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Two-Sided Estimations for Nonlinear Parabolic Systems with Functionals

W. VOIGT

Es werden nichtlineare parabolische Systeme untersucht, bei denen sowohl die Differentialgleichungen als auch die nichtlinearen Randbedingungen zusätzlich Funktionalterme enthalten. Auf unbeschränkten Gebieten werden zweiseitige gleichmäßige Abschätzungen für die Lösungen von Differential-Ungleichungen, die obigen Systemen entsprechen, hergeleitet. wobei das asymptotische Wachstum der Lösungen vorgegeben ist. Aus den Abschätzungen folgen Eindeutigkeits- und Stabilitätsaussagen.

Исследуются нелинейные параболические системы, у которых и дифференциальные уравнения и нелинейные граничные условия содержат дополнительно функциональные выражения. Для неограниченных областей доказываются двухсторонние равномерные оценки для решений дифференциальных неравенств, соответствующих названным системам, причём асимптотический рост решений задан. Из оценок следуют утверждения об единственности и устойчивости.

Nonlinear parabolic systems are investigated. The differential equations as well as the nonlinear boundary conditions involve additionally functional terms. On unbounded domains, two-sided uniform estimates are proved for the solutions of differential inequalities corresponding to the mentioned systems, while the asymptotic growth of these solutions is given. The estimates provide uniqueness and stability.

0. Problems with coupling of a partial differential equation or a boundary condition with functional terms become more and more significant. One of the domains where mathematical models of this kind can be derived is the theory of epidemics (cf. PAO [4]), where integral terms are involved in the system of the Kermack-McKendrick equations. If thermoelastic displacements are investigated (cf. DAY [2]), then integral terms occur in the boundary conditions. Generalizing these problems, we obtain functional-differential equations with functional-boundary conditions. Then other types of functionals, for instance delayed arguments, are also admitted. For functional-differential equations of parabolic type many results have been published by REDHEFFER and WALTER (cf. [5, 6] and other papers). Functional-boundary conditions have been investigated by REDHEFFER and WALTER [5] and VOIGT [8] for bounded domains and by VOIGT [7, 9] for unbounded domains. Results for functional-differential equations of elliptic type are due to AVANTAGGIATI and MALEC [1] and MALEC (cf. [3] and other papers). In the present paper the results from [9] related to two-sided estimates for parabolic problems with functionals are carried out for nonlinear parabolic systems with functionals. Let us remark that here, like in other papers in this field, only weakly coupled systems and boundary conditions are admitted.

1. For $i \in I = \{1, \ldots, m\}$ let $G^i \subset \mathbf{R}^{n+1}$ $(n \geqq 1)$ be any connected and closed set of all points $(t, x) \in G^i$, $x = (x_1, \ldots, x_n)$, for which real numbers T_i and T exist with the properties

$$-\infty < \inf \{t : (t, x) \in G^i\} = -T_i \leqq 0 < \sup \{t : (t, x) \in G^i\} = T < \infty.$$

Let $\bar{G} = \cap \{G^i : i \in I\}$. By G we denote all the points $(t, x) \in \bar{G}$ for which a number $\varkappa = \chi(t, x) > 0$ exists with

$$\{(\tau, x) : t - \chi(t, x) < \tau \leqq t\} \subset G \quad \text{and} \quad \{(t, \bar{x}) : |x - \bar{x}| < \chi(t, x)\} \subset G.$$

Here the notation $|x|^2 = |x_1|^2 + \cdots + |x_n|^2$ is used. Let

$$\inf \{t : (t, x) \in G\} = 0 \quad \text{and} \quad \sup \{t : (t, x) \in G\} = T.$$

With G we define $\Gamma^i = G^i \setminus G$. If for any point $(t, x) \in \Gamma^i$ a sequence $\{(t, x^{(j)})\} \subset G$ exists with $(t, x^{(j)}) \to (t, x)$, we say that a (generalized) normal $\nu_i(t, x)$ in (t, x) exists (cf. [10]). We denote all the points of Γ^i, for which such a normal $\nu_i(t, x)$ exists, by $\Gamma_2{}^i$. Eventually, we define $\Gamma_1{}^i = \Gamma^i \setminus \Gamma_2{}^i$, $J_0 = (0, T)$, $J = [0, T]$, and $J_k{}^i = \{t : (t, x) \in \Gamma_k{}^i\}$ $(i \in I;\ k = 1, 2)$.

