On the Singular Behaviour of Fluid in a Vertical Wedge

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Dedicated to the memory of Johannes Maul

Es werden Lösungen der Gleichung für Kapillarflächen über Gebieten mit Ecken betrachtet. Dabei wird angenommen, daß die Ecke durch zwei Kurven begrenzt wird, die einen inneren Winkel $2\alpha$ haben mit $0 < 2\alpha < \pi$ und $\alpha + \gamma < \pi/2$, wobei $0 < \gamma < \pi/2$, der Kontaktwinkel zwischen der Fläche und der Containerwand ist. Es wird eine asymptotische Formel für Lösungen in der Umgebung der Ecke angegeben.

Solutions of capillary surface equation over domains with corners are considered. It is assumed that the corner is bounded by curves which make an interior angle $2\alpha$ with $0 < 2\alpha < \pi$ and $\alpha + \gamma < \pi/2$, where $0 < \gamma < \pi/2$ is the contact angle between the surface and the container wall. An asymptotic formula for the solutions near the corner is given.

1. Introduction. We consider the non-parametric capillary problem in the presence of gravity. One seeks a surface $S: u = u(x), x = (x_1, x_2)$, defined over a bounded base domain $\Omega \subset \mathbb{R}^2$, such that $S$ meets vertical cylinder walls over the boundary $\partial \Omega$ in a prescribed constant angle $\gamma$, $0 \leq \gamma \leq \pi/2$. The problem when a tube of cross-section $S$ is placed into an infinite reservoir leads to the equations (see Finn [3])

\begin{align}
\text{div } T u &= x u \quad \text{in } \Omega, \\
v \cdot T u &= \cos \gamma \quad \text{on the smooth parts of } \partial \Omega,
\end{align}

where $T u = Du/\sqrt{1 + |Du|^2}$, $x = \text{const} > 0$ and $v$ is the exterior unit normal on $\partial \Omega$. By $Du$ we denote the gradient of $u$.

Let the origin $x = 0$ be a corner of $\Omega$ with the interior angle $2\alpha$ satisfying $0 < 2\alpha < \pi$. We assume that the corner is bounded by two sufficiently regular curves and that each curve makes an angle $\alpha$ with the positive $x_1$-axis, see Figure 1.

In fact, it is enough that the curves belong to $C^{\mu}$ for some $\mu \in (0, 1)$. When the curves are lines near the origin, then Concus and Finn [2] have shown that $u$ is unbounded at the origin if and only if $\alpha + \gamma < \pi/2$ holds. In this paper we are interested in this singular case. Thus, we suppose that $\alpha + \gamma < \pi/2$ in what follows. Let $r, \theta$ be polar coordinates centred at $x = 0$, set $k = \sin \alpha/\cos \gamma$ and define

$$u_0(r, \theta) = \left(\cos \theta - \sqrt{k^2 - \sin^2 \theta}\right)/kr.$$
Then, using a method of Concus and Finn, we have shown in [4] that

\[ u(x) = u_0(r, \theta) + O(r^\epsilon) \]  

(1.4)

holds near the corner for an \( \epsilon > 0 \) when the corner is bounded by lines near the origin. That is, \( x_2 = \tan \alpha \cdot x_1 \) is the upper curve and \( x_2 = -\tan \alpha \cdot x_1 \) the lower one which define the corner. The leading singular term \( u_0(r, \theta) \) was discovered by Concus and Finn ([2] and [3, Theorem 5.5]). The expansion (1.4) shows that for fixed \( \theta \) the function \( u(r) \) is asymptotically a hyperbola. For \( \theta = \pm \alpha \) one obtains the curves of contact on the container wall, compare Finn [3, Note 4, p. 131] with respect to an experiment performed by Taylor [5].

The aim of this note is to obtain an expansion like (1.4) in the case when the corner is bounded by curves instead by lines. Under the stronger assumption \( 0 < \gamma < \pi/2 \) we obtain by the same method that

\[ u(x) = u_0(s, \theta) + q(\theta) + O(s') \]  

(1.5)

holds. Here \( s, \theta \) denote curvilinear coordinates and \( q \) is the (unique) solution of a two-point boundary value problem for a regular second order ordinary differential equation, see the next sections.

