J. Eur. Math. Soc. 16, 1915–1935

DOI 10.4171/JEMS/479

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Complex one-frequency cocycles

Received August 17, 2013

Abstract. We show that on a dense open set of analytic one-frequency complex valued cocycles in arbitrary dimension Oseledets filtration is either *dominated* or *trivial*. The underlying mechanism is different from that of the Bochi–Viana Theorem for continuous cocycles, which links non-domination with discontinuity of the Lyapunov exponent. Indeed, in our setting the Lyapunov exponents are shown to depend continuously on the cocycle, even if the initial irrational frequency is allowed to vary. On the other hand, this last property provides a good control of the periodic approximations of a cocycle, allowing us to show that *domination* can be characterized, in the presence of a gap in the Lyapunov spectrum, by additional regularity of the dependence of sums of Lyapunov exponents.

Keywords. Analytic cocycles, dominated splittings, Lyapunov exponents

1. Introduction

In dynamical systems and ergodic theory, the fundamental notion of hyperbolicity appears under many guises, which are usually split into two broad categories. Generally speaking, *uniform* notions of hyperbolicity play a major role in the description of robust behavior. For instance, the strongest such notion is called simply *uniform hyperbolicity* and is closely associated to structural stability, while a weakening of this concept, *partial hyperbolicity*, has been intensively developed in particular for its connection with stable ergodicity. The weakest form of uniform hyperbolicity, sometimes called projective hyperbolicity, demands merely the presence of a continuous *dominated* decomposition of the tangent dynamics, and has been linked to robust transitivity as well as robustness of positive entropy.

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Mathematics Subject Classification (2010): Primary 37D30; Secondary 34D08, 37F99, 37C55

On the other hand, *non-uniform* notions of hyperbolicity are developed around the Oseledets Theorem, which provides a decomposition of the tangent dynamics at almost every point with non-trivial Lyapunov spectrum. Of course, such a decomposition is *a priori* only measurable, and it may depend wildly on parameters, but the flexibility afforded by getting rid of continuity requirements makes for much greater potential applicability. For instance, while there are manifolds (such as even-dimensional spheres) that do not support any non-trivial continuous decomposition of the tangent bundle, any manifold supports ergodic non-uniformly hyperbolic conservative dynamics [DP].

In his address at the 1983 ICM [M], Mañé suggested that the apparent gap between uniform and non-uniform notions of hyperbolicity can be bridged in the case of generic conservative dynamical systems in the C^1-topology. This program was eventually developed by Bochi–Viana [BV], who proved that for almost every orbit, either all Lyapunov exponents are zero or the Oseledets splitting is dominated, and hence either there is no hyperbolicity at all (even non-uniform), or uniform projective hyperbolicity takes place. Moreover, those results were also obtained in the setting of continuous cocycles over measure-preserving transformations.

In full generality, the Bochi–Viana Theorem is certainly dependent on low regularity considerations: For instance, there are open sets of (sufficiently smooth) ergodic conservative diffeomorphisms for which the Oseledets splitting is not dominated. However, as far as we know, all such examples currently rely on some underlying uniform form hyperbolicity (see, e.g., [AV], [ASV]).

It would seem that similar considerations apply to the case of cocycles over hyperbolic transformations: Indeed, non-zero Lyapunov exponents tend to appear robustly already for Hölder regularity, even in the presence of topological obstructions to domination [V]. But we will show in this paper that Mañé's picture turns out to hold unexpectedly in very large regularity in one important setup.

1.1. Bochi–Viana Theorem for analytic one-frequency complex cocycles

Let $\mathcal{L}(\mathbb{C}^d, \mathbb{C}^d)$ denote the set of linear operators from $\mathbb{C}^d$ to $\mathbb{C}^d$, i.e. the set of $d \times d$ complex matrices. A complex one-frequency cocycle is given by a pair (α, A), where $\alpha \in \mathbb{R}$ is the frequency and $A \in C^0(\mathbb{R}/\mathbb{Z}, \mathcal{L}(\mathbb{C}^d, \mathbb{C}^d))$ is a continuous function from $\mathbb{R}/\mathbb{Z}$ to $\mathcal{L}(\mathbb{C}^d, \mathbb{C}^d)$, understood as a map $(\alpha, A) : (x, w) \mapsto (x + \alpha, A(x) \cdot w)$. The cocycle iterates are given by $(\alpha, A)^n = (n\alpha, A_n)$, where the A_n are given by

$$A_n(x) = \prod_{j=n-1}^{0} A(x + j\alpha). \tag{1.1}$$

If we want to emphasize the dependence on the frequency, then we write $A_n(\alpha, x)$. We will be mostly interested in the case of irrational frequencies, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. In this case, the dynamics is ergodic and the Oseledets Theorem provides us with a strictly decreasing sequence of Lyapunov exponents $\gamma_j \in [-\infty, \infty)$ of multiplicity $m_j \in \mathbb{N}$, $1 \le j \le k \le d$, such that $\sum_{j=1}^{k} m_j = d$ and for almost every $x \in \mathbb{R}/\mathbb{Z}$ there exists a filtration $\mathbb{C}^d = \tilde{E}_x^1 \supset \cdots \supset \tilde{E}_x^k$ with $\dim \tilde{E}_x^j = m_j + \cdots + m_k$, depending measurably on x, that is invariant in the sense that $A(x) \cdot \tilde{E}_x^j \subset \tilde{E}_{x+\alpha}^j$, $j = 1, \ldots, k$,[1] and for every $w \in \tilde{E}_x^j \setminus \tilde{E}_x^{j+1}$ we have $\limsup_{n\to\infty} \frac{1}{n} \ln \|A_n(x) \cdot w\| = \gamma_j$. Such a filtration often (always, in the invertible case) is associated with an invariant[2] continuous decomposition $\mathbb{C}^d = E_x^1 \oplus \cdots \oplus E_x^k$ with $\dim E_x^j = m_j$ and $\tilde{E}_x^j = E_x^j \oplus \cdots \oplus E_x^k$, also depending measurably on x, with $\limsup_{n\to\infty} \frac{1}{n} \ln \|A_n(x) \cdot w\| = \gamma_j$ for every $w \in E_x^j \setminus \{0\}$ [R].

An invariant continuous decomposition $\mathbb{C}^d = E_x^1 \oplus \cdots \oplus E_x^k$ is called *dominated* if there exists $n \geq 1$ such that for any unit vectors $w_j \in E_x^j$ we have $\|A_n(x) \cdot w_j\| > \|A_n(x) \cdot w_{j+1}\|$. It can be shown that such a dominated decomposition is robust, in the sense that small perturbations of the cocycle will still display a dominated decomposition which will be a small perturbation of the original one. We will say that a filtration is dominated if it is associated with a dominated decomposition.

The Bochi–Viana Theorem, specified to this setting, establishes that for each $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, there exists a residual subset of $A \in C^0(\mathbb{R}/\mathbb{Z}, \mathrm{GL}(d, \mathbb{C}))$ such that the Oseledets splitting is dominated. Our main result shows that even a significantly stronger statement is true in the analytic category.

Theorem 1.1. *Fix $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. There exists a dense open subset $\mathcal{V} \subset C^\omega(\mathbb{R}/\mathbb{Z}, \mathcal{L}(\mathbb{C}^d, \mathbb{C}^d))$ such that for every $A \in \mathcal{V}$ the Oseledets filtration of (α, A) is either trivial[3] or dominated.*

Here we endow the space $C^\omega(\mathbb{R}/\mathbb{Z}, \mathcal{L}(\mathbb{C}^d, \mathbb{C}^d))$ with the usual inductive limit topology. We will actually show a somewhat stronger version of this result, namely with $C^\omega(\mathbb{R}/\mathbb{Z}, \mathcal{L}(\mathbb{C}^d, \mathbb{C}^d))$ replaced by a Banach space $C^\omega_\delta(\mathbb{R}/\mathbb{Z}, \mathcal{L}(\mathbb{C}^d, \mathbb{C}^d))$ of analytic functions $A : \mathbb{R}/\mathbb{Z} \to \mathcal{L}(\mathbb{C}^d, \mathbb{C}^d)$ admitting a holomorphic extension to $\{|\mathrm{Im}\, z| < \delta\}$ which is continuous up to the boundary.

1.2. Regularity and domination

The proof of the Bochi–Viana Theorem given in [BV] centers around the idea that an absence of domination in the Oseledets splitting can be exploited to "mix" different Lyapunov exponents through suitable perturbations, and hence it leads to discontinuity of the Lyapunov spectrum. On the other hand, a very general Baire category reasoning guarantees that the Lyapunov exponents must be continuous at a *generic* cocycle.

At a *very* rough level, something similar is taking place in our setting, in that we do show that some (verified on an open and dense set) regularity of the dependence of Lyapunov exponents on parameters implies domination (or triviality) of the Oseledets splitting. The actual details are however completely different, starting with the fact that the regularity property which is related to domination is not merely continuity of the Lyapunov exponent. In fact, it turns out to involve the holomorphic extension of the cocycle

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1 If $A(x)$ is invertible, then $A(x) \cdot \tilde{E}_x^j = \tilde{E}_{x+\alpha}^j$; if $A(x)$ has a kernel, then $\ker A(x) \subset \tilde{E}_x^k$.

2 In the sense that $A(x) \cdot E_x^j = E_{x+\alpha}^j$, $j = 1, \ldots, k-1$, $A(x) \cdot E_x^k \subset E_{x+\alpha}^k$.

3 We do not know whether the set with trivial Oseledets filtration (all Lyapunov exponents are equal) contains an open set or not within the set of analytic complex cocycles.

dynamics, and was first introduced (in the particular case of $SL(2, \mathbb{C})$-cocycles) by Avila in [Av1].

