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Implicit time discretization of the mean curvature flow with a discontinuous forcing term

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We consider an implicit time discretization for the motion of a hypersurface driven by its anisotropic mean curvature. We prove some convergence results of the scheme under very general assumptions on the forcing term, which include in particular the case of a typical path of the Brownian motion. We compare this limit with other available solutions, whenever they are defined. As a by-product of the analysis, we also provide a simple proof of the coincidence of the limit flow with the regular evolutions, defined for small times, in the case of a regular forcing term.

1. Introduction

Mean curvature flow has attracted a lot of attention in the past few years. Although it is one of the simplest evolutions of hypersurfaces of $\mathbb{R}^n$, in its analysis many difficult issues arise, mainly related to the formation of singularities, which sometimes lead to changes of the topology. To deal with this phenomenon, several notions of weak solutions have been proposed, such as (to mention but a few) the varifold theory of Brakke [10], the level-set solution defined through viscosity theory [18, 19, 15], the minimal barrier method of De Giorgi [16], the limit of a reaction-diffusion equations [14, 21] and the minimizing movements method [1, 25, 2], which corresponds to an implicit time-discrete scheme.

Each of these methods has different features and presents advantages and disadvantages. In particular, the level-set method always provides a unique solution, globally defined in time in the class of compact subsets of $\mathbb{R}^n$, but it is often very difficult to prove that such a solution is a regular hypersurface. There are even some singular situations in which this solution becomes a compact set with nonempty interior, showing the so-called *fattening phenomenon*. The minimal barrier method is a geometric counterpart of the level-set method and produces essentially the same solution [6].

On the contrary, the minimizing movements method produces a solution, called the *flat flow*, which may be nonunique but is always a (possibly nonsmooth) hypersurface. One of the difficulties in this approach is to show that the solution coincides with the classical smooth solution, whenever the latter exists, a property which is very easy to prove in the context of level-set viscosity solutions. One faces similar difficulties in proving that the flat flow is always contained in the level-set solution.

In this paper we study the (anisotropic) mean curvature flow with a possibly discontinuous driving force, by adapting the minimizing movements method, which has been originally developed

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without any forcing term. More precisely, we consider the evolution $E(t)$ of a set whose boundary is driven by the velocity

$$V(x,t) = -(\kappa_\phi(x,t) + g(x,t))n_\phi(x,t) \tag{1}$$

for any $x \in \partial E(t)$, where $\kappa_\phi(x,t)$ and $n_\phi(x,t)$ are respectively the ϕ-curvature and ϕ-normal to $\partial E(t)$ at x (see Section 2 for the precise definitions, and [9] for a general introduction on curvatures in Finsler geometry).

The purpose of this paper is twofold:

1. We extend the method of minimizing movements (and the proofs of consistency in [13]) to evolutions with a driving force, providing simple proofs of the coincidence with regular solutions and the inclusion in the level-set solution (see Cor. 4.6 and Props. 4.8, 4.9).
2. Our approach applies to the case where the forcing term is discontinuous. One important example is a forcing term which is the time derivative of a Hölder continuous function $G(t)$, e.g., a typical path of the Brownian motion dW/dt. It also covers the case of spatially correlated Brownian motion, typically of the form $g(x,t) = dW/dt(t) + g^0(x,t)$ where g^0 is Lipschitz-continuous in x and continuous in t (see (18) for a precise formulation). A theory yielding existence and uniqueness for such evolutions, based on a level-set formulation in the framework of viscosity theory, has recently been developed in [23, 24], and a corresponding theory in the framework of minimal barriers, valid only for x-independent forcing terms, has been proposed in [17]. We also refer to [26, 27] for a similar approach to a related problem, which still uses an implicit time discretization procedure.

We do not address in the present paper the issue of continuity in time (in a suitable topology) of the limit flat flow, even if we prove some weaker continuity results with respect to the Hausdorff distance (see Props. 4.3, 4.4 and Remark 4.5).

The plan of this paper is as follows. In Section 2, we define the appropriate notion of sub- and superflow associated to the evolution equation (1), and we recall the definition of maximal and minimal barriers in the sense of De Giorgi. These allow one to define generalized evolutions, which are essentially equivalent to the evolutions defined in terms of level sets of viscosity solutions, when the forcing term is regular enough.

Then our implicit time-discretization scheme is defined in Section 3, and we show that it is consistent with our sub- and superflows (Thm. 3.3). As a corollary, we obtain a comparison results for sub- and superflows, which follows from the monotonicity of our scheme.

This consistency result is used in Section 4 to study the convergence of our time-discrete scheme, as the time step goes to zero. We define a notion of weak solution starting from an initial surface ∂E (given by $\Gamma(t) = E^*(t) \setminus E_*(t)$, where $E_*(t) \subset E^*(t)$ are two evolving sets with $E_*(0) = \mathrm{int}(E)$, $E^*(0) = \overline{E}$), which coincides with the barrier solution as long as the latter is unique (Cor. 4.6). Under additional assumptions on the evolution law, we deduce that it is contained in the zero level-set of the viscosity solution (Prop. 4.8). This is also true for a particular class of stochastic evolutions (Prop. 4.9).

In Sections 4.2–4.3, under some further assumption on the forcing term (which still allows a stochastic forcing), we build a level-set evolution $u(t)$ starting from any bounded, uniformly continuous function u_0. In particular, for all initial data $\{u_0 = s\}$ but a countable number, it shows that we can define a generalized flow $\{u(t) = s\}$ which remains a continuous hypersurface (with possible singularities). We do not prove that it is unique, though. If the forcing is regular, $u(t)$ is the same as the unique viscosity solution of the geometric equation associated to the flow. In general, we expect it to coincide with the solution defined by Lions and Souganidis in [23, 24].

2. Preliminary definitions and results

Let $\phi : \mathbb{R}^N \to \mathbb{R}$ be a *norm* on $\mathbb{R}^N$ (that is, an even, convex, one-homogeneous function) such that $\phi \in C^2(\mathbb{R}^N \setminus \{0\})$ (we shall simply say that ϕ is *smooth*) and $\nabla^2(\phi^2) \geqslant c\,\mathrm{Id}$ for some $c > 0$, so that ϕ is uniformly convex or *elliptic*. Most of our results could be extended to more general norms (or even possibly noneven convex one-homogeneous functions), but the proof of the consistency theorem 3.3, in the form presented here, needs such a regularity. Moreover, providing a clear and sound definition of a sub/superflow, as in Definition 2.1 below, is more difficult if the anisotropy is only Lipschitz-continuous, or nonelliptic.

Let ϕ° be the *polar norm*, that is, $\phi^\circ(\xi) := \sup_{\phi(\eta)\leqslant 1} \xi \cdot \eta$ for all $\xi \in \mathbb{R}^N$. It turns out that ϕ° is smooth and elliptic. The couple (ϕ, ϕ°) will be referred as the *anisotropy*. A ball of radius $r > 0$ centred at $x_0 \in \mathbb{R}^N$ for the norm ϕ, i.e., the set $W_\phi(x_0, r) := \{\phi(x - x_0) \leqslant \rho\}$, will be called a *Wulff shape* (we set for simplicity $W_\phi := W_\phi(0, 1)$).

When $E, F \subset \mathbb{R}^N$, we denote by $\mathrm{dist}_\phi(E, F)$ the distance between E and F with respect to ϕ:

$$\mathrm{dist}_\phi(E, F) := \inf_{x\in E,\, y\in F} \phi(x - y).$$

Given a set $E \subset \mathbb{R}^N$, we also define $d_E(x)$, the *signed distance* function to ∂E (with respect to the norm ϕ), by

$$d_E(x) := \inf_{y\in E} \phi(x - y) - \inf_{y\in\mathbb{R}^N\setminus E} \phi(y - x).$$

We let $n_\phi(x) := \nabla\phi^\circ(\nabla d_E(x))$ and $\kappa_\phi(x) := \mathrm{div}\, n_\phi(x)$ be respectively the *ϕ-normal* and *ϕ-curvature* of ∂E at x. Notice that if ∂E is of class C^2, then the functions n_ϕ and κ_ϕ are defined and continuous in an open neighbourhood of ∂E. We refer to [9] for a general introduction to the anisotropic curvature flow.

We say that E satisfies an *interior* (resp. *exterior*) *εW_ϕ-condition*, $\varepsilon > 0$, if $E = \{d_E < -\varepsilon\} + \varepsilon W_\phi$ (resp. $\mathbb{R}^N \setminus E = \{d_E > \varepsilon\} + \varepsilon W_\phi$), which is equivalent to requiring that at each point of ∂E, there is a Wulff shape of radius ε inside E (resp., outside E) that is tangent to ∂E at x.

