Generalized Solutions of the Cauchy Problem for a Nonlinear Functional Partial Differential Equation

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Es wird ein Satz über die Existenz und Eindeutigkeit der verallgemeinerten Lösung (in Sinne „fast überall“) des Cauchy-Problems für nichtlineare Funktional-Differential-Gleichungen mit partiellen Ableitungen erster Ordnung bewiesen.

An existence and uniqueness theorem for the generalized solution (in the sense “almost everywhere”) of the Cauchy problem for a nonlinear functional partial differential equation of first order is proved.

1. Introduction. Let us consider the Cauchy problem

\[ D_x u(x, y) = F(x, y, u(x, y), (Vu)(x, y), D_y u(x, y)) \]

(a.e. in \([0, a]; y \in \mathbb{R}\)),

\[ u(0, y) = \varphi(y) \quad (y \in \mathbb{R}), \]  

where \( D_x = \frac{\partial}{\partial x}, D_y = \frac{\partial}{\partial y} \), and \( V \) is an operator of Volterra type.

Equation (1) contains as particular cases \((Vu)(x, y) = u(a(x, y), \beta(x, y))\) the differential equations with a retarded argument, the special cases of which arise in the theory of the distribution of wealth [5]. A few kinds of integral-differential equations can be obtained from (1) by specializing the operator \( V \). For instance, problems arising from laser problems in Nonlinear Optics are also particular cases of the problem with \((Vu)(x, y) = \int_{-\infty}^{y} K(y - t) u(x, t) \, dt \) [1].

In recent papers P. Brandi and R. Ceppitelli [2], Z. Kamont [6, 7], and A. Salvadori [8] have considered the existence and uniqueness of continuously differentiable solutions of problem (1), (2) under the assumption that \( F \) is differentiable. The aim of the present paper is to extend these results to a more general case where the given function \( F \) is not necessarily continuous and solutions of problem (1), (2) are generalized (or weak) in the sense “almost everywhere” (a.e.). The method applied is of fixed point type. It is based on defining an operator, whose range consists of solutions of suitable equations without functional argument. By applying differential inequalities (see Lemma 1) it is proved that this operator is a contraction, and its fixed point is a solution of problem (1), (2). The solution is local in \( x \) and global in \( y \), and it is unique in a class of bounded functions, absolutely continuous in \( x \) and possessing Lipschitzian derivatives in \( y \).

2. The auxiliary results. In the sequel we will use the existence theorem for the nonlinear partial differential equation

\[ D_x u(x, y) = f(x, y, u(x, y), D_y u(x, y)) \quad (a.e. \text{ in } [0, a]; y \in \mathbb{R}). \]
Assumption $H_1$: Suppose that

$1^o$ $f(x, \cdot): \mathbb{R}^3 \to \mathbb{R}$ is continuous, and derivatives $D_1f(x, \cdot), D_2f(x, \cdot), D_3f(x, \cdot): \mathbb{R}^3 \to \mathbb{R}$ exist and are continuous for every $x \in [0, a_0], a_0 > 0$;

$2^o$ $f(\cdot, y, z, q), D_1f(\cdot, y, z, q), D_2f(\cdot, y, z, q), D_3f(\cdot, y, z, q): [0, a_0] \to \mathbb{R}$ are measurable for every $(y, z, q) \in \mathbb{R}^3$;

$3^o$ there are measurable and integrable functions $M_i, L_i: [0, a_0] \to [0, +\infty)$ $(i = 0, 1, 2, 3)$ such that, a.e. in $[0, a_0]$,

$$|f(x, y, z, q)| \leq M_0(x), \quad |D_1f(x, y, z, q)| \leq M_1(x),$$
$$|D_2f(x, y, z, q)| \leq M_2(x), \quad |D_3f(x, y, z, q)| \leq M_3(x)$$

and

$$|f(x, y, z, q) - f(x, \bar{y}, \bar{z}, \bar{q})| \leq L_0(x) (|y - \bar{y}| + |z - \bar{z}| + |q - \bar{q}|),$$
$$|D_1f(x, y, z, q) - D_1f(x, \bar{y}, \bar{z}, \bar{q})| \leq L_1(x) (|y - \bar{y}| + |z - \bar{z}| + |q - \bar{q}|),$$
$$|D_2f(x, y, z, q) - D_2f(x, \bar{y}, \bar{z}, \bar{q})| \leq L_2(x) (|y - \bar{y}| + |z - \bar{z}| + |q - \bar{q}|),$$
$$|D_3f(x, y, z, q) - D_3f(x, \bar{y}, \bar{z}, \bar{q})| \leq L_3(x) (|y - \bar{y}| + |z - \bar{z}| + |q - \bar{q}|)$$

for all $(y, z, q), (\bar{y}, \bar{z}, \bar{q}) \in \mathbb{R}^3$.

