A Class of Nonlinear Singular Integro-Differential Equations

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-Dedicated to S. G. Mikhlin on the occasion of his 80th birthday-

Mit Hilfe des Hauptsatzes der Theorie pseudo-monotoner Operatoren wird ein Existenzsatz für eine Klasse von nichtlinearen singulären Integrodifferentialgleichungen vom Cauchyschen Typ und zwei zugehörige Klassen von nichtlinearen Integralgleichungen bewiesen.

С помощью основной теоремы теории псевдомонотонных операторов доказывается теорема существования для одного класса нелинейных сингулярных интегро-дифференциальных уравнений типа Коши и двух связанных с ним классов нелинейных интегральных уравнений.

By means of the main theorem of the theory of pseudo-monotone operators, an existence theorem is proved for a class of nonlinear singular integro-differential equations of Cauchy type and two related classes of nonlinear integral equations.

Introduction

In recent papers (cf. [3] and [5] for an overview) methods of monotone operator theory were applied for proving the existence of a solution to various classes of nonlinear singular integral and integro-differential equations of Cauchy type. In particular, in [4] by means of the theory of pseudo-monotone operators, the author proves an existence theorem for a class of singular integro-differential equations of second order with linear main part. In the present paper we extend this approach to a corresponding class of second-order equations with nonlinear main part. Moreover, a class of nonlinear integral equations occurring in contact problems of elasticity theory [2] are reduced to special singular integro-differential equations of this type.

1. Formulation of problem

We deal with the nonlinear singular integro-differential equation

\[-(p(u'))' + \delta Q(u') + \alpha S[u] - \beta S[u'] + \varphi(u) u' + \lambda \psi(u) = f \quad \text{on} \quad [-a, a]\]

under the boundary conditions \(u(-a) = u(a) = 0\), where \(S\) denotes the Cauchy operator

\[S[u](x) = \frac{1}{\pi} \int_{-a}^{a} \frac{u(y)}{y-x} \, dy.\]
The data fulfil the basic Assumptions I:
(i) \( P, Q \) are continuous functions on \( \mathbb{R} \) satisfying the growth conditions
\[
|P(U)| \leq c_1 |U|^{p-1} + d_1, \quad |Q(U)| \leq c_2 |U|^p + d_2
\]
for any \( U \in \mathbb{R} \) with positive constants \( c_i, d_i \), \( i = 1, 2 \), and some \( p \in [2, \infty) \).
(ii) \( \gamma, \delta \in L_\infty(-a, a) \).
(iii) \( f, \theta \in L_1(-a, a) \).
(iv) \( \alpha, \beta \in \mathbb{R} \).
(v) \( \varphi, \psi \in C(\mathbb{R}) \).

In the sequel we are looking for generalized solutions \( u \in W_p^1(-a, a) \) of (1) which are defined by the integral identity
\[
a_0(u, v) + a_1(u, v) + a_2(u, v) = b(v)
\]
for any \( v \in W_p^1(-a, a) \), where
\[
a_0(u, v) := \int_{-a}^a \gamma P(u') v' \, dx + \int_{-a}^a \delta Q(u') v \, dx,
\]
\[
a_1(u, v) := \alpha \int_{-a}^a S[u] v \, dx - \beta \int_{-a}^a S[u'] v \, dx,
\]
\[
a_2(u, v) := \int_{-a}^a \varphi(u) u' v \, dx + \int_{-a}^a \lambda \psi(u) v \, dx
\]
\[
= - \int_{-a}^a \Phi(u) v' \, dx + \int_{-a}^a \lambda \psi(u) v \, dx
\]
with \( \Phi \) a primitive of \( \varphi \) and
\[
b(v) := \int_{-a}^a f v \, dx.
\]

