \rightarrow & -\frac{t_0^j}{|\mathbf{t}_0|^2} \left(\frac{\partial u_0^j}{\partial x^k}\right)^\pm w_0^k - \frac{t_0^j}{|\mathbf{t}_0|^2} \left(\frac{\partial u_0^j}{\partial t}\right)^\pm \\
& - \frac{t_0^k}{|\mathbf{t}_0|^2} \left(\frac{\partial g(\rho_0)}{\partial x^k}\right)^\pm - (u_{v_0}^\pm - w_{v_0}) - \frac{t_0^j}{|\mathbf{t}_0|^2} \left(\frac{\partial u_0^j}{\partial x^k}\right)^\pm v_0^k. \quad (7.80)
\end{aligned}$$

On the other hand, ρ_0, \mathbf{u}_0 satisfy (4.5) in the bulk, such that

$$\frac{t_0^j}{|\mathbf{t}_0|^2} \left(\left(\frac{\partial u_0^j}{\partial t}\right)^\pm + (u_0^k)^\pm \left(\frac{\partial u_0^j}{\partial x^k}\right)^\pm + \left(\frac{\partial g(\rho_0)}{\partial x^j}\right)^\pm \right) = 0. \quad (7.81)$$

Inserting (7.81) in (7.80) we get

$$f_3 \rightarrow -\frac{t_0^j}{|\mathbf{t}_0|^2} \left((u_0^k)^\pm - w_0^k - (u_{v_0}^\pm - w_{v_0}) v_0^k \right) \left(\frac{\partial u_0^j}{\partial x^k}\right)^\pm = 0. \quad (7.82)$$

To obtain a solvability criterion, which has the form of the Young–Laplace law, we will deal with the solvability criteria (7.24) and (7.25) simultaneously. As v_0 and \mathbf{t}_0 are linearly independent equations (7.24) and (7.25) are equivalent to

$$\int_{-\infty}^{\infty} R_0 f_3[R_0, \mathbf{U}_0, \Phi, \Psi] t_0^i + \frac{j_0}{R_0} f_1[R_0, \mathbf{U}_0, \Phi, \Psi] v_0^i + R_0 f_2[R_0, \mathbf{U}_0, \Phi, \Psi] v_0^i dz = 0. \quad (7.83)$$

As f_1, f_2, f_3 go to zero superlinearly for $|z| \rightarrow \infty$, we have

$$\lim_{z \rightarrow \pm\infty} R_0 f_3[R_0, \mathbf{U}_0, \Phi, \Psi] t^i + \frac{j_0}{R_0} f_1[R_0, \mathbf{U}_0, \Phi, \Psi] v^i + R_0 f_2[R_0, \mathbf{U}_0, \Phi, \Psi] v^i = 0. \quad (7.84)$$

Inserting (7.47)–(7.49) in (7.83) yields

$$\begin{aligned}
0 = \int_{-\infty}^{\infty} & \left[\frac{J_0}{R_0} v_0^i \left(-\frac{1}{|\mathbf{t}_0|} (w_{t_0} |\mathbf{t}_0|)_s R_0 - R_{0,\tau} + \kappa_0 R_0 U_{v_0} - \left(\Phi \frac{J_0}{R_0} + R_0 (\Psi_v - w_{v_1}) \right)_z \right) \right. \\
& + R_0 v_0^i \left(-v_0^j (U_0^j)_\tau + (\lambda + 2\mu) \frac{1}{R_0} (U_{v_0})_{zz} - \gamma \kappa_0 R_{0,zz} \right) \\
& - R_0 v_0^i \left(\frac{J_0}{R_0} (\Psi_v - w_{v_1}) + g'(R_0) \Phi - \gamma \Phi_{zz} \right)_z \\
& \left. - R_0 t_0^i \left(\frac{t_0^j}{|\mathbf{t}_0|^2} (U_0^j)_\tau + \frac{1}{|\mathbf{t}_0|^2} (g(R_0) - \gamma R_{0,zz})_s + \frac{J_0}{R_0} (\Psi_t)_z \right) \right] dz. \quad (7.85)
\end{aligned}$$

The terms containing w_{v_1} cancel out and using integration by parts, (7.85) becomes

$$\begin{aligned}
0 = \int_{-\infty}^{\infty} & \left[-J_0 v_0^i \frac{1}{|\mathbf{t}_0|} (w_{t_0} |\mathbf{t}_0|)_s - \frac{J_0}{R_0} v_0^i R_{0,\tau} + \kappa_0 v_0^i J_0 U_{v_0} \right. \\
& - v_0^i \left(J_0 \Psi_v + \frac{J_0^2}{R_0^2} \Phi + J_0 \Psi_v + p'(R_0) \Phi - \gamma (\Phi_{zz} R_0 - \Phi_z R_{0,z} + \Phi R_{0,zz}) \right)_z \\
& + \frac{J_0}{R_0} \left(\frac{J_0}{R_0} \right)_z v_0^i \Phi + \left(\frac{J_0}{R_0} \right)_z v_0^i R_0 \Psi_v - R_0 (U_0^i)_\tau + (\lambda + 2\mu) (U_{v_0})_{zz} v_0^i \\
& - \gamma \kappa_0 v_0^i R_0 R_{0,zz} + v_0^i R_{0,z} \frac{J_0}{R_0} \Psi_v + g(R_0)_z \Phi v_0^i - \gamma R_{0,zzz} \Phi v_0^i \\
& \left. - \frac{t_0^i}{|\mathbf{t}_0|^2} p(R_0)_s + \gamma \frac{t_0^i}{|\mathbf{t}_0|^2} R_0 R_{0,zzz} - J_0 t_0^i (\Psi_t)_z \right] dz. \quad (7.86)
\end{aligned}$$

We observe that (6.9) and (7.13) imply

$$\frac{J_0}{R_0} v_0^i R_{0,\tau} + R_0 (U_0^i)_\tau = (J_0 v_0^i)_\tau + R_0 (w_0^i)_\tau \quad \text{and} \quad \left(\frac{J_0}{R_0} \right)_z R_0 + \left(\frac{J_0}{R_0} \right) R_{0,z} = 0.$$