$4^o$ the initial function $\varphi$ in (2) belongs to $C^1(\mathbb{R}, \mathbb{R}) (C^1(\mathbb{R}, \mathbb{R})$ denotes the set of all continuously differentiable functions on $\mathbb{R}$ into $\mathbb{R}$), and there exist constants $k_1, k_2 > 0$ such that $|\varphi(y)| \leq k_1$ and $|\varphi(y) - \varphi(\bar{y})| \leq k_2 |y - \bar{y}| (y, \bar{y} \in \mathbb{R})$.

Let us define the constants

$$K_1 = k_1 + (1 + k_1 + k_2) \int_0^a \left((L_0 + \Omega(a_0)L_3 + M_3) \exp \int_0^z G dt \right) dx,$$
$$K_2 = k_2 + (1 + k_1 + k_2) \int_0^a \left((L_1 + \Omega(a_0)L_2 + M_2) \exp \int_0^z G dt \right) dx,$$
$$g_0 = 1 - (1 + k_1 + k_2) \int_0^a \left(L_3 \exp \int_0^z G dt \right) dx,$$

where

$$G = L_0 + L_1 + L_2 + \Omega(a_0) (L_2 + L_3) + M_2 + M_3,$$
$$\Omega(a_0) = \exp \int_0^a M_2 dt \left\{k_1 + \int_0^a \left(M_1 \exp \left(-\int_0^z M_2 ds \right) \right) dt \right\}.$$

Theorem 1 [3, 4]: If Assumption $H_1$ is satisfied, then there are a constant $a \in (0, a_0)$ and a function $u: E_a = [0, a] \times \mathbb{R} \to \mathbb{R}$ satisfying equation (3) and condition (2). This solution is unique in that class of functions $u: E_a \to \mathbb{R}$ for which $u(\cdot, y), D_1u(\cdot, y): [0, a] \to \mathbb{R}$ are absolutely continuous for every $y \in \mathbb{R}$, and

$$|D_1u(x, y)| \leq \Omega(a_0),$$

$$|u(x, y) - u(x, \bar{y})| \leq \frac{K_1}{g_0} |y - \bar{y}|, \quad |D_1u(x, y) - D_1u(x, \bar{y})| \leq \frac{K_2}{g_0^2} |y - \bar{y}|,$$

for every $(x, y), (x, \bar{y}) \in E_a$.

Remark: In the above theorem $a$ is chosen sufficiently small such that $g_0 > 0$. 


Set, for \(a, b > 0, M \geq 0\); and \(Ma \leq b\),

\[
E_{ab} = \{(x, y) : 0 \leq x \leq a, |y| \leq b - Mx\},
\]

\[
\tilde{E}_{ab} = \{(s, y) \in E_{ab} : s \leq x\}, \quad S_x = \{(y : (x, y) \in E_{ab}\}.
\]

We shall need the following

**Lemma 1:** Suppose that

1° \(u : E_{ab} \to \mathbb{R}\) is continuous, and \(D_xu\) exists for a.e. \(x \in [0, a]\) and every \(y \in [-b + Mx, b - Mx]\);

2° for every \(x \in [0, a]\), \(u(x, \cdot)\) fulfills a Lipschitz condition;

3° there are constants \(c_1, c_2 \geq 0\) such that

\[
|D_xu(x, y)| \leq c_1 |u(x, y)| + c_2 + M |D_yu(x, y)| \quad \text{(a.e. in } [0, a]\};
\]

\[
|y| \leq b - Mx).
\]

Then the derivative \(\gamma'\) of the function \(\gamma\),

\[
\gamma(x) = \max \{|u(s, t) : (s, t) \in E_{ab}\} \quad (x \in [0, a]),
\]

exists a.e. and \(\gamma' (x) \leq c_1 \gamma(x) + c_2 \) a.e. in \([0, a]\).