The problem (3) is equivalent to the operator equation
\[
Au = b \quad \text{for} \quad u \in X := W_p^1(-a, a),
\]
where \( A := A_0 + A_1 + A_2 \), the operators \( A_k : X \to X^* \), \( k = 0, 1, 2 \), are defined by
\[
\langle A_k u, v \rangle := a_k(u, v) \quad \text{for} \quad u, v \in X \quad \text{and} \quad b \in X^* \quad \text{is defined by (4)}. \]
Namely, since \( f \in L_1(-a, a) \) and the Sobolev space \( W_p^1(-a, a) \) is continuously imbedded in the space \( C[-a, a] \) of continuous functions on \( [-a, a] \), we have \( |b(v)| \leq \|v\|_C \leq C_p \|f\|_{L_1} \|v\| \)
where \( \|\cdot\| \) denotes the norm in \( X \) defined by \( \|u\| := \|u\|_{W_p^1} = \left( \int_{-a}^a \left[ |u'|^p + |u|^p \right] \, dx \right)^{1/p} \)
and \( C_p \) is the imbedding constant of \( W_p^1(-a, a) \) in \( C[-a, a] \). Analogously one proves that under Assumptions I for any fixed \( u \in X \) the expressions \( a_k(u, \cdot) \), \( k = 0, 1, 2 \), represent bounded linear functionals on \( X \). On account of (i), (ii) we have
\[
|a_0(u, v)| \leq \|v\|_{L_\infty} \left\| \int_{-a}^a \left[ c_1 |u'|^{p-1} + d_1 \right] |v'| \, dx + \|\delta\|_{L_\infty} \int_{-a}^a \left[ c_2 |u'|^p + d_2 \right] |v| \, dx \right.
\]
\[
\leq \|v\|_{L_\infty} \left[ c_1 \|u'|_{L_p}^{p/q} + d_1(2a)^{1/q} \right] \|v'\|_{L_p}
\]
\[
+ \|\delta\|_{L_\infty} \left[ c_2 \|u'\|_{L_p}^p + d_22a \right] \|v\|_C \leq h_0(\|u\|) \|v\|.
\]
with
\[ h_0(\|u\|) := \|\gamma\|_{L_\infty} \left[ c_1 \|u\|^{p/q} + (2a)^{1/q} d_1 \right] + \|\delta\|_{L_\infty} C_p [c_2 \|u\|^p + 2ad_2] , \]

where \( q = p/(p-1) \) is the exponent conjugate to \( p \). By (iv) and the boundedness of the Cauchy operator \( S \) in \( L_p(-a, a) \) there holds
\[
|a_1(u, v)| \leq |\alpha| \|S[u]\|_{L_p} + |\beta| \|S[u']\|_{L_q} \|v\|_{L_q} \\
\leq |\alpha| + |\beta| B_p D_q \|u\| \|v\| ,
\]

where \( B_p \) is the norm of \( S \) in \( L_p(-a, a) \) and \( D_q \) the imbedding constant of \( W^{1}_p(-a, a) \) in \( L_q(-a, a) \). Finally, in view of (v), \( u \in C[-a, a] \), and \( \lambda \in L_1(-a, a) \) we have
\[
|a_2(u, v)| \leq \|\Phi(u)\|_{L_q} \|v\|_{L_p} + \|\lambda\|_{L_1} \|\psi(u)\|_C \|v\|_C \\
\leq \|\Phi(u)\|_{L_q} + C_p \|\lambda\|_{L_1} \|\psi(u)\|_C \|v\|_C .
\]

2. Existence theorem

At first we state the needed boundedness and continuity properties of the operators \( A_k, k = 0, 1, 2 \).

The operator \( A_0 \) is bounded since by (6) we have \( \|A_0 u\| \leq h_0(\|u\|) \). Further \( A_0 \) is continuous as follows from the estimations
\[
|a_0(u, v) - a_0(u_n, v)| \leq \|\gamma\|_{L_\infty} \|P(u') - P(u_n')\|_{L_q} \|v'\|_{L_p} \\
+ \|\delta\|_{L_\infty} \|Q(u') - Q(u_n')\|_{L_1} \|v\|_C \\
\leq \|\gamma\|_{L_\infty} \|P(u') - P(u_n')\|_{L_q} + C_p \|\delta\|_{L_\infty} \|Q(u') - Q(u_n')\|_{L_1} ,
\]

and
\[
\|A_0 u - A_0 u_n\| = \sup \{|a_0(u, v) - a_0(u_n, v)| : \|v\| \leq 1\} \\
\leq \|\gamma\|_{L_\infty} \|P(u') - P(u_n')\|_{L_q} + C_p \|\delta\|_{L_\infty} \|Q(u') - Q(u_n')\|_{L_1} ,
\]

Under assumptions (i) the Nemitskyi operators of \( P \) and \( Q \) are continuous from \( L_p(-a, a) \) to \( L_q(-a, a) \) and \( L_1(-a, a) \), respectively. Since \( u_n' \to u' \) in \( L_p(-a, a) \) if \( u_n \to u \) in \( X = W^1_p(-a, a) \), the assertion follows.