Let $L_1(\alpha, A) \geq \cdots \geq L_d(\alpha, A)$ be the Lyapunov exponents of (α, A) repeated according to their multiplicity, i.e.,

$$L_k(\alpha, A) = \lim_{n\to\infty} \frac{1}{n} \int_{\mathbb{R}/\mathbb{Z}} \ln(\sigma_k(A_n(\alpha, x)))\, dx, \tag{1.2}$$

where for a matrix $B \in \mathcal{L}(\mathbb{C}^d, \mathbb{C}^d)$ we denote by $\sigma_1(B) \geq \cdots \geq \sigma_d(B)$ its *singular values* (eigenvalues of $\sqrt{B^*B}$). Since the k-th exterior product $\Lambda^k B$ of B satisfies $\prod_{j=1}^k \sigma_j(B) = \sigma_1(\Lambda^k B) = \|\Lambda^k B\|$, $L^k(\alpha, A) = \sum_{j=1}^k L_j(\alpha, A)$ satisfies

$$L^k(\alpha, A) = L_1(\alpha, \Lambda^k A) = \lim_{n\to\infty} \frac{1}{n} \int_{\mathbb{R}/\mathbb{Z}} \ln \|\Lambda^k A_n(\alpha, x)\|\, dx. \tag{1.3}$$

By analyticity, one can extend $A(x)$ to a strip $|\mathrm{Im}\, x| < \delta$ in the complex plane. Then, by subharmonicity and constancy in $\mathrm{Re}\, x$, $L^k(\alpha, A(\cdot + it)) = L_1(\alpha, \Lambda^k A(\cdot + it))$ is a *convex* function of $t \in (-\delta, \delta)$[4] unless it is identically[5] equal to $-\infty$. We say that (α, A) is *k-regular* if $t \mapsto L^k(\alpha, A(\cdot + it))$ is an *affine* function of t in a neighborhood of 0.

Let us say that (α, A) is *k-dominated* (for some $1 \leq k \leq d - 1$) if there exists a dominated decomposition $\mathbb{C}^d = E^+ \oplus E^-$ with $\dim E^+ = k$. If $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, then it follows from the definitions that the Oseledets splitting is dominated if and only if (α, A) is k-dominated for each k such that $L_k(\alpha, A) > L_{k+1}(\alpha, A)$.

The next two results give the basic relation between regularity and domination and show that regularity is fairly frequent.

Theorem 1.2. *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $A \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathcal{L}(\mathbb{C}^d, \mathbb{C}^d))$. If $1 \leq k \leq d - 1$ is such that $L_k(\alpha, A) > L_{k+1}(\alpha, A)$ then (α, A) is k-regular if and only if (α, A) is k-dominated.*

Theorem 1.3. *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $A \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathcal{L}(\mathbb{C}^d, \mathbb{C}^d))$ and $L_k(\alpha, A) > -\infty$. Then for every $t \neq 0$ small enough, $(\alpha, A(\cdot + it))$ is k-regular.*

The last result means that the convex functions $t \mapsto L^k(\alpha, A(\cdot + it))$ are in fact piecewise affine. As in [Av1], this behavior is connected to a quantization phenomenon which we now describe. If $L^k(\alpha, A) \neq -\infty$, define the accelerations

$$\omega^k = \lim_{\epsilon\to 0^+} \frac{1}{2\pi\epsilon} \big(L^k(\alpha, A(\cdot + i\epsilon)) - L^k(\alpha, A)\big), \qquad \omega_k = \omega^k - \omega^{k-1}. \tag{1.4}$$

It is easy to see that ω^k is an integer for k-dominated cocycles (also, ω^d is always an integer if $L^d(\alpha, A) \neq -\infty$). The next result shows that this topological phenomenon manifests itself also in the general case:

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4 This can be viewed as a corollary of Hadamard's three-circle theorem.

5 Convexity implies that right derivatives exist and the graph lies above the tangent line. Hence, $L^k(\alpha, A(\cdot + it))$ is either always or never $-\infty$.

Theorem 1.4. *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $A \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathcal{L}(\mathbb{C}^d, \mathbb{C}^d))$. Then the acceleration is quantized: there exists $1 \le l \le d$, $l \in \mathbb{N}$, such that $l\omega^k$ and $l\omega_k$ are integers.*[6] *If $A \in C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}(d, \mathbb{C}))$ then $1 \le l \le d-1$.*

Theorems 1.2–1.4 generalize earlier results of [Av1] (also extended in [JM]) obtained for $d = 2$.

At this point, we must note a fundamental distinction between the analytic and the continuous setups. The Bochi–Viana Theorem (specified to cocycles over irrational translations) is proved by showing that if $L^k > L^{k+1}$ and (α, A) is not k-dominated then A is not a continuity point of L^k on $C^0(\mathbb{R}/\mathbb{Z}, \mathcal{L}(\mathbb{C}^d, \mathbb{C}^d))$. It turns out that for analytic cocycles, L^k is continuous everywhere. Moreover, we may even perturb the *frequency,* and this indeed plays a fundamental role in our analysis.

Theorem 1.5. *The functions $\mathbb{R} \times C^\omega(\mathbb{R}/\mathbb{Z}, \mathcal{L}(\mathbb{C}^d, \mathbb{C}^d)) \ni (\alpha, A) \mapsto L_k(\alpha, A) \in [-\infty, \infty)$ are continuous at any (α', A') with $\alpha' \in \mathbb{R} \setminus \mathbb{Q}$.*

Theorem 1.5 is optimal in that L_k can be discontinuous at rational frequencies[7] or in lower (even C^∞) regularity [WY, JM].

This result generalizes earlier results [BJ], [JM] for the case $d = 2$. Bourgain [B] also obtained joint continuity for non-singular $d = 2$ cocycles over rotations of higher-dimensional tori. The extension to higher d has been open for over a decade.

A somewhat related theme is quantitative continuity results for (mainly non-singular) analytic cocycles with a *fixed* Diophantine frequency ([GS] for $\mathrm{SL}(2, \mathbb{R})$), more recently extended to $\mathrm{GL}(d, \mathbb{C})$ in [Sch, DK]).[8] Those also hold for the multi-frequency case.

However we are particularly concerned with the dependence on the frequency (and especially the behavior of the Lyapunov exponents of rational approximations); among other things it is the key ingredient in the proofs of Theorems 1.1–1.4.

We also note that all the other past and recent continuity results: both [BJ, B, JM] that provide joint continuity in the cocycle and frequency and [GS, Sch, DK] that are for a fixed Diophantine frequency, are based on some form of the Avalanche principle (originally in [GS]) and large deviation theorem (see [BB]). Here we develop a different strategy which is indeed intimately related to the proof of the connection of regularity and domination: it focuses on the direct construction of invariant cone fields for certain complex phases. This allows us to cover all irrational frequencies without the need to delve into arithmetic considerations.

Our approach of selecting complex phases for which such an analysis can be carried out is ideologically close to the new proof of the result of [BJ] developed in the appendix of [B]. On the technical level, our key analytic argument, given in Section 2, borrows some important ingredients from [Av3].

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6 We note that Theorems 1.3, 1.4 do not in general hold for $\alpha \in \mathbb{Q}$ (see a simple counterexample in Remark 5 of [Av1]).

7 They are for the almost Mathieu cocycles $A(x) = \begin{pmatrix} E-\lambda\cos 2\pi x & -1 \\ 1 & 0 \end{pmatrix}$ as follows from [K], or see an example in Remark 5 of [Av1].

8 It should be noted that the results of the present paper preceded the independent recent work [Sch, DK] (e.g. [J1]).

Finally, the extension of various continuity results originally obtained for $\mathrm{SL}(2, \mathbb{C})$ to the singular case has been achieved gradually, by overcoming a significant number of technical challenges [JKS, T, JM2, JM]. In our current approach singularity of cocycles does not present an additional difficulty.

2. A Brownian motion argument

A quick motivation for the main theorem of this section (which is of general nature) is the following. Consider $\psi = \ln |P(x)|$ where P is a trigonometric polynomial of degree n. Then, by the Lagrange interpolation trick, that has been used in the proofs of localization for the almost Mathieu operator, for any $\epsilon > 0$, $\psi(x)$ cannot be smaller than $\sup \psi(x) - \epsilon$ at $n + 1$ uniformly distributed points, for large n. The same cannot be said of course about an arbitrary subharmonic function. The tool that has been used in the proofs of localization for analytic potentials and continuity arguments is Large Deviation Theorems, showing that "almost invariant" subharmonic functions deviate from the mean only on sets of small measure. In the present argument the key idea is that complexifying the argument leads to many values of the imaginary part where the situation is as nice as for the n-th degree polynomial.

We start with what we call the Big Obstacle Lemma.

Lemma 2.1. *There exists $c > 0$ with the following property. Let $B \subset \mathbb{R}^2$ be a Borel set with non-empty intersection with $(-1, 1) \times \{t\}$ for a subset of $t \in (-1, 1)$ of Lebesgue measure at least ρ. Run Brownian motion starting at the origin until it escapes from $(-2, 2) \times (-2, 2)$. Then the probability of hitting B before escape is at least $c\rho$.*

Proof. We will first need some notions from potential theory. Let μ be a continuous probability measure supported on a Borel subset $A \subset \mathbb{R}^2$. Given a *kernel* K, i.e. a measurable function $K : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, \infty]$ such that $K(x, y)$ is a continuous and decreasing function of $|x - y|$, one can define the *energy* of μ with respect to K by

$$I_K(\mu) = \int_A \int_A K(x, y)\, d\mu(x)\, d\mu(y). \tag{2.1}$$

The corresponding capacity $\mathrm{Cap}_K(A)$ is defined as $1/\inf_\mu I_K(\mu)$. Note that the standard logarithmic capacity which we denote $\mathrm{Cap}(A)$ is defined differently than $\mathrm{Cap}_{-\ln|x-y|}(A)$, namely, $\mathrm{Cap}(A) = e^{-\inf_\mu I_K(\mu)}$. The inf of (2.1) is achieved at a unique measure. In the case of the logarithmic kernel it is called the *equilibrium measure* for A.