Using this (7.86) yields

$$\begin{aligned}
0 = \int_{-\infty}^{\infty} & \left[-J_0 v_0^i \frac{1}{|\mathbf{t}_0|} (w_{t_0} |\mathbf{t}_0|)_s - (J_0 v_0^i)_\tau - R_0 (w_0^i)_\tau + \kappa_0 v_0^i J_0 U_{v_0} \right. \\
& - v_0^i \left(2J_0 \Psi_v + \frac{J_0^2}{R_0^2} \Phi + p'(R_0) \Phi - \gamma (\Phi_{zz} R_0 - \Phi_z R_{0,z} + \Phi R_{0,zz}) \right)_z \\
& + (\lambda + 2\mu) (U_{v_0})_{zz} v_0^i - \gamma \kappa_0 v_0^i R_0 R_{0,zz} - \frac{t_0^i}{|\mathbf{t}_0|^2} p(R_0)_s + \gamma \frac{t_0^i}{|\mathbf{t}_0|^2} R_0 R_{0,zzz} - J_0 t_0^i (\Psi_t)_z \\
& \left. + \Phi v_0^i \left(\frac{J_0}{R_0} \left(\frac{J_0}{R_0} \right)_z + g(R_0)_z - \gamma R_{0,zzz} \right) \right] dz. \quad (7.87)
\end{aligned}$$

We remark that the last line of (7.87) vanishes due to (6.10).

Now we will consider some of the terms from (7.87) separately. We start with multiplying (6.10) with R_0 and subtracting (6.9) times w_0^j . We obtain

$$\left(R_0 U_{\nu 0}(U_0^j - w_0^j)\right)_z - \underbrace{\left(R_0 w_{\nu 0}(U_0^j - w_0^j)\right)_z}_{=0} = -v_0^j p(R_0)_z + v_0^j \gamma R_0 R_{0,zzz}. \quad (7.88)$$

So there exists D_1^i independent of z such that

$$v_0^i j_0 U_{\nu 0} = -v_0^i p(R_0) + \gamma v_0^i R_0 R_{0,zz} - \gamma v_0^i \frac{1}{2} (R_{0,z})^2 + D_1^i. \quad (7.89)$$

With (5.3) we compute

$$\begin{aligned} p(R_0)_s \frac{t_0^j}{|\mathbf{t}_0|^2} &= \frac{1}{|\mathbf{t}_0|} \left(p(R_0) \frac{t_0^j}{|\mathbf{t}_0|} \right)_s - p(R_0) \frac{1}{|\mathbf{t}_0|} \left(\frac{t_0^j}{|\mathbf{t}_0|} \right)_s \\ &= \frac{1}{|\mathbf{t}_0|} \left(\frac{t_0^j}{|\mathbf{t}_0|} p(R_0) \right)_s - v_0^j \kappa_0 p(R_0). \end{aligned} \quad (7.90)$$

Additionally we observe that due to (6.9)

$$D_2^j := (R_0(U_0^j - w_0^j))_\tau = (j_0 v_0^j)_\tau \quad (7.91)$$

and

$$D_3^j := \frac{1}{|\mathbf{t}_0|} (|\mathbf{t}_0| w_{\tau 0})_s R_0(U_0^j - w_0^j) = \frac{1}{|\mathbf{t}_0|} (|\mathbf{t}_0| w_{\tau 0})_s j_0 v_0^j \quad (7.92)$$

are independent of z . Now we insert (7.89) and (7.90) in (7.87) and we end up with

$$\begin{aligned} 0 &= \int_{-\infty}^{\infty} \left[-D_3^i - D_2^i - R_0(w_0^i)_\tau - \kappa_0 \gamma v_0^i \frac{1}{2} (R_{0,z})^2 + \kappa_0 D_1^i \right. \\ &\quad \left. - \left(2J_0 \Psi_\nu v_0^i + J_0 \Psi_\tau t_0^i + v_0^i \frac{J_0^2}{R_0^2} \Phi + p'(R_0) \Phi v_0^i - \gamma v_0^i (\Phi_{zz} R_0 - \Phi_z R_{0,z} + \Phi R_{0,zz}) \right)_z \right. \\ &\quad \left. + (\lambda + 2\mu) (U_{\nu 0})_{zz} v_0^i - \frac{1}{|\mathbf{t}_0|} \left(\frac{t_0^i}{|\mathbf{t}_0|} p(R_0) \right)_s + \gamma \frac{t_0^i}{|\mathbf{t}_0|^2} R_0 R_{0,zzz} \right] dz. \end{aligned} \quad (7.93)$$

By (7.84) the integrand in (7.93) goes to zero for $|z| \rightarrow \infty$, i.e.,

$$\begin{aligned} 0 &= -D_2^i - D_3^i + \kappa_0 D_1^i - \rho_0^\pm (w_0^i)_\tau - 2j_0 \left(\frac{\partial u_0^j}{\partial x^k} \right)^\pm v_0^i v_0^j v_0^k - j_0 \left(\frac{\partial u_0^j}{\partial x^k} \right)^\pm \frac{t_0^i t_0^j}{|\mathbf{t}_0|^2} v_0^k \\ &\quad - \frac{j_0^2}{(\rho_0^\pm)^2} v_0^i \left(\frac{\partial \rho_0}{\partial x^k} \right)^\pm v_0^k - p'(\rho_0^\pm) v_0^i \left(\frac{\partial \rho_0}{\partial x^k} \right)^\pm v_0^k - \frac{1}{|\mathbf{t}_0|} \left(p(\rho_0^\pm) \frac{t_0^i}{|\mathbf{t}_0|} \right)_s. \end{aligned} \quad (7.94)$$