2. For a function $u^i : G^i \to \mathbf{R}$, let $u_t{}^i$ be the one-sided derivative from below with respect to t, $u_x{}^i$ the gradient and u^i_{xx} the Hessian. For $(t, x) \in \Gamma_2{}^i$ we define a derivative in the following sense:

$$u^i_{\nu_i}(t, x) = \lim_{j \to \infty} \big(u^i(t, x^{(j)}) - u^i(t, x)\big) / |x^{(j)} - x|,$$

where the sequence $\{(t, x^{(j)})\}$ provides a normal $\nu_i(t, x)$. For the sake of brevity we write $u_\nu{}^i(t, x) = u^i_{\nu_i}(t, x)$. By U^i we denote the set of functions u^i defined on G^i with the properties:

$$u^i \in C(G^i),\ u^i_{x_j x_k} \in C(G), \text{ the derivatives } u_t{}^i \text{ and } u_\nu{}^i \text{ exist in } G \text{ and on } \Gamma_2{}^i, \text{ respectively } (i \in I;\ j, k = 1, 2, \ldots, n).$$

Finally, we set $U = U^1 \times \cdots \times U^m$ and write $\mathbf{u} = (u^1, \ldots, u^m)$.

Now let us introduce operators

$$\begin{aligned} P^i\mathbf{u} &= u_t{}^i(t, x) - f^i\big(t, x, \mathbf{u}, u_x{}^i, u^i_{xx}, \mathbf{u}(\cdot)\big), \\ R_1{}^i\mathbf{u} &= u^i(t, x) - g_1{}^i\big(t, x, \mathbf{u}, \mathbf{u}(\cdot)\big), \qquad (i \in I) \\ R_2{}^i\mathbf{u} &= u^i(t, x) - g_2{}^i\big(t, x, \mathbf{u}, u_\nu{}^i, \mathbf{u}(\cdot)\big), \end{aligned}$$

where

$$f^i : G \times \mathbf{R}^m \times \mathbf{R}^{n+n} \times U \to \mathbf{R},$$

$$g_1{}^i : \Gamma_1{}^i \times \mathbf{R}^m \times U \to \mathbf{R}, \qquad g_2{}^i : \Gamma_2{}^i \times \mathbf{R}^m \times \mathbf{R} \times U \to \mathbf{R}.$$

The already mentioned Kermack-McKendrick equations, for instance, are of this type. We consider the $P^i\mathbf{u}$ as components of the vector $P\mathbf{u} := (P^1\mathbf{u}, \ldots, P^m\mathbf{u})$. Analogously, $\mathbf{R}_1\mathbf{u}$ and $\mathbf{R}_2\mathbf{u}$ are defined. U in the domains of definition of f^i and $g_k{}^i$ expresses that f^i and $g_k{}^i$ are functionals of Volterra type with respect to $\mathbf{u}(\cdot)$, i.e. for fixed $(t, x) \in G^i$ the function u^i can be used on the set $\{(\tau, x) \in G^i : \tau \leqq t\}$. Therefore retarded t, decaying x, integrals of u^i and so on are allowed. If in the case $G^i \neq G^j$ the function u^i is not defined for certain (t, x) belonging to the domain of definition of u^j $(i \neq j)$, we consider $R_k{}^j\mathbf{u}$ to be independent of u^i in this (t, x). Inequalities between vector functions are to be understood componentwise and pointwise, i.e. for $\mathbf{u}, \mathbf{v} \in U$ holds $\mathbf{u} \leqq \mathbf{v}$ if and only if $u^i(t, x) \leqq v^i(t, x)$ $\big((t, x) \in G^i, i \in I\big)$.

For $\mathbf{u} \in U$ we denote by $\mathcal{P}_\mathbf{u}{}^m$ the set of all $\mathbf{f} = (f^1, \ldots, f^m)$ such that for any fixed $(t, x) \in G$, $\gamma = \pm 1$ and $i \in I$ with all real symmetric matrices $q \geqq 0$

$$\gamma f^i\big(t, x, \mathbf{u}, u_x{}^i, u^i_{xx} + \gamma q, \mathbf{u}(\cdot)\big) \geqq \gamma f^i\big(t, x, \mathbf{u}, u_x{}^i, u^i_{xx}, \mathbf{u}(\cdot)\big).$$

If $\mathbf{f} \in \mathcal{P}_{\mathbf{u}}^m$, then the Operator $\mathbf{P}$ is called *parabolic with respect to* $\mathbf{u}$. For $\mathbf{u} \in U$ we denote by $\mathcal{R}_{\mathbf{u}}^m$ the set of all $\mathbf{g}_2 = (g_2^1, \ldots, g_2^m)$ such that for any fixed $(t, x) \in \Gamma_2^i$, $\gamma = \pm 1$ and $i \in I$ with all real numbers $s \geqq 0$

$$\gamma g_2^i(t, x, \mathbf{u}, u_\nu^i + \gamma s, \mathbf{u}(\cdot)) \geqq \gamma g_2^i(t, x, \mathbf{u}, u_\nu^i, \mathbf{u}(\cdot)).$$

Definition 1 (cf. [6]): $(t_0, x_0) \in G^i$ will be called *Nagumo-point* for the function $u^i \in U^i$ if

$$u^i(t_0, x_0) = 0, \quad u^i(t_0, x) \leqq 0, \quad \text{and} \quad u^i(t, x) < 0 \qquad ((t, x) \in G^i,\ t < t_0).$$

Lemma 1: *If* (t_0, x_0) *is a Nagumo-point for* u^i, *then*
(i) *for* $(t_0, x_0) \in G$ *follows* $u_x^i(t_0, x_0) = 0$, $u_{xx}^i(t_0, x_0) \leqq 0$ *and* $u_t^i(t_0, x_0) \geqq 0$;
(ii) *for* $(t_0, x_0) \in \Gamma_2^i$ *follows* $u_\nu^i(t_0, x_0) \leqq 0$.