We will need two kernels: the Green kernel $G(x, y)$ and the Martin kernel $M(x, y)$. Here G is the Green's function of Brownian motion stopped at exit from $(-2, 2)\times(-2, 2)$, namely

$$G(x, y) = -\frac{1}{\pi}(\ln |x - y| - \mathbb{E}_x \ln |B(T) - y|) \tag{2.2}$$

where $B(T)$ is the Brownian motion at the time of first hit of the boundary of $(-2, 2) \times (-2, 2)$ and $\mathbb{E}_x$ stands for the expectation over Brownian motion started at x. Moreover, $M(x, y)$ is defined as $G(x, y)/G(0, y)$ for $x \neq y$, and $M(x, x) = \infty$.

We will use the fact that for compact sets $A \subset (-2,2) \times (-2,2)$,

$$\mathbb{P}\{\text{hitting } A \text{ before escape from } (-2,2)\times(-2,2)\} \geq \tfrac{1}{2}\operatorname{Cap}_M(A)$$

(see [MP, Theorem 8.24]). Since $G(0,y) > c > 0$, this implies that for any probability measure μ, $I_M(\mu) < cI_G(\mu)$. From the explicit form of G given in (2.2) it follows that for $A \subset (-1,1)\times(-1,1)$, $I_G(\mu) < I_{-\pi^{-1}\ln|x-y|}(\mu) + 1$.

Thus, with μ_0 an equilibrium measure of a closed $A \subset (-1,1)\times(-1,1)$,

$$I_M(\mu_0) \leq C(I_{-\ln|x-y|}(\mu_0) + 1) = C(1 - \ln \operatorname{Cap}(A)).$$

This implies that

$$\mathbb{P}\{\text{hitting } A \text{ before escape from } (-2,2)\times(-2,2)\} \geq \frac{c}{1-\ln\operatorname{Cap}(A)} \geq c\operatorname{Cap}(A).$$

Since for Borel $B \subset \mathbb{R}^2$,

$$\operatorname{Cap}(B) = \sup_K \operatorname{Cap}(K) \tag{2.3}$$

where sup runs over compact subsets of B, and the probability of hitting B before escape is bounded below by the probability of hitting A for $A \subset B$, it is enough to estimate the logarithmic capacity of B by $c\rho$.

Note that for subspaces $V \subset \mathbb{R}^2$,

$$|\operatorname{Proj}_V A| = \sup_K |\operatorname{Proj}_V K| \tag{2.4}$$

where sup runs over compact subsets of B and $|\cdot|$ stands for the Lebesgue measure in V. To prove the non-trivial inequality in (2.4) observe that by the measurable selection theorem one can find a measurable function $f : \operatorname{Proj}_V A \to A$ such that $\operatorname{Proj}_V f(y) = y$. Then by Luzin's theorem, for any $\epsilon > 0$, f is continuous on a compact $C \subset \operatorname{Proj}_V A$ of measure at least $|\operatorname{Proj}_V A| - \epsilon$, and thus $f(C) \subset A$ is a compact set with the desired measure of projection.

We now use the fact that for compact sets, capacity coincides with transfinite diameter:

$$\operatorname{Cap} K = \lim_{n\to\infty} \max_{z_1,\dots,z_n \in K} \Big(\prod_{1\leq j<k\leq n} |z_j - z_k| \Big)^{\frac{2}{n(n-1)}}. \tag{2.5}$$

Clearly, for any compact $K \subset A$ the RHS of (2.5) is minorized by the same quantity with K replaced by the $\operatorname{Proj}_V K$.

It remains to note that for any Borel $D \subset [0,1]$ of Lebesgue measure ρ and any $b < \rho/n$, there exist $z_1,\dots,z_n \in D$ that belong to an arithmetic progression with step b, or equivalently with $|z_i - z_j| = k/b$ for some k (see e.g. [J]). Estimating the RHS of (2.5) for such $z_1,\dots,z_n$ leads to the claim. □

We can now move to the main lemma of this section.

Lemma 2.2. *Let $\epsilon, \delta > 0$. Let $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ and let $p/q \in \mathbb{Q}$ be a continued fraction approximant, and q' the denominator of the previous approximant. Let ψ be a subharmonic function on $|\operatorname{Im} z| < \epsilon$ satisfying $\psi(z) \leq 1$ and $\underline{\psi} = \inf_{|t|<\epsilon}\sup_{x\in\mathbb{R}/\mathbb{Z}} \psi(x+it) \geq 0$,* *and let T be the set of all $t \in (-\epsilon, \epsilon)$ such that*

$$\inf_{x\in\mathbb{R}/\mathbb{Z}} \sup_{0\le k\le q+q'-1} \psi(x+k\alpha+it) \le \underline{\psi} - \delta. \tag{2.6}$$

Then $|T| \le c_q$, where $c_q = c_q(\epsilon, \delta) > 0$ satisfies $\lim_{q\to\infty} c_q = 0$.

Proof. Let M be maximal with $M/q + 1/(2q) < \epsilon$. We say that some $j \in [-M, M]$ is *ρ-bad* if $|T \cap (j/q - 1/(2q), j/q + 1/(2q))| > \rho/q$.

Let B be the set of all z with $|\mathrm{Im}\, z| < \epsilon$ such that $\psi(z) \le \underline{\psi} - \delta$. Notice that if $t \in T$ then there exists $y \in \mathbb{R}/\mathbb{Z}$ such that $y + k\alpha + it \in B$ for $0 \le k \le q + q' - 1$. Notice that $\{y + k\alpha + it\}$ is $1/q$-dense in the circle $\mathrm{Im}\, z = t$.

Let us consider any point of the form $x_0 + ij/q$ where $x_0 \in \mathbb{R}/\mathbb{Z}$ and j is ρ-bad. Let us start Brownian motion at $x_0 + ij/q$, and run it until it escapes $|\mathrm{Im}\, z - j/q| < 1/q$. Then the probability that the Brownian motion does not hit B before escaping is at most $e^{-\kappa}$ for some $\kappa = \kappa(\rho) > 0$. Indeed, this probability is at most that of escaping the square $\left(x_0 - \frac{1}{2q}, x_0 + \frac{1}{2q}\right) \times (\frac{j}{2q} - \frac{1}{2q}, \frac{j}{q} + \frac{1}{q})$ without hitting B. One easily sees that B is a big obstacle for the Brownian motion in this rectangle by noticing that the projection of $B \cap \left(x_0 - \frac{1}{2q}, x_0 + \frac{1}{2q}\right) \times \left(\frac{j}{q} - \frac{1}{2q}, \frac{j}{q} + \frac{1}{2q}\right)$ on the second coordinate has measure at least ρ/q. Therefore, by Lemma 2.1, $\kappa(\rho) \ge -\ln(1 - c\rho) \ge c\rho$.

Assume that the number of ρ-bad j's is either $2l$ or $2l - 1$. Let j_0 be such that there are at least $l - 1$ ρ-bad j's greater than j_0 and at least $l - 1$ ρ-bad j's smaller than j_0.

Fix $x_0 \in \mathbb{R}/\mathbb{Z}$ such that $\psi(x_0 + ij_0/q) \ge \underline{\psi}$. Let us start Brownian motion from $x_0 + ij_0/q$, and run it until it escapes $|\mathrm{Im}\, z| < \epsilon$. Then

$$\underline{\psi} \le \psi(x_0 + ij_0/q) \le p_0 + (1 - p_0)(\underline{\psi} - \delta), \tag{2.7}$$

where p_0 is the probability that the Brownian motion escapes without hitting any point in B. Since $\underline{\psi} \ge 0$, we have

$$p_0/(1 - p_0) \ge \delta. \tag{2.8}$$

When the Brownian motion escapes, it has to go at least through $l - 1$ layers of ρ-bad j's, therefore $p_0 \le e^{-(l-1)\kappa}$, implying

$$l - 1 \le \frac{\ln((\delta + 1)/\delta)}{\kappa} \le \frac{C \ln(1 + 1/\delta)}{\rho}.$$

Since one has at most $2l$ ρ-bad j's and $\epsilon - (M/q + 1/(2q)) < 1/q$, one finds that

$$|T| \le \frac{2l + 2}{q} + 2\rho\epsilon \le C\frac{-\ln\delta}{\rho q} + 2\epsilon\rho.$$

Optimizing for ρ gives $|T| \le C\epsilon^{1/2} q^{-1/2} (\ln(1/\delta))^{1/2}$. □

3. A criterion for domination

We will need a few well known properties of dominated cocycles. The discussion below is parallel to the $\mathrm{SL}(2, \mathbb{R})$ case[9] carried out in detail in Section 2.1 of [Av2], so we omit the proofs.

[9] In this case, 1-domination is the same as uniform hyperbolicity.

The set of k-dimensional subspaces of $\mathbb{C}^d$ is a compact Grassmannian manifold with a holomorphic structure (cf. Appendix) and will be denoted by $G(k,d)$. A *k-conefield* is an open set $U \subset \mathbb{R}/\mathbb{Z} \times G(k,d)$ such that for every $x \in \mathbb{R}/\mathbb{Z}$ there exist $w \in G(k,d)$ and $w' \in G(d-k,d)$ such that $(x,w) \in U$ and $(x,\tilde{w}) \notin \overline{U}$ whenever $\tilde{w}$ is not transverse to w'. If (α, A) is k-dominated, then it is easy to construct a k-conefield U such that for every $(x,w) \in \overline{U}$, w is transverse to the kernel of $A(x)$ and $(x+\alpha, A(x)\cdot w) \in U$. Conversely, k-domination can be detected by a *conefield criterion*: there exist $n \geq 1$ and a k-conefield U such that for every $(x,w) \in \overline{U}$, $(x+n\alpha, A_n(x)\cdot w) \in U$. The conefield criterion implies that k-domination holds through an open set of $(\alpha, A) \in \mathbb{R} \times C^0(\mathbb{R}/\mathbb{Z}, \mathcal{L}(\mathbb{C}^d, \mathbb{C}^d))$.