Using the matching conditions and (7.43)–(7.45) equation (7.93) is equivalent to

$$\begin{aligned}
0 = & \lim_{a \rightarrow \infty} \left[-2aD_2^i - 2aD_3^i + 2a\kappa_0 D_1^i - \int_{-a}^a R_0(w_0^i)_\tau + \frac{\gamma}{2}\kappa_0 v_0^i (R_{0,z})^2 dz \right. \\
& - 2j_0 v_0^i \left((u_0^j)^+ v_1^j + (u_1^j)^+ v_0^j + \left(\frac{\partial u_0^j}{\partial x^k} \right)^+ v_0^j (r_1^k + av_0^k) \right) \\
& + 2j_0 v_0^i \left((u_0^j)^- v_1^j + (u_1^j)^- v_0^j + \left(\frac{\partial u_0^j}{\partial x^k} \right)^- v_0^j (r_1^k - av_0^k) \right) \\
& - j_0 t_0^i \left((u_1^j)^+ \frac{t_0^j}{|\mathbf{t}_0|^2} + \left(\frac{\partial u_0^j}{\partial x^k} \right)^+ \frac{t_0^j}{|\mathbf{t}_0|^2} (r_1^k + av_0^k) + (u_0^k)^+ \left(\frac{t_1^k}{|\mathbf{t}_0|^2} - 2 \frac{t_0^k t_0^j t_1^j}{|\mathbf{t}_0|^4} \right) \right) \\
& + j_0 t_0^i \left((u_1^j)^- \frac{t_0^j}{|\mathbf{t}_0|^2} + \left(\frac{\partial u_0^j}{\partial x^k} \right)^- \frac{t_0^j}{|\mathbf{t}_0|^2} (r_1^k - av_0^k) + (u_0^k)^- \left(\frac{t_1^k}{|\mathbf{t}_0|^2} - 2 \frac{t_0^k t_0^j t_1^j}{|\mathbf{t}_0|^4} \right) \right) \\
& - v_0^i \left(\frac{j_0^2}{(\rho_0^+)^2} + p'(\rho_0^+) \right) \left(\rho_1^+ + \left(\frac{\partial \rho_0}{\partial x^k} \right)^+ (r_1^k + av_0^k) \right) \\
& + v_0^i \left(\frac{j_0^2}{(\rho_0^-)^2} + p'(\rho_0^-) \right) \left(\rho_1^- + \left(\frac{\partial \rho_0}{\partial x^k} \right)^- (r_1^k - av_0^k) \right) \\
& \left. - \int_{-a}^a \frac{1}{|\mathbf{t}_0|} \left(p(R_0) \frac{t_0^i}{|\mathbf{t}_0|} \right)_s - \gamma \frac{t_0^i}{|\mathbf{t}_0|^2} R_0 R_{0,szz} dz \right].
\end{aligned} \tag{7.95}$$

We subtract a times (7.94)⁺ and a times (7.94)⁻ before letting $a \rightarrow \infty$ which yields

$$\begin{aligned}
0 = & - \int_0^\infty (R_0 - \rho_0^+) (w_0^i)_\tau + \frac{\gamma}{2}\kappa_0 v_0^i (R_{0,z})^2 dz - \int_{-\infty}^0 \dots dz \\
& - 2j_0 v_0^i \left((u_0^j)^+ v_1^j + (u_1^j)^+ v_0^j + \left(\frac{\partial u_0^j}{\partial x^k} \right)^+ v_0^j r_1^k \right) \\
& + 2j_0 v_0^i \left((u_0^j)^- v_1^j + (u_1^j)^- v_0^j + \left(\frac{\partial u_0^j}{\partial x^k} \right)^- v_0^j r_1^k \right) \\
& - j_0 t_0^i \left((u_1^j)^+ \frac{t_0^j}{|\mathbf{t}_0|^2} + \left(\frac{\partial u_0^j}{\partial x^k} \right)^+ \frac{t_0^j}{|\mathbf{t}_0|^2} r_1^k + (u_0^k)^+ \left(\frac{t_1^k}{|\mathbf{t}_0|^2} - 2 \frac{t_0^k t_0^j t_1^j}{|\mathbf{t}_0|^4} \right) \right) \\
& + j_0 t_0^i \left((u_1^j)^- \frac{t_0^j}{|\mathbf{t}_0|^2} + \left(\frac{\partial u_0^j}{\partial x^k} \right)^- \frac{t_0^j}{|\mathbf{t}_0|^2} r_1^k + (u_0^k)^- \left(\frac{t_1^k}{|\mathbf{t}_0|^2} - 2 \frac{t_0^k t_0^j t_1^j}{|\mathbf{t}_0|^4} \right) \right) \\
& - v_0^i \left(\frac{j_0^2}{(\rho_0^+)^2} + p'(\rho_0^+) \right) \left(\rho_1^+ + \left(\frac{\partial \rho_0}{\partial x^k} \right)^+ r_1^k \right) \\
& + v_0^i \left(\frac{j_0^2}{(\rho_0^-)^2} + p'(\rho_0^-) \right) \left(\rho_1^- + \left(\frac{\partial \rho_0}{\partial x^k} \right)^- r_1^k \right)
\end{aligned} \tag{7.96}$$

$$- \int_0^\infty \frac{1}{|\mathbf{t}_0|} \left((p(R_0) - p(\rho_0^+)) \frac{t_0^j}{|\mathbf{t}_0|} \right)_s - \gamma \frac{t_0^j}{|\mathbf{t}_0|^2} R_0 R_{0,zz} dz - \int_{-\infty}^0 \dots dz.$$

As explained in the appendix, (7.96) can be written as

$$\begin{aligned} & \llbracket \rho_\varepsilon (u_\nu)_\varepsilon - (w_\nu)_\varepsilon (\mathbf{u}_\varepsilon - \mathbf{w}_\varepsilon) + p(\rho_\varepsilon) \mathbf{v}_\varepsilon \rrbracket_1 = \\ & \quad - (w_0^j)_\tau \int_0^\infty (R_0 - \rho_0^+) dz - \int_{-\infty}^0 \dots dz \\ & \quad - \int_0^\infty \left[\frac{1}{|\mathbf{t}_0|} \left(p(R_0) \frac{t_0^j}{|\mathbf{t}_0|} \right)_s - \frac{1}{|\mathbf{t}_0|} \left(p(\rho_0^+) \frac{t_0^j}{|\mathbf{t}_0|} \right)_s \right] dz - \int_{-\infty}^0 \dots dz \\ & \quad - \kappa_0 \gamma \frac{1}{2} v_0^j \int_{-\infty}^\infty (R_{0,z})^2 dz - \gamma \frac{1}{2} \frac{t_0^j}{|\mathbf{t}_0|^2} \int_{-\infty}^\infty ((R_{0,z})^2)_s dz \quad (7.97) \end{aligned}$$

or by (5.3) equivalently

$$\begin{aligned} & \llbracket \rho_\varepsilon (u_\nu)_\varepsilon - (w_\nu)_\varepsilon (\mathbf{u}_\varepsilon - \mathbf{w}_\varepsilon) + p(\rho_\varepsilon) \mathbf{v}_\varepsilon \rrbracket_1 = \\ & \quad - (w_0^j)_\tau \int_0^\infty (R_0 - \rho_0^+) dz - \int_{-\infty}^0 \dots dz \\ & \quad - \int_0^\infty \left[\frac{1}{|\mathbf{t}_0|} \left(p(R_0) \frac{t_0^j}{|\mathbf{t}_0|} \right)_s - \frac{1}{|\mathbf{t}_0|} \left(p(\rho_0^\pm) \frac{t_0^j}{|\mathbf{t}_0|} \right)_s \right] dz - \int_{-\infty}^0 \dots dz \\ & \quad - \gamma \frac{1}{2} \frac{1}{|\mathbf{t}_0|} \int_{-\infty}^\infty \left(\frac{t_0^j}{|\mathbf{t}_0|} (R_{0,z})^2 \right)_s dz. \quad (7.98) \end{aligned}$$