For $\mathbf{s} \in \mathbf{R}^m$ we set $[\mathbf{s}] = (|s^1|, \ldots, |s^m|)$. With continuous functions $\varkappa^i = \varkappa^i(x)$, defined for all x with $(t, x) \in G^i$ and with the property $\varkappa^i(x) \geqq 1$, we define

$$\|u^i\|_{\varkappa}(t) = \sup \{|u^i(\bar{t}, \bar{x})|/\varkappa^i(\bar{x}) : (\bar{t}, \bar{x}) \in G^i, \bar{t} \leqq t\},$$

$$|[\mathbf{u}]|_{\varkappa}(t) = (\|u^1\|_{\varkappa}(t), \ldots, \|u^m\|_{\varkappa}(t)).$$

For vectors $p \in \mathbf{R}^n$ and symmetric $n \times n$-matrices q we denote by $|p|$ an arbitrary vector-norm and by $|q|$ an arbitrary matrix-norm, respectively. It can easily be seen from the context which kind of norm is meant.

3. For n_0 with $1 \leqq n_0 \leqq n$ let the domain G be unbounded with respect to the co-ordinates x_i, $i \leqq n_0$, and be bounded with respect to x_i, $i > n_0$ (if they exist). Further, let a be a positive constant, which can be chosen so large that for $n_0 < n$, $(t, x) \in G^1 \cup \cdots \cup G^m$ and $|x| > a$ it yields

$$|x|^2 \leqq 4(|x_1|^2 + \cdots + |x_{n_0}|^2). \tag{1}$$

In the following we need for $i \in I$ functions $\lambda^i = \lambda^i(x)$ with the domain $D(\lambda^i) = \{x : (t, x) \in G^i, |x| > a\}$ and $d^i = d^i(t, x)$ with the domain $D(d^i) = \{(t, x) \in G^i : |x| > a, x_{n_0+1} = \cdots = x_n = 0\}$. The functions λ^i and d^i and the derivatives d_t^i and d_{xx}^i are required to be continuous and $\lambda^i(x) \geqq 1$, $d^i(t, x) \geqq 1$. For $|x| \to \infty$ and uniformly with respect to t the asymptotic relations

$$\lambda^i(x) = O(\varkappa^i(x)) \quad \text{and} \quad \lambda^i(x) = o(d^i(t, x)) \tag{2}$$

are to be fulfilled. Now we denote by $U^i(\lambda^i)$ the set of functions $u^i \in U^i$ with $u^i(t, x) = O(\lambda^i(x))$ for $|x| \to \infty$ and uniformly with respect to t such that (2) is fulfilled. Then we can define

$$U(\lambda) = U^1(\lambda^1) \times \cdots \times U^m(\lambda^m).$$

From now we denote by $p \in \mathbf{R}^n$ vectors and by q symmetric $n \times n$-matrices with $p_i = q_{ij} = 0$ for $i, j > n_0$. Let $\mathbf{s} \in \mathbf{R}^m$, $z \in \mathbf{R}$, and let $\mathbf{\Phi} \in U(\boldsymbol{\lambda})$ be any arbitrary function. Now we can formulate the first assumption

($\mathbf{A}_1$) *Let* $\mathbf{v} \in U(\boldsymbol{\lambda})$ *be any given function and* $i \in I$. *Assume the existence of functions*

$$\omega_0^i : J_0 \times \mathbf{R}_+^{m+1} \to \mathbf{R}, \qquad \omega_k^i : J_k^i \times \mathbf{R}_+^{m+1} \to \mathbf{R} \qquad (k = 1, 2),$$

$$K_k^i : \{x : (t, x) \in G\} \to \mathbf{R} \qquad (k = 0, 1), \qquad K_2^i : \{x : (t, x) \in \Gamma_2^i\} \to \mathbf{R}.$$

such that the following inequalities for each (t, x) from the domains of definition of the related functions are fulfilled:

$$\operatorname{sgn} s^i\big(f^i(t, x, \mathbf{v} + \mathbf{s}, v_x^i + p, v_{xx}^i + q, \mathbf{v}(\cdot) + \boldsymbol{\Phi}(\cdot)) - f^i(t, x, \mathbf{v}, v_x^i, v_{xx}^i, \mathbf{v}(\cdot))\big)$$
$$\leqq \omega_0^i\big(t, [\mathbf{s}], \|[\boldsymbol{\Phi}]\|_\varkappa (t)\big) + |p|\, K_1^i(x) + |q|\, K_0^i(x),$$
$$\operatorname{sgn} s^i\big(g_1^i(t, x, \mathbf{v} + \mathbf{s}, \mathbf{v}(\cdot) + \boldsymbol{\Phi}(\cdot)) - g_1^i(t, x, \mathbf{v}, \mathbf{v}(\cdot))\big)$$
$$\leqq \omega_1^i\big(t, [\mathbf{s}], \|[\boldsymbol{\Phi}]\|_\varkappa (t)\big),$$
$$\operatorname{sgn} s^i\big(g_2^i(t, x, \mathbf{v} + \mathbf{s}, v_\nu^i + z, \mathbf{v}(\cdot) + \boldsymbol{\Phi}(\cdot)) - g_2^i(t, x, \mathbf{v}, v_\nu^i, \mathbf{v}(\cdot))\big)$$
$$\leqq \omega_2^i\big(t, [\mathbf{s}], \|[\boldsymbol{\Phi}]\|_\varkappa (t)\big) + |z|\, K_2^i(x).$$

One can easily see that some assumptions are formulated for a function $\mathbf{u}$ (parabolicity), other assumptions like (A_1) are formulated for a function $\mathbf{v}$. Usually $\mathbf{u}$ is an unknown function which is in some sense comparable with $\mathbf{v}$, the so-called *comparison function*. Because the function $\mathbf{v}$ is given, the assumptions with $\mathbf{v}$ are easily verified.

The necessity of introducing different functions $\boldsymbol{\lambda}$, differing in their asymptotic growth, follows from (A_1). On the other hand, this assumption allows us to determine that function $\boldsymbol{\lambda}$ which has an asymptotic growth as fast as possible such that the given investigations are possible yet. This is in a certain sense the "optimal" function $\boldsymbol{\lambda}$. If all the functional terms are absent, it is not necessary to use functions $\boldsymbol{\lambda}$ with different asymptotic growth. Then it suffices to take $\lambda^i(x) = \exp(L|x|^2)$ (cf. [9, 10]). For examples with respect to λ^i, K_k^i and $\varkappa^i$ we refer to [7, 9].

In that what follows we denote $\boldsymbol{\omega} = (\boldsymbol{\omega}_0, \boldsymbol{\omega}_1, \boldsymbol{\omega}_2)$ with $\boldsymbol{\omega}_k = (\omega_k^1, \ldots, \omega_k^m)$ for $k = 0, 1, 2$.

Definition 2: Let $\mathscr{E}$ be the set of all $\boldsymbol{\omega}$ for which a vector $\boldsymbol{\rho}: J \to \mathbf{R}_+^m$ exists such that for $i \in I$, $k = 1, 2$ with given functions $\bar{\delta}_k^i: J_k^i \to \mathbf{R}_+$, $\bar{\delta}_0^i: J \to \mathbf{R}_+$ the following holds:

(i) $\boldsymbol{\rho}(t) > 0$, ϱ^i is differentiable from below on J_0 with $\boldsymbol{\rho}'(t) \geqq 0$.
(ii) $\boldsymbol{\rho}'(t) \geqq \boldsymbol{\omega}_0(t, \boldsymbol{\rho}, \boldsymbol{\rho}) + \bar{\boldsymbol{\delta}}_0(t)$, $t \in J_0$.
(iii) $\varrho^i(t) \geqq \omega_k^i(t, \boldsymbol{\rho}, \boldsymbol{\rho}) + \bar{\delta}_k^i(t)$, $t \in J_k^i$; $\varrho^i(t) = \varrho^i(0)$, $t < 0$.

Definition 3: The vector $\boldsymbol{\omega}$ is called *admissible* if it has the following properties $(i \in I;\ k = 0, 1, 2)$:

(i) For $\mathbf{s}_1, \mathbf{s}_2 \in \mathbf{R}^m$ the function $\omega_k^i(t, \mathbf{s}_1, \mathbf{s}_2)$ is increasing with respect to s_2^j for $j \in I$ as well as s_1^j for $j \in I \setminus \{i\}$.
(ii) For $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3, \mathbf{s}_4 \in \mathbf{R}_+^m$ with $0 \leqq \mathbf{s}_1 \leqq \mathbf{s}_2$ and $0 \leqq \mathbf{s}_3 \leqq \mathbf{s}_4$ there exist bounded functions $\mu_k^{ij} = \mu_k^{ij}(t)$, $\bar{\mu}_k^{ij} = \bar{\mu}_k^{ij}(t)$ with

$$\omega_k^i(t, \mathbf{s}_2, \mathbf{s}_4) - \omega_k^i(t, \mathbf{s}_1, \mathbf{s}_3) \leqq \sum_{j \in I} \big(\mu_k^{ij}(t)\,(s_2^j - s_1^j) + \bar{\mu}_k^{ij}(t)\,(s_4^j - s_3^j)\big).$$

(iii) It holds

$$c := \sup \left\{ \sum_{j \in I} \big(\mu_k^{ij}(t) + \bar{\mu}_k^{ij}(t)\big) : t \in J_k^i, i \in I, k = 1, 2 \right\} < 1.$$

The property (i) with respect to $\mathbf{s}_1$ is usually called *quasimonotonicity* of ω_k^i (cf. [10]).