The dominated splitting for a k-dominated cocycle will be typically denoted by $\mathbb{C}^d = u(x) \oplus s(x)$, $u(x) \in G(k,d)$, $s(x) \in G(d-k,d)$.

In the particular case where A admits a holomorphic extension through $|\mathrm{Im}\, z| < \epsilon_0$, we see that there exists $0 < \epsilon < \epsilon_0$ such that $(\alpha, A(\cdot + it))$ is k-dominated for $|t| < \epsilon$ and with invariant sections of the form $u(\cdot + it)$ and $s(\cdot + it)$, with u and s holomorphic through $|\mathrm{Im}\, z| < \epsilon$ (cf. Theorem 6.1).

Before we can state our criterion we need the following lemma. If for a matrix B, $\sigma_k(B) > \sigma_{k+1}(B)$, then we denote by $E_k^+(B) \in G(k,d)$ the k-dimensional subspace of $\mathbb{C}^d$ associated with the first k singular values. Moreover, $P_{E_k^+(B)}$ denotes the orthogonal projection on that subspace.

Lemma 3.1. *Let $0 < \rho \leq 1/4$ be such that A, B satisfy $\sigma_2(A) \leq \rho^2 \sigma_1(A)$, $\sigma_2(B) \leq \rho^2 \sigma_1(B)$ and $\sigma_1(BA) \geq 4\rho\sigma_1(B)\sigma_1(A)$. If $w \in \mathbb{PC}^d$ satisfies $\|P_{E_1^+(A)}|w\| \geq \rho$ then $\|P_{E_1^+(B)}|(A\cdot w)\| \geq 2\rho$.*

Proof. Let $\gamma = \|P_{E_1^+(B)}|A \cdot E_1^+(A)\|$. Let $v \in E_1^+(BA)$ be a unit vector, and let $y = P_{E_1^+(A)} \cdot v$ and $z = v - y$. Then $\sigma_1(BA) = \|BA\cdot v\| \leq \|BA \cdot y\| + \|BA\cdot z\|$. Clearly

$$\|BA\cdot z\| \leq \sigma_1(B)\sigma_2(A) \quad \text{and} \quad \|BA\cdot y\| \leq \gamma\sigma_1(A)\sigma_1(B) + \sigma_1(A)\sigma_2(B).$$

It follows that $\gamma \geq 4\rho - 2\rho^2 \geq 3.5\rho$, as $\rho \leq 1/4$.

Let now $w \in \mathbb{C}^d$ be a unit vector such that $\|P_{E_1^+(A)} w\| \geq \rho$, write $w = u + x$ with $u = P_{E_1^+(A)}(w)$. Then $\|u\| \geq \rho$ and hence

$$\begin{aligned}\frac{\|P_{E_1^+(B)} A w\|}{\|Aw\|} &\geq \frac{\gamma\|Au\| - \|Ax\|}{\|Au\| + \|Ax\|} \geq \frac{\gamma\sigma_1(A)\|u\| - \sigma_2(A)}{\sigma_1(A)\|u\| + \sigma_2(A)} \\ &\geq \frac{\gamma - \rho^2/\|u\|}{1+\rho^2/\|u\|} \geq \frac{\gamma-\rho}{1+\rho} \geq \frac{4}{5} \cdot \frac{5}{2}\rho = 2\rho.\end{aligned}$$

Thus $\|P_{E_1^+(B)}|(A\cdot w)\| \geq 2\rho$. ◻

Now we can formulate a criterion for domination.

Lemma 3.2. *Assume that there exist $n \in \mathbb{N}$ and $0 < \rho \leq 1/4$ such that for every $x \in \mathbb{R}/\mathbb{Z}$, $\sigma_2(A_n(x)) \leq \rho^2\sigma_1(A_n(x))$, $\sigma_2(A_n(x+n\alpha)) \leq \rho^2\sigma_1(A_n(x+n\alpha))$ and $\sigma_1(A_{2n}(x)) \geq 4\rho\sigma_1(A_n(x+n\alpha))\sigma_1(A_n(x))$. Then the cocycle (α, A) is 1-dominated.*

Proof. The set $U = \{(x,w) : \|P_{E_1^+(A_n(x))}|w\| > \rho\}$ is a conefield and satisfies the conefield criterion for domination. ◻

4. Continuity of the Lyapunov exponents

Recall that $L_j(\alpha, A)$ denotes the j-th Lyapunov exponent of the cocycle (α, A) and $L^k = \sum_{j=1}^k L_j$, $L^k(\alpha, A) = L_1(\alpha, \Lambda^k A)$.

From now on we consider cocycles (α, A) with A analytic.

Lemma 4.1. *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Assume that $L_k(\alpha, A) > L_{k+1}(\alpha, A)$. Then there exists $\epsilon > 0$ such that for almost every $|t| < \epsilon$ the cocycle $x \mapsto A(x + it)$ is k-dominated.*

Proof. Taking exterior products, we reduce to the case $k = 1$. Let $L_1 = L_1(\alpha, A)$ and $L_2 = \max\{L_2(\alpha, A), L_1(\alpha, A) - 1\}$, $L^1 = L_1$, $L^2 = L_1 + L_2$.

Fix $0 < \kappa < (L_1 - L_2)/24$. By unique ergodicity,[10] there exists $n_0 \in \mathbb{N}$ such that

$$\sup_{x \in \mathbb{R}/\mathbb{Z}} \frac{1}{n} \ln \|\Lambda^j A_n(x)\| \leq L^j + \kappa, \qquad j = 1, 2, \tag{4.1}$$

holds for $n \geq n_0$. Fix $\epsilon_0 > 0$ sufficiently small so that

$$\sup_{|\mathrm{Im}\, z| < \epsilon} \frac{1}{n} \ln \|\Lambda^j A_n(z)\| \leq L^j + 2\kappa, \qquad j = 1, 2, \tag{4.2}$$

holds for $n_0 \leq n \leq 2n_0 - 1$, and hence (by subadditivity) for all $n \geq n_0$.

The function $t \mapsto L^j(\alpha, A(\cdot + it))$ is convex, and hence continuous. Take $0 < \epsilon < \epsilon_0$ such that $L_1(\alpha, A(\cdot + it)) \geq L_1 - \kappa$ for $|t| < \epsilon$. In particular, for $|t| < \epsilon$ we have

$$\sup_{x \in \mathbb{R}/\mathbb{Z}} \frac{1}{n} \ln \|A_n(x + it)\| \geq L_1 - \kappa. \tag{4.3}$$

Fix a continued fraction approximant p/q of α and let q' be the denominator of the previous approximant. For any $n \geq n_0$, let $\phi_n(z) = \frac{1}{n} \ln \|A_n(z)\|$, which is subharmonic in $|\mathrm{Im}\, z| < \epsilon$. Notice that

$$\sup_{0 \leq k \leq q+q'-1} \phi_{n+q+q'}(z - q\alpha + k\alpha) \leq \frac{n}{n+q+q'} \phi_n(z) + \frac{q+q'}{n+q+q'} \sup_{|\mathrm{Im}\, z| < \epsilon} \ln \|A(z)\|. \tag{4.4}$$

Let T_n be the set of all $|t| < \epsilon$ such that there exists $x \in \mathbb{R}/\mathbb{Z}$ with $\phi_n(x + it) \leq L_1 - 3\kappa$, and let $T_{n,q}$ be the set of all $|t| < \epsilon$ such that there exists $x \in \mathbb{R}/\mathbb{Z}$ with $\phi_n(x + it + k\alpha) \leq L_1 - 2\kappa$ for all $k = 0, \ldots, q + q' - 1$. By (4.4), there exists $n(q) \in \mathbb{N}$ such that for $n \geq n(q)$ we have $T_n \subset T_{n+q+q',q}$.

By Lemma 2.2 (applied to the function $\psi = \frac{\phi_n - (L_1 - \kappa)}{3\kappa}$ and $\delta = 1/3$), for $n \geq n_0$ we have $|T_{n,q}| \leq c_q$, with $\lim c_q = 0$. Thus for $n \geq \max\{n_0, n(q)\}$ we have $|T_n| \leq c_q$ as well. It follows that $\lim |T_n| = 0$.

[10] See a more detailed argument in the proof of Lemma 5.1.

In particular, for almost every $|t| < \epsilon$, there exist arbitrarily large n with $t \notin T_n \cup T_{2n}$. Fix such t and n. Then for every $x \in \mathbb{R}/\mathbb{Z}$, letting $W_1 = A_n(x+it)$ and $W_2 = A_n(x+it+n\alpha)$, so that $W_2 W_1 = A_{2n}(x+it)$, we have, by (4.2) and (4.3),

$$\sigma_1(W_2 W_1) \geq e^{-8\kappa n} \sigma_1(W_2)\sigma_1(W_1), \tag{4.5}$$

as well as

$$\sigma_2(W_j) = \frac{\|\Lambda^2 W_j\|}{\|W_j\|^2}\sigma_1(W_j) \leq e^{(L_2-L_1+8\kappa)n}\sigma_1(W_j), \qquad j=1,2. \tag{4.6}$$

For large n (such that $e^{(L_2-L_1+24\kappa)n} \leq 1/16$), we can apply Lemma 3.2 with $\rho = \frac{1}{4}e^{-8\kappa n}$, to conclude that $(\alpha, A(\cdot+it))$ is 1-dominated. Note that the whole argument also works if A is not invertible and even if $L_2(\alpha, A) = -\infty$. By taking exterior products, the case where $L_k(\alpha, A)$ is finite but $L_{k+1}(\alpha, A) = -\infty$ is also covered. □

Now we are ready to prove Theorem 1.5.