2.1 *Evolution law*

DEFINITION 2.1 Let $E(t) \subset \mathbb{R}^N$, $t \in [t_0, t_1]$, be closed sets. We say that $E(t)$ is a *superflow* of (1) if there exist a bounded open set $A \subset \mathbb{R}^N$ with $\bigcup_{t_0\leqslant t\leqslant t_1} \partial E(t) \times \{t\} \subset A \times [t_0, t_1]$ and $\delta > 0$ such that $d(x, t) = d_{E(t)}(x) \in C^0([t_0, t_1]; C^2(A))$, and

$$d(x, s) - d(x, t) \geqslant \int_t^s \mathrm{div}\, \nabla\phi^\circ(\nabla d)(x, \tau)\, \mathrm{d}\tau + G(x, s) - G(x, t) + \delta(s - t) \tag{2}$$

for a.e. $x \in A$ and any t, s with $t_0 \leqslant t \leqslant s \leqslant t_1$, where $G(x, t) := \int_0^t g(x, s)\, \mathrm{d}s$.

We say that $E(t)$ is a *subflow* whenever there exist $A \subset \mathbb{R}^N$ as above and $\delta < 0$ such that the reverse inequality holds in (2).

We denote by $\mathcal{F}^+$ (resp. $\mathcal{F}^-$) the family of all superflows (resp. subflows) of (1).

We observe that if g is continuous in (x,t), and d is C^1 in t, condition (2) is equivalent to

$$\frac{\partial d}{\partial t} > \operatorname{div} \nabla\phi^\circ(\nabla d) + g \quad \text{in } A \times [t_0, t_1].$$

On the other hand, Definition 2.1 still makes sense if the driving term is the "time-derivative" of a function $G \in C^0([t_0,t_1]; L^\infty(A))$, even when G is nondifferentiable with respect to t.

2.2 *Barriers*

We recall the definition of minimal and maximal barrier in the sense of De Giorgi. We refer to [8, 5, 7] for a more general introduction to this topic.

DEFINITION 2.2 We say that a function $\Phi : [t_0,+\infty) \to \mathcal{P}(\mathbb{R}^N)$ ($\mathcal{P}(\mathbb{R}^N)$ is the set of all subsets of $\mathbb{R}^N$) is a *barrier* with respect to $\mathcal{F}^+$ if for any $\Sigma(t) \in \mathcal{F}^+$ with $t \in [a,b] \subset [t_0,+\infty)$, $\Sigma(a) \subseteq \Phi(a)$ implies $\Sigma(b) \subseteq \Phi(b)$.

Similarly, we say that Φ is a *barrier* with respect to $\mathcal{F}^-$ if for any $\Sigma(t) \in \mathcal{F}^-$ with $t \in [a,b] \subset [t_0,+\infty)$, $\Sigma(a) \supseteq \Phi(a)$ implies $\Sigma(b) \supseteq \Phi(b)$.

In the following we denote by $\mathcal{B}^\pm_{t_0}$ the class of all barriers with respect to $\mathcal{F}^\pm$, defined on $[t_0,+\infty)$.

DEFINITION 2.3 Let $E \subseteq \mathbb{R}^N$, $t_0 \in \mathbb{R}$. The *minimal barrier* $\mathcal{M}(E,t_0) : [t_0,+\infty) \to \mathcal{P}(\mathbb{R}^N)$ starting from E at time t_0 is defined as

$$\mathcal{M}(E,t_0)(t) := \bigcap \{\Phi(t) : \Phi \in \mathcal{B}^+_{t_0},\ \Phi(t_0) \supseteq E\}.$$

We define the *maximal barrier* $\mathcal{N}(E,t_0) : [t_0,+\infty) \to \mathcal{P}(\mathbb{R}^N)$ starting from E at time t_0 as

$$\mathcal{N}(E,t_0)(t) := \bigcup \{\Phi(t) : \Phi \in \mathcal{B}^-_{t_0},\ \Phi(t_0) \subseteq E\}.$$

We also define the *upper* and *lower regularized barriers as*

$$\mathcal{M}_*(E,t_0)(t) := \bigcup_{\rho>0} \mathcal{M}(E^-_\rho,t_0)(t), \qquad \mathcal{M}^*(E,t_0)(t) := \bigcap_{\rho>0} \mathcal{M}(E^+_\rho,t_0)(t),$$
$$\mathcal{N}_*(E,t_0)(t) := \bigcup_{\rho>0} \mathcal{N}(E^-_\rho,t_0)(t), \qquad \mathcal{N}^*(E,t_0)(t) := \bigcap_{\rho>0} \mathcal{N}(E^+_\rho,t_0)(t),$$

where $E^\pm_\rho = \{d_E \leqslant \pm\rho\}$.

We recall the following result, proved in [6] (see also [22] for the case of the motion by mean curvature).

THEOREM 2.4 Assume that $G(x,t) = \int_0^t g(x,s)\,\mathrm{d}s$ with g continuous. Then $\mathcal{M}^*(E,t_0)(t) = \mathcal{N}^*(E,t_0)(t)$ and $\mathcal{M}_*(E,t_0)(t) = \mathcal{N}_*(E,t_0)(t)$ for any $E \subset \mathbb{R}^N$ and $t \geqslant t_0$. Moreover, the set $\mathcal{M}^*(E,t_0)(t) \setminus \mathcal{M}_*(E,t_0)(t)$ coincides with the zero level-set of the viscosity solution of the parabolic equation corresponding to (1).

The parabolic equation mentioned here is equation (21) or (22). In the following, we shall omit the explicit dependence of barriers on t_0 whenever $t_0 = 0$.

2.3 *Anisotropic total variation*

The *total variation* of a function $w \in L^1(\Omega)$ is defined as

$$\sup\left\{\int_\Omega u(x)\,\mathrm{div}\,\psi(x)\,\mathrm{d}x : \psi \in C_0^1(\Omega;\mathbb{R}^N),\ |\psi(x)| \leqslant 1\ \forall x \in \Omega\right\}.$$

It turns out that it is finite if and only if the distributional derivative Dw is a bounded Radon measure. In this case, the total variation is equal to the variation $|Dw|(\Omega) = \int_\Omega |Dw|$ of the measure Dw, and w belongs to the space $BV(\Omega)$ of functions with bounded variation.

Given a couple (ϕ, ϕ°) of mutually polar norms in $\mathbb{R}^N$ (an anisotropy), one defines in the same way the *anisotropic total variation*

$$\int_\Omega \phi^\circ(Dw) = \sup\left\{\int_\Omega u(x)\,\mathrm{div}\,\psi(x)\,\mathrm{d}x : \psi \in C_0^1(\Omega;\mathbb{R}^N),\ \phi(\psi(x)) \leqslant 1\ \forall x \in \Omega\right\}.$$

Clearly, it is finite if and only if $w \in BV(\Omega)$. If $w = \chi_E$, the characteristic function of a measurable set E, then $w \in BV(\Omega)$ if and only if E is a set of finite perimeter in Ω (a *Caccioppoli set*). In this case, one can define a reduced boundary ∂^*E (which is $\mathcal{H}^{N-1}$-equivalent to the measure-theoretical boundary, that is, the set of points where E has Lebesgue density neither 0 nor 1), on which a normal unit vector $\nu_E(x)$ is well defined, and $D\chi_E = \nu_E \mathcal{H}^{N-1} \llcorner \partial^*E$. Then

$$\int_\Omega |D\chi_E| = \mathcal{H}^{N-1}(\partial^*E) \quad \text{and} \quad \int_\Omega \phi^\circ(D\chi_E) = \int_{\partial^*E} \phi^\circ(\nu_E(x))\,\mathrm{d}\mathcal{H}^{N-1}(x).$$

See [20, 3] for more details.

3. The implicit time discretization

Let Ω be a bounded, convex, open subset of $\mathbb{R}^N$. Let $G \in C^0([0,+\infty); L^\infty(\Omega))$ and let $\omega_{G,T}$ be its modulus of continuity in $[0,T]$. Let (ϕ,ϕ°) be the anisotropy, which we assume to be smooth and elliptic. Let $E \subseteq \mathbb{R}^N$. Given $s > t \geqslant 0$, let w denote the unique solution of

$$\min_{w \in L^2(\Omega)} \int_\Omega \phi^\circ(Dw) + \frac{1}{2(s-t)} \int_\Omega (w(x) - d_E(x) - G(x,s) + G(x,t))^2\,\mathrm{d}x. \tag{3}$$

We let $T_{t,s}(E) = \{x \in \Omega : w(x) < 0\}$. The existence and uniqueness of w minimizing (3) does not raise any difficulty, since the energy which is minimized is trivially strictly convex, and lower-semicontinuous in $L^2(\Omega)$.

Notice that the set $T_{t,s}(E)$ is the minimizer of a prescribed curvature problem, with bounded mean curvature. Indeed, reasoning as in [12, 11, 4], one can check that this set is a solution of the variational problem

$$\min\left(\int_{\Omega\cap\partial^*F} \phi^\circ(\nu_F(x))\,\mathrm{d}\mathcal{H}^{N-1}(x) + \frac{1}{s-t}\int_F (d_E(x) + G(x,s) - G(x,t))\,\mathrm{d}x\right),$$

where the minimum is taken over the subsets F of Ω of finite perimeter. It follows that the set $T_{t,s}(E)$ has boundary of class $C^{1,\alpha}$ inside Ω, outside a compact singular set of zero $\mathcal{H}^{N-1}$-dimension [1] (when $N = 2$, the set $T_{t,s}(E)$ has boundary of class $C^{1,1}$). The variational problem above is a generalization of the approach proposed in [1, 25], for building mean curvature flows without driving terms, through an implicit time discretization.