In view of (7) the linear operator \( A_1 \) is bounded and continuous.

Finally, the operator \( A_2 \) is completely continuous in the sense that it maps weakly convergent sequences (towards \( u \in X \)) into strongly convergent ones (towards \( A_2 u \) \( \in X^* \)). Namely, let \( u_n \to u \) in \( X \). Then \( \|u_n\| \leq \text{Const} \) and, due to the compact embedding of \( X = W^1_p(-a, a) \) in \( C[-a, a] \), we have \( u_n \to u \) in \( C[-a, a] \). By assumption (v) then also \( \psi(u_n) \to \psi(u) \) and \( \Phi(u_n) \to \Phi(u) \) in \( C[-a, a] \). Therefore, \( \|A_2 u - A_2 u_n\| \) tends to zero as \( n \to \infty \). As a completely continuous operator, \( A_2 \) is bounded, too. Hence also \( A = A_0 + A_1 + A_2 \) is a bounded operator.

For proving the monotonicity of the operator \( B := A_0 + A_1 \) we make the additional assumptions II:

(i) There exist constants \( c_0 > 0 \) and \( d_0 \geq 0 \) such that
\[
|P(U_1) - P(U_2)| \|U_1 - U_2\| \geq c_0 (U_1 - U_2)^2 , \\
|Q(U_1) - Q(U_2)| \|U_1 - U_2\| \leq d_0 |U_1 - U_2| \quad \text{for } U_1, U_2 \in \mathbb{R} .
\]
(ii) \( \gamma(x) \geq \gamma_0 > 0, \beta \geq 0. \)

(iii) There holds the inequality, with \( \Delta := \|\delta\|_{L^\infty}, \)

\[
\gamma_{\delta_0} \geq \begin{cases} 
\frac{\Delta^2 d_0^2 a/4}{\beta} & \text{if } \beta \geq (\pi/4) \Delta d_0, \\
2\Delta d_0 a/\pi + 4a\beta/\pi^2 & \text{if } \beta < (\pi/4) \Delta d_0.
\end{cases}
\]

Then we have

\[
D := (B u_1 - B u_2, u_1 - u_2)_x
\]

\[
= [a_0(u_1, u_1 - u_2) - a_0(u_2, u_1 - u_2)] + a_1(u_1 - u_2, u_1 - u_2)
\]

\[
= \int_a^b [P(u_1') - P(u_2')] (u_1' - u_2') \, dx + \int_a^b \delta([Q(u_1') - Q(u_2')]) (u_1 - u_2) \, dx
\]

\[
+ \int_a^b (\alpha S(u_1 - u_2) - \beta S(u_1' - u_2')) (u_1 - u_2) \, dx
\]

\[
\geq \gamma_{\delta_0} \int_a^b (u_1' - u_2')^2 \, dx - \Delta d_0 \int_a^b |u_1' - u_2'| |u_1 - u_2| \, dx + \frac{\beta}{a} \int_a^b (u_1 - u_2)^2 \, dx
\]

since (cf. [3])

\[
\int_a^b S[u] \, u \, dx = 0, \quad -\int_a^b S[u'] \, u \, dx \geq 1 \int_a^b u^2 \, dx \quad \text{for } u \in W_p^1(-a, a).
\]