**Proof:** Since \(\delta(x) = \max \{|u(x, y) : y \in S_x\}\) is continuous on \([0, a]\) (see \([9]\)) and \(\gamma(x) = \max \{|\delta(s) : s \in [0, x]\}\), \(\gamma\) is continuous on \([0, a]\) and \(\gamma'\) exists a.e. in \([0, a]\). Suppose that \((\bar{x}, \bar{y}) \in \tilde{E}_{ab}\) is such that

\[
\gamma(x) = |u(\bar{x}, \bar{y})|, \quad \text{(4)}
\]

and \(u_+(\bar{x}, \bar{y}), u_-(\bar{x}, \bar{y}), \gamma'(x)\) exist. Thus we have \(0 \leq \bar{x} \leq x\). First, let \(\bar{x} \neq 0\). From the definitions of \(\gamma\) and \(\delta\) it follows that \(\gamma(x) = \delta(\bar{x})\). For \(h < 0\), we get

\[
\gamma(x + h) \geq \delta(x + h) \quad \text{and} \quad \frac{\gamma(x + h) - \gamma(x)}{h} \leq \frac{\delta(\bar{x} + h) - \delta(\bar{x})}{h}.
\]

Hence, by \(h \to 0^-\), we obtain

\[
\gamma'(x) \leq D_x \delta(x) \quad \text{(5)}
\]

(\(D_\cdot\) is the left-hand lower Dini derivative). For \(\bar{x} = 0\) inequality (5) is certainly satisfied, since \(\gamma\) is constant in \([0, \bar{x}]\). If \((\bar{x}, \bar{y})\) is an interior point of \(E_{ab}\), then we have

\[
[9] D_x \delta(\bar{x}) \leq |D_xu(\bar{x}, \bar{y})|, \quad \text{and also} \quad |D_yu(\bar{x}, \bar{y})| = 0.
\]

Hence, by (5) and assumption 3° it follows that

\[
\gamma'(x) \leq D_x \delta(\bar{x}) \leq |D_xu(x, y)| \leq c_1 |u(\bar{x}, \bar{y})| + c_2 = c_1 \gamma(x) + c_2.
\]

Now, suppose that \((\bar{x}, \bar{y})\) is not an interior point of \(E_{ab}\). Then \(\bar{y} = b - M\bar{x}\) or \(\bar{y} = -b + M\bar{x}\). We consider only the first case. Assume also, that in (4) we have \(\gamma(x) = u(\bar{x}, \bar{y})\) (for \(\gamma(x) = -u(\bar{x}, \bar{y})\) the proof is quite similar). Let us consider the function \(\tilde{m}\) defined by \(\tilde{m}(x) = u(x, b - Mx)\). Since \(\tilde{m}(x) \leq \delta(x), x \in [0, \bar{x}]\), and \(\tilde{m}(\bar{x}) = \delta(\bar{x})\),

\[
\frac{\tilde{m}(\bar{x} + h) - \tilde{m}(\bar{x})}{h} \geq \frac{\delta(\bar{x}) + h) - \delta(\bar{x})}{h} \quad \text{for } h < 0.
\]

Hence, we get

\[
\tilde{m}'(x) \geq D_x \delta(\bar{x}). \quad \text{(6)}
\]

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From assumption 2° it follows that \( \bar{m}' \) exists a.e. in \([0, a]\) and \( \bar{m}'(x) = D_xu(x, b - Mx) - M.D_xu(x, b - Mx) \). In particular, we have \( \bar{m}'(\bar{x}) = D_xu(\bar{x}, \bar{y}) - M.D_xu(\bar{x}, \bar{y}) \). Hence, and by (6), we obtain \( D_x\delta(\bar{x}) \leq D_xu(\bar{x}, \bar{y}) - M.D_xu(\bar{x}, \bar{y}) \). The inequality together with assumption 3° yields

\[
D_x\delta(\bar{x}) \leq D_xu(\bar{x}, \bar{y}) - M.D_xu(\bar{x}, \bar{y}) \leq c_1 |u(\bar{x}, \bar{y})| + c_2 = c_1 \gamma(x) + c_2.
\]

Hence and by (5) we get the assertion of the lemma 1.