The functions d^i are connected with the functions K_k^i by inequalities. If $\sup\{|x| : (t, x) \in \Gamma_2^i\}$ is finite for some i, then the constant a is to be chosen larger than the values of these suprema. Then the inequalities in the second assumption belonging to Γ_2^i are to be regarded as not present.

(A_2) *There exists a constant $K \geqq 0$ such that for $i \in I$*

$$\sum_{j\in I, j\neq i} \mu_0^{ij} d^j + |d_x^i| K_1^i + |d_{xx}^i| K_0^i - d_t^i \leqq K \text{ on } \{(t, x) \in G : |x| > a\},$$

$$\sum_{j\in I, j\neq i} \mu_2^{ij} d^i + |d_x^i| K_2^i - d^i \leqq K \text{ on } \{(t, x) \in \Gamma_2^i : |x| > a\}.$$

By $\tilde{\mathbf{f}}, \tilde{\mathbf{g}}_k$ ($k = 1, 2$) we denote "disturbed" mappings with the same domains of definition as $\mathbf{f}$ and the $\mathbf{g}_k$, respectively. They define the operators $\tilde{\mathbf{P}}$ and $\tilde{\mathbf{R}}_k$.

(A_3) *For any given function $\mathbf{v} \in U(\lambda)$ and $i \in I$ there exist functions η_k^i ($k = 0, 1, 2$) such that the following inequalities on the domains of definition of the functions $\mathbf{f}$ and $\mathbf{g}_k$ are fulfilled:*

$$\left|f^i(t, x, \mathbf{v}, v_x^i, v_{xx}^i, \mathbf{v}(\cdot)) - \tilde{f}^i(t, x, \mathbf{v}, v_x^i, v_{xx}^i, \mathbf{v}(\cdot))\right| \leqq \eta_0^i(t);$$

$$\left|g_1^i(t, x, \mathbf{v}, \mathbf{v}(\cdot)) - \tilde{g}_1^i(t, x, \mathbf{v}, \mathbf{v}(\cdot))\right| \leqq \eta_1^i(t),$$

$$\left|g_2^i(t, x, \mathbf{v}, v_\nu^i, \mathbf{v}(\cdot)) - \tilde{g}_2^i(t, x, \mathbf{v}, v_\nu^i, \mathbf{v}(\cdot))\right|' \leqq \eta_2^i(t).$$

4. Some given functions of t are to be investigated more detailed. In the case of admissibility of $\boldsymbol{\omega}$ we can derive further properties of the functions μ_k^{ij} and $\bar{\mu}_k^{ij}$.

Lemma 2: *If the vector $\boldsymbol{\omega}$ is admissible, then*

(i) $\bar{\mu}_k^{ij}(t) \geqq 0$ *for all* i, j, k, t; $\mu_k^{ij}(t) \geqq 0$ *for* $i \neq j$, *all* k, t.

(ii) $\sum_{j\in I} \mu_k^{ij} \leqq c$ *for all i and k.*

(iii) $c_0 := \sup \left\{ \sum_{j\in I} \bar{\mu}_k^{ij}(t) \Big/ \left(1 - \sum_{j\in I} \mu_k^{ij}(t)\right) : t \in J_k^i, i \in I, k = 1, 2 \right\} < 1.$

Proof: (i) follows from Definition 3/(i). With (i) and Definition 3/(iii) we obtain (ii). Finally, (iii) follows because of $c < 1$ and the inequality

$$\sum_{j\in I} \bar{\mu}_k^{ij} \Big/ \left(1 - \sum_{j\in I} \mu_k^{ij}\right) \leqq \left(c - \inf_{t,k,i} \sum_{j\in I} \mu_k^{ij}\right) \Big/ \left(1 - \inf_{t,k,i} \sum_{j\in I} \mu_k^{ij}\right) < 1 \quad \blacksquare$$

Let $\bar{\boldsymbol{\delta}}_k$ be the functions in Definition 2. Further, let $\xi^i : J_1^i \cup J \to \mathbf{R}_+$ ($i \in I$) be bounded functions and $\boldsymbol{\rho}$ an arbitrary solution of the inequalities in Definition 2.

Definition 4: By Σ we denote the set of functions $\boldsymbol{\zeta}: J \to \mathbf{R}_+^m$, $\boldsymbol{\zeta}(t) = \boldsymbol{\zeta}(0)$ for $t < 0$, in J_0 differentiable from below, with the properties

$$\vartheta_0 = \inf \{\zeta^{i\prime}(t) - \omega_0^i(t, \boldsymbol{\zeta}, \boldsymbol{\xi}) - \bar{\delta}_0^i(t) : t \in J_0, i \in I\} > 0,$$

$$\vartheta_k = \inf \{\zeta^i(t) - \omega_k^i(t, \boldsymbol{\zeta}, \boldsymbol{\xi}) - \bar{\delta}_k^i(t) : t \in J_k^i, i \in I\} > 0 \qquad (k = 1, 2),$$

$$\inf \{\boldsymbol{\zeta}'(t) - \boldsymbol{\rho}'(t) : t \in J_0\} > 0 \quad \text{and} \quad \inf \{\boldsymbol{\zeta}(t) - \boldsymbol{\rho}(t) : t \in J_0\} > 0.$$

Hence, $\vartheta = \min \{\vartheta_0, \vartheta_1, \vartheta_2\} > 0$. The structure of the following lemma is due to [6]. Here a generalized version is used for functional-boundary conditions.