For $s = t + h$, the Euler–Lagrange equation for w at a point $x \in \partial T_{t,t+h}(E)$ formally reads

$$d_E(x) = -h\left(\kappa_\phi(x) + \frac{G(x, t+h) - G(x,t)}{h}\right),$$

with κ_ϕ being the ϕ-curvature of $\partial T_{t,t+h}(E)$ at x, so that it corresponds to an implicit time discretization of (1). Observe also that this approximation is monotone: indeed, if $E \subseteq E'$ then $d_E \geqslant d_{E'}$, which yields $w \geqslant w'$, where w and w' are the solutions of (3) for the distance functions d_E and $d_{E'}$ respectively. We deduce that $\{w < 0\} \subseteq \{w' < 0\}$, that is, $T_{t,s}(E) \subseteq T_{t,s}(E')$.

We will soon show (Thm. 3.3) that this scheme is also consistent, in some sense, with the evolution (1). Before this, let us prove that it is independent of Ω, in the sense that if $\partial E \subset \Omega$, then for $s - t$ small enough the set $T_{t,s}(E)$ is also the zero sublevel-set of any function w' solving (3) in any larger open set $\Omega' \supseteq \Omega$. This justifies ignoring the dependence on Ω in our notation. Here and in the rest of the paper we shall assume that G is defined in the whole space: $G \in C^0([0, +\infty); L^\infty(\mathbb{R}^N))$.

PROPOSITION 3.1 For any $\delta, T > 0$, there exists $h_0 > 0$ such that if E is a closed set with compact boundary $\partial E \subset \Omega$ such that $\mathrm{dist}_\phi(\partial\Omega, \partial E) \geqslant \delta$, then for any $h \leqslant h_0$ and $t \leqslant T$, the set $T_{t,t+h}(E)$ is the same whether computed in Ω or in any larger open set $\Omega' \supseteq \Omega$.

Before proving this proposition, we show a result that allows us to control in some uniform way the speed at which an initial Wulff shape $\{\phi(x - x_0) \leqslant \rho\}$ decreases in an iteration of the algorithm. The convexity of Ω is needed in the proof of this result.

LEMMA 3.2 Let $x_0 \in \Omega$ and $\rho > 0$, and let $t \geqslant 0$. Let w solve

$$\min_{w \in L^2(\Omega)} \int_\Omega \phi^\circ(Dw) + \frac{1}{2h}\int_\Omega (w(x) - (\phi(x - x_0) - \rho) - G(x, t+h) + G(x,t))^2\, \mathrm{d}x. \tag{4}$$

Then

$$w(x) \leqslant \begin{cases} \phi(x - x_0) + h\dfrac{N-1}{\phi(x - x_0)} + \Delta_h(t) - \rho & \text{if } \phi(x - x_0) \geqslant \sqrt{h(N+1)}, \\ \sqrt{h}\dfrac{2N}{\sqrt{N+1}} + \Delta_h(t) - \rho & \text{otherwise}, \end{cases} \tag{5}$$

where $\Delta_h(t) := \|G(\cdot, t+h) - G(\cdot, t)\|_{L^\infty(\Omega)}$.

Proof. The Euler–Lagrange equation for (4) can be written as follows: there exists a field $z \in L^\infty(\Omega; \mathbb{R}^N)$, with $z \in \partial\phi^\circ(\nabla w)$ a.e. and $z \cdot \nu_\Omega = 0$ on $\partial\Omega$, such that

$$w(x) - \phi(x - x_0) + \rho - G(x, t+h) + G(x,t) - h\, \mathrm{div}\, z(x) = 0 \tag{6}$$

(see for instance [11, 4]).

Let $\overline{w}$ denote the function on the right-hand side of (5). Let $\overline{z}$ be the field given by

$$\overline{z}(x) = \begin{cases} \dfrac{x - x_0}{\phi(x - x_0)} & \text{if } \phi(x - x_0) \geqslant \sqrt{h(N+1)}, \\ \left(1 - \left(\dfrac{\phi(x - x_0)}{\sqrt{h(N+1)}} - 1\right)^2\right)\dfrac{x - x_0}{\phi(x - x_0)} & \text{otherwise.} \end{cases}$$

One checks, as in [11, App. B], that $\overline{z} \in \partial\phi^\circ(\nabla\overline{w}(x))$ a.e., and

$$\frac{\overline{w}(x) - \phi(x - x_0) + \rho}{h} - \operatorname{div}\overline{z}(x) = \frac{\Delta_h(t)}{h}$$

a.e. in Ω. Moreover, if $x \in \partial\Omega$, then $\overline{z}(x)\cdot\nu_\Omega(x)$ has the sign of $(x - x_0)\cdot\nu_\Omega(x)$, which is nonnegative since Ω is convex. By definition of $\Delta_h(t)$, we deduce that $\overline{w}$ is a supersolution for (6). It follows that $\overline{w} \geqslant w$ a.e. in Ω: indeed, we have

$$\begin{aligned}\int_\Omega [(w - \overline{w})^+]^2 &= \int_{\{w>\overline{w}\}} (w - \overline{w})(\operatorname{div} z - \operatorname{div}\overline{z}) \\ &= \int_{\partial\Omega\cap\{w>\overline{w}\}} (w - \overline{w})(z - \overline{z})\cdot\nu_\Omega - \int_{\{w>\overline{w}\}} (\nabla w - \nabla\overline{w})\cdot(z - \overline{z}) \\ &\leqslant -\int_{\partial\Omega\cap\{w>\overline{w}\}} (w - \overline{w})\overline{z}\cdot\nu_\Omega \leqslant 0,\end{aligned}$$

which shows the desired inequality. □

Proof of Proposition 3.1. We assume $E \subset \Omega$, the proof in the case $\mathbb{R}^N \setminus E \subset \Omega$ being identical. Let w solve

$$\min_{w\in L^2(\Omega)} \int_\Omega \phi^\circ(Dw) + \frac{1}{2h}\int_\Omega (w(x) - d_E(x) - G(x, t + h) + G(x, t))^2\,\mathrm{d}x,$$

and let $x \in \Omega$ with $d_E(x) \geqslant \delta/2$. One has $d_E \geqslant \delta/2 - \phi(\cdot - x)$ in Ω. Invoking Lemma 3.2, we deduce that $w(x) \geqslant \delta/2 - \Delta_h(t) - 2N\sqrt{h}/\sqrt{N+1}$. Hence if $h_0 \leqslant 1$ is such that $\omega_{G,T+1}(h_0) + 2N\sqrt{h_0}/\sqrt{N+1} \leqslant \delta/4$, we find that when $h \leqslant h_0$ we have $w(x) \geqslant \delta/4$.

Let now $\Omega' \supseteq \Omega$. If $h \leqslant h_0$, we have in particular $d_E(x) + G(x, t + h) - G(x, t) \geqslant \delta/4$ for any $x \in \Omega' \setminus \Omega$, $t \leqslant T$. We can hence reproduce the proof of Corollary A.2 in [12], which shows that if w' is the solution of the same problem as w, but in Ω' instead of Ω, then $w' \wedge (\delta/4)$ is equal to $w \wedge (\delta/4)$ in Ω and to $\delta/4$ in $\Omega' \setminus \Omega$. Thus $\{w < 0\} = \{w' < 0\}$. Observe that in this proof, the larger domain Ω' does not need to be convex. □

The previous proposition allows us to define, in a unique and intrinsic way, the evolution $T_{t,t+h}(E)$ in $\mathbb{R}^N$, for any $t \geqslant 0$ and $h > 0$, of a set E with compact boundary $\partial E \Subset \mathbb{R}^N$, by considering the corresponding set computed in a ball with radius large enough. Therefore, from now on we shall assume $\partial E \Subset \mathbb{R}^N$ and we shall omit the dependence on Ω in the construction of the limit flow. We now prove our main consistency result.

THEOREM 3.3 Let $E(t)$, $t \in [t_0, t_1]$, be a superflow of (1). Then there exists h_0 such that $T_{t,t+h}(E(t)) \supseteq E(t + h)$ for any $h < h_0$ and t with $t_0 \leqslant t < t + h \leqslant t_1$. Moreover, if $E(t)$ is a subflow of (1), then $T_{t,t+h}(E(t)) \subseteq E(t + h)$ for h small enough.

Proof. Let $A \subset \mathbb{R}^N$ be the open set associated to the superflow $E(t)$ (cf. Definition 2.1) and let Ω be a bounded, convex open set with $A \Subset \Omega$.

