A Characterization of Dobrushin's Coefficient of Ergodicity

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It is proved that the ergodicity coefficient \( r \) corresponding to any vector norm on \( \mathbb{R}^n \) fulfills the inequality \( r(P) \leq 1 \) for all \( n \times n \) stochastic matrices \( P \) iff it is the Dobrushin ergodicity coefficient.

Let \( \| \cdot \| \) be an arbitrary norm on \( \mathbb{R}^n \) and \( S_n \) (\( n \geq 2 \)) the set of all \( n \times n \) stochastic matrices. In [4] E. Seneta has introduced a general concept of coefficients of ergodicity \( r \) for \( P \in S_n \) with respect to \( \| \cdot \| \):

\[
\tau(P) = \sup \{ \| xP \| : x \in H, \| x \| = 1 \} \quad (P \in S_n),
\]

where \( H = \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 + \cdots + x_n = 0 \} \). For the \( l_1 \)-norm \( \| \cdot \|_1 \), \( \| x \|_1 = |x_1| + \cdots + |x_n| \) (\( x \in \mathbb{R}^n \)), the ergodicity coefficient denoted by \( \tau_1 \) is the well-known Dobrushin coefficient

\[
\tau_1(P) = \frac{1}{2} \max_{ij} \sum_{k=1}^{n} |p_{ik} - p_{jk}| \quad (P \in S_n)
\]

with \( P = (p_{ij})_{i,j=1}^{n} \) (see the remark following Lemma 2). The coefficients \( \tau(P) \) are bounds on all non-unit eigenvalues of \( P \). For \( \tau_1 \), the inequality \( \tau_1(P) \leq 1 \) \((P \in S_n)\) holds. In our note we show that \( \tau_1 \) is the only ergodicity coefficient \( \tau \) satisfying this inequality. The proof points out the role of certain extremal points.

Denote \( K, S, K_1 \) and \( S_1 \) the set of all \( x \in \mathbb{R}^n \) with \( \| x \| \leq 1, \| x \|_1 = 1, \| x \|_1 \leq 1 \) and \( \| x \|_1 = 1 \), respectively. For \( i, j \in \{1, \ldots, n\}, i \neq j \), let \( e_{ij} = (x_1, \ldots, x_n) \) with \( x_i = -x_j = 1/2 \) and \( x_i = 0 \) for \( i \neq i, j \), and denote \( E_1 = \{ e_{ij} \} \). Let \( x^+ = \max(x, 0) \) and \( x^- = \max(-x, 0) \) for \( x \in \mathbb{R} \). Finally, for a linear subspace \( L \subseteq \mathbb{R}^n, L \neq \{0\} \), and norms \( \| \cdot \| \) on \( L \) denote \( B_i \) the corresponding unit balls and \( \text{Ex } B_i \) the set of their extremal points (\( i = 1, 2 \)).

Lemma 1: For \( \lambda > 0 \) the following assertions are equivalent:

(i) \( |x|_1 = \lambda |x|_2 \) (\( x \in L \));

(ii) \( \lambda = \sup_{B_i} |x|_1 \) and \( |e|_2 = 1/\lambda \) \((e \in \text{Ex } B_i)\).

Proof: The implication (i) \( \Rightarrow \) (ii) is obvious. (ii) \( \Rightarrow \) (i): The relation \( \lambda = \sup_{B_i} |x|_1 \) yields \( |x|_1 \leq \lambda |x|_2 \) (\( x \in L \)). On the other hand, if \( x \in L \) by the Krein-Milman Theorem, there are \( e_i \in \text{Ex } B_1 \) and \( \lambda_i \in \mathbb{R} \) \((i = 1, \ldots, k) \) with \( \lambda_i \geq 0, \lambda_1 + \cdots + \lambda_k = 1 \) such that \( x = |x|_1 \sum \lambda_i e_i \). Thus, by \( |e|_2 \geq 1/\lambda \) the inequality \( |x|_2 \leq (1/\lambda) |x|_1 \) follows.
Lemma 2: The equality $\text{Ex}(K \cap H) = E_1$ is true.

Proof: Clearly, $\text{Ex}(K \cap H) \subseteq S_1 \cap H$. Since $x = \left( \sum x_i^+ \right)^{-\frac{1}{2}} \sum x_i^+ x_i^- e_{ij} \ (x \in S_1 \cap H)$, each $x \in S_1 \cap H$ is a convex combination of elements of $E_1$. Therefore, we have $\text{Ex}(K \cap H) \subseteq E_1$. On the other hand, let $e_{ij} = \lambda x + (1 - \lambda) y$ with $x, y \in K \cap H$ and $0 < \lambda < 1$. Since $\lambda x_k + (1 - \lambda) y_k = 1/2$, $\lambda x_i + (1 - \lambda) y_i = -1/2$ and $\sum x_i^+ - \sum x_i^- = 1/2 \|x\|_1 \leq 1/2$, $\sum y_i^+ - \sum y_i^- = 1/2 \|y\|_1 \leq 1/2$ we have $x_k = y_k = 1/2, x_i = y_i = -1/2$, and therefore $x = y = e_{kl}$. Thus, $E_1 \subseteq \text{Ex}(K \cap H)$.

Remark: Using Lemma 2 and (1) one can easily prove (2). Indeed, for each $P \in S_n$ the norm $\|P\|_1$ is a convex functional on $K \cap H$ assuming its maximum at a point of $\text{Ex}(K \cap H)$. Therefore, one obtains $\tau_1(P) = \sup_{K \cap H} \|xP\|_1 = \max_{x \in K \cap H} \|x\|_1 = 1/2$.

Theorem: Let $\tau$ be an ergodicity coefficient with respect to the norm $\|\|_1$ satisfying $\tau(P) \leq 1$ for all $P \in S_n$. Then

(i) $\tau(P) = \tau_1(P)$ for all $P \in S_n$;
(ii) $\lambda \|x\|_1 = \|x\|_1 \ (x \in H)$ for some $\lambda > 0$.

Proof: Since $\|\|_1$ is a continuous functional on the compact set $S \cap H$, there is a $y \in S \cap H$ with $\|\|_1 = \max_{x \in S \cap H} \|x\|_1 = \lambda$. For $j, k \in \{1, \ldots, n\}, j \neq k$, let $P_{j,k} = (p_{mk}) \in S_n$ be defined by $y_{mj} = 1$ for $y_{mk} > 0$ and $y_{mk} = 1$ for $y_{mk} > 0$. Then $yP_{j,k} = 2 \sum y_{ik} e_{jk} = \|y\|_1 e_{jk}$. Therefore, $\tau(P) \leq 1 \ (P \in S_n)$ yields $\|e_{jk}\|_1 \leq 1/\lambda$. By definition of $\lambda$, $1 \leq \|e_{jk}\|_1 \leq \lambda \|e_{jk}\|_1$, so that $\|e_{jk}\|_1 = 1/\lambda$. Thus, Lemmata 2 and 1 imply (ii), and (i) directly follows.

Remark: After we had finished the first form of this paper we have been informed by A. Lešanovský that he has obtained independently the result of the theorem [2]. However his proof does not use the aspect of extremal points.

REFERENCES


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