Proof of Theorem 1.5. Consider first the case $L_k(\alpha', A') = -\infty$. As $L^k = L_k + L^{k-1}$, this means $L^k(\alpha', A') = -\infty$. By upper-semicontinuity, L^k is continuous at (α', A'). As $kL_k \leq L^k$ we find for $(\alpha_n, A_n) \to (\alpha', A')$ that $L_k(\alpha_n, A_n) \to -\infty$ as well, showing continuity.

Let $L_k(\alpha', A') > -\infty$. By the definition of the inductive topology, it is enough to consider the restriction to $C^\omega_{\epsilon_0}(\mathbb{R}/\mathbb{Z}, \mathcal{L}(\mathbb{C}^d, \mathbb{C}^d))$ for arbitrary $\epsilon_0 > 0$. Let $\alpha' \in \mathbb{R} \setminus \mathbb{Q}$.

If $L_k(\alpha', A') > L_{k+1}(\alpha', A')$, we choose $\epsilon > 0$ small such that $(\alpha', A'(\cdot+it))$ is k-dominated for $t = \pm\epsilon, \pm 2\epsilon$. Then $(\alpha, A) \mapsto L^k(\alpha, A(\cdot+it))$ is continuous in a neighborhood of (α', A') for $t = \pm\epsilon, \pm 2\epsilon$. Let $s_{(\alpha,A)}(a,b) = (b-a)^{-1}[L^k(\alpha, A(\cdot+ib)) - L^k(\alpha, A(\cdot+ia))]$ be the slope of the secant of the function $t \mapsto L^k(\alpha, A(\cdot+it))$ from a to b. By convexity, for $|t| < \epsilon$ one finds

$$s_{(\alpha,A)}(-\epsilon, -2\epsilon) \leq s_{(\alpha,A)}(0,t) \leq s_{(\alpha,A)}(\epsilon, 2\epsilon). \tag{4.7}$$

Since $(\alpha, A) \mapsto s_{(\alpha,A)}(\pm\epsilon, \pm 2\epsilon)$ is continuous at (α', A'), we find a neighborhood $\mathcal{U}$ of (α', A') and a uniform constant C such that $|L^k(\alpha, A(\cdot+it)) - L^k(\alpha, A)| \leq C|t|$ for $(\alpha, A) \in \mathcal{U}$ and $|t| < \epsilon$. Considering a sequence $t_n \to 0$ for which $(\alpha', A'(\cdot+it_n))$ is k-dominated, and hence $(\alpha, A) \mapsto L^k(\alpha, A(\cdot+it))$ is continuous on a neighborhood of (α', A') (possibly decreasing with n), we conclude that L^k is continuous at (α', A').

Assume now that $L_j(\alpha', A') = L_k(\alpha', A') > -\infty$ for j in a maximal interval $[a,b]$ containing k. Then L^{a-1} and L^b are continuous at (α', A').[11] Since L^a and L^{b-1} are upper-semicontinuous, L_a is upper-semicontinuous at (α', A') and L_b is lower-semicontinuous at (α', A'). Since $L_a \geq L_j \geq L_b$ for $a \leq j \leq b$ and $L_a(\alpha', A') = L_b(\alpha', A')$ by hypothesis, we conclude that L_j is continuous at (α', A') for $a \leq j \leq b$. The result follows. □

[11] If $b = d$ then this follows as well since $L^d = \int \ln|\det A|\,dx$, which is continuous on $C^\omega(\mathbb{R}/\mathbb{Z}, \mathcal{L}(\mathbb{C}^d, \mathbb{C}^d))$ (see e.g. [JM2]). For $a = 1$ we just define $L^0 = 0$ so that $L_1 = L^1 - L^0$.

5. Regularity and approximation through rationals

Recall that

$$\omega^k = \lim_{\epsilon\to 0+} \frac{1}{2\pi\epsilon}\big(L^k(\alpha, A(\cdot + i\epsilon)) - L^k(\alpha, A)\big). \tag{5.1}$$

We let the frequency α be irrational from now on. Furthermore we assume that A extends to a complex analytic function in a neighborhood of $|\mathrm{Im}\, z| \leq \delta$.

Recall that (α, A) is k-regular if $t \mapsto L^k(\alpha, A(\cdot + it))$ is an affine function for $|t| < \epsilon$.

Let

$$\mathbb{R}\setminus\mathbb{Q} \ni \alpha = \lim_{n\to\infty} \frac{p_n}{q_n} \quad \text{with} \quad p_n, q_n \in \mathbb{Z}_+,\ (p_n, q_n) = 1,$$

and define, for $z = x + it$ and $p/q \in \mathbb{Q}$,

$$L^k(p/q, A, x) := \lim_{n\to\infty} \frac{1}{n} \ln \|\Lambda^k A_n(p/q, x)\|.$$

Clearly, the limit exists for all $x \in \mathbb{T}$ and we have $L^k(p/q, A, x) = \frac{1}{q}\ln \rho(\Lambda^k A_q(p/q, x))$ where $\rho(A_*)$ is the spectral radius of $A_* \in \mathcal{L}(\mathbb{C}^d, \mathbb{C}^d)$. By definition,

$$L^k(p/q, A) = \int_{\mathbb{R}/\mathbb{Z}} L^k(p/q, A, x)\, dx.$$

We start with the following crucial lemma.

Lemma 5.1. *If $L^k(\alpha, A) > -\infty$ then uniformly for small t and all x,*

$$L^k(p_n/q_n, A(\cdot + it), x) \leq L^k(\alpha, A(\cdot + it)) + o(1). \tag{5.2}$$

More precisely, the estimate being uniform means that for some $\delta > 0$,

$$\limsup_{n\to\infty} \sup_{x\in\mathbb{R}/\mathbb{Z}} \sup_{|t|\leq\delta} [L^k(p_n/q_n, A(\cdot + it), x) - L^k(\alpha, A(\cdot + it))] \leq 0. \tag{5.3}$$

If $L^k(\alpha, A) = -\infty$ then $L^k(p_n/q_n, A(\cdot + it), x)$ converges to $-\infty$ as $n \to \infty$, uniformly for all x.

Proof. By taking exterior products we may just consider the case $k = 1$. Let $\Phi(t) = L_1(\alpha, A(\cdot + it))$. We first assume $\Phi(0) > -\infty$. By Theorem 1.4, $\Phi(t)$ is piecewise affine (note that the proof of Theorem 1.4 depends on Lemmas 4.1 and 6.4 which do not depend on this lemma). Take $\delta > 0$ such that $\Phi(t)$ is affine on $[-\delta, 0]$ and $[0, \delta]$ with a possible corner at 0. Choose n such that $\frac{1}{n}\int_{\mathbb{R}/\mathbb{Z}} \ln \|A_n(\alpha, x + it)\| dx < \Phi(t) + \epsilon$ for $t \in \{-\delta, 0, \delta\}$. By unique ergodicity we get uniform upper bounds in the ergodic theorem applied to $\ln \|A_n(\alpha, x + it)\|$, so there exists j such that for all x and $t \in \{-\delta, 0, \delta\}$,

$$\frac{1}{jn}\sum_{k=0}^{j-1} \ln \|A_n(\alpha, x + kn\alpha + it)\| < \frac{1}{n}\left(\int_{\mathbb{R}/\mathbb{Z}} \ln \|A_n(\alpha, x + it)\| dx + \epsilon\right) < \Phi(t) + 2\epsilon.$$

Thus, for $m = jn$ we have, by subadditivity, $\frac{1}{m} \ln \|A_m(\alpha, x+it)\| - \Phi(t) < 2\epsilon$ for any x and $t \in \{-\delta, 0, \delta\}$. By continuity and compactness we find $N > 0$ such that for $n > N$, $\frac{1}{m} \ln \|A_m(p_n/q_n, x+it)\| - \Phi(t) < 3\epsilon$ for all x and $t \in \{-\delta, 0, \delta\}$. The left hand side is subharmonic for $t \in (-\delta, 0)$ and $t \in (0, \delta)$. Therefore, by the maximum principle, the last estimate holds for all $|t| \le \delta$. By subadditivity, for K large enough and any $r = 0, \dots, m-1$ one has uniformly for $|t| \le \delta$

$$\frac{1}{Km+r} \ln \|A_{Km+r}(p_n/q_n, x)\| \le \frac{1}{Km+r}(Km(\Phi(t)+3\epsilon)+Cr) < \Phi(t)+4\epsilon\ .$$

This proves the claim. If $\Phi(0) = -\infty$ then by continuity and convexity this happens for all t where the holomorphic extension $A(x+it)$ is defined and we can change $\Phi(t)$ to $-1/\epsilon$ in the estimates. □

If the cocycle is k-regular, then one can approximate the Lyapunov exponent by using rational frequencies and any phase x. This is the main result in this section.