We first observe that there exists $\varepsilon > 0$ such that $C := \{(x, t) : t_0 \leqslant t \leqslant t_1,\ |d(x, t)| \leqslant \varepsilon\} \subset A \times [t_0, t_1]$. Since $d(\cdot, t)$ is uniformly bounded in $C^2(A)$, we can also assume, possibly reducing ε, that $E(t)$ satisfies for all t an interior and exterior εW_ϕ-condition. Given t, h with $t_0 \leqslant t < t + h \leqslant t_1$, we build from $d(\cdot, t + h)$ a supersolution for problem (3). Consider a smooth increasing function $\psi : \mathbb{R} \to \mathbb{R}$ with $\psi(s) \geqslant s$ and $\psi(s) = s$ for $|s| \leqslant \varepsilon/2$. We set $v(x) := \psi(d(x, t+h))$ for $x \in B = \{|d(\cdot, t)| < \varepsilon\}$. Then, for $x \in B$, from (2) it follows

$$\frac{v(x) - d_{E(t)}(x) - G(x, t+h) + G(x, t)}{h} \geqslant \frac{d(x, t+h) - d(x, t) - G(x, t+h) + G(x, t)}{h}$$
$$\geqslant \frac{1}{h}\int_t^{t+h} \operatorname{div} \nabla\phi^\circ(\nabla d)(x, s)\, \mathrm{d}s + \delta.$$

Let now ω be a modulus of continuity for $\operatorname{div} \nabla\phi^\circ(\nabla d)$ in C. We find

$$\frac{v(x) - d_{E(t)}(x) - G(x, t+h) + G(x, t)}{h} \geqslant \operatorname{div} \nabla\phi^\circ(\nabla d)(x, t+h) + \delta - \omega(h).$$

Observe that for any $x \in B$ we have $\nabla v(x) = \psi'(d(x, t+h))\nabla d(x, t+h)$, so that (recall that $\nabla\phi^\circ$ is 0-homogeneous) $\nabla\phi^\circ(\nabla v(x)) = \nabla\phi^\circ(\nabla d(x, t+h))$, hence $\operatorname{div} \nabla\phi^\circ(\nabla d)(x, t+h) = \operatorname{div} \nabla\phi^\circ(\nabla v)(x)$. Therefore, if h is small enough so that $\omega(h) \leqslant \delta$, we get

$$\frac{v(x) - d_{E(t)}(x) - G(x, t+h) + G(x, t)}{h} \geqslant \operatorname{div} \nabla\phi^\circ(\nabla v)(x).$$

Let w solve (3), with $E = E(t)$ and $s = t + h$. We will show that we may choose ψ in order to have $v \geqslant w$ on ∂B, so that v is a supersolution for the problem

$$\min\left\{\int_B \phi^\circ(Du) + \frac{1}{2h}\int_B (u(x) - d_{E(t)}(x) - G(x, t+h) + G(x, t))^2\, \mathrm{d}x : u = w \text{ on } \partial B\right\}$$

(which is solved by w). We will deduce that $v \geqslant w$ in B, hence $\{w < 0\} \supseteq \{v < 0\} = \{d(\cdot, t+h) < 0\}$, that is, $T_{t,t+h}(E(t)) \supseteq E(t+h)$.

First of all, d is uniformly continuous in time, so that if h is small enough, one has $d(x, t+h) \geqslant 3\varepsilon/4$ if $d(x, t) = \varepsilon$. If $M > \operatorname{diam} \Omega$, then $M \geqslant w$ in Ω. We may choose a function ψ with $\psi(3\varepsilon/4) \geqslant M$, so that $v(x) \geqslant M \geqslant w(x)$ if $d(x, t) = \varepsilon$.

On the other hand, since $E(t)$ satisfies the interior εW_ϕ-condition, one deduces from Lemma 3.2 that $w(x) \leqslant 2N\sqrt{h}/\sqrt{N+1} + \Delta_h(t) - \varepsilon$ whenever $d(x, t) = -\varepsilon$. We observe that $\Delta_h(t) \to 0$ as $h \to 0$ uniformly in $[t_0, t_1]$. Hence if h is small enough, we find that $w(x) \leqslant -3\varepsilon/4$. We can choose ψ such that $\psi(s) \geqslant -3\varepsilon/4$ for any s, so that $v(x) \geqslant w(x)$ if $d(x, t) = -\varepsilon$. We conclude that $v \geqslant w$ on ∂B. Hence v is a supersolution for (3), which implies $T_{t,t+h}(E(t)) \supseteq E(t+h)$.

If $E(t)$ is a subflow, we can reproduce the same proof to show that $T_{t,t+h}(E(t)) \subseteq E(t+h)$. □

We deduce the following comparison result for sub/superflows.

COROLLARY 3.4 Assume that $E_1(t)$, $E_2(t)$ are respectively a superflow and subflow of (1) on $[t_0, t_1]$ such that $E_1(t_0) \subseteq E_2(t_0)$. Then $E_1(t) \subseteq E_2(t)$ for all $t \in [t_0, t_1]$.

Proof. By the previous theorem, there exists h_0 such that $T_{t,t+h}(E_1(t)) \supseteq E_1(t+h)$ and $T_{t,t+h}(E_2(t)) \subseteq E_2(t+h)$ for any $t \in [t_0, t_1 - h]$, as soon as $h \leqslant h_0$. Hence, if $t \in [t_0, t_1]$, we just let $n \geqslant 1$ be such that $(t - t_0)/n = h \leqslant h_0$. Then, letting $t_k = t_0 + kh$, one can easily check by induction that $E_1(t_k) \subseteq T_{t_{k-1},t_k}(E_1(t_{k-1})) \subseteq T_{t_{k-1},t_k}(E_2(t_{k-1})) \subseteq E_2(t_k)$ for any $1 \leqslant k \leqslant n$, which implies the assertion since $t = t_n$. □

REMARK 3.5 It could be interesting, from a numerical analysis point of view, to modify slightly the algorithm presented in this paper by introducing a threshold $S > 0$ and replace in problem (3) the distance function d_E with a truncated distance function $(-S \wedge d^E) \vee S$. Almost all of the results presented in this paper would remain identical (in particular, the consistency still holds). Only the comparison results in Section 4.2 are not completely clear, since they rely on comparisons of the true distance functions. However, it is reasonable to believe that in the limit $h \to 0$ also these results hold.

4. Convergence of the algorithm

In this section, we study the limit of the iterates of our variational algorithm as the time-step goes to zero. First (Section 4.1), under some local boudedness assumption on the forcing term, we find a limit which has some continuity properties (Props. 4.3 and 4.4), and we show that it is consistent with other notions of weak solutions (Cor. 4.6, and with further assumptions on the forcing term, Props. 4.8 and 4.9).

Then in 4.2, assuming some spatial regularity (18) of the forcing term, we find a comparison principle for our limits (however, with some limitation), which allows us to build level-set solutions starting from an arbitrary bounded uniformly continuous function (Section 4.3).

4.1 *The discrete flow and its limit*

Given $E \subset \mathbb{R}^N$, closed with compact boundary, and $h > 0$, we define the "tube" $E_h \subset \mathbb{R}^N \times [0, +\infty)$ as follows:

$$E_h(t) := T_{[t/h]h-h,[t/h]h} \cdots T_{2h,3h} T_{h,2h} T_{0,h}(E), \tag{7}$$

where $[x]$ denotes the integer part of x. We then define $E_h := \bigcup_{t \geqslant 0} E_h(t) \times \{t\}$.

There exists a sequence $(h_n)_{n \geqslant 1}$ such that both E_{h_n} and $\mathbb{R}^N \times [0, +\infty) \setminus E_{h_n} = {}^c E_{h_n}$ converge in the Hausdorff distance (locally in time) to E^* and ${}^c E_*$ respectively. Such convergence is equivalent to the locally uniform convergence, in $\mathbb{R}^N \times [0, +\infty)$, of the distance functions $\operatorname{dist}((x, t), E_{h_n})$ and $\operatorname{dist}((x, t), {}^c E_{h_n})$ to the distance functions $\operatorname{dist}((x, t), E^*)$ and $\operatorname{dist}((x, t), {}^c E_*)$ (see [11, App. A]). In particular, for any $(x, t) \in E^*$ (resp., ${}^c E_*$), there exist $(x_n, t_n) \in E_{h_n}$ (resp., ${}^c E_{h_n}$) such that $(x_n, t_n) \to (x, t)$, and if $(x_n, t_n) \in E_{h_n}$ (resp., ${}^c E_{h_n}$) and converge to some point $(x, t) \in \Omega \times [0, +\infty)$, then $(x, t) \in E^*$ (resp., ${}^c E_*$). Below, we denote by $(h)_{h>0}$ the sequence $(h_n)_{n \geqslant 1}$.

Clearly, E_* is open while E^* is closed, and $E_* \subset E^*$. For any $t \geqslant 0$, we denote by $E^*(t)$ (resp. $E_*(t)$) the section $\{x : (x, t) \in E^*\}$ (resp. $\{x : (x, t) \in E_*\}$). For any $t \geqslant 0$, we let $\Gamma(t) = E^*(t) \setminus E_*(t)$, which in some sense is our generalized evolution starting from ∂E.