By means of the elementary inequality \( 2wz \leq \mu w^2 + z^2/\mu \) there follows

\[
D \geq \left( \gamma_{\delta_0} - \frac{\Delta d_0 \mu}{2} \right) \int_a^b (u_1' - u_2')^2 \, dx + \left( \frac{\beta}{a} - \frac{\Delta d_0}{2\mu} \right) \int_a^b (u_1 - u_2)^2 \, dx
\]

for any \( \mu > 0. \) In the sequel we choose

\[
\mu = \begin{cases} 
(\Delta d_0/2\beta) a, & \text{if } \beta \geq (\pi/4) \Delta d_0, \\
(2/\pi) a & \text{if } \beta < (\pi/4) \Delta d_0.
\end{cases}
\]

and obtain

\[
D \geq \left( \gamma_{\delta_0} - \frac{\Delta^2 d_0^2 a}{4\beta} \right) \int_a^b (u_1' - u_2')^2 \, dx \quad \text{if } \beta \geq \frac{\pi}{4} \Delta d_0,
\]

\[
D \geq \left( \gamma_{\delta_0} - \frac{\Delta d_0 a}{\pi} \right) \int_a^b (u_1' - u_2')^2 \, dx - \left( \frac{\pi}{4} \Delta d_0 - \beta \right) \frac{1}{a} \int_a^b (u_1 - u_2)^2 \, dx
\]

\[
\geq \left( \gamma_{\delta_0} - \frac{2\Delta d_0 a}{\pi} + 4a\beta/\pi^2 \right) \int_a^b (u_1' - u_2')^2 \, dx \quad \text{if } \beta < \frac{\pi}{4} \Delta d_0.
\]
in virtue of Wirtinger's inequality
\[ \int_{-a}^{a} u^2 \, dx \leq \left( \frac{4a^2}{\pi^2} \right) \int_{-a}^{a} u^2 \, dx \quad \text{for } u \in \dot{W}_p^1(-a, a). \]
This yields \( D \geq 0 \) if the inequality (9) is fulfilled. Therefore, \( B = A_0 + A_1 \) is a continuous monotone operator and since \( A_2 \) is a completely continuous operator, the operator \( A = B + A_2 \) is pseudo-monotone.

Finally we show the coercivity of the operator \( A \) under the following additional Assumptions III:

(i) There exist constants \( c_3 > 0, d_3 \geq 0 \) and \( c_4 \geq 0, d_4 \geq 0 \) such that
\[ P(U) U \geq c_3 |U|^p - d_3, \quad |Q(U)| \leq c_4 |U|^r + d_4 \quad \text{for } U \in \mathbb{R}, \]
where \( 0 < r \leq p - 1 \) and in case of \( r = p - 1 \) there holds the inequality
\[ \gamma c_3 \geq \alpha c_2 d c_4, \]
where \( \alpha_2 = 2/\pi \) is the constant in the generalized Wirtinger inequality below.

(ii) \( \lambda(x) \geq 0 \) and there exists a constant \( \nu \geq 0 \) such that
\[ \psi(x, u) \geq -\nu \quad \text{for } u \in \mathbb{R}. \]

Remark: Obviously, the conditions (2) and (10) for \( P \) are fulfilled for functions of type \( P(U) = |U|^{p-2} U \) being moreover monotonically increasing. Further the condition (8) for \( P \) is satisfied if \( P \) possesses a derivative greater than a positive constant. Therefore the conditions (2), (8), and (10) for \( P \) in case of \( p > 2 \) are especially fulfilled for functions of the form
\[ P(U) = |U|^{p-2} U + c_0 U, c_0 > 0. \]
The condition (10) for \( Q \) implies the condition (2) for \( P \) and the Lipschitz condition (8) for \( Q \) is satisfied if \( Q \) possesses a bounded derivative. Therefore the conditions (2), (8), and (10) for \( Q \) are fulfilled if \( Q \) possesses a bounded derivative and grows at most as the power \( |U|^{p-1} \) at infinity, for instance for the functions \( Q(U) = \arctan U + c_0 U, c_0 \in \mathbb{R} \), to mention a concrete example.

