Elliptic Oscillation Theory

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Using an appropriate version of Gårding's inequality, we show how to deduce new non-oscillation theorems for the general even-order elliptic equation

\[
\sum_{|\alpha|,|\beta|\leq m} (-1)^{|\alpha|} D^{\alpha}[A_{\alpha\beta}(x) D^{\beta}u] = 0 \quad (x \in \Omega)
\]

from known non-oscillation theorems for the more frequently studied equation \((-1)^m A^m u + h(x) u = 0 \quad (x \in \Omega)\), where \(\Omega\) is an unbounded, open subset of \(\mathbb{R}^n\).

1. Introduction. Several writers (see, e.g., [2, 3, 5, 8, 9]), have obtained non-oscillation theorems for various forms of the elliptic partial differential equation

\[
(-1)^m \sum_{|\alpha| = |\beta| = m} D^\alpha[u_{\alpha\beta}(x) D^\beta u] + a_0(x) u = 0 \quad (x \in \Omega \subseteq \mathbb{R}^n)
\]

in an unbounded open set \(\Omega\). In a recent paper [6], by using a version of Poincaré's inequality, the author obtained non-oscillation theorems for the more general equation

\[
(-1)^m \sum_{|\alpha| = |\beta| = m} D^\alpha[A_{\alpha\beta}(x) D^\beta u] + \sum_{|\alpha| \leq m} B_{\alpha}(x) D^\alpha u = 0.
\]

In the present paper, by using an appropriate version of Gårding's inequality, we will extend the results in [6] to the equation

\[
Lu := \sum_{|\alpha|,|\beta|\leq m} (-1)^{|\alpha|} D^{\alpha}[A_{\alpha\beta}(x) D^{\beta}u] = 0 \quad (x \in \Omega \subseteq \mathbb{R}^n); \quad (1)
\]
where the coefficient functions $A_{x\beta}$ are real-valued and sufficiently smooth. (The multi-index notation employed here is the same as in [1].) Our main result is a comparison theorem, whose proof, based on a suitable version of Gårding's inequality, will show that every known non-oscillation theorem for the equation

$$(-1)^m \Delta^mu + h(x) u = 0 \quad (x \in \Omega \subseteq \mathbb{R}^n),$$

gives rise to a new non-oscillation theorem for (1).

2. Definitions and preliminary results. Throughout this paper, $G$ will denote any non-empty, open (possibly unbounded) subset of $\Omega$. If $k$ is any non-negative integer, we define the 'seminorm' $|\cdot|_{k,G}$, the weighted seminorm $|\cdot|_{k,G,w}$, and the norm $\|\cdot\|_{k,G}$ as follows:

$$|u|_{k,G} = \left[ \sum_{\alpha} \int |D^{\alpha} u|^2 \, dx \right]^{1/2},$$

$$|u|_{k,G,w} = \left[ \sum_{\alpha} \int \left( k!/\alpha! \right) |D^{\alpha} u|^2 \, dx \right]^{1/2},$$

$$\|u\|_{k,G} = \left[ \sum_{j=0}^k |u_j|^2 \right]^{1/2}.$$

The definition of $|\cdot|_{k,G,w}$ is motivated by the following formula, which is valid for all real-valued $\Phi$ in $C_0^{\infty}(G)$:

$$(-1)^m \int G \Phi \Delta^m \Phi \, dx = (-1)^m \int G \Phi \left( \sum_{k=1}^m D_k^2 \right)^m \Phi \, dx$$

$$= (-1)^m \int G \Phi \sum_{m=1}^n \Phi D^2 \Phi \, dx$$

$$= \sum_{m=1}^n \int G \Phi D^2 \Phi \, dx.$$ 

To compare the seminorms $|\cdot|_{m,G}$ and $|\cdot|_{m,G,w}$, we let

$$c_0 = \max \{|m!/\alpha!| \, : \, |\alpha| = m\}. \quad (6)$$

Then it is easily seen that

$$|u|_{m,G} \leq |u|_{m,G,w} \leq c_0^{1/2} |u|_{m,G}.$$ 

(7)

We also note that, in (3) and (5), when there is no danger of confusion, we omit the subscript $G$. Let $C^k_G = \{ u \in C^k(G) \, : \, |u|_{k,G} < \infty \}$, and let $H_k(G)$ and $H^\infty_k(G)$ denote the completions of $C^k_G$ and $C^\infty_G$, respectively, with respect to the norm $\|\cdot\|_{k,G}$.