From the definition of E_*, E^* it follows that

$$E_*(0) \subseteq \operatorname{int}(E), \qquad E \subseteq E^*(0), \tag{8}$$

in particular $\Gamma(0) \supseteq \partial E$. If $F(t)$ is a superflow on $[t_0, t_1]$ such that $F(t_0) \subset E_*(t_0)$, since $\operatorname{dist}(F(t_0) \times \{t_0\}, {}^c E_*) > 0$ (as $F(t)$ is assumed to be closed for any time t, and E_* is open), one sees that for h small enough, $F(t_0) \subset E_h([t_0/h]h) \cap E_h([t_0/h]h + h)$. It then follows from Theorem 3.3 that (if h is enough small) $F(t) \subset E_h(t)$ for any $t \in [t_0, t_1]$, and passing to the limit we get $F \subset E_* \cap (\mathbb{R}^N \times [t_0, t_1])$. Hence E_* satisfies a comparison principle for superflows that start inside, and analogously E^* satisfies a comparison principle for subflows starting outside, so that we have shown the following:

PROPOSITION 4.1 The set-valued functions $E_*(\cdot)$ and $E_*(\cdot)$ are barriers on $[0, +\infty)$ with respect to $\mathcal{F}^+$ and $\mathcal{F}^-$ respectively, that is, $E_* \in \mathcal{B}^+$ and $E^* \in \mathcal{B}^-$. In particular, from Definition 2.3 that for all $t \geqslant 0$,

$$E_*(t) \supseteq \mathcal{M}_*(E_*(0))(t) \quad \text{and} \quad E^*(t) \subseteq \mathcal{M}^*(E^*(0))(t). \tag{9}$$

On the other hand, it is not clear if we also have $E_* \in \mathcal{B}^-$ and $E^* \in \mathcal{B}^+$.

Let us now show that $E_*(0) = \text{int}(E)$ and $E^*(0) = E$, so that we can substitute $E_*(0)$ and $E^*(0)$ with E in (9). In order to do so, we further require that the function G satisfies the following regularity assumption: for any $T > 0$ there exists a constant $C(T)$ such that

$$\left| \frac{G(x,s) - G(y,s) - G(x,t) + G(y,t)}{s-t} \right| \leqslant C(T) \tag{10}$$

for any $s, t \leqslant T$. Note that this is equivalent to requiring that G can be written as $G(x,t) = G_1(t) + G_2(x,t)$ with $G_1 \in C^0([0,+\infty))$ and $G_2 \in \text{Lip}_{\text{loc}}([0,+\infty); L^\infty(\mathbb{R}^N))$.

We first construct explicit super/subflows starting from a Wulff shape $W_\phi(x_0, r)$ of radius $r > 0$ (or its complement), at time $t \geqslant 0$. More precisely, we construct superflows $W^+_{x_0,t,r}(s)$, with $s \in [t, +\infty)$, starting from $W_\phi(x_0, r)$ at time t, which are smooth on $[t, t+\tau]$ and vanish after time $t+\tau$, where the duration τ depends only on r, and such that $T_{s,s+h}(W^+_{x_0,t,r}(s)) \supseteq W^+_{x_0,t,r}(s+h)$ for all $h > 0$ and $s \geqslant t$.

LEMMA 4.2 Let $x_0 \in \mathbb{R}^N$ and $r > 0$. Define

$$d_\pm(x,s) := \pm\left(\phi(x-x_0) - r + \frac{s-t}{2\tau} r\right) + G(x_0,s) - G(x_0,t) \tag{11}$$

for $(x,s) \in \mathbb{R}^n \times [t, t+\tau]$, where τ is such that

$$\omega_{G,t+\tau}(\tau) \leqslant \frac{r}{4}, \tag{12}$$

$$\tau \leqslant \frac{r^2}{2(C(t+\tau) + 4(N-1))} \wedge \frac{r^2}{16(N+1)}, \tag{13}$$

where $\omega_{G,t+\tau}$ is as before a modulus of continuity of G on $[0, t+\tau]$, and $C(\cdot)$ is the constant appearing in (10). Let $W^+_{x_0,t,r}(s) := \{d_+(\cdot, s) \leqslant 0\}$ when $s \in [t, t+\tau]$, and $W^+_{x_0,t,r}(s) := \emptyset$ for $s > t+\tau$. Then $T_{s,s+h}(W^+_{x_0,t,r}(s)) \supseteq W^+_{x_0,t,r}(s+h)$ for any $s \geqslant t$ and $h > 0$. On the other hand, if $W^-_{x_0,t,r}(s) := \{d_-(\cdot,s) \leqslant 0\}$ when $s \in [t, t+\tau]$, and $W^-_{x_0,t,r}(s) := \mathbb{R}^N$ for $s > t+\tau$, then $T_{s,s+h}(W^-_{x_0,t,r}(s)) \subseteq W^-_{x_0,t,r}(s+h)$ for any $s \geqslant t$ and $h > 0$.

Notice that, letting $\tau(r)$ be the maximal time τ satisfying (12) and (13) for a given $r > 0$, we have $\tau(r) > 0$ and

$$\lim_{r\to\infty} \tau(r) = +\infty.$$

Notice also that the condition $\omega_{G,t+\tau}(\tau) \leqslant r/4$ ensures the inclusion $W_\phi(x_0, r/4) \subseteq W^+_{x_0,t,r}(s)$ for $s \in [t, t+\tau]$, hence in particular the set $W^+_{x_0,t,r}(s)$ is nonempty. In fact, one could check that $W^+_{x_0,t,r}$ is a superflow (in the sense of Definition 2.1) on $[t, t+\tau]$, while $W^-_{x_0,t,r}$ is a subflow. Thus, the conclusion would follow from Theorem 3.3, at least for h small enough. The statement of Lemma 4.2 is slightly more precise, as it holds without any restriction on h so that the superflow property which is shown is, in particular, uniform in x_0.

Proof. Let $s \in [t, t+\tau]$ and $h > 0$. If $s + h > t + \tau$, then $W^+_{x_0,t,r}(s) = \emptyset$ so that the statement is obvious, hence we may assume $s + h \leqslant t + \tau$. For any $x \in \mathbb{R}^N$, by (10) we have

$$\begin{aligned} &d_+(x,s) + G(x,s+h) - G(x,s) \\ &\quad = \phi(x-x_0) - r + G(x_0,s) - G(x_0,t) + G(x,s+h) - G(x,s) + \frac{s-t}{2\tau}r \\ &\quad \leqslant \phi(x-x_0) - r + G(x_0,s+h) - G(x_0,t) + \frac{s-t}{2\tau}r + C(t+\tau)h. \end{aligned}$$

Let now Ω be an open bounded subset of $\mathbb{R}^N$ which is big enough to guarantee that the set $T_{s,s+h}(W^+_{x_0,t,r}(s))$ does not depend on Ω (Prop. 3.1). Hence the solution w of (3), with E replaced by $W^+_{x_0,t,r}(s)$ and (t,s) replaced by $(s, s+h)$, is less than the solution of

$$\min_{v \in BV(\Omega)} \Bigg(\int_\Omega \phi^\circ(Dv) + \frac{1}{2h}\int_\Omega \Big(v - \phi(x-x_0) + r - G(x_0,s+h) + G(x_0,t) - \frac{s-t}{2\tau}r - C(t+\tau)h\Big)^2 \,\mathrm{d}x \Bigg),$$

which in turn (as shown in the proof of Lemma 3.2) is less than the function

$$\begin{cases} \phi(x-x_0) + h\dfrac{N-1}{\phi(x-x_0)} - r + G(x_0,s+h) - G(x_0,t) + \dfrac{s-t}{2\tau}r + C(t+\tau)h \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } \phi(x-x_0) \geqslant \sqrt{h(N+1)}, \\ \sqrt{h}\dfrac{2N}{\sqrt{N+1}} - r + G(x_0,s+h) - G(x_0,t) + \dfrac{s-t}{2\tau}r + C(t+\tau)h \quad \text{otherwise.} \end{cases}$$

Hence, we see that

$$w(x) \;\leqslant\; d_+(x,s+h) + h\frac{N-1}{\phi(x-x_0)} - h\frac{r}{2\tau} + C(t+\tau)h$$

when $\phi(x-x_0) \geqslant \sqrt{h(N+1)}$. Now, since $\tau \leqslant r^2/(16(N+1))$ and $h \leqslant \tau$, we get $r/4 \geqslant \sqrt{h(N+1)}$ so that we can replace the last condition with the stronger condition $\phi(x-x_0) \geqslant r/4$. On the other hand, if both $\phi(x-x_0) \geqslant r/4$ and $\tau \leqslant r^2/(2(C(t+\tau)+4(N-1)))$, then

$$\frac{r}{2\tau} \geqslant \frac{C(t+\tau)+4(N-1)}{r} \geqslant C(t+\tau) + \frac{N-1}{\phi(x-x_0)},$$

so that $w(x) \leqslant d_+(x, s+h)$. This shows that $T_{s,s+h}(W^+_{x_0,t,r}(s)) \supseteq W^+_{x_0,t,r}(s+h)$.