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2. Curvilinear coordinates. We use curvilinear coordinates \( x_1 = x_1(s, \theta) \) and \( x_2 = x_2(s, \theta) \) \((-\alpha \leq \theta \leq \alpha; 0 \leq s \leq s_0, s_0 \) small enough). Here \( \theta = \text{const} \) yield the curves passing through the origin and \( s \) denotes the arc length on these curves measured from the origin, see Figure 1. More precisely, let

\[ x_2 = f_1(x_1) = \tan \alpha \cdot x_1 + a_1 x_1^2 + O(x_1^3) \]  

(2.1)

be the upper curve and

\[ x_2 = f_2(x_1) = -\tan \alpha \cdot x_1 + a_2 x_1^2 + O(x_1^3) \]  

(2.2)

the lower one which define the corner. We set

\[ x_2(x_1, \theta) = \frac{1}{2} \left( 1 + \frac{\tan \theta}{\tan \alpha} \right) f_1(x_1) + \frac{1}{2} \left( 1 - \frac{\tan \theta}{\tan \alpha} \right) f_2(x_1) \]  

(2.3)

and introduce the arc length instead of \( x_1 \) through

\[ s = \int_1^{x_1} \sqrt{1 + x_2^2(x_1, \theta)} \, d\xi \]
which defines \( x_1 = x_1(s, \theta) \) and \( x_2 = x_2(s, \theta) \), where we denote \( x_2(x_1(s, \theta), \theta) \) by \( x_2(s, \theta) \) again. We find the coefficients \( g_1, g_2 \) in the expansions

\[
\begin{align*}
x_1(s, \theta) &= s \cos \theta + s^2 g_1(\theta) + O(s^3), \\
x_2(s, \theta) &= s \sin \theta + s^2 g_2(\theta) + O(s^3)
\end{align*}
\]  

as follows: Inserting (2.1) and (2.2) into (2.3) and then (2.4) for \( x_1 \), we obtain \( x_2(s, \theta) \). Comparison of coefficients with (2.5) yields

\[
\begin{align*}
ge_1(\theta) &= g_1(\theta) \tan \theta + 2 s \cos^2 \theta \cdot G(a_1, a_2, \alpha, \theta), \\
ge_2(\theta) &= 2 s \cos \theta \cdot G(a_1, a_2, \alpha, \theta).
\end{align*}
\]  

where \( G \) is defined by

\[
G(a_1, a_2, \alpha, \theta) = (1 + \tan \theta / \tan \alpha) a_1 + (1 - \tan \theta / \tan \alpha) a_2.
\]  

From (2.4), (2.5) and \( x_1 \cdot x_2 = 1 \) it follows that

\[
ge_1(\theta) = -g_2(\theta) \tan \theta
\]  

holds. Combining this equation with (2.6), we obtain

\[
ge_1(\theta) = -2^{-1} \sin \theta \cos^3 \theta \cdot G(a_1, a_2, \alpha, \theta)
\]  

and

\[
ge_2(\theta) = 2 \cos \theta \cdot G(a_1, a_2, \alpha, \theta).
\]  

Set \( x = (x_1, x_2) \) and \( D = \det \begin{pmatrix} x_1(s, \theta) \\ x_2(s, \theta) \end{pmatrix} \). From (2.4), (2.5) we see that

\[
\begin{align*}
x_1 \cdot x_0 &= s^2 + 2 e(\theta) s^3 + O(s^4), \\
x_2 \cdot x_0 &= f(\theta) s^2 + O(s^3),
\end{align*}
\]

\[
D = s + e(\theta) s^2 + O(s^3),
\]

where

\[
\begin{align*}
e(\theta) &= -g_1(\theta) \sin \theta + g_2(\theta) \cos \theta, \\
f(\theta) &= -g_1(\theta) \sin \theta + g_2(\theta) \cos \theta.
\end{align*}
\]

We mention that \( e = f' \) holds because (2.8). Finally, we obtain from (2.9), (2.10) for \( f \) and \( e \)

\[
\begin{align*}
f &= (1/2) \cos^3 \theta \cdot G(a_1, a_2, \alpha, \theta), \\
e &= -(3/2) \sin \theta \cos^2 \theta \cdot G(a_1, a_2, \alpha, \theta) + (\cos \theta / 2 \tan \alpha) (a_1 - a_2),
\end{align*}
\]

where \( G \) is defined by (2.7).