Because of (7.11) we have

$$\frac{j_0^2}{R_0} - \frac{j_0^2}{\rho_0^\pm} - \gamma R_0 R_{0,zz} = p(\rho_0^\pm) - p(R_0) - \frac{\gamma}{2} R_{0,z}^2.$$

Inserting this into (7.98) we can, in view of (3.4), identify the surface stress vector σ_Γ as

$$\sigma_\Gamma = \sigma_\Gamma^j = \left(\int_0^\infty \left(\frac{j_0^2}{R_0} - \frac{j_0^2}{\rho_0^\pm} + \gamma R_{0,z}^2 \right) dz + \int_{-\infty}^0 \dots dz \right) \frac{t_0^j}{|\mathbf{t}_0|^2} + \mathcal{O}(\varepsilon). \quad (7.99)$$

□

8. Proof of Theorem 6.7

Now we will establish Theorem 6.7.

Proof. To prove the theorem we will compute the first non-vanishing order of

$$\llbracket j_\varepsilon \rrbracket \left\{ W'(\rho_\varepsilon) + \frac{|\mathbf{u}_\varepsilon - \mathbf{w}_\varepsilon|^2}{2} \right\} + \{j_\varepsilon\} \left[\left[W'(\rho_\varepsilon) + \frac{|\mathbf{u}_\varepsilon - \mathbf{w}_\varepsilon|^2}{2} \right] \right] \quad (8.1)$$

by the jump conditions in Theorem 6.4.

As we see from (6.13) and (6.14) there are no $\mathcal{O}(1)$ contributions in the jump terms in (8.1), so the leading order terms are of $\mathcal{O}(\varepsilon)$. They are given as products of the order $\mathcal{O}(\varepsilon)$ -terms in the jumps and the order $\mathcal{O}(1)$ -terms in the mean values, i.e., the leading order of (8.1) is given by

$$\begin{aligned} & \llbracket \rho_\varepsilon((u_v)_\varepsilon - (w_v)_\varepsilon) \rrbracket_1 \left\{ W'(\rho_0) + \frac{|\mathbf{u}_0 - \mathbf{w}_0|^2}{2} \right\} \\ & \quad + \{ \rho_0(u_{v0} - w_{v0}) \} \llbracket W'(\rho_\varepsilon) + \frac{|\mathbf{u}_\varepsilon - \mathbf{w}_\varepsilon|^2}{2} \rrbracket_1. \end{aligned} \quad (8.2)$$

By (7.13) we have $u_{t0} = w_{t0}$ such that $u_0^j - w_0^j = (u_{v0} - w_{v0})v_0^j$, whence

$$|\mathbf{u}_0 - \mathbf{w}_0|^2 = (u_{v0} - w_{v0})^2 \quad \text{and} \quad (u_0^j - w_0^j)u_1^j = (u_{v0} - w_{v0})u_{v1}. \quad (8.3)$$

Hence, (8.2) equals

$$\begin{aligned} & \llbracket \rho_\varepsilon((u_v)_\varepsilon - (w_v)_\varepsilon) \rrbracket_1 \left\{ W'(\rho_0) + \frac{|\mathbf{u}_0 - \mathbf{w}_0|^2}{2} \right\} \\ & \quad + \{ \rho_0(u_{v0} - w_{v0}) \} \llbracket W'(\rho_\varepsilon) + \frac{|(u_v)_\varepsilon - (w_v)_\varepsilon|^2}{2} \rrbracket_1. \end{aligned} \quad (8.4)$$

Now we use (6.13) and (6.14) to get an equality

$$\begin{aligned} & \llbracket \rho_\varepsilon((u_v)_\varepsilon - (w_v)_\varepsilon) \rrbracket_1 \left\{ W'(\rho_0) + \frac{|\mathbf{u}_0 - \mathbf{w}_0|^2}{2} \right\} \\ & \quad + \{ \rho_0(u_{v0} - w_{v0}) \} \llbracket W'(\rho_\varepsilon) + \frac{|\mathbf{u}_\varepsilon - \mathbf{w}_\varepsilon|^2}{2} \rrbracket_1 \\ & = -j_0 \left(\int_0^\infty v_0^j (U_0^j - (u_0^j)^+) dz + \int_{-\infty}^0 v_0^j (U_0^j - (u_0^j)^+) dz \right) \\ & \quad - \left(g(\rho_0^+) + \frac{j_0^2}{2(\rho_0^+)^2} \right) \int_0^\infty (R_0 - \rho_0^+) dz - \left(g(\rho_0^-) + \frac{j_0^2}{2(\rho_0^-)^2} \right) \int_{-\infty}^0 (R_0 - \rho_0^-) dz \\ & \quad - \left(\frac{1}{|\mathbf{t}_0|} (|\mathbf{t}_0|w_{t0})_s - \kappa_0 w_{v0} \right) \left(g(\rho_0^+) + \frac{j_0^2}{2(\rho_0^+)^2} \right) \int_0^\infty (R_0 - \rho_0^+) dz \\ & \quad - \left(\frac{1}{|\mathbf{t}_0|} (|\mathbf{t}_0|w_{t0})_s - \kappa_0 w_{v0} \right) \left(g(\rho_0^-) + \frac{j_0^2}{2(\rho_0^-)^2} \right) \int_{-\infty}^0 (R_0 - \rho_0^-) dz \\ & \quad - (\lambda + 2\mu)j_0^2 \int_{-\infty}^\infty \left(\frac{1}{R_0} \right)_z dz. \end{aligned} \quad (8.5)$$