The proof of the similar assertion for $W^-_{x_0,t,r}$ is analogous. □

By the previous lemma, if $E_h(t) \supseteq W_\phi(x_0, r)$, then $E_h(t+nh) \supseteq W^+_{x_0,t,r}(t+nh)$ for any $n \geqslant 1$, and in particular $E_h(t+nh) \supseteq W_\phi(x_0, r/4)$ as long as $nh \leqslant \tau(r)$. In other words, for any $r > 0$, the inequality $nh \leqslant \tau(r)$ yields

$$E_h(t+nh) \supseteq \{x : d_{E_h(t)}(x) \leqslant -r\}.$$

This shows that the (discrete) evolution will not vanish "suddenly", in the sense that any subset in the interior of $E_h(t)$ remains in the interior of $E_h(s)$ for $s > t$ sufficiently close to t.

We can easily deduce the same "semicontinuity" property for E_*: indeed, if $d_{E_*(t)}(x) = -r$, then, for any $r' < r$, $W_\phi(x, r') \subset E_h(t)$ as soon as h is small enough, so that $W_\phi(x, r'/4) \subset E_h(t + [\tau/h]h)$ for all $\tau < \tau(r')$. Letting first $h \to 0$ and then $r' \to r$, we find that if $\tau < \tau(r)$, then

$$E_*(t+\tau) \supseteq \{x : d_{E_*(t)}(x) \leqslant -r\}. \tag{14}$$

In the same way we obtain

$$E^*(t+\tau) \subseteq \{x : d_{E^*(t)}(x) < r\}. \tag{15}$$

Moreover, one can easily verify that the same properties hold at $t = 0$ with $E^*(t)$ replaced with E and $E_*(t)$ replaced with $\mathrm{int}(E)$, where E is the initial set. From (14) and (15) we also get

$$\Gamma(t+\tau) \ \subseteq \ \{\mathrm{dist}_\phi(\cdot, \Gamma(t)) < r\}. \tag{16}$$

As a consequence, we obtain the following semicontinuity property for the tubes E_*, E^*.

PROPOSITION 4.3 Assume that G satisfies (10). Let E be a closed subset of $\mathbb{R}^N$ with compact boundary. Let O, F be an open and a closed subset of $\mathbb{R}^N$ respectively. Let $t \geqslant 0$ and let $(\tau_n)_{n\geqslant 0}$ be a sequence of nonnegative numbers going to 0. Then

- If ${}^cE_*(t+\tau_n) \to {}^cO$ in the Hausdorff sense, then $E_*(t) \subseteq O$,
- If $E^*(t+\tau_n) \to F$ in the Hausdorff sense, then $F \subseteq E^*(t)$,

In particular, any Hausdorff limit of a sequence $\Gamma(t+\tau_n)$ is contained in $\Gamma(t)$. Moreover, if $t = 0$, we can replace $E_*(t)$ with $\mathrm{int}(E)$ in the first statement and $E^*(t)$ with E in the second. In particular, choosing $\tau_n \equiv 0$, we get

$$\mathrm{int}(E) \subseteq E_*(0) \subseteq E^*(0) \subseteq E,$$

which implies, recalling (8), that

$$E_*(0) = \mathrm{int}(E) \quad \text{and} \quad E^*(0) = E. \tag{17}$$

Notice that (17) shows that $\Gamma(0) = \partial E$.

Since $O \subset F$ in the above proposition, we also see that if $E^*(t) = \overline{E_*(t)}$, then $E^*(t+\tau) \to E^*(t)$ in the Hausdorff sense as $\tau \to 0$, whereas if $E_*(t) = \mathrm{int}(E^*(t))$, then ${}^cE_*(t+\tau) \to {}^cE_*(t)$, and if both are true, then $\Gamma(t+\tau) = E^*(t+\tau) \setminus E_*(t+\tau) \to \partial E^*(t)$ as $\tau \to 0$. To show this, one just needs to show that for any $x \in \partial E^*(t)$, there exist $x_\tau \in \Gamma(t+\tau)$ that converge to x as $\tau \to 0$. We know that there exist $y_\tau \in E^*(t+\tau)$ and $z_\tau \notin E_*(t+\tau)$ such that both y_τ and z_τ converge to x. Then the segment $[y_\tau, z_\tau]$ must intersect $\Gamma(t+\tau)$ and any point x_τ in this intersection will have the desired property.

Notice also that if $E = \overline{\mathrm{int}(E)}$, then $\Gamma(t) = E^*(t) \setminus E_*(t)$ converges to ∂E as $t \to 0$, in the Hausdorff sense.

The left continuity of the tubes E_*, E^* is given by the following proposition.

PROPOSITION 4.4 Assume that G satisfies (10). Let E be a closed subset of $\mathbb{R}^N$ with compact boundary, and let $t > 0$. Then ${}^cE_*(t-\tau) \to {}^cE_*(t)$ in the Hausdorff sense as $\tau \to 0$ with $\tau \geqslant 0$, while $E^*(t-\tau) \to E^*(t)$. Moreover, $\Gamma(t-\tau) \to \Gamma(t)$.

Proof (sketch). As for the previous proposition, one will deduce from (14) that if ${}^cE_*(t-\tau_n) \to {}^cO$ in the Hausdorff sense, along a subsequence τ_n going to 0, then $O \subseteq E_*(t)$. On the other hand, since ${}^cE_*$ is closed in $\Omega \times [0, +\infty)$, one must have ${}^cO \subseteq {}^cE_*(t)$. Thus $O = E_*(t)$ and the conclusion follows. In the same way, (15) implies that if $E^*(t-\tau_n) \to F$ in the Hausdorff sense, then $E^*(t) \subseteq F$. From the closedness of F^* we conclude in the same way that $F = E^*(t)$.

The last assertion follows from (16): first, any Hausdorff limit F of a subsequence $\Gamma(t-\tau_n) = E^*(t-\tau_n) \setminus E_*(t-\tau_n)$ is inside $\Gamma(t) = E^*(t) \setminus E_*(t)$, by the previous results. Now, since $\mathrm{dist}_\phi(\cdot, \Gamma(t-\tau_n))$ converges uniformly to $\mathrm{dist}_\phi(\cdot, F)$ as $n \to \infty$, $\Gamma(t) \subseteq \{\mathrm{dist}_\phi(\cdot, F) \leqslant r\}$ for any $r > 0$, hence it lies in F. Thus $F = \Gamma(t)$. □

REMARK 4.5 Notice that in general we cannot expect the maps $t \mapsto E_*(t)$ and $t \mapsto E^*(t)$ to be continuous in the Hausdorff distance: indeed, this would prevent small disconnected parts from disappearing in finite time, a phenomenon which is known to happen even when $G \equiv 0$. On the other hand, these maps are likely to be continuous in the L^1-topology, under suitable assumptions on G (when $G \equiv 0$ it is proved in [1, Thm. 4.4]).

From Propositions 4.1 and 4.3, and in particular (9) and (17), we get the following corollary.

COROLLARY 4.6 If G satisfies condition (10), then

$$\Gamma(t) \subseteq \mathcal{N}^*(E,0)(t) \setminus \mathcal{M}_*(E,0)(t).$$

In particular, as long as $\mathcal{N}^*(E,0)(t) \setminus \mathcal{M}_*(E,0)(t)$ has no interior (*nonfattening* condition), the motions $E^*(t)$ and $E_*(t)$ are uniquely defined and do not depend on the sequence along which the limits are obtained.

REMARK 4.7 Notice that, as long as the set E has compact boundary, all the resuls of this section can be easily extended to functions G which are only locally bounded in x, i.e. $G(x,t) = G_1(t) + G_2(x,t)$ with $G_1 \in C^0([0,+\infty))$ and $G_2 \in \mathrm{Lip}_{\mathrm{loc}}([0,+\infty); L^\infty_{\mathrm{loc}}(\mathbb{R}^N))$.

PROPOSITION 4.8 If $G(x,t) = \int_0^t g(x,s)\,\mathrm{d}s$ with g continuous, then $\Gamma(t)$ is contained in the zero level-set of the corresponding viscosity solution.

Proof. This follows immediately from Theorem 2.4 and Corollary 4.6. □

From Corollary 4.6 and [17, Section 3] we also have the following consistency result in the case of an x-independent forcing term.

PROPOSITION 4.9 Let $G(x,t) = G(t) \in C^0([0,+\infty))$ and let $\phi(x) = |x|$ (i.e. isotropic mean curvature flow). Then $\Gamma(t)$ is contained in the minimal barrier solution defined in [17]. In particular, if ∂E is of class $C^{2,\alpha}$, then $E^*(t) = \overline{E_*(t)}$ and $\Gamma(t) = \partial E^*(t)$ coincides with the unique (local in time) solution of (1) given in [17].

REMARK 4.10 As already pointed out in the introduction, viscosity theory can be applied under more general assumptions on G than what is required in Proposition 4.8 (see [23, 24]). However, it is still not clear what is the relation between the limit set $\Gamma(t)$ and the zero level-set of such viscosity solutions, except for the particular case of an x-independent forcing term, where the equality holds for small times as a consequence of Proposition 4.9 (if ∂E is regular enough).