3. The existence theorem. We denote by \( K(a, P, Q) \) the class of all continuous and bounded functions \( u: E_a \to R \) satisfying the following conditions:

(i) \( u(\cdot, \bar{y}): [0, a] \to R \) are absolutely continuous for every \( y \in R \);

(ii) there are constants \( P, Q \geq 0 \) such that

\[
|u(x, y) - u(x, \bar{y})| \leq P |y - \bar{y}|, \quad |D_xu(x, y) - D_xu(x, \bar{y})| \leq Q |y - \bar{y}|
\]

for all \((x, y), (x, \bar{y}) \in E_a\).

Assumption \( H_2 \): Suppose that

1° \( F(x, \cdot): R^4 \to R \) is continuous, and the derivatives \( D_yF(x, \cdot), D_zF(x, \cdot), D_pF(x, \cdot): R^4 \to R \) exist and are continuous for every \( x \in [0, a_0] \);

2° \( F(\cdot, y, z, p, q), D_yF(\cdot, y, z, p, q), D_zF(\cdot, y, z, p, q), D_pF(\cdot, y, z, p, q), D_qF(\cdot, y, z, p, q): [0, a_0] \to R \) are measurable for every \((y, z, p, q) \in R^4\);

3° there are a constant \( \ell_0 \geq 0 \), and measurable and integrable functions \( m_i, l_i: [0, a_0] \to R_+ (i = 0, 1, 2, 3, 4; j = 1, 2, 3, 4) \) such that a.c. in \([0, a_0]\)

\[
|F(x, y, z, p, q)| \leq m_0(x),
\]

\[
|D_yF(x, y, z, p, q)| \leq m_1(x), \quad |D_zF(x, y, z, p, q)| \leq m_2(x),
\]

\[
|D_pF(x, y, z, p, q)| \leq m_3(x), \quad |D_qF(x, y, z, p, q)| \leq m_4(x),
\]

and

\[
|F(x, y, z, p, q) - F(x, \bar{y}, \bar{z}, \bar{p}, \bar{q})| \leq \ell_0(|y - \bar{y}| + |z - \bar{z}| + |p - \bar{p}| + |q - \bar{q}|); 
\]

\[
|D_yF(x, y, z, p, q) - D_yF(x, \bar{y}, \bar{z}, \bar{p}, \bar{q})| \leq \ell_1(x) (|y - \bar{y}| + |z - \bar{z}| + |p - \bar{p}| + |q - \bar{q}|), 
\]

\[
|D_zF(x, y, z, p, q) - D_zF(x, \bar{y}, \bar{z}, \bar{p}, \bar{q})| \leq \ell_2(x) (|y - \bar{y}| + |z - \bar{z}| + |p - \bar{p}| + |q - \bar{q}|), 
\]

\[
|D_pF(x, y, z, p, q) - D_pF(x, \bar{y}, \bar{z}, \bar{p}, \bar{q})| \leq \ell_3(x) (|y - \bar{y}| + |z - \bar{z}| + |p - \bar{p}| + |q - \bar{q}|), 
\]

\[
|D_qF(x, y, z, p, q) - D_qF(x, \bar{y}, \bar{z}, \bar{p}, \bar{q})| \leq \ell_4(x) (|y - \bar{y}| + |z - \bar{z}| + |p - \bar{p}| + |q - \bar{q}|)
\]

for all \((y, z, p, q), (\bar{y}, \bar{z}, \bar{p}, \bar{q}) \in R^4\);

4° \( V_u(\cdot, y): [0, a_0] \to R \) is measurable for \( y \in R \), \( u \in K(a_0, P, Q) \), there exists \( D_y(V_u) \in C(E_{u_0}, R) \) for each \( u \in K(a_0, P, Q) \), and there are measurable and integrable
functions \( p, r : [0, a_0] \to \mathbb{R}^+ (i = 0, 1) \) such that a.e. in \([0, a_0]\)

\[
(Vu)(x, y) - (Vu)(x, \bar{y}) \leq r_0(x) |y - \bar{y}|
\]

\[
|D_p(Vu)(x, y)| \leq p(x), \quad |D_p(Vu)(x, y) - D_p(Vu)(x, \bar{y})| \leq r_1(x) |y - \bar{y}|
\]

for each \( u \in K(a_0, P, Q) \);