If $G$ is bounded, and if there exists a non-trivial function $u$ in $H^\infty_k(G) \cap C^{2m}(G)$ such that (1) holds, then $G$ is called a nodal domain for $L$ or a nodal domain for (1). If for all positive $r$ the region $\{x \in \Omega \, : \, |x| > r\}$ contains a nodal domain for $L$, then (1) is said to be nodally oscillatory (or strongly oscillatory) in $\Omega$.

Using integration by parts, we can easily show that if $G$ is any non-empty, open (possibly unbounded) subset of $\Omega$, then for every real-valued $\Phi$ in $C_0^{\infty}(G)$ we have

$$\int G \Phi L \Phi \, dx = \sum_{|\alpha| = |\beta| = m} \int G A_{x\beta}(x) D^\alpha \Phi D^\beta \Phi \, dx + \sum_{|\alpha| = 2m} \int G \Phi^2 A_{00}(x) \, dx$$

$$+ \sum_{|\alpha| + |\beta| = 1} \int G A_{x\beta} D^\alpha \Phi D^\beta \Phi \, dx.$$ 

(8)
The standard proof of the global version of Garding's inequality [1: Theorem 7.6] now yields the following result.

**Lemma 2.1**: Let $E_0$ denote the ellipticity constant of the differential operator $L$; in other words, let

$$E_0 = \inf \left\{ \sum_{|\alpha| = |\beta| = m} A_{\alpha\beta}(x) \xi^\alpha + \xi^\beta: 0 + \xi \in \mathbb{R}^n, \xi \in \Omega \right\}.$$ 

Suppose that the principal coefficients $A_{\alpha\beta} (|\alpha| = |\beta| = m)$ are uniformly continuous on $\Omega$, and that the remaining coefficients $A_{\alpha\beta} (|\alpha| + |\beta| \leq 2m - 1)$ are bounded and measurable on $\Omega$. Let $G$ be any non-empty, open subset of $\Omega$. Then there exist constants $c_1 \in (0, \infty)$ and $c_2 \in [0, \infty)$ such that, for every real-valued $\Phi \in C_0^\infty(\Omega)$, we have

$$\sum_{|\alpha| = |\beta| = m |\alpha| + |\beta| = 1} \int \sum_{|x|=|\beta|=m} A_{\alpha\beta}(x) \Delta^{\alpha} \Delta^{\beta} \Phi \, dx + \sum_{|\alpha| + |\beta| = 2m - 1} \int A_{\alpha\beta}(x) \Delta^{\alpha} \Delta^{\beta} \Phi \, dx \geq c_1 E_0 \|\Phi\|_{m,G}^2 - c_2 \|\Phi\|_{0,G}^2.$$ 

The constant $c_1$ depends only on $m$ and $n$; the constant $c_2$ depends only on $m$, $n$, $E_0$, $\sup |A_{\alpha\beta}(x)|: x \in \Omega; 1 \leq |\alpha| + |\beta| \leq 2m - 1'$ and the modulus of continuity for the principal coefficients.

3. The main results. Using Lemma 2.1, we will first obtain a comparison theorem, which we can then employ to obtain new non-oscillation theorems for (1) from all known non-oscillation theorems for (2).