Lemma 3: *If the vector $\boldsymbol{\omega} \in \mathscr{E}$ is admissible and $\xi^i \leqq \bar{\xi} =$ const, $\bar{\xi} > 0$ ($i \in I$), then*

(i) $\sum$ *is not empty,*

(ii) $\boldsymbol{\xi} \leqq \boldsymbol{\zeta}$ *for all $\boldsymbol{\zeta} \in \Sigma$ implies $\boldsymbol{\xi} \leqq \boldsymbol{\rho}$.*

Proof: Let $c_1 \in (c_0, 1)$ be an arbitrary, but fixed constant. With c_1 we define the following expressions:

$$\gamma = \sup \left\{ \sum_{j\in I} \left(c_1 \mu_0^{ij}(t) + \bar{\mu}_0^{ij}(t)\right) : t \in J_0, i \in I \right\},$$

$$\varrho_k^1(t) = \begin{cases} c_1^k \bar{\xi} \exp(2\gamma t / c_1), & \gamma > 0 \\ c_1^k \bar{\xi} \exp(t / c_1), & \gamma \leqq 0, \end{cases} \quad \varrho_k^j(t) = \varrho_k^1(t) \qquad (j = 2, \ldots, m).$$

We regard only the case $\gamma > 0$, the case $\gamma \leqq 0$ is to be treated analogously. Then we can easily derive $\varrho_k^j = c_1\varrho_{k-1}^j$, $\varrho_k^j = 2\gamma\varrho_{k-1}^j$, $\varrho_k^j \geqq c_1^k\bar{\xi} > 0$, and

$$\xi^j(t) \leqq \bar{\xi} \leqq \varrho_0^j(t) \leqq \varrho_0^j(t) + \varrho^j(t). \tag{3}$$

Now we show that

$$\boldsymbol{\xi} \leqq \boldsymbol{\rho}_{k-1} + \boldsymbol{\rho} \quad \text{provides} \quad \boldsymbol{\rho}_k + \boldsymbol{\rho} \in \Sigma. \tag{4}$$

With (3) this yields $\boldsymbol{\rho}_1 + \boldsymbol{\rho} \in \Sigma$, i.e. Σ is not empty, and (i) is proved. For the proof of (4) we denote $\boldsymbol{\sigma} = \boldsymbol{\rho}_k + \boldsymbol{\rho}$. Using the admissibility of $\boldsymbol{\omega}$, we can estimate

$$\sigma^i - \omega_0^i(t, \boldsymbol{\sigma}, \boldsymbol{\xi}) - \tilde{\delta}_0^i \geqq 2\gamma\varrho^i - \sum_{j\in I} \mu_0^{ij}\varrho_k^j - \sum_{j\in I} \bar{\mu}_0^{ij}\varrho_{k-1}^j \geqq \gamma c_1^{k-1}\bar{\xi} > 0.$$

Analogously, we get on J_k^i

$$\sigma^i - \omega_k^i(t, \boldsymbol{\sigma}, \boldsymbol{\xi}) - \tilde{\delta}_k^i \geqq c_1^k\bar{\xi}(1 - c_0/c_1)(1 - c) > 0.$$

The remaining properties of $\boldsymbol{\sigma}$ (cf. Def. 4) can easily be shown. Because of $\boldsymbol{\rho}_k \to 0$ uniformly on J follows $\boldsymbol{\xi} \leqq \boldsymbol{\rho}$ ∎

5. Now the central theorem of the paper can be proved.

Theorem 1: *Let the following assumptions be fulfilled:*

(i) $\mathbf{u}, \mathbf{v} \in U(\boldsymbol{\lambda})$.
(ii) $\mathbf{f} \in \mathcal{P}_{\mathbf{u}}^m$, $\mathbf{g}_2 \in \mathcal{R}_{\mathbf{u}}^m$.
(iii) $(\mathbf{A}_1)$ *holds for* $|x| > a$, *for* $|x| \leqq a$ *with* $p = 0$ *and* $q = 0$. *Further,* $(\mathbf{A}_2)$ *and* $(\mathbf{A}_3)$ *hold.*
(iv) $|\mathbf{Pu} - \tilde{\mathbf{P}}\mathbf{v}| \leqq \boldsymbol{\delta}_0(t)$ *and* $|\mathbf{R}_k\mathbf{u} - \tilde{\mathbf{R}}_k\mathbf{v}| \leqq \boldsymbol{\delta}_k(t)$ $(k = 1, 2)$ *on the domains of definition.*
(v) $\boldsymbol{\omega} \in \mathcal{E}$ *is admissible with* $\tilde{\boldsymbol{\delta}}_k = \boldsymbol{\delta}_k + \boldsymbol{\eta}_k$ $(k = 0, 1, 2)$.