4.2 *An inclusion principle*

Let us now consider the case where the driving term is the "time-derivative" of a function $G(x,t)$ that satisfies

$$\left| \frac{G(x,s) - G(y,s) - G(x,t) + G(y,t)}{s-t} \right| \leqslant C(T)|x-y|. \tag{18}$$

This condition is stronger than (10) (see also Remark 4.7) and is for instance true whenever $G(x,t) = G_1(t) + G_2(x,t)$ with $G_1 \in C^0([0,+\infty))$ and $G_2 \in C^1([0,+\infty); \mathrm{Lip}(\mathbb{R}^N))$. In particular, all the results of Section 4.1 still hold under assumption (18).

Given a closed set $E \subset \mathbb{R}^N$ with nonempty compact boundary ∂E, we define the maximal existence time $T_E^* \in [0,+\infty]$ for the flow E^* as the supremum of all times t such that $E^*(t) \neq \emptyset$ and $E_*(t) \neq \mathbb{R}^N$. The fact that $T_E^* > 0$ is ensured by Proposition 4.3, whenever $\mathrm{int}(E) \neq \emptyset$.

Consider now two closed sets E^1 and E^2 with nonempty compact boundary, and assume $E^1 \subset E^2$ and $D := \mathrm{dist}_\phi(\partial E^1, \partial E^2) > 0$. Notice that if G depends only on time, then for each z such that $\phi(z) \leqslant D$, we have $z + E^1 \subset E^2$, so that $T_{0,h}(z + E^1) \subset T_{0,h}(E^2)$ for any $h > 0$. Since G does not depend on x, we get $T_{0,h}(z + E^1) = z + T_{0,h}(E^1)$. It follows that $W_\phi(0, D) + T_{0,h}(E^1) \subset T_{0,h}(E^2)$, which implies $\mathrm{dist}_\phi(\partial T_{0,h}(E^1), \partial T_{0,h}(E^2)) \geqslant D$. By induction, we deduce that $\mathrm{dist}_\phi(\partial E_h^1(t), \partial E_h^2(t)) \geqslant D$ for any $t \geqslant 0$ (where we set the distance equal to $+\infty$ if one of the two sets disappears).

For a general G the estimate is slightly trickier, even if it follows the same idea. Assume $T^* = \min\{T_{E^1}^*, T_{E^2}^*\} > 0$ and let $T < T^*$. By Proposition 3.1, we can find a "large" bounded open set $\Omega \subset \mathbb{R}^N$ such that the sets $E_h^1(t)$ and $E_h^2(t)$ defined in (7) do not depend on Ω for $t \in [0, T]$ and h small enough. In particular, we can assume that $\partial E_h^1(t)$ and $\partial E_h^2(t)$ remain at a positive distance from $\partial\Omega$ for any $t \in [0, T]$. Let w_1, w_2 be the solutions of the variational problem (3) for $t = 0$ and $s = h$ with d_E replaced by d_{E^1}, d_{E^2} respectively. Notice that, for $z \in \mathbb{R}^N$, the set $z + T_{0,h}(E^1) = z + \{w_1 < 0\}$ coincides with $\{x \in z + \Omega : w_1(x - z) < 0\}$, and the function $\tilde{w}_1(x) = w_1(x - z)$, defined in $z + \Omega$, is the solution of

$$\min_{w \in L^2(z+\Omega)} \left(\int_{z+\Omega} \phi^\circ(Dw) + \frac{1}{2h} \int_{z+\Omega} (w(x) - d_{E^1}(x - z) - G(x - z, h) + G(x - z, 0))^2 \, \mathrm{d}x \right).$$

Possibly enlarging Ω, we can assume that both w_2 and $\tilde{w}_1$ are solutions of their respective variational problems in the same domain (for instance, $\Omega \cup (z + \Omega)$). Then, since $d_{E^1}(x - z) \geqslant d_{E^1}(x) - \phi(z) \geqslant d_{E^2}(x) + D - \phi(z)$ and, by (18), $-G(x - z, h) + G(x - z, 0) \geqslant -G(x, h) + G(x, 0) - C(T)|z|h$, one finds that $\tilde{w}_1 \geqslant w_2 + D - \phi(z) - C(T)|z|h$. In particular, if $\phi(z) \leqslant D/(1 + hC'(T))$ with $C'(T) = C(T) \sup_{z \neq 0} |z|/\phi(z)$, we get $\{\tilde{w}_1 \leqslant 0\} \subset \{w_2 \leqslant 0\}$, which in turn implies $\mathrm{dist}_\phi(\partial T_{0,h}(E^1), \partial T_{0,h}(E^2)) \geqslant D/(1 + hC'(T))$. By an induction argument, we deduce that $\mathrm{dist}_\phi(\partial E_h^1(t), \partial E_h^2(t)) \geqslant D(1 + hC'(T))^{-[t/h]}$ for any $t \in [0, T - h]$ and $h > 0$ small enough. Observe that, as $h \to 0$, we have $D(1 + hC'(T))^{-[t/h]} \to De^{-C'(T)t}$. We will show that this estimate also holds in the limit, for the motions $(E^1)^*$ and $(E^2)_*$ obtained along the *same* subsequence $(h_k)_{k \geqslant 1}$ (which we will still denote by $(h)_{h>0}$).

Fix $\delta < De^{-C'(T)T}$. If h is small enough, we have $\delta \leqslant \mathrm{dist}_\phi(E_h^1(t), \Omega \setminus E_h^2(t))$ for any $t \in [0, T)$. Given a fixed $t < T$, choose a subsequence (h_k) such that both Hausdorff limits of $E_{h_k}^1(t)$ and $\Omega \setminus E_{h_k}^2(t)$ exist in $\overline{\Omega}$, and denote them by K and L, respectively. Since $\mathrm{dist}_\phi(E_h^1(t), \Omega \setminus E_h^2(t)) \geqslant \delta$, in the limit we find $\mathrm{dist}_\phi(K, L) \geqslant \delta$. We also have $K \subseteq (E^1)^*(t)$ and $L \subseteq \overline{\Omega} \setminus (E^2)_*(t)$. Define now $K^{\delta/2} = K + W_\phi(0, \delta/2)$, which has its boundary between ∂K and ∂L and lies at distance at least $\delta/2$ from both boundaries. Let $\delta' < \delta$ and set $\delta'' = (\delta + \delta')/2$. If $x \in \partial K^{\delta/2}$, then $W_\phi(x, \delta''/2) \Subset \overline{\Omega} \setminus (K \cup L)$ so that if h_k is small enough, then $W_\phi(x, \delta''/2) \subset E_{h_k}^2(t)$ and $W_\phi(x, \delta''/2) \cap E_{h_k}^1(t) = \emptyset$. By Lemma 4.2 there exists $\tau > 0$, depending only on δ'' and $\delta' < \delta''$, such that $W_\phi(x, \delta'/2) \subset E_{h_k}^2(s)$ and $W_\phi(x, \delta'/2) \cap E_{h_k}^1(s) = \emptyset$ for all $t \leqslant s < t + \tau$. In the limit, this implies that for $t \leqslant s < t + \tau$,

$W_\phi(x, \delta'/2) \subset (E^2)_*(s)$ and $(E^1)^*(s) \cap W_\phi(x, \delta'/2) = \emptyset$. Since x is an arbitrary point of $\partial K^{\delta/2}$, this implies that both $\mathrm{dist}_\phi(\partial K^{\delta/2}, \partial(E^1)^*(s)) \geqslant \delta'/2$ and $\mathrm{dist}_\phi(\partial K^{\delta/2}, \partial(E^2)_*(s)) \geqslant \delta'/2$. We deduce that for any s in $(t, t+\tau)$, $\mathrm{dist}_\phi(\partial(E^1)^*(s), \partial(E^2)_*(s)) \geqslant \delta'$. Since t is arbitrary in $[0, T)$ and τ does not depend on t, we deduce that in fact for any $t \in [0, T)$, $\mathrm{dist}_\phi(\partial(E^1)^*(t), \partial(E^2)_*(t)) \geqslant \delta'$. (The case $t = 0$ follows directly from Proposition 4.3.) We may let $\delta' \to \delta$ to see that the inequality holds with δ instead of δ'. In fact, we can deduce from the previous argument that the distance between the two sets decreases at most like $De^{-C'(T)t}$, $t \in [0, T)$.

We have obtained the following result.

PROPOSITION 4.11 Let $E^1 \subset E^2 \subset \mathbb{R}^N$ be two closed sets with nonempty compact boundary, and assume $\mathrm{dist}_\phi(\partial E^1, \partial E^2) > 0$. Denote by E_h^1 and E_h^2, $h > 0$, the corresponding discrete evolutions in $\mathbb{R}^N \times [0, +\infty)$. Let (h_k) be a subsequence such that $E_{h_k}^1 \to (E^1)^*$ and ${}^c E_{h_k}^2 \to {}^c(E^2)_*$ in the Hausdorff sense. Assume also that $E_{h_k}^2 \to (E^2)^*$. Then $(E^1)^*(t) \subseteq (E^2)_*(t)$ for any $t \geqslant 0$.