To simplify the notation we define

$$A^\pm := \mp j_0 \int_0^{\pm\infty} v_0^j (U_0^j - (u_0^j)^\pm) dz \mp \left(g(\rho_0^\pm) + \frac{j_0^2}{2(\rho_0^\pm)^2} \right) \int_0^{\pm\infty} (R_0 - \rho_0^\pm) dz, \quad (8.6)$$

$$B^\pm := \mp \left(\frac{1}{|\mathbf{t}_0|} (|\mathbf{t}_0|w_{t0})_s - \kappa_0 w_{v0} \right) \left(g(\rho_0^\pm) + \frac{j_0^2}{2(\rho_0^\pm)^2} \right) \int_0^{\pm\infty} (R_0 - \rho_0^\pm) dz. \quad (8.7)$$

We will first deal with A^\pm . Due to (5.4) there exists $c \in \mathbb{R}$, which is independent of z as \mathbf{v}_0 and \mathbf{t}_0 are independent of z , such that

$$(v_0^j)_\tau = ct_0^j.$$

Therefore we get

$$\begin{aligned} \int_0^{\pm\infty} v_0^j (U_0^j - (u_0^j)^\pm)_\tau dz &= \int_0^{\pm\infty} (U_{v_0} - (u_{v_0})^\pm)_\tau dz - \int_0^{\pm\infty} (v_0^j)_\tau (U_0^j - (u_0^j)^\pm) dz \\ &= \int_0^{\pm\infty} (U_{v_0} - (u_{v_0})^\pm)_\tau dz - c \int_0^{\pm\infty} (t_0^j U_0^j - t_0^j (u_0^j)^\pm) dz \\ &\stackrel{(7.6)}{=} \int_0^{\pm\infty} ((U_{v_0} - w_{v_0})_\tau - (u_{v_0}^\pm - w_{v_0})_\tau) dz. \end{aligned} \quad (8.8)$$

Using (7.9) yields

$$\begin{aligned} &\left(g(\rho_0^\pm) + \frac{j_0^2}{2(\rho_0^\pm)^2} \right) \int_0^{\pm\infty} (R_0 - \rho_0^\pm)_\tau dz \\ &= \int_0^{\pm\infty} \left(g(\rho_0^\pm) + \frac{j_0^2}{2(\rho_0^\pm)^2} \right) (R_0 - \rho_0^\pm)_\tau dz \\ &= \int_0^{\pm\infty} \left(g(R_0) + \frac{j_0^2}{2R_0^2} - \gamma R_{0,zz} \right) R_{0,\tau} - \left(g(\rho_0^\pm) + \frac{j_0^2}{2(\rho_0^\pm)^2} \right) (\rho_0^\pm)_\tau dz. \end{aligned} \quad (8.9)$$

Inserting (8.8) and (8.9) in (8.6) implies

$$\begin{aligned} A^\pm &= \mp \int_0^{\pm\infty} \left(j_0 \left(\frac{j_0}{R_0} \right)_\tau - j_0 \left(\frac{j_0}{\rho_0^\pm} \right)_\tau \right) dz \\ &\mp \int_0^{\pm\infty} \left(g(R_0) + \frac{j_0^2}{2R_0^2} - \gamma R_{0,zz} \right) R_{0,\tau} - \left(g(\rho_0^\pm) + \frac{j_0^2}{2(\rho_0^\pm)^2} \right) (\rho_0^\pm)_\tau dz \\ &= \mp \int_0^{\pm\infty} \left(W(R_0) - W(\rho_0^\pm) + \frac{j_0^2}{2R_0} - \frac{j_0^2}{2\rho_0^\pm} \right)_\tau dz \pm \gamma \int_0^{\pm\infty} R_{0,zz} R_{0,\tau} dz. \end{aligned} \quad (8.10)$$

We observe

$$\int_{-\infty}^{\infty} R_{0,zz} R_{0,\tau} = - \int_{-\infty}^{\infty} \left(\frac{1}{2} (R_{0,z})^2 \right)_\tau dz,$$

whence (8.10) implies

$$\begin{aligned} A^+ + A^- &= - \int_0^\infty \left(W(R_0) - W(\rho_0^+) + \frac{j_0^2}{2R_0} - \frac{j_0^2}{2\rho_0^+} + \frac{\gamma}{2} R_{0,z}^2 \right)_\tau dz \\ &\quad - \int_{-\infty}^0 \left(W(R_0) - W(\rho_0^-) + \frac{j_0^2}{2R_0} - \frac{j_0^2}{2\rho_0^-} + \frac{\gamma}{2} R_{0,z}^2 \right)_\tau dz. \end{aligned} \quad (8.11)$$

This finishes our calculations for A^\pm and we turn to B^\pm .

First we observe that due to (7.9)

$$\begin{aligned}
& \left(g(\rho_0^\pm) + \frac{j_0^2}{2(\rho_0^\pm)^2} \right) \int_0^{\pm\infty} (R_0 - \rho_0^\pm) dz \\
&= \int_0^{\pm\infty} \left(g(R_0) + \frac{j_0^2}{2R_0^2} - \gamma R_{0,zz} \right) R_0 - \left(g(\rho_0^\pm) + \frac{j_0^2}{2(\rho_0^\pm)^2} \right) \rho_0^\pm dz \\
&= \int_0^{\pm\infty} \left(p(R_0) + W(R_0) + \frac{j_0^2}{2R_0} - \gamma R_0 R_{0,zz} - p(\rho_0^\pm) - W(\rho_0^\pm) - \frac{j_0^2}{2\rho_0^\pm} \right) dz.
\end{aligned} \tag{8.12}$$

From (7.11) and (7.12) we already know

$$p(R_0) - p(\rho_0^\pm) - \gamma R_0 R_{0,zz} = -\frac{j_0^2}{R_0} + \frac{j_0^2}{\rho_0^\pm} - \frac{\gamma}{2} R_{0,z}^2. \tag{8.13}$$

Inserting (8.12) and (8.13) in (8.7) we find

$$\begin{aligned}
B^+ + B^- &= \\
&- \left(\frac{1}{|\mathbf{t}_0|} (|\mathbf{t}_0| w_{t0})_s - \kappa_0 w_{v0} \right) \int_0^\infty \left(W(R_0) - W(\rho_0^+) - \frac{j_0^2}{2R_0} + \frac{j_0^2}{2\rho_0^+} - \frac{\gamma}{2} R_{0,z}^2 \right) dz \\
&- \left(\frac{1}{|\mathbf{t}_0|} (|\mathbf{t}_0| w_{t0})_s - \kappa_0 w_{v0} \right) \int_{-\infty}^0 \left(W(R_0) - W(\rho_0^-) - \frac{j_0^2}{2R_0} + \frac{j_0^2}{2\rho_0^-} - \frac{\gamma}{2} R_{0,z}^2 \right) dz.
\end{aligned} \tag{8.14}$$