Then

$$|u^i(t, x) - v^i(t, x)| \leqq \varrho^i(t) \qquad \big((t, x) \in G^i, i \in I\big).$$

Proof: We denote $\boldsymbol{\xi}(t) = |[\mathbf{u} - \mathbf{v}]|_{\varkappa}(t)$. Then $\boldsymbol{\xi}(T) < \infty$. Let $\boldsymbol{\zeta} \in \Sigma$ and with arbitrary $\alpha_0 > 0$

$$\bar{\boldsymbol{\Phi}}(t, x) = \begin{cases} \boldsymbol{\zeta}(0), & t < 0 \\ \boldsymbol{\zeta}(t), & t \geqq 0, \ |x| \leqq a \qquad \big((t, x) \in \bar{G}\big) \\ \boldsymbol{\zeta}(t) + \alpha_0\mathbf{d}(t, x), & t \geqq 0, \ |x| > a \qquad \big((t, x) \in \bar{G}\big). \end{cases}$$

At first the inequality $|\mathbf{u} - \mathbf{v}| < \bar{\boldsymbol{\Phi}}$ is to be proved. On Γ_1^i even the inequality

$$|u^i - v^i| < \zeta^i \qquad (i \in I) \tag{5}$$

is valid. To show this, we suppose that there are $i \in I$ and $(t, x) \in \Gamma_1^i$ with $|u^i(t, x) - v^i(t, x)| \geqq \zeta^i(t)$. For such a fixed (t, x) let

$$|u^i(t, x) - v^i(t, x)| - \zeta^i(t) := \max_{j\in I} \big(|u^j(t, x) - v^j(t, x)| - \zeta^j(t)\big)$$

and $|u^i(t, x) - v^i(t, x)| = u^i(t, x) - v^i(t, x)$. Then follows, with $(\mathbf{A}_3)$ and $(\mathbf{A}_1)$,

$$\begin{aligned} |u^i - v^i| &\leqq g_1^i\big(t, x, \mathbf{u}, \mathbf{u}(\cdot)\big) - g_1^i\big(t, x, \mathbf{v}, \mathbf{v}(\cdot)\big) + \tilde{\delta}_1^i(t) \\ &\leqq \omega_1^i(t, [\mathbf{u} - \mathbf{v}], \boldsymbol{\xi}) + \tilde{\delta}_1^i(t). \end{aligned}$$

Because of $\boldsymbol{\zeta} \in \Sigma$, this yields

$$|u^i - v^i| - \zeta^i < \omega_1{}^i(t, [\mathbf{u} - \mathbf{v}], \boldsymbol{\xi}) - \omega_1{}^i(t, \boldsymbol{\zeta}, \boldsymbol{\xi}).$$

Every component of the vector $[\mathbf{u} - \mathbf{v}]$ belongs to one of the classes

a) $|u^j - v^j| = \zeta^j$, b) $|u^j - v^j| < \zeta^j$, c) $|u^j - v^j| > \zeta^j$.

In a) we substitute $|u^j - v^j|$ by ζ^j. In b) we use the quasimonotonicity of $\omega_1{}^i$ for getting ζ^j. Writing $I_0 = \{j : |u^j(t, x) - v^j(t, x)| > \zeta^j\}$ and applying Definition 3/(ii), we obtain

$$|u^i - v^i| - \zeta^i < \sum_{j \in I_0} \hat{\mu}_1{}^{ij}(t) \, (|u^j - v^j| - \zeta^j) \leq c(|u^i - v^i| - \zeta^i)$$

because of Lemma 2/(ii). This leads to a contradiction, and (5) is proved.

Now suppose the existence of $(t, x) \in \bar{G} \setminus (\cup\{\Gamma_1{}^j : j \in I\})$ and $j \in I$ with $|u^j(t, x) - v^j(t, x)| \geq \bar{\Phi}^j(t, x)$. By I_1 we denote the set of j for which such (t, x) exist. Then for each $j \in I_1$ a Nagumo-point $(t^j_{\alpha_0}, x^j_{\alpha_0})$ exists for the function $|u^j - v^j| - \bar{\Phi}^j$. Let I_2 be the set of $j \in I_1$ with $|x^j_{\alpha_0}| > a$. Then, with ϑ from Definition 4 and K from ($\mathbf{A}_2$), follows