5° there is a constant \( s \geq 0 \) such that \( \|Vu - V\bar{u}\|_2 \leq s \|u - \bar{u}\|_2 \) for any \( u, \bar{u} \in K(a_0; P, Q) \), where \( \|u\|_2 = \sup \{ |u(s, y)| : (s, y) \in E_x, E_x = [0, x] \times \mathbb{R} \};

6° the initial function \( \varphi \) in (2) belongs to \( C^1(\mathbb{R}, \mathbb{R}) \), and there exist constants \( k_0, k_1 > 0 \) and \( k_2 \geq 0 \) such that \( |\varphi(y)| \leq k_0, |\varphi'(y)| \leq k_1 \) and \( |\varphi'(y) - \varphi'(...)| \leq k_2 |y - \bar{y}| \) \( (y, \bar{y} \in \mathbb{R}) \).

**Lemma 2:** If Assumption \( H_2 \) is satisfied, then there exist constants \( a \in (0, a_0] \) and \( P, Q \geq 0 \), such that for every \( w \in K(a, P, Q) \) there is a unique solution \( u[w] \in K(a, P, Q) \) of the equation

\[
D_2u(x, y) = F(x, y, u(x, y); (Vu)(x, y), D_yu(x, y)) \quad (a.e. in [0, a]; y \in \mathbb{R}) \tag{7}
\]

satisfying condition (2).

**Proof:** In order to prove this lemma, we show that all assumptions of Theorem 1 are satisfied with \( f(x, y, z, q) = F(x, y, z; (Vu)(x, y), q) \), where \( w \in K(a, P, Q) \). From \( H_2'/1^o, 2^o \) and \( 4^o \) it follows that \( H_1'/1^o, 2^o \) are satisfied, \( H_1'/3^o \) is satisfied with

\[
\begin{align*}
M_0 &= m_0, \quad M_1 = m_1 + m_3p, \quad M_2 = m_2, \quad M_3 = m_4, \\
L_0 &= l_0(1 + r_0), \quad L_1 = (1 + r_0)(l_1 + pl_0) + m_3r_1, \\
L_2 &= l_2(1 + r_0), \quad L_3 = l_4(1 + r_0).
\end{align*}
\]

At last, \( H_1'/4^o \) is covered by \( H_2'/6^o \). By Theorem 1 it follows that there is a unique function \( u[w] \) satisfying equation (7) and condition (2). Moreover, this solution satisfies the conditions

\[
|u[w](x, y) - u[w](x, \bar{y})| \leq \frac{\bar{K}_1}{\mathcal{g}_0} |y - \bar{y}|,
\]

\[
|D_yu[w](x, y) - D_yu[w](x, \bar{y})| \leq \frac{\bar{K}_2}{\mathcal{g}_0} |y - \bar{y}|,
\]

where

\[
\begin{align*}
\bar{K}_1 &= k_1 + (1 + k_1 + k_2) \int_0^a \left\{ (1 + r_0)(l_0 + \tilde{\mathcal{G}}(a_0) l_4) + m_4 \right\} \exp \int_0^a \tilde{\mathcal{G}} \, dt \, dx, \\
\bar{K}_2 &= k_2 + (1 + k_1 + k_2) \int_0^a \left\{ (1 + r_0)(l_1 + pl_3 + \tilde{\mathcal{G}}(a_0) l_4) + Pm_3r_1 + m_2 \right\} \\
&\quad \times \exp \int_0^a \tilde{\mathcal{G}} \, dt \, dx, \\
\mathcal{g}_0 &= 1 - (1 + k_1 + k_2) \int_0^a \left( l_4(1 + r_0) \exp \int_0^a \tilde{\mathcal{G}} \, dt \right) \, dx, \\
\tilde{\mathcal{G}}(a_0) &= \exp \int_0^a m_2 \, dt \left\{ k_1 + \int_0^a \left( m_1 + m_3p \exp \left( -\int_0^a m_2 \, ds \right) \right) \, dt \right\}, \\
\tilde{\mathcal{G}} &= (1 + r_0)(l_0 + l_1 + pl_3 + l_4 + \tilde{\mathcal{G}}(a_0)(l_2 + l_4)) + m_3r_1 + m_2 + m_4.
\end{align*}
\]