3. The asymptotic formula. For \( 0 < \varepsilon < \varepsilon_0, \varepsilon_0 \) small enough, we set \( \Omega_\varepsilon = \Omega \cap B_\varepsilon, \Sigma_\varepsilon = (\partial \Omega \cap \partial B_\varepsilon) \setminus \{0\} \) and \( \Gamma_\varepsilon = \Omega \cap \partial B_\varepsilon \). Here \( B_\varepsilon \) denotes a disc with radius \( \varepsilon \) and the centre at the origin. The proof of the asymptotic formulas (1.4) and (1.5) is based on a method of Concus and Finn, see [3, proof of Theorem 5.5], which relies on the following comparison principle. We give here a special version which we need in our case. For the constant \( \chi > 0 \) let \( N_\varepsilon = \text{div} \, T_\nu - \chi v \).

**Theorem 3.1 (Concus and Finn [1]):** Suppose that \( N_\varepsilon \geq N_\varepsilon \) in \( \Omega_\varepsilon, v \geq w \) on \( \Gamma_\varepsilon \) and \( v \cdot T_\nu \geq v \cdot T_\nu \) on \( \Sigma_\varepsilon \) hold. Then \( v \geq w \) in \( \Omega_\varepsilon \).
With the abbreviation
\[ R = D^2 + \omega R^2 + x_\theta = x_\theta w_\theta - 2 x_\theta \cdot x_\theta w_\theta \]
we have in curvilinear coordinates \( s, \theta \)
\[
\text{div } T_w = \frac{1}{|D|} \left[ \left( \frac{x_\theta \cdot x_\theta w_\theta - x_\theta \cdot x_\theta w_\theta}{\sqrt{R}} \right)_s + \left( \frac{w_\theta - x_\theta \cdot x_\theta w_\theta}{\sqrt{R}} \right)_\theta \right]
\]
and
\[
\nu \cdot T_w = \begin{cases} 
(w_\theta - x_\theta \cdot x_\theta w_\theta) / \sqrt{R} & \text{on the upper curve } (\theta = \alpha), \\
(-w_\theta + x_\theta \cdot x_\theta w_\theta) / \sqrt{R} & \text{on the lower curve } (\theta = -\alpha). 
\end{cases}
\]
For
\[
h(\theta) = (\cos \theta - \sqrt{k^2 - \sin^2 \theta}) / k_x,
\]
where \( k = \sin \alpha / \cos \gamma \), let \( w = s^{-1}h(\theta) + q(\theta) - As^4 \) for a function \( q \in C^2[-\alpha, \alpha] \), a constant \( A \neq 0 \) and a constant \( \lambda > 0 \). We define
\[
Lq = (A_2(\theta) q')' + A_1(\theta) q',
\]
where \( A_2 = h^2(h^2 + h'^2)^{-3/2} \) and \( A_1 = 2hh'(h^2 + h'^2)^{-3/2} \), and set
\[
F(\theta) = -2(h^2 + h'^2)^{-3/2} (eh^3 + 2ehh'^2 + fh'h^2)
\]
\[
+ [(h^2 + h'^2)^{-3/2} (fh'^3 - eh'h^2)]' + eh(h^2 + h'^2)^{-1/2}
\]
\[
- e[h'(h^2 + h'^2)^{-1/2}]' \]
with \( e \) and \( f \) from (2.12) and (2.11). After some calculation, we obtain that
\[
\text{div } T_w = xw + xAs^4 + O(As^4) + O(s) \]
holds, provided that \( \lambda \leq 1 \) and \( |A| \lambda \leq K_0 \) are satisfied for a constant \( K_0 > 0 \). Hence, since \( s^{-1}h(\theta) = w - q(\theta) + As^4 \), it follows
\[
\text{div } T_w = xw + xAs^4 + O(As^4) + O(s) \]
if \( q \) is a solution of
\[
Lq - xq + F = 0 \quad \text{on } (-\alpha, \alpha).
\]
Again, after some calculation, one finds
\[
\nu \cdot T_w = \cos \gamma - A\lambda hh'(h^2 + h'^2)^{-3/2} s^{1+\lambda} + O(s^2)
\]
on the upper curve \( (\theta = \alpha) \) and
\[
\nu \cdot T_w = \cos \gamma + A\lambda hh'(h^2 + h'^2)^{-3/2} s^{1+\lambda} + O(s^2)
\]
on the lower one \( (\theta = -\alpha) \), provided that \( q \) satisfies the boundary conditions
\[
q' + fh - eh' = 0 \quad \text{for } \theta = -\alpha \text{ and } \theta = \alpha.
\]
As above, we assume also here that \( |A| \lambda \leq K_0 \) and \( \lambda \leq 1 \) hold.