Theorem 5.2. *Assume that $L^k(\alpha, A) > -\infty$ and (α, A) is k-regular. Then uniformly for small t and all $x \in \mathbb{R}/\mathbb{Z}$,*

$$L^k(p_n/q_n, A(\cdot+it), x) = L^k(\alpha, A(\cdot+it)) + o(1). \tag{5.4}$$

Proof. Again by using exterior products it is enough to consider $k = 1$. Assume that A admits a holomorphic extension to $|\mathrm{Im}(x)| < \delta_1$ bounded by $C > 0$ and that $\Phi(t) = L_1(\alpha, A(\cdot+it))$ is affine for $|t| \le \delta_0 < \delta_1$. Up to multiplying A by a sufficiently large constant, we may also assume that $\Phi(t) > 1$ for $|t| \le \delta_0$. We are going to show that

$$\frac{1}{q_n} \ln \rho(A_{q_n}(p_n/q_n, \cdot+it)) \ge \Phi(t) + o(1), \qquad |t| \le \delta_0/2, \tag{5.5}$$

This concludes the proof, since (5.2) can be rewritten as $\frac{1}{q_n} \ln \rho(A_{q_n}(p_n/q_n, \cdot+it)) \le \Phi(t) + o(1)$, so (5.5) implies $\frac{1}{q_n} \ln \rho(A_{q_n}(p_n/q_n, \cdot+it)) = \Phi(t) + o(1)$, which is just (5.4) for $k = 1$.

It is easy to see that there exists $c_d > 0$ such that for any $A_* \in \mathcal{L}(\mathbb{C}^d, \mathbb{C}^d)$ there exists $1 \le k \le d$ such that $|\mathrm{tr}\, A_*^k|^{1/k} \ge c_d \rho(A_*)$.[12] Let $1 \le k_n \le d$ and $x \in \mathbb{R}/\mathbb{Z}$ be such that $|\mathrm{tr}\, A_{q_n}(p_n/q_n, x)^{k_n}|^{1/k_n}$ is maximal. Let

$$\phi_n(t) = \max_{x \in \mathbb{R}/\mathbb{Z}} \frac{1}{k_n q_n} \ln |\mathrm{tr}\, A_{q_n}(p_n/q_n, x+it)^{k_n}|.$$

[12] Indeed, by homogeneity and compactness, this inequality holds with the constant $c_d = \min \max_{1 \le k \le d} |\sum_{j=1}^d \lambda_j^k|$, where the minimum is taken over all sequences $\lambda_j \in \overline{\mathbb{D}}$, $1 \le j \le d$, such that $\max_j |\lambda_j| = 1$, and we just have to check that $c_d > 0$. But if $c_d = 0$ then there exist $\lambda_j \in \mathbb{C}$, $1 \le j \le d$, not all zero, such that $\sum_j \lambda_j^k = 0$ for $1 \le k \le d$. Let $J \subset \{1, \dots, d\}$ be the set of all j such that $\lambda_j \ne 0$. Then letting $p(z) = \prod_{j \in J}(z - \lambda_j)$ we have $p(\lambda_j) = 0$ for each $j \in J$, while $\frac{1}{\#J} \sum_{j \in J} p(\lambda_j) = p(0) \ne 0$ (since the contributions corresponding to each non-constant monomial add up to zero), a contradiction.

Then $\phi_n(0) \geq L_1(p_n/q_n, A) + \frac{1}{q_n} \ln c_d$, and using $L_1(p_n/q_n, A) = L_1(\alpha, A) + o(1)$ (Theorem 1.5), we get $\phi_n(0) \geq \Phi(0) + o(1)$.

On the other hand, by Lemma 5.1 we have $\phi_n(t) \leq \Phi(t) + o(1)$ for $|t| \leq \delta_0$. Since $\phi_n(t)$ is clearly a convex function of t, and $\Phi(t)$ is affine for $|t| \leq \delta_0$, it follows that $\phi_n(t) = \Phi(t) + o(1)$ for $|t| \leq \delta_0$.

Write $\operatorname{tr} A_{q_n}(x)^{k_n} = \sum_{j \in \mathbb{Z}} a_{j,n} e^{2\pi i j q_n x}$. Then $|a_{j,n}| \leq dC^{q_n} e^{-2\pi |j| q_n \delta_1}$. Thus we can choose $m_0 > 0$ such that $\sum_{|j| > m_0} |a_{j,n}| e^{2\pi |j| \, q_n \delta_0} \leq 1$ for every n. It follows that

$$\phi_n(t) = \max_{|j| \leq m_0} \left(\frac{1}{k_n q_n} \ln |a_{j,n}| - \frac{2\pi j}{k_n} t \right) + o(1), \qquad |t| \leq \delta_0,$$

and since $\phi_n(t) = \Phi(t) + o(1)$ with Φ affine, we see that there exist $|j_n| \leq m_0$ such that the slope of Φ is $-2\pi j_n / k_n$ and we have

$$\phi_n(t) = \frac{1}{k_n q_n} \ln |a_{j_n,n}| - \frac{2\pi j_n}{k_n} t + o(1),$$

while for each $|t| \leq \delta_0/2$ and $|j| \leq m_0$ we have

$$\frac{1}{k_n q_n} \ln |a_{j,n}| - \frac{2\pi j}{k_n} t \leq \frac{1}{k_n q_n} \ln |a_{j_n,n}| - \frac{2\pi j_n}{k_n} t - \frac{\delta_0 \pi |j - j_n|}{k_n} + o(1).$$

It follows that

$$\frac{\operatorname{tr} A_{q_n}(z, p_n/q_n)^{k_n}}{a_{j,n} e^{2\pi i j q_n z}} = 1 + o(1), \qquad z = x + it, \ |t| \leq \delta_0/2,$$

so that

$$\frac{1}{k_n q_n} \ln |\operatorname{tr} A_{q_n}(z, p_n/q_n)^{k_n}| \geq \Phi(t) + o(1), \qquad |t| \leq \delta_0/2.$$

Thus $\frac{1}{q_n} \ln \rho(A_{q_n}(p_n/q_n, z)) \geq \Phi(t) + o(1)$ for $|t| \leq \delta_0/2$, as desired. □

6. Holomorphic dependence and convergence

In this section we will finally prove the main theorems. In order to obtain the equivalence of regularity and domination as stated in Theorem 1.2 we will use approximation of the unstable and stable directions by rational frequencies and convergence of holomorphic functions. As before, $G(k, d)$ denotes the Grassmannian of k-dimensional subspaces of $\mathbb{C}^d$. As described in the Appendix, this is a holomorphic manifold. An important fact is the holomorphic dependence of dominated splittings:

Theorem 6.1. *Let $(\alpha, A(\cdot + it))$ be k-dominated for $t \in (t_-, t_+)$ and let $u(x + it) \oplus s(x + it)$ be the corresponding dominated splitting. Then $z \mapsto u(z) \in G(k, d)$ and $z \mapsto s(z) \in G(d - k, d)$ are holomorphic for $z = x + it$, $t \in (t_-, t_+)$.*

We first consider just the more unstable directions in the dominated splitting and start with an analogue to Lemma 2.1 in [Av2] showing holomorphic dependence. This means that in the splitting $\mathbb{C}^d = u(x) \oplus s(x)$ considered, we assume that for some n, any x and any unit vectors $w \in u(x)$, $v \in s(x)$ we have $\|A_n(x)w\| > \|A_n(x)v\|$. As a corollary we will obtain Theorem 6.1 for rational frequencies. The holomorphic dependence of $s(z)$ for irrational frequencies will be deduced in the proof of Theorem 1.2.[13]

Lemma 6.2. *Let $\mathcal{DO}_k(\alpha, \mathbb{C}^d)$ denote the set of k-dominated analytic cocycles on $\mathbb{C}^d$ with frequency α. For any $x \in \mathbb{R}/\mathbb{Z}$ the map $A \mapsto u_A(x)$ is a holomorphic function of $A \in \mathcal{DO}_k(\alpha, \mathbb{C}^d)$. Here, $u_A(x)$ denotes the corresponding unstable subspace.*

In particular, an immediate corollary is

Corollary 6.3. (i) *The unstable subspace $u(x+it) \in G(k,d)$ depends holomorphically on $x + it$.*

(ii) *If $\alpha \in \mathbb{Q}$ is rational, then the stable subspace $s(x+it)$ depends holomorphically on $x + it$.*

Proof. Holomorphic dependence of $u_1 \wedge \cdots \wedge u_k \in \mathbb{P}(\Lambda^k\mathbb{C}^d)$ implies holomorphic dependence of the subspace spanned by $u_1, \ldots, u_k$. In fact, $G(k,d)$ can be considered as a closed submanifold[14] of the projective space $\mathbb{P}(\Lambda^k\mathbb{C}^d)$. Therefore, we may consider $\Lambda^k A$ and can assume $k = 1$. Now let ϵ_0 be the infimum of the distance between $u_A(x)$ and unit vectors in $s_A(x)$. Let $0 < \epsilon < \epsilon_0/2$ and consider the conefield $U = \{x, m\}$, $m \in \mathbb{PC}^d$, such that m is ϵ-close to $u(x)$. Here we use the spherical metric on $\mathbb{PC}^d$. Note that A acts on $\mathbb{PC}^d$ in a natural way. Take n large enough such that $(x + n\alpha, A_n(x)\cdot m) \in U$ for every $(x,m) \in \overline{U}$. Let $\mathcal{V} \subset \mathcal{DO}_1(\alpha, \mathbb{C}^d)$ be the set of all (α, A') such that $(x + n\alpha, A_n(x)\cdot m) \in U$ for every $(x,m) \in \overline{U}$. $\mathcal{V}$ is an open neighborhood of A and for $A' \in \mathcal{V}$ we find that $u_{A'}(x)$ is the limit as $k \to \infty$ of $u^k_{A'}(x) = A'_{kn}(x - kn\alpha)\cdot u_A(x - kn\alpha)$. For each $k \geq 1$ this is a holomorphic function of A' taking values in the hemisphere of $\mathbb{PC}^d$ centered at $u_A(x)$. By Montel's Theorem, the limiting function $A' \mapsto u_{A'}(x)$ is holomorphic.

Part (i) of the corollary follows by holomorphy in Δz for $A'_{\Delta z}(z) = A(z + \Delta z)$. Then $u_{A'_{\Delta z}}(z) = u_A(z + \Delta z)$.