Let \( P \) and \( Q \) in the definition of \( K(a, P, Q) \) be constants satisfying the inequalities \( \bar{K}_1 \leq P\mathcal{g}_0, \bar{K}_2 \leq Q\mathcal{g}_0 \) which are certainly satisfied for \( a \) sufficiently small (for \( a \to 0 \).
these inequalities reduce to \( k_1 \leq P, k_2 \leq Q \). Thus we have
\[
|u[w] (x, y) - u[w] (x, \bar{y})| \leq P |y - \bar{y}|
\]
\[
|D_{y}u[w] (x, y) - D_{y}u[w] (x, \bar{y})| \leq Q |y - \bar{y}|
\]
Since \( u[w] \) is generated by characteristics \([3]\), then writing the characteristic system for \((7)\) it is easy to show that \( u[w] \) is bounded. Hence, we have \( u[w] \in K(a, P, Q) \) for every \( w \in K(a, P, Q) \).}

**Theorem 2:** If Assumption \( H_2 \) is satisfied, then there are constants \( a, P, Q, 0 < a \leq a_0 \), and a function \( u: E_0 \to \mathbb{R} \), satisfying equation \((1)\) and condition \((2)\). Furthermore, \( u \) is unique in the class \( K(a, P, Q) \).

**Proof:** Let us define the operator \( T \) on \( K(a, P, Q) \) by \((T w)(x, y) = u[w](x, y)\), where \( a, P, Q \) are given in Lemma 2. It follows by Lemma 2 that \( T \) maps \( K(a, P, Q) \) into itself. Let us introduce the norm
\[
||z|| = \sup_{(x, y) \in E_a} |z(x, y)| \exp (-\lambda x), \quad \text{where} \quad \lambda > l_0(1 + s).
\]
By \( H_2/5^\circ \) we have
\[
||(V u)(x, y) - (V \bar{u})(x, y)|| \exp (-\lambda x) \leq s \sup_{(s, y) \in E_a} |u(s, y) - \bar{u}(s, y)| \exp (-\lambda s) \leq s ||u - \bar{u}||.
\]
Hence \( ||V u - V \bar{u}|| \leq s ||u - \bar{u}|| \). Now, we prove that \( T \) is a contraction. Indeed, for any \( w, \bar{w} \in K(a, P, Q) \), we have
\[
||D_{y}[(T w)(x, y) - (T \bar{w})(x, y)]|| = ||D_{y}[u[w](x, y) - u[\bar{w}](x, y)]||
\]
\[
\leq l_0 ||u[w](x, y) - u[\bar{w}](x, y)|| + l_0 \sup_{(s, y) \in E_a} |(V w)(x, y) - (V \bar{w})(x, y)||
\]
\[
\leq l_0 ||u[w](x, y) - u[\bar{w}](x, y)|| + l_0 s ||w - \bar{w}|| \exp (\lambda x)
\]
and \( u[w](0, y) - u[\bar{w}](0, y) = 0 \). Hence, and by Lemma 1, we obtain
\[
\gamma(x) \leq l_0 \gamma(x) + l_0 s ||w - \bar{w}|| \exp (\lambda x), \quad \text{where} \quad \gamma(x) = \sup_{(s, t) \in E_{2b}} ||u[w](s, t) - u[\bar{w}](s, t)||: (s, t) \in E_{2b}
\]
and consequently, by the extended Peano inequality \([3]\), \( \gamma(x) \leq l_0 s(\lambda - l_0)^{-1} ||w - \bar{w}|| \exp (\lambda x) \). Since this inequality is satisfied for every \( b \), it follows that
\[
||T w - T \bar{w}|| \leq l_0 s(\lambda - l_0)^{-1} ||w - \bar{w}||, \quad l_0 s(\lambda - l_0)^{-1} < 1.
\]
Thus, \( T \) is a contraction. It is easily seen that the fixed point of the operator \( T \) satisfies condition \((2)\) and equation \((1)\).

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