Under the additional assumptions (10) and (12), with \( \gamma \geq 0 \) besides \( \gamma \geq \gamma_0 > 0, \beta \geq 0 \) we have
\[
\mathcal{D}_0 := \langle Au, u \rangle_X
= \int_{-a}^{a} \gamma P(u') \, u' \, dx + \int_{-a}^{a} \delta Q(u') \, u \, dx
+ \alpha \int_{-a}^{a} S(u) \, u \, dx - \beta \int_{-a}^{a} S(u') \, u' \, dx - \int_{-a}^{a} \Phi(u) \, u' \, dx + \int_{-a}^{a} \lambda \psi(u) \, u \, dx
\geq \gamma_0 \int_{-a}^{a} [c_3 |u'|^p - d_3] \, dx - \Delta \int_{-a}^{a} [c_4 |u'|^r + d_4] \, |u| \, dx - \nu_0,
\]
where \( \nu_0 = \nu \int_{-a}^{a} \lambda \, dx \) and we further used that \( \int_{-a}^{a} \Phi(u) \, u' \, dx = 0 \) for \( u \in \dot{W}_p^1(-a, a) \).

In the following we introduce the equivalent norm \( \| \cdot \|_0 \) in \( \dot{W}_p^1(-a, a) \) defined by
\[ \|u_0\| := \|u''\|_{L_p}. \]
If \( r < p - 1 \), we put \( s = \frac{p}{r} \) and \( t \) the exponent conjugate to \( s \) and obtain
\[ \int_{-a}^{a} |u'|^r \, |u| \, dx \leq \|u''\|_{L_p} \|u\|_{L_s} = \|u''\|_{L_p} \|u\|_{L_s} \leq E_t \|u\|_0^{r+1}, \]
where $E_t$, $t \geq 1$, is the imbedding constant of $\bar{W}^1_p(-a, a)$ in $L_t(-a, a)$. Therefore,

$$D_0 \geq \gamma_0 c_3 \|u\|_p - \gamma_0 d_2 \alpha a - \Delta c_4 E_t \|u\|^{r+1}_p - \Delta d_4 E_t \|u\|_p - \nu_0$$

from which in view of $r + 1 < p$ the coercivity of $A$ follows. In case $r = p - 1$ we have

$$\int_a^a |u'|^{p-1} |u| \, dx \leq \|u'\|^{p-1}_p \leq \alpha_p \|u\|_p$$

in virtue of the generalized Wirtinger inequality (cf. [1])

$$\int_a^a |u|^p \, dx \leq \alpha_p \int_a^a |u'|^p \, dx \quad \text{for } u \in \bar{W}^1_p(-a, a).$$

Under the assumption (11) again the operator $A$ is coercive.

The main theorem of the theory of pseudo-monotone operators by Brézis (cf. [6: Theorem 27.2]) now yields the existence of a solution $u$ to the operator equation (5).

**Theorem 1:** Under Assumptions I—III the integro-differential equation (1) possesses a generalized solution $u \in \bar{W}^1_p(-a, a)$, $2 \leq p < \infty$.

**Remark 2:** If $\varphi = \psi = 0$ and (9) is fulfilled as true inequality, the operator $A = B$ is strictly monotone and the solution $u \in \bar{W}^1_p(-a, a)$ of (1) is unique.

**Remark 3:** Theorem 1 also holds if in the left-hand side of (1) an additional term of the form $K_1[u] + K_2[u']$ is present, where $K_1$ is a positive linear bounded operator in $L_p(-a, a)$ (or, more generally, from $L_\infty(-a, a)$ into $L_1(-a, a)$) and $K_2$ is a linear bounded operator in $L_p(-a, a)$ satisfying the condition $\int_a^a K_2[u'] \, dx \geq 0$ for $u \in \bar{W}^1_p(-a, a)$.

3. Application to integral equations

We consider the integral equation

$$-\gamma P(u') - \alpha N[u] - \beta S[u] + \Phi(u) = F + \epsilon$$

with a free constant $\epsilon \in \mathbb{R}$, where $N$ denotes the operator

$$N[u](x) = \frac{1}{\pi} \int_a^a u(y) \ln |y - x| \, dy,$$

$\Phi$ is a continuously differentiable function on $\mathbb{R}$ and $F$ an absolutely continuous function on $[-a, a]$. This equation is equivalent to the equation (1) with $Q \equiv 0$, $\varphi = \Phi'$, $\psi \equiv 0$, $\ell = F'$ obtained by differentiating (13). As a corollary to Theorem 1 we therefore get

**Theorem 2:** Let there be $\gamma \in L_\infty(-a, a)$ with $\gamma(x) \geq \gamma_0 > 0$, $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}_{+}$, and $P$ a continuous function on $\mathbb{R}$ satisfying the conditions (2), (8), and (10). Then the equation (13) possesses a solution $u \in \bar{W}^1_p(-a, a)$, $2 \leq p < \infty$, for some $\epsilon \in \mathbb{R}$.