Plugging (8.11) and (8.14) in (8.5) we get

$$\begin{aligned}
& \llbracket \rho_\varepsilon((u_v)_\varepsilon - (w_v)_\varepsilon) \rrbracket_1 \left\{ W'(\rho_0) + \frac{|\mathbf{u}_0 - \mathbf{w}_0|^2}{2} \right\} \\
& \quad + \{ \rho_0((u_{v0}) - w_{v0}) \} \llbracket W'(\rho_\varepsilon) + \frac{|\mathbf{u}_\varepsilon - \mathbf{w}_\varepsilon|^2}{2} \rrbracket_1 \\
&= - \int_0^\infty \left(W(R_0) - W(\rho_0^+) + \frac{j_0^2}{2R_0} - \frac{j_0^2}{2\rho_0^+} + \frac{\gamma}{2} R_{0,z}^2 \right) dz \\
& \quad - \int_{-\infty}^0 \left(W(R_0) - W(\rho_0^-) + \frac{j_0^2}{2R_0} - \frac{j_0^2}{2\rho_0^-} + \frac{\gamma}{2} R_{0,z}^2 \right) dz \\
& \quad - \left(\frac{1}{|\mathbf{t}_0|} (|\mathbf{t}_0| w_{t0})_s - \kappa_0 w_{v0} \right) \int_0^\infty \left(W(R_0) - W(\rho_0^+) - \frac{j_0^2}{2R_0} + \frac{j_0^2}{2\rho_0^+} - \frac{\gamma}{2} R_{0,z}^2 \right) dz \\
& \quad - \left(\frac{1}{|\mathbf{t}_0|} (|\mathbf{t}_0| w_{t0})_s - \kappa_0 w_{v0} \right) \int_{-\infty}^0 \left(W(R_0) - W(\rho_0^-) - \frac{j_0^2}{2R_0} + \frac{j_0^2}{2\rho_0^-} - \frac{\gamma}{2} R_{0,z}^2 \right) dz \\
& \quad - (\lambda + 2\mu) j_0^2 \int_{-\infty}^\infty \left(\left(\frac{1}{R_0} \right)_z \right)^2 dz. \tag{8.15}
\end{aligned}$$

Recalling equation (3.10) and definitions (6.17), (6.18) we can identify W_Γ as

$$\begin{aligned} W_\Gamma = & \int_0^\infty \left(W(R_0) - W(\rho_0^+) + \frac{1}{2} \frac{j_0^2}{R_0} - \frac{1}{2} \frac{j_0^2}{\rho_0^+} + \frac{\gamma}{2} R_{0,z}^2 \right) dz \\ & + \int_{-\infty}^0 \left(W(R_0) - W(\rho_0^-) + \frac{1}{2} \frac{j_0^2}{R_0} - \frac{1}{2} \frac{j_0^2}{\rho_0^-} + \frac{\gamma}{2} R_{0,z}^2 \right) dz. \end{aligned}$$

and obtain the assertion of Theorem 6.7. \square

9. Kinetic relation

We will show that the leading order jump conditions stated in Theorem 6.4 consist of the two conservation laws for mass and momentum and in addition of a kinetic relation. In particular, when the direction of the mass flux is given, the normal velocity of the interface is determined by the state on one side of the interface.

LEMMA 9.1 Let $\rho_0^+, \rho_0^-, u_{v0}^+, u_{v0}^-, u_{t0}^+, u_{t0}^-, w_{v0}$ satisfy the jump conditions from Lemma 7.1, such that $\rho_0^- \notin [\alpha, \beta]$, where α, β are the Maxwell points of W , and the equal area construction

$$0 = - \int_{\frac{1}{\rho_0^-}}^{\frac{1}{\rho_0^+}} p(\rho_0) d \left(\frac{1}{\rho_0} \right) + \{p(\rho_0)\} \llbracket \frac{1}{\rho_0} \rrbracket \quad (9.1)$$

is possible. If $\rho_0^-, u_{v0}^-, u_{t0}^-$ and the sign of j_0 are specified, then $\rho_0^+, u_{v0}^+, u_{t0}^+, w_{v0}$ are determined.

Proof. Due to (7.3) we know u_{t0}^+ equals u_{t0}^- . Using (7.2) we get the following expression for the mass flux

$$j_0^2 = - \frac{\llbracket p(\rho_0) \rrbracket}{\llbracket \frac{1}{\rho_0} \rrbracket}. \quad (9.2)$$

Inserting (9.2) in (7.4) we find

$$\llbracket g(\rho_0) \rrbracket = - \frac{1}{2} j_0^2 \llbracket \left(\frac{1}{\rho_0} \right)^2 \rrbracket = \frac{1}{2} \llbracket p(\rho_0) \rrbracket \frac{\left(\frac{1}{\rho_0^+} \right)^2 - \left(\frac{1}{\rho_0^-} \right)^2}{\frac{1}{\rho_0^+} - \frac{1}{\rho_0^-}} = \llbracket p(\rho_0) \rrbracket \left\{ \frac{1}{\rho_0} \right\}. \quad (9.3)$$

This is equivalent to

$$0 = \llbracket g(\rho_0) - \frac{p(\rho_0)}{\rho_0} \rrbracket + \{p(\rho_0)\} \llbracket \frac{1}{\rho_0} \rrbracket. \quad (9.4)$$

From equation (2.5) we conclude

$$\frac{d}{d \frac{1}{\rho_0}} \left(g(\rho_0) - \frac{p(\rho_0)}{\rho_0} \right) = -p(\rho_0). \quad (9.5)$$

Inserting (9.5) in (9.4) we obtain

$$0 = - \int_{\frac{1}{\rho_0^-}}^{\frac{1}{\rho_0^+}} p(\rho_0) d \left(\frac{1}{\rho_0} \right) + \{p(\rho_0)\} \llbracket \frac{1}{\rho_0} \rrbracket. \quad (9.6)$$

Note that (9.6) is an equal area rule which determines ρ_0^+ . Then (9.2) prescribes j_0 which we can, in turn, use to obtain w_{v0} and u_{v0}^+ from (7.1). \square

As shown in [5] the kinetic relation derived in Lemma 9.1 guarantees the solvability of the free boundary value problem for ρ_0, \mathbf{u}_0 . Hence, when we want to determine $\rho_1^+, u_{v1}^+, u_{t1}^+$ and w_{v1} from ρ_1^-, u_{v1}^- and u_{t1}^- , we can assume that we can, at least numerically, calculate ρ_0, \mathbf{u}_0 and \mathbf{w}_0 . Furthermore R_0 and \mathbf{U}_0 are traveling wave solutions of the non-viscous problem. Their phase portraits can be found in [4].