**Theorem 3.1**: Let $M$ be the differential operator defined by

$$Mu = (-1)^m c_4 \Delta^m u + \left[ A_{00}(x) - c_2 \right] u,$$

where

$$c_4 = c_1 E_0 / c_0,$$

and $c_0$ is defined by (6). If (1) is nodally oscillatory in $\Omega$, then the differential equation

$$Mu = 0$$

is also nodally oscillatory in $\Omega$.

**Proof**: If (1) is nodally oscillatory in $\Omega$, then for every positive number $r$ the region $\{x \in \Omega: |x| > r\}$ contains a nodal domain $G$ for the differential operator $L$. Thus, there exists a non-trivial function $u \in H_m^0(\Omega) \cap C^m(\Omega)$ such that (1) holds. Furthermore, (8), Lemma 2.1, (9), integration by parts, (4), (7) and (10) imply that, for every $\Phi$ in $C^\infty_0(\Omega)$, we have

$$\int_G \Phi L \Phi \, dx - \int_G \Phi M \Phi \, dx$$

$$\geq [c_1 E_0 \|\Phi\|_{m,G}^2 - c_2 \|\Phi\|_{0,G}^2 + A_{00} \|\Phi\|_{0,G}^2] - [c_4 \|\Phi\|_{m,G,G}^2 + (A_{00} - c_2) \|\Phi\|_{0,G}^2]$$

$$= c_1 E_0 \|\Phi\|_{m,G}^2 - c_4 \|\Phi\|_{m,G,G}^2 \geq c_1 E_0 \|\Phi\|_{m,G}^2 - c_4 \|\Phi\|_{m,G,G}^2 \geq [c_1 E_0 / c_0 - c_2] \|\Phi\|_{m,G,G}^2 = 0.$$ 

(12)

Using (1), (12) and a continuity argument, we obtain $0 = \int_G \Phi M \Phi \, dx \geq \int_G Mu \, dx$.

Therefore, the smallest eigenvalue of the eigenvalue problem $Mv = \lambda v$, $v \in H_m^0(\Omega) \cap C^m(\Omega)$ is non-positive. Hence, we can apply a known monotonicity principle [4] to show that $G$ has a non-empty open subset $G'$ such that zero is the smallest eigenvalue of the eigenvalue problem $Mv = \mu v$, $v \in H_m^0(\Omega') \cap C^m(\Omega')$. Thus, we have shown that, for every positive number $r$, the equation $Mu = 0$ has a non-trivial solution $w$, with a nodal domain $G' \subset G \subset \{x \in \Omega: |x| > r\}$.  \[\square\]
To illustrate how Theorem 3.1 may be employed to obtain new non-oscillation theorems for (1) from known non-oscillation theorems for (2), we now generalize the non-oscillation portion of [8: Theorem 1]. (In [7] we showed how to obtain new oscillation theorems for (1) from known oscillation theorems for (2).)

Theorem 3.2: Consider the polynomial
\[
\prod_{j=0}^{m-1} \left[r + \frac{(n - 2m + 4j)/2}{j!}\right] = \sum_{k=0}^{m} b_k r^k.
\]
If \( n \geq 2m \), or if \( n < 2m \) and \( n \) is odd, then (1) is nodally non-oscillatory in \( \Omega \) if there exists a positive number \( r_0 \) such that for every \( x \) in the region \( \{x \in \Omega: |x| > r_0\} \) we have
\[
[A_{\Omega}(x) - c_2]/c_4 > -|x|^{-2m} \sum_{k=0}^{m} \left[ (2k - 1)!! \right] b_k/4^k \log^{2k} |x|.
\] (13)

Proof: Suppose to the contrary that (1) is nodally oscillatory in \( \Omega \). Then it follows from Theorem 3.1 that (11) is nodally oscillatory in \( \Omega \), contrary to the fact, proved in [8: Theorem 1], that (11) is nodally non-oscillatory in \( \Omega \) whenever (13) holds.

We invite the reader to formulate appropriate generalizations of other known non-oscillation criteria.

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REFERENCES