Further we deal with the integral equation

$$-\gamma P(v) + \int_a^a \delta Q(v) \, d\xi + \beta N[v] = F + \epsilon$$

(14)
with a free constant $c \in \mathbb{R}$ and the additional condition
\begin{equation}
\int_{-a}^{a} v(x) \, dx = d \tag{15}
\end{equation}
with a given constant $d \in \mathbb{R}$. Again $F$ is a given absolutely continuous function on $[-a, a]$.

Equation (14) (with constant coefficients $\delta, \gamma$) occurs in a plane contact problem of denting a stamp into a plate with rough surface, where $v$ is the sought function of the contact pressure, $F$ the function describing the form of the basis of the stamp, $P$ the tangential pressure on the surface of the plate as function of $v$ and $Q$ a characteristic quantity for the roughness of the surface of the plate ("dislocation of micro-roughness") also as a function of $v$ (cf. [2]).

We again differentiate the equation (14) and introduce the new unknown function
\[ u(x) = \int_{-a}^{x} v(\xi) \, d\xi - d(x + a)/2a \]
satisfying the conditions $u(-a) = u(a) = 0$ in view of (15). This function $u$ is a solution of the equation
\begin{equation}
-(\gamma P(u' + D))' + \delta Q(u' + D) - \beta S[u'] = f \tag{16}
\end{equation}
with $D := d/2a$ and $f := F' + \beta D S[1] = F' + \beta D \ln [(a - x)/(a + x)]$. This is a special case of the equation (1). If $u$ is a solution of (16), $v(x) = u'(x) + d/2a$ is a solution of the equation (14) with (15). Theorem 1 and Remark 2 imply the following

Theorem 3: Let there be $\gamma \in L_{\infty}(-a, a)$ with $\gamma(x) \geq \gamma_0 > 0$, $\delta \in L_{\infty}(-a, a)$, $\beta \in \mathbb{R}_+$, $P$ and $Q$ continuous functions on $\mathbb{R}$ satisfying the conditions (2), (8) with (9), and (10) with (11). Then the equation (14) with (15) possesses a solution $v \in L_p(-a, a)$, $2 \leq p < \infty$, which is uniquely determined if in (9) the strict inequality sign holds.

Remark 4: If $Q$ is a linear function and $\delta$ is a constant, the term with $Q$ in (16) can be viewed as a term of the form $\phi(u) u'$ in (1) (with an additional given function). Then the existence of a solution $v \in L_p(-a, a)$ to (14) with (15) follows without assuming the inequality (9). Also the solution is unique since $\int_{-a}^{a} u' u \, dx = 1/2[u^2]_{-a}^{a} = 0$ so that the corresponding operator $A$ is strictly monotone.

Remark 5: In [2], without proof, KUDIesch states an existence and uniqueness theorem for nonnegative solutions of the equation (14) with (15) under certain monotonicity assumptions on the functions $P$ and $Q$ without assuming an inequality of the form (9). In particular, he assumes that, if $\delta$ is a negative constant, $Q$ is a nonnegative increasing continuous function on $\mathbb{R}_+$ with $Q(0) = 0$ and the Nemitskyi operator of the integral of $Q$ in (14) is a monotone operator on a certain class of nonnegative functions.

The corresponding assumption in our treatment of (solutions with arbitrary sign of) the equation (14) in case of a positive constant $\delta$ would be
\[ \int_{-a}^{a} (Q(u_1) - Q(u_2)) (u_1 - u_2) \, dx \geq 0 \quad \text{for} \quad u_1, u_2 \in W_p^1(-a, a). \]

But as is easily seen this requirement on $Q$ is only fulfilled if $Q$ is a linear function so that the foregoing Remark 4 applies.

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