**Lemma 3.1** There exists a unique solution to the two-point boundary value problem (3.5), (3.8).

**Proof:** It is enough to show that the homogeneous problem has only the solution \( q = 0 \). Let \( q_0 \) be a solution to the homogeneous problem associated to (3.5), (3.8) and \( u \) a solution to (1.1), (1.2), when the origin is a corner which is bounded by lines and
each line makes an angle \( \alpha \) with the positive \( x_1 \)-axis. By the same argument as in [4] one finds \( u = r^{-1}h(\theta) + q_0(\theta) + O(r) \). Here \( r, \theta \) denote polar coordinates. Thus, since (1.4) holds too, it follows that \( q_0(\theta) = 0 \) on \([-\alpha, \alpha]\).

**Theorem 3.2:** Let \( u \) be a solution to (1.1), (1.2) and suppose that \( 0 < 2\alpha < \pi \), \( 0 < \gamma < \pi/2 \) and \( \alpha + \gamma < \pi/2 \). Then, for an \( \epsilon > 0 \), \( u = u_0(s, \theta) + q(\theta) + O(s^\epsilon) \) near the corner, where \( q \) is the solution to the boundary value problem (3.5), (3.8) and \( u_0 \) is defined through (1.3).

**Proof:** Since \( 0 < \gamma < \pi/2 \) holds it follows from the definition (3.1) of \( h \) that \( h \in C^\infty([-\alpha, \alpha]) \) and \( h'(\alpha) = -h'(-\alpha) > 0 \). Let \( w = s^{-1}h(\theta) + q(\theta) - A s^\lambda \), where the constant \( A \) is positive, then one obtains from (3.4), (3.6) and (3.7) by the same argument as in [4, proof of the theorem] that there are positive constants \( A, \rho \) and \( \lambda \) not depending on the particular solution \( u \) considered such that \( \text{div} Tw - \pi \omega \geq 0 \) in \( \Omega_\rho \), \( w \leq u \) on \( \Gamma_\rho \) and \( v \cdot Tw \leq \cos \gamma \) on \( \Sigma_\rho \) hold. Then, Theorem 3.1 implies \( u \geq u_0(s, \theta) + q(\theta) - A s^\lambda \) in \( \Omega_\rho \). By the same argument it follows \( u \leq u_0(s, \theta) + q(\theta) + A s^\lambda \) for possibly other positive constants \( A, \rho \) and \( \lambda \). Here the comparison function \( w = s^{-1}h(\theta) + q(\theta) + A s^\lambda \), \( A > 0 \), is used. Thus, the theorem is proved.

*Note added in proof.* More recently, the correction term \( q(\theta) \) was being calculated numerically by Dr. Berndt and Dr. Janassary from the University of Leipzig.

**REFERENCES**


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