Lemma 4 [9]: *If $(t^j_{\alpha_0}, x^j_{\alpha_0})$ is a Nagumo-point for $|u^j - v^j| - \bar{\Phi}^j$ with $|x^j_{\alpha_0}| > a$, then there exists an $\alpha_1{}^j \leq \alpha_0$ such that for every $\tilde{\alpha} \leq \alpha_1{}^j$ a Nagumo-point $(t_{\tilde{\alpha}}{}^j, x_{\tilde{\alpha}}{}^j)$ with $|x_{\tilde{\alpha}}| > a$ exists, and*

$$\tilde{\alpha} \leq \vartheta/(4K) \quad \text{and} \quad \tilde{\alpha} d^j(t_{\tilde{\alpha}}{}^j, x_{\tilde{\alpha}}{}^j) \leq \vartheta/4. \tag{6}$$

For $K = 0$, the first inequality in (6) is to be omitted. If I_2 is not empty, we can set

$$\alpha = \min_{j \in I_2} \alpha_1{}^j \quad \text{and} \quad t_0 = \min_{j \in I_1} t_\alpha{}^j = t_\alpha{}^k.$$

Then $(t_0, x_0) := (t_\alpha{}^k, x_\alpha{}^k)$ is Nagumo-point for $|u^k - v^k| - \bar{\Phi}^k$.

Now let $|x_0| > a$ and $(t_0, x_0) \in \Gamma_2{}^k$. With $|u^k - v^k| = u^k - v^k$ (the reverse case is to be treated analogously) and $\boldsymbol{\zeta} \in \Sigma$ we obtain

$$\begin{aligned} \vartheta &\leq \zeta^k - \omega_2{}^k(t_0, \boldsymbol{\zeta}, \boldsymbol{\xi}) - \bar{\delta}_2{}^k \\ &\leq \omega_2{}^k(t_0, [\mathbf{u} - \mathbf{v}], \boldsymbol{\xi}) - \omega_2{}^k(t_0, \boldsymbol{\zeta}, \boldsymbol{\xi}) + |\bar{\Phi}^k| \, K_2{}^k - \alpha d^k. \end{aligned}$$

Using the quasimonotonicity of $\omega_2{}^k$ with respect to $[\mathbf{u} - \mathbf{v}]$ and following Definition 3/(ii), the right-hand side can be further estimated:

$$\begin{aligned} \vartheta &\leq \mu_2{}^{kk}(t_0) \, \alpha d^k(t_0, x_0) + \alpha \Big(\sum_{i \in I, i \neq k} \mu_2{}^{ki}(t_0) \, d^i(t_0, x_0) + |d_x{}^k| \, K_2{}^k(x_0) - d^k(t_0, x_0) \Big) \\ &\leq c\vartheta/4 + \vartheta/4 < \vartheta/2 \end{aligned}$$

(the inequality $\mu_2{}^{kk} \leq c$ follows from Lemma 2/(i)). This contradiction provides $(t_0, x_0) \notin \Gamma_2{}^k$.

Suppose $(t_0, x_0) \in G$: Applying analogous methods and assumptions, we obtain

$$\begin{aligned} \vartheta &\leq \zeta^{k'} - \omega_0{}^k(t_0, \boldsymbol{\zeta}, \boldsymbol{\xi}) - \bar{\delta}_0{}^k \\ &\leq \omega_0{}^k(t_0, [\mathbf{u} - \mathbf{v}], \boldsymbol{\xi}) - \omega_0{}^k(t_0, \boldsymbol{\zeta}, \boldsymbol{\xi}) + \alpha(|d_x{}^k| \, K_1{}^k + |d^k_{xx}| \, K_0{}^k - d_t{}^k) \\ &\leq \alpha c d^k(t_0, x_0) + \alpha \Big(\sum_{i \in I, i \neq k} \mu_0{}^{ki}(t_0) \, d^i + |d_x{}^k| \, K_1{}^k + |d^k_{xx}| \, K_0{}^k - d_t{}^k \Big) < \vartheta/2. \end{aligned}$$

Hence $(t_0, x_0) \notin G$. Therefore we do not have any Nagumo-point with $|x_0| > a$. The assumption that a Nagumo-point (t_0, x_0) with $|x_0| \leqq a$ exists leads in an analogous way to a contradiction.

Due to Lemma 4 we do not have any Nagumo-point for α_0, i.e. the inequality $|\mathbf{u} - \mathbf{v}| < \bar{\mathbf{\Phi}}$ holds for any (t, x) and arbitrary $\alpha \leqq \alpha_0$. For $\alpha \to 0^+$ in it, the inequality $|\mathbf{u} - \mathbf{v}| \leqq \boldsymbol{\zeta}$ follows. Taking the supremum with respect to x and $\tau \leqq t$, we obtain $\boldsymbol{\xi}_1(t) := |[\mathbf{u} - \mathbf{v}]|_1 (t) \leqq \boldsymbol{\zeta}(t)$. For $\boldsymbol{\varkappa}^i \equiv 1$ $(i \in I)$, the assertion of Theorem 1 follows with Lemma 3. For the other $\boldsymbol{\varkappa}$, the assumptions of Theorem 1 are fulfilled with $\boldsymbol{\xi}_1