For the following lemma we assume that we have solved, at least numerically, the inner and outer problem of zeroth order. Hence, we know the right hand side of the first order jump conditions in Theorem 6.4.

LEMMA 9.2 Let $\rho_1^+, \rho_1^-, u_{v1}^+, u_{v1}^-, u_{t1}^+, u_{t1}^-, w_{v1}$ satisfy the first order of the jump conditions in Theorem 6.4. If $\rho_1^-, u_{v1}^-, u_{t1}^-$ are given and $j_0 \neq 0$, then $\rho_1^+, u_{v1}^+, u_{t1}^+, w_{v1}$ are determined.

Proof. We start by determining u_{t1}^+ . We multiply (7.77) by \mathbf{t}_0 and obtain

$$j_0 u_{t1}^+ = h_1, \quad (9.7)$$

where h_1 comprises terms depending only on $\rho_1^-, u_{v1}^-, u_{t1}^-$ and terms of zeroth order and their derivatives. As $j_0 \neq 0$ equation (9.7) prescribes the value of u_{t1}^+ . To determine $\rho_1^+, u_{v1}^+, w_{v1}$ we consider (7.58), (7.68) and (7.77) times \mathbf{v}_0 . We obtain a system of linear equations for $\rho_1^+, u_{v1}^+, w_{v1}$ which reads

$$\underbrace{\begin{pmatrix} \frac{j_0}{\rho_0^+} & \rho_0^+ & -\rho_0^+ + \rho_0^- \\ g'(\rho_0^+) & \frac{j_0}{\rho_0^+} & -\frac{j_0}{\rho_0^+} + \frac{j_0}{\rho_0^-} \\ \left(\frac{j_0}{\rho_0^+}\right)^2 + p'(\rho_0^+) & 2j_0 & 0 \end{pmatrix}}_{=:A} \begin{pmatrix} \rho_1^+ \\ u_{v1}^+ \\ w_{v1} \end{pmatrix} = \begin{pmatrix} h_2 \\ h_3 \\ h_4 \end{pmatrix}, \quad (9.8)$$

where h_2, \dots, h_4 only contain terms depending on $\rho_1^-, u_{v1}^-, u_{t1}^-$, terms of zeroth order and their derivatives and r_1 . An easy calculation shows

$$\det A = \frac{j_0}{\rho_0^-} (\rho_0^+ - \rho_0^-)^2 \left(g'(\rho_0^+) - \frac{j_0^2}{(\rho_0^+)^3} \right) = \frac{j_0}{\rho_0^- (\rho_0^+)^3} (\rho_0^+ - \rho_0^-)^2 \left(-\frac{\partial p}{\partial \left(\frac{1}{\rho_0}\right)}(\rho_0^+) + \frac{\llbracket p(\rho_0) \rrbracket}{\llbracket \frac{1}{\rho_0} \rrbracket} \right).$$

The last expression vanishes if and only if the tangent at ρ_0^+ and the secant connecting ρ_0^+ to ρ_0^- coincide. However this is not possible due to the equal area rule (9.1).

Thus we have a relation $w_{v1}(r_1)$ such that we end up with an ODE to determine the interface position up to the first order. \square

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A. Appendix: List of the ε -expansions of the jump brackets and proof of Lemma 7.3

In this section we provide identities for the first two orders of the jump brackets of mass, Gibbs free energy and pressure.

A.1 Mass flux

We have

$$\begin{aligned} & \llbracket \rho_\varepsilon((u_\nu)_\varepsilon - (w_\nu)_\varepsilon) \rrbracket \\ &= \lim_{z \rightarrow 0, z > 0} \rho_\varepsilon(\mathbf{r}_\varepsilon + z\mathbf{v}_\varepsilon) (v_\varepsilon^i u_\varepsilon^i(\mathbf{r}_\varepsilon + z\mathbf{v}_\varepsilon) - v_\varepsilon^i w_\varepsilon^i) - \\ & \quad \lim_{z \rightarrow 0, z < 0} \rho_\varepsilon(\mathbf{r}_\varepsilon + z\mathbf{v}_\varepsilon) (v_\varepsilon^i u_\varepsilon^i(\mathbf{r}_\varepsilon + z\mathbf{v}_\varepsilon) - v_\varepsilon^i w_\varepsilon^i). \quad (\text{A.1}) \end{aligned}$$

To determine the first orders of the jump we will look at the ε -expansion of ρ_ε :

$$\begin{aligned} & \rho_\varepsilon(\mathbf{r}_\varepsilon + z\mathbf{v}_\varepsilon) (v_\varepsilon^i u_\varepsilon^i(\mathbf{r}_\varepsilon + z\mathbf{v}_\varepsilon) - v_\varepsilon^i w_\varepsilon^i) \\ &= \rho_0(\mathbf{r}_0 + z\mathbf{v}_0) (v_0^i u_0^i(\mathbf{r}_0 + z\mathbf{v}_0) - v_0^i w_0^i) \\ & \quad + \varepsilon \left[\left(\rho_1(\mathbf{r}_0 + z\mathbf{v}_0) + \left(\frac{\partial \rho_0}{\partial x^k} \right) (\mathbf{r}_0 + z\mathbf{v}_0) (r_1^k + z v_1^k) \right) (v_0^i u_0^i(\mathbf{r}_0 + z\mathbf{v}_0) - v_0^i w_0^i) \right. \\ & \quad \left. + \rho_0(\mathbf{r}_0 + z\mathbf{v}_0) \left(v_1^i u_0^i(\mathbf{r}_0 + z\mathbf{v}_0) + v_0^i u_1^i(\mathbf{r}_0 + z\mathbf{v}_0) + v_0^i \left(\frac{\partial u_0^i}{\partial x^k} \right) (\mathbf{r}_0 + z\mathbf{v}_0) (r_1^k + z v_1^k) \right. \right. \\ & \quad \left. \left. - v_1^i w_0^i - v_0^i w_1^i \right) \right] + \mathcal{O}(\varepsilon^2) \end{aligned}$$