For part (ii) first note that taking $\alpha = 0$ shows that the eigenvector corresponding to the largest modulus of the eigenvalues of a holomorphic matrix valued function $B(z)$ with a gap between the largest and second largest eigenvalues depends holomorphically on z. Using tensor products and inverses $(\Lambda^k B(z) + \epsilon\mathbf{1})^{-1}$ we find that the direct sums of generalized eigenspaces[15] (corresponding to Jordan blocks) of eigenvalues of modulus greater or smaller than a constant c also depend holomorphically on z in a neighborhood where no eigenvalue has modulus c. For rational $\alpha = p/q$, the subspace $s(z)$ is locally characterized as such a subspace, where c is between the k-th and $k+1$-st largest modulus of eigenvalues of $A_q(z)$. □

Using the analyticity of u we obtain the following.

[13] If $A(z)$ is always invertible, then the holomorphic dependence of $s(z)$ follows directly from Lemma 6.2 by considering the inverse cocycle, but the singular case requires approximation by rational frequencies.

[14] Being precisely those elements that can be written as $v_1 \wedge \cdots \wedge v_k$.

[15] The generalized eigenspace for a $d \times d$ matrix B and an eigenvalue λ is the kernel of $(B - \lambda)^d$.

Lemma 6.4. *If* (α, A) *is* k*-dominated then* ω^k *is a constant integer in a neighborhood of* (α, A)*. Moreover, if* $\det A(x) \neq 0$ *for all* x*, then* ω^d *is a constant integer in a neighborhood of* (α, A).

Proof. It is enough to consider the case $k = 1$. As in Theorem A.1(vi) we lift $u(z) \in \mathbb{PC}^d$ to a one-periodic, holomorphic function $u(z) \in \mathbb{C}^d \setminus 0$. Then $A(z)u(z) = \lambda(z)u(z+\alpha)$ for a one-periodic, holomorphic function $\lambda(z)$. Note that $u(z)$ and $\lambda(z)$ also depend holomorphically on A. Thus, for $z = x + it$, $L^1(\alpha, A(\cdot + it)) = \int_0^1 \ln \|A(z)u(z)\| - \ln \|u(z)\| \, dx = \int_0^1 \ln |\lambda(z)| \, dx$. A direct computation (see e.g. [JM2]) shows that $\omega^1(\alpha, A) = \frac{d}{d\epsilon}\big|_{\epsilon=0} \int_0^1 \ln |\lambda(x + i\epsilon)| \, dx$ is minus the winding number of $\lambda(x)$ around 0, so it is an integer and locally constant. As $L^d(\alpha, A) = \int_0^1 \ln |\det A(x)| \, dx$, one obtains the same result for ω^d by the same argument. □

Before proving the main theorems we need another lemma that will guarantee the convergence of the unstable and stable directions when approaching α by rationals.

Lemma 6.5. *Let* $D = \{z \in \mathbb{C} : t_- \leq \operatorname{Im} z \leq t_+\}$ *and let* $u : D \to G(k, d)$, $s : D \to G(d-k, d)$ *be holomorphic functions on the interior* $\mathring{D}$ *and continuous on* D*. Assume that* u *is transverse to* s *at every point and the angle is minorized by* ϵ *at the boundary* ∂D*. Then it is minorized by* ϵ *in the whole strip. Moreover, for any compact subset* $K \subset \mathring{D}$ *of the open strip,* u *and* s *are* C*-Lipschitz where* C *depends only on* ϵ *and* K.

Proof. Let P be the projection on u along s, i.e. P is the unique matrix with $\ker P = s$ and $P|u = \mathrm{id}|u$. By Theorem A.1(v) we can locally lift the pair (u, s) to a holomorphic function $B \in \mathrm{GL}(d, \mathbb{C})$ where the first k vectors represent u and the last $d - k$ column vectors represent s. Then $P = BP_kB^{-1}$ where P_k projects on the first k coordinates in $\mathbb{C}^k$, and hence P is holomorphic. Now, $\|P\| = \sup_{\|w\|=1} \|Pw\|$ is a decreasing function[16] of the angle θ between u and s, going to ∞ if the angle goes to zero. However, as P is holomorphic, $\|P\| = \max_{\|w\|=1} \|Pw\|$ is maximized in D on the boundary ∂D.

For the second part, note that by Cauchy's formula, the partial derivatives of P at $z_0 \in \mathring{D}$ are bounded by $C/\mathrm{dist}(z_0, \partial D)$ for some constant C only depending on ϵ. Now, choose an orthonormal basis $w_1, \ldots, w_k$ for u at z_0 (they are fixed, independent of z) and consider the projections Pw_j as one varies the base point z. Those are Lipschitz near z_0 and the space they generate (which is u) depends in a Lipschitz way on z near z_0. Using the uniform bounds of P and of its derivatives on compact sets $K \subset \mathring{D}$ we obtain a Lipschitz constant C only depending on K and ϵ. □

Now we are ready to prove the main theorems.

Proof of Theorem 1.2. It is enough to consider the case $k = 1$. Let $L_1(\alpha, A) > L_2(\alpha, A)$ and let (α, A) be 1-regular. By Lemma 6.4 it is only left to prove that regularity implies domination.

[16] In fact, the maximum of $\|Pw\|$ occurs if w lies in the plane with the minimal angle and is perpendicular to s. Moreover, $\|P\| = 1/\sin(\theta)$ (see e.g. [GK]).

We let p_n/q_n be rational approximants with $p_n/q_n \to \alpha$. By Lemma 5.1, uniformly in x and $|t| < \epsilon$ we have $L_1(p_n/q_n, A(\cdot + it), x) = L_1(\alpha, (\cdot + it)) + o(1)$ and $L^2(p_n/q_n, A(\cdot + it), x) \leq L^2(\alpha, A(\cdot + it)) + o(1)$ if $L^2(\alpha, A) > -\infty$. If $L^2(\alpha, A) = -\infty$ then $L^2(p_n/q_n, A(\cdot + it), x)$ approaches $-\infty$ uniformly in x and $|t| < \epsilon$. Therefore, either $L_2(p_n/q_n, A(\cdot + it), x) \leq L_2(\alpha, A(\cdot + it)) + o(1)$ or it approaches $-\infty$ and it follows that for large n, $L_2(p_n/q_n, A(\cdot + it), x) < L_1(p_n/q_n, A(\cdot + it), x)$ for every $x \in \mathbb{R}/\mathbb{Z}$ and every $|t| < \epsilon$. Thus, for n large, $(p_n/q_n, A(\cdot + it))$ is 1-dominated throughout the band $|\mathrm{Im}\, z| = |t| < \epsilon$.

Select $t_- < 0 < t_+$ in this band, so that $(\alpha, A(\cdot + t_\pm))$ is 1-dominated. By robustness of domination, the cocycles $(p_n/q_n, A(\cdot + t_\pm))$ are uniformly 1-dominated.

By Lemma 6.2 the unstable and stable subspaces $u_n(x + it)$, $s_n(x + it)$ depend holomorphically on $z = x + it$ for t in a neighborhood of $\{z : t_- \leq \mathrm{Im}\, z \leq t_+\}$. By Lemma 6.5 for each n, the smallest angle occurs at some point z at the boundary $\mathrm{Im}\, z = t_\pm$. But since the cocycles $(p_n/q_n, A(\cdot + t_\pm))$ are uniformly 1-dominated, we find a uniform, non-zero lower bound for the angle between $u_n(x + it)$ and $s_n(x + it)$. Again, by Lemma 6.5 the functions u_n and s_n are uniformly Lipschitz on compact subsets of $\{z : \mathrm{Im}\, z \in (t_-, t_+)\}$. Therefore, there is a convergent subsequence such that u_{n_k} and s_{n_k} converge (uniformly on compacts) to holomorphic functions u and s, satisfying $A(z)s(z) = s(z + \alpha)$, $A(z)u(z) = u(z + \alpha)$. Since $L_2 < L_1$, in the limit $p_n/q_n \to \alpha$ the one-dimensional bundle $u(z)$ is associated to the top Lyapunov exponent almost everywhere and unique ergodicity implies domination. □

Note that the limits $u(z)$ and $s(z)$ are holomorphic functions and therefore we have also proved Theorem 6.1. Next, we show the quantization of acceleration.

Proof of Theorem 1.4. We only need to consider the case $k < d$ and $L^k > -\infty$. Assume that $L_k(\alpha, A(\cdot + it)) - L_{k+1}(\alpha, A(\cdot + it))$ is not identically zero on $t \in [0, \epsilon]$ for any $\epsilon > 0$. Then using Lemma 4.1 one obtains a sequence $t_n \to 0$ where $(\alpha, A(\cdot + it_n))$ is k-dominated. At any such t_n, $\omega^k(\alpha, A(\cdot + it_n))$ is an integer by Lemma 6.4. By convexity of L^k in t, ω^k must be right-continuous and constant for $t \geq 0$ small, hence $\omega^k \in \mathbb{Z}$.

Consider the case $L_k(\alpha, A(\cdot + it)) = L_{k+1}(\alpha, A(\cdot + it)) > -\infty$ for $t \geq 0$ small. Let $[a, b]$ be the maximal interval such that there exists $\epsilon > 0$ with $L_j(\alpha, A(\cdot + it)) = L_k(\alpha, A(\cdot + it))$ for $a \leq j \leq b$ and for every $t \in [0, \epsilon)$. Let us define $L^0 = 0$ and $\omega^0 = 0$. Then, by the arguments above or Lemma 6.4 (in case $b = d$) we find that ω^{a-1} and ω^b are integers. Moreover, $L^k = L^{a-1} + (L^b - L^{a-1})\frac{k-a+1}{b-a+1}$ for every $0 \leq t < \epsilon$. Hence,

$$\omega^k = \omega^{a-1} + (\omega^b - \omega^{a-1})\frac{k - a + 1}{b - a + 1} \in \frac{1}{b - a + 1}\mathbb{Z}.$$

As $\omega^{k-1} \in \frac{1}{b-a+1}\mathbb{Z}$ as well,[17] one also has $\omega_k = \omega^k - \omega^{k-1} \in \frac{1}{b-a+1}\mathbb{Z}$. If $A(z) \in \mathrm{SL}(d, \mathbb{C})$ for all z, then $l\omega_k$, $l\omega^k \in \mathbb{Z}$ for an integer $1 \leq l \leq d - 1$.[18] □

[17] This is clear for $k \geq a + 1$ and if $k = a$ then one even has $\omega^{k-1} \in \mathbb{Z}$.