In particular, for any $t \in [0, T)$ with $T := \max\{T_{E^1}^*, T_{E^2}^*\}$, and any t' with $t < t' < T$,

$$\mathrm{dist}_\phi(\partial(E^1)^*(t), \partial(E^2)_*(t)) \geqslant \mathrm{dist}_\phi(\partial E^1, \partial E^2)e^{-C'(t')t} > 0 \tag{19}$$

where C' is proportional to the constant in (18).

REMARK 4.12 Notice that the inclusions $(E^1)^*(t) \subseteq (E^2)^*(t)$ and $(E^1)_*(t) \subseteq (E^2)_*(t)$ always hold (without any assumption on $\mathrm{dist}_\phi(\partial E^1, \partial E^2)$).

REMARK 4.13 We remark that we do not know if the conclusion of Proposition 4.11 still holds if $(E^1)^*$ and $(E^2)^*$ are limits of $E_{h_k}^1$ and $E_{h_j}^2$ (respectively) along different subsequences. This would be an important result, yielding for instance the uniqueness of the level-set solution $u(x, t)$, defined in Section 4.3.

4.3 *The level-set approach*

Consider now a function $u_0 \in BUC(\mathbb{R}^N)$ such that for each $t \in \mathbb{R}$ the level-set $\partial\{u_0 > t\}$ is bounded. For all $q \in \mathbb{Q}$ consider the level-sets $E^q := \{u_0 \geqslant q\}$ and let $E_h^q \subset \mathbb{R}^N \times [0, +\infty)$ be the discrete evolutions of E^q. Then a diagonal argument shows that, along a subsequence $(h_k)_{k\geqslant 1}$, we have $E_{h_k}^q \to (E^q)^*$ and ${}^c E_{h_k}^q \to {}^c(E^q)_*$ locally in the Hausdorff sense, i.e. the distance functions $\mathrm{dist}(\cdot, E_{h_k}^q)$ and $\mathrm{dist}(\cdot, \mathbb{R}^N \times [0, +\infty) \setminus E_{h_k}^q)$ converge to $\mathrm{dist}(\cdot, (E^q)^*)$ and $\mathrm{dist}(\cdot, \mathbb{R}^N \times [0, +\infty) \setminus (E^q)_*)$ respectively, uniformly in $\mathbb{R}^N \times [0, T]$ for any $T > 0$.

Observe that (Remark 4.12) for all $q, r \in \mathbb{Q}$ with $q \geqslant r$, we have $(E^q)^*(t) \subseteq (E^r)^*(t)$ and $(E^q)_*(t) \subseteq (E^r)_*(t)$ for any $t \geqslant 0$. Hence we can define two functions $u^*, u_* : \mathbb{R}^N \times 0, +\infty) \to \mathbb{R}$ by letting

$$u^*(x, t) := \sup\{q \in \mathbb{Q} : x \in (E^q)^*(t)\}, \quad u_*(x, t) := \sup\{q \in \mathbb{Q} : x \in (E^q)_*(t)\}.$$

By Proposition 4.11, we know that $(E^q)^*(t) \subset (E^r)_*(t)$ for any $t \geqslant 0$ whenever $q > r$, which implies $u^*(x, t) = u_*(x, t)$ for any $(x, t) \in \mathbb{R}^N \times [0, +\infty)$. Indeed, if $q > u^*(x, t)$, then $x \notin (E^q)^*(t)$, so that also $x \notin (E^q)_*(t)$, hence $u_*(x, t) \leqslant u^*(x, t)$; on the other hand, if $q > u_*(x, t)$, then if $q' \in (u_*(x, t), q) \cap \mathbb{Q}$, we have $x \notin (E^{q'})_*(t) \supset (E^q)^*(t)$, hence $x \notin (E^q)^*(t)$, and we deduce $u^*(x, t) \leqslant u_*(x, t)$. We simply denote by $u(x, t)$ this common value.

Let us observe that, for each $t \geqslant 0$, from Proposition 4.11 (more exactly from the estimate (19)) it follows that $u(\cdot, t)$ is uniformly continuous on $\mathbb{R}^N$ (with the same modulus of continuity as u_0 if $C(T) = 0$ in (18)). It also follows easily from Propositions 4.3 and 4.4 that if $(x_n, t_n) \to (x, t)$, then $u(x_n, t_n) \to u(x, t)$: indeed, for instance, one sees that if $u(x_n, t_n) < q$ for n large enough, then $x_n \notin (E^q)_*(t_n)$, hence in the limit $x \notin (E^q)_*(t)$ so that $u(x, t) \leqslant q$. This means that u is globally continuous on $\mathbb{R}^N \times [0, +\infty)$. In particular, $(E^q)^*(t) \subset (E^s)_*(t)$ for any $t \geqslant 0$ whenever $q > s$, $q, s \in \mathbb{R}$. We deduce easily that (letting now $\Gamma^s(t) := (E^s)^*(t) \setminus (E^s)_*(t)$)

$$\bigcup_{t \geqslant 0} \Gamma^s(t) = (E^s)^* \setminus (E^s)_* \subseteq \{(x, t) : u(x, t) = s\}. \tag{20}$$

Now, let $N := \{s \in \mathbb{R} : |\{(x, t) \in \mathbb{R}^N \times [0, +\infty) : u(x, t) = s\}| = 0\}$. The set N is at most countable. If $s \notin N$, then (20) is in fact an equality: one has $\{u > s\} = (E^s)_*$ and $\{u \geqslant s\} = (E^s)^*$. One can deduce that $\partial E^s_{h_k}$ converges to $\{u = s\}$ in the local Hausdorff sense. For these values of s, the flow defined by our algorithm is a "true" evolution of hypersurfaces. Indeed, at any time $t \geqslant 0$, we can show that $\{u(\cdot, t) = s\}$ has empty interior. Otherwise, there would exist $W_\phi(x, \rho) \subseteq \{u(\cdot, t) = s\}$. In particular, if $q > s > q'$ with $q, q' \in \mathbb{Q}$, we would have $W_\phi(x, \rho) \subseteq (E^{q'})_*(t)$ while $W_\phi(x, \rho) \cap (E^q)^*(t) = \emptyset$. By (14) and (15), it would follow that if $t \leqslant t' < t + \tau_{t+1}(\rho/2)$, then $W_\phi(x, \rho/2) \subset (E^{q'})_*(t)$ and $W_\phi(x, \rho/2) \cap (E^q)^*(t) = \emptyset$, which would imply that $\{u = s\}$ has nonempty interior, leading to a contradiction. We cannot prove in general the uniqueness of the flow $(E^s)^*$, since it could depend on the subsequence (h_k) along which the first limits have been taken. If on the contrary $s \in N$, then a *fattening* of the corresponding level-set happens, and we can only deduce the inclusion (20). As in the case of classical level-set solutions, we expect nonuniqueness of the limit flow in this situation (and *only* then).

If $G(x, t) = \int_0^t g(x, s)\, \mathrm{d}s$ with g continuous, then (by Proposition 4.8) u is the unique viscosity solution [15] of

$$\frac{\partial u}{\partial t} = \phi^\circ(\nabla u)(\operatorname{div} \nabla\phi^\circ(\nabla u) + g). \tag{21}$$

In particular, in this case, the limit u is the same along any subsequence. We can then deduce that $\partial E^s_h(t) \to \{u(\cdot, t) = s\}$ as $h \to 0$, for each level $s \notin N$, or each time t before the moment the level $s \in N$ fattens.

In case G is an arbirary driving term satisfying (18), we conjecture that our u is still the viscosity solution of

$$\frac{\partial u}{\partial t} = \phi^\circ(\nabla u)\left(\operatorname{div} \nabla\phi^\circ(\nabla u) + \frac{\partial G}{\partial t}\right) \tag{22}$$

built by Lions and Souganidis [23, 24]. However, to show this, we would need either to show the stability of our construction under small perturbations of G (that would allow us to approximate G with smooth functions), or a comparison result like Theorem 2.4 between barriers and viscosity solutions (in the sense of [23, 24]) and then use Corollary 4.6.

Let us finally make a few remarks. We first observe that our construction can still be performed if ϕ and ϕ° are nonsmooth: typically, in the *crystalline* case, where the Wulff shape $\{\phi \leqslant 1\}$ is a polyhedron. In this case, our proof of consistency does not hold (nor is it clear how to extend the definition of a sub/superflow). On the other hand, most of the results are still valid, including the comparison principle in Proposition 4.11, and the construction of the level-set function u starting from u_0 still makes sense.

We also mention that in the convex case, if $G = G(t)$, by the same arguments as in [11] we can show that the evolution (defined in $\mathbb{R}^N$) remains convex for all time, including when the anisotropy is nonsmooth. We also expect that the results in [4] still hold with similar proofs, and that a unique "regular" evolution can be defined for small times as the unique limit of our algorithm, when the initial convex set satisfies an interior εW_ϕ-condition. This would in turn yield the uniqueness of the level-set function defined above.

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