Hence we obtain the following two jump conditions:

$$\llbracket \rho_\varepsilon((u_\nu)_\varepsilon - (w_\nu)_\varepsilon) \rrbracket_0 = [\rho_0(u_{\nu 0} - w_{\nu 0})] \quad (\text{A.2})$$

$$\begin{aligned} \llbracket \rho_\varepsilon((u_\nu)_\varepsilon - (w_\nu)_\varepsilon) \rrbracket_1 &= \left(\rho_1^+ + \left(\frac{\partial \rho_0}{\partial x^k} \right)^+ r_1^k \right) (v_0^i (u_0^i)^+ - v_0^i w_0^i) \\ & \quad + \rho_0^+ \left(v_1^i (u_0^i)^+ + v_0^i (u_1^i)^+ + v_0^i \left(\frac{\partial u_0^i}{\partial x^k} \right)^+ r_1^k - v_1^i w_0^i - v_0^i w_1^i \right) \\ & \quad - \left(\rho_1^- + \left(\frac{\partial \rho_0}{\partial x^k} \right)^- r_1^k \right) (v_0^i (u_0^i)^- - v_0^i w_0^i) \\ & \quad - \rho_0^- \left(v_1^i (u_0^i)^- + v_0^i (u_1^i)^- + v_0^i \left(\frac{\partial u_0^i}{\partial x^k} \right)^- r_1^k - v_1^i w_0^i - v_0^i w_1^i \right) \end{aligned} \quad (\text{A.3})$$

A.2 Balance of Gibbs free energy

$$\left[\left[\frac{1}{2}((u_v)_\varepsilon - (w_v)_\varepsilon)^2 + g(\rho_\varepsilon) \right] \right]_0 = \left[\left[\frac{1}{2}(u_{v0} - w_{v0})^2 + g(\rho_0) \right] \right] \quad (\text{A.4})$$

$$\begin{aligned} \left[\left[\frac{1}{2}((u_v)_\varepsilon - (w_v)_\varepsilon)^2 + g(\rho_\varepsilon) \right] \right]_1 &= - (u_{v0}^+ - w_{v0})w_{v1} + (u_{v0}^- - w_{v0})w_{v1} \\ &+ (u_{v0}^+ - w_{v0}) \left((u_0^j)^+ v_1^j + (u_1^j)^+ v_0^j + \left(\frac{\partial u_0^j}{\partial x^k} \right)^+ v_0^j r_1^k \right) \\ &- (u_{v0}^- - w_{v0}) \left((u_0^j)^- v_1^j + (u_1^j)^- v_0^j + \left(\frac{\partial u_0^j}{\partial x^k} \right)^- v_0^j r_1^k \right) \\ &+ g'(\rho_0^+) \rho_1^+ + g'(\rho_0^+) \left(\frac{\partial \rho_0}{\partial x^k} \right)^+ r_1^k \\ &- g'(\rho_0^-) \rho_1^- - g'(\rho_0^-) \left(\frac{\partial \rho_0}{\partial x^k} \right)^- r_1^k \end{aligned} \quad (\text{A.5})$$

The derivation of (A.4) and (A.5) is analogous to the derivation of (A.2) and (A.3).

A.3 Momentum balance

$$\left[\rho_\varepsilon ((u_v)_\varepsilon - (w_v)_\varepsilon) (\mathbf{u}_\varepsilon - \mathbf{w}_\varepsilon) + p(\rho_\varepsilon) \mathbf{v}_\varepsilon \right]_0 = \left[\rho_0 (u_{v0}) - w_{v0} (\mathbf{u}_0 - \mathbf{w}_0) + p(\rho_0) \mathbf{v}_0 \right] \quad (\text{A.6})$$

$$\left[\rho_\varepsilon ((u_v)_\varepsilon - (w_v)_\varepsilon) (\mathbf{u}_\varepsilon - \mathbf{w}_\varepsilon) + p(\rho_\varepsilon) \mathbf{v}_\varepsilon \right]_1 \quad (\text{A.7})$$

$$\begin{aligned} &= 2j_0 v_0^i \left((u_0^j)^+ v_1^j + (u_1^j)^+ v_0^j + \left(\frac{\partial u_0^j}{\partial x^k} \right)^+ v_0^j r_1^k \right) \\ &- 2j_0 v_0^i \left((u_0^j)^- v_1^j + (u_1^j)^- v_0^j + \left(\frac{\partial u_0^j}{\partial x^k} \right)^- v_0^j r_1^k \right) \\ &+ j_0 t_0^i \left((u_1^j)^+ \frac{t_0^j}{|\mathbf{t}_0|^2} + \left(\frac{\partial u_0^j}{\partial x^k} \right)^+ \frac{t_0^j}{|\mathbf{t}_0|^2} r_1^k + (u_0^k)^+ \left(\frac{t_1^k}{|\mathbf{t}_0|^2} - 2 \frac{t_0^k t_0^j t_1^j}{|\mathbf{t}_0|^4} \right) \right) \\ &- j_0 t_0^i \left((u_1^j)^- \frac{t_0^j}{|\mathbf{t}_0|^2} + \left(\frac{\partial u_0^j}{\partial x^k} \right)^- \frac{t_0^j}{|\mathbf{t}_0|^2} r_1^k + (u_0^k)^- \left(\frac{t_1^k}{|\mathbf{t}_0|^2} - 2 \frac{t_0^k t_0^j t_1^j}{|\mathbf{t}_0|^4} \right) \right) \\ &+ v_0^i \left(\frac{j_0^2}{(\rho_0^+)^2} + p'(\rho_0^+) \right) \left(\rho_1^+ + \left(\frac{\partial \rho_0}{\partial x^k} \right)^+ r_1^k \right) \\ &- v_0^i \left(\frac{j_0^2}{(\rho_0^-)^2} + p'(\rho_0^-) \right) \left(\rho_1^- + \left(\frac{\partial \rho_0}{\partial x^k} \right)^- r_1^k \right) \end{aligned}$$

The derivation of (A.6) and (A.7) is analogous to the derivation of (A.2) and (A.3).

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