[18] The case $b - a + 1 = d$ implies $a = 1$, $b = d$ and hence $\omega_k = (1/d)\omega^d$. But if $\det(A(z)) = 1$ then $\omega^d = 0$, and hence all ω^k, ω_k are zero.

Proof of Theorem 1.3. As a consequence it follows immediately that $L^k(\alpha, A(\cdot + it))$ is piecewise affine. Hence, for $t \neq 0$ small enough, L^k is affine in a neighborhood of t. By definition, this means that $(\alpha, A(\cdot + it))$ is k-regular for $t \neq 0$ small enough, which proves Theorem 1.3. □

Now we have everything to prove the main theorem.

Proof of Theorem 1.1. By Theorem 1.5, the continuity of the Lyapunov exponents, there is an open and dense subset $U \subset C^\omega(\mathbb{R}/\mathbb{Z}, \mathcal{L}(\mathbb{C}^d, \mathbb{C}^d))$ such that for $A \in U$ the number of distinct Lyapunov exponents is locally constant. Within U the set where Oseledets filtration is dominated or trivial is automatically open. By Theorem 1.3 the set of cocycles that are k-regular for all k with $L_k > -\infty$ is dense in U, and by Theorem 1.2 all such cocycles with not all Lyapunov exponents equal have dominated Oseledets splitting. □

Appendix. Holomorphic quotients, submersions and lifts

In this appendix we want to briefly explain the holomorphic structure of the Grassmannians $G(k, d)$ and show the existence of local holomorphic lifts to representing matrices.

Let us define the following subgroup of $\mathrm{GL}(d)$:

$$\mathrm{GL}(k, d) := \left\{ \begin{pmatrix} A & C \\ 0 & D \end{pmatrix} : A \in \mathrm{GL}(k),\ D \in \mathrm{GL}(d-k),\ C \in \mathbb{C}^{k \times (d-k)} \right\}. \tag{A.1}$$

Furthermore, let $\mathcal{M}_k(d)$ denote the set of $d \times k$ matrices of rank k.

Theorem A.1. (i) *The Grassmannian $G(k, d)$ can be considered as the quotient*

$$G(k, d) \cong \mathrm{GL}(d)/\mathrm{GL}(k, d) \qquad (\textit{left cosets } G\,\mathrm{GL}(k, d)).$$

(ii) *Let $p : \mathrm{GL}(d) \to G(k, d)$ be the natural projection. Then $G(k, d)$ has a unique holomorphic structure such that p is a holomorphic submersion (meaning p' has full possible rank everywhere). Moreover, the left action of $\mathrm{GL}(d)$ is holomorphic.*

(iii) *There is a natural projection $\tilde{p} : \mathcal{M}_k(d) \to G(k, d)$ which is also a holomorphic submersion.*

(iv) *Locally, for each $G \in \mathrm{GL}(d)$ and $M \in \mathcal{M}_k(d)$ there exist neighborhoods U_G of $p(G)$ and U_M of $\tilde{p}(M)$ and holomorphic injections $i_G : U_G \to \mathrm{GL}(d)$ and $i_M : U_M \to \mathcal{M}_k(d)$ such that $p \circ i_G = \mathrm{id}|U_G$ and $\tilde{p} \circ i_M = \mathrm{id}|U_M$.*

(v) *A holomorphic function $u : D \to G(k, d)$ can be locally lifted in a small neighborhood U_z of $z \in D$ to a holomorphic function $G : U_z \to \mathrm{GL}(d)$ or $B : U_z \to \mathcal{M}_k(d)$ such that $p \circ G = u$ or $\tilde{p} \circ B = u$, respectively.*

(vi) *An analytic function $u \in C^\omega(\mathbb{R}/\mathbb{Z}, G(k, d))$ can be lifted to a one-periodic holomorphic function $\tilde{u} : D_\delta \to \mathcal{M}_k(d)$ such that $\tilde{p} \circ \tilde{u} = u$, for some $\delta > 0$. Here $D_\delta = \{\mathrm{Im}\, z < \delta\}$.*

Proof. $G(k, d)$ denotes the set of k-dimensional subspaces of $\mathbb{C}^d$. A k-dimensional subspace $u \in G(k, d)$ can be represented by a basis, hence by a $d \times k$ matrix $B(z)$ of full rank k where the k column vectors span u. Hence, we obtain a natural projection $\tilde{p} : \mathcal{M}_k(d) \to G(k,d)$. We also have a natural projection $\hat{p} : \mathrm{GL}(d) \to \mathcal{M}_k(d)$ by simply selecting the first k column vectors. It is clear that $\hat{p}$ is holomorphic and we find local holomorphic injections i from small neighborhoods in $\mathcal{M}_k(d)$ to $\mathrm{GL}(d)$ such that $\hat{p} \circ i = \mathrm{id}$. Therefore, statement (iii) follows from (ii) and the statement about i_M in (iv) follows from the one about i_G in (iv).

Two matrices G_1, G_2 in $\mathrm{GL}(d)$ represent the same element in $G(k,d)$ if and only if the first k column vectors span the same space. This is equivalent to $G_1 = G_2 G$ where $G \in \mathrm{GL}(k,d)$. In other words, the set $\mathrm{GL}(d)/\mathrm{GL}(k,d)$ of left cosets is equivalent to $G(k,d)$ and there is a natural, transitive left action of $\mathrm{GL}(d)$ on it. We want to make p a holomorphic submersion. Therefore, consider the exponential chart $P \mapsto G\exp(P)$ around $G \in \mathrm{GL}(d)$. If p is a submersion, then the kernel of $p'(G)$ must precisely be given by the Lie algebra $\mathfrak{gl}(k,d)$ of $\mathrm{GL}(k,d)$. The Killing form $\mathrm{Tr}(A^*B)$ defines a natural metric on $\mathfrak{gl}(d)$ and we can consider the orthogonal complement $\mathfrak{gl}(k,d)^\perp$. Consider the map $p_G(C) = p(G\exp(C))$ for $C \in \mathfrak{gl}(k,d)^\perp$. For small C, these maps are injective. Now, if p is a holomorphic submersion, then p_G is holomorphic and the derivative at 0 must have full rank and hence p_G is locally invertible, i.e. p_G defines a chart for small C. On the other hand, using small C, the maps p_G for $G \in \mathrm{GL}(d)$ clearly define an atlas giving $G(k,d)$ a holomorphic structure such that p is a holomorphic submersion. Moreover, the left action of $\mathrm{GL}(d)$ is also clearly holomorphic. This proves (ii).

For (iv) note that using the charts p_G, the maps i_G defined by $i_G(p_G(C)) = G\exp(C)$ fulfill the requirement. Clearly, (v) follows from (iv).

To obtain (vi) let us consider first the case $k = 1$ for simplicity. Then $G(1,d) = \mathbb{PC}^d$ and $\mathcal{M}_1(d) = \mathbb{C}^d \setminus \{0\}$. It is enough to find $v \in \mathbb{C}^d$ such that v is never orthogonal to $u(x)$, i.e. $v^*u(x) \neq 0$ for all $x \in [0,1]$, because the canonical projection $p_v : \{w \in \mathbb{C}^d : v^*w = 1\} \to \mathbb{PC}^d$ defines a chart and the inverse gives the desired 1-periodic lift $\tilde{u} = p_v^{-1} \circ u$.

So let $W(x) = \{w \in \mathbb{C}^d : w^*u(x) = 0\}$; then $W(x) \cong \mathbb{C}^{d-1} \cong \mathbb{R}^{2d-2}$ defines a real, $2d-2$-dimensional fiber bundle over the torus $\mathbb{R}/\mathbb{Z}$ and $\mathcal{M} = \bigcup_{x \in \mathbb{R}/\mathbb{Z}} \{x\} \times W(x)$ can be seen as a real $2d-1$-dimensional submanifold of $(\mathbb{R}/\mathbb{Z}) \times \mathbb{C}^d$. The map $f : \mathcal{M} \to \mathbb{C}^d$, $f(x,w) = w$, is differentiable. As $\mathbb{C}^d \cong \mathbb{R}^{2d}$ is real $2d$-dimensional, f is not surjective. Take v not in the image of f.

For general k one needs to find $V \in \mathcal{M}_k(d)$ such that $\det(V^*u(x)) \neq 0$[19] for all $x \in [0,1]$. Then the projection $p_V : \{W \in \mathcal{M}_k(d) : V^*W = 1\} \to G(k,d)$ is a chart and $\tilde{u} = p_V^{-1} \circ u$ will be the desired one-periodic lift. The existence of V can be obtained by similar arguments.[20] □

Acknowledgements. We would like to thank Christoph Marx and Adriana Sanchez for their useful comments on this manuscript.

The work of S.J. was supported by the NSF Grant DMS 1101578 and the work of C.S. was supported by the NSERC Discovery grant 92997-2010 RGPIN.

[19] The condition is independent of the representative of $u(x)$ in $\mathcal{M}_k(d)$.

[20] Associating V and $u(x)$ with the exterior products of their column vectors, this is equivalent to the case $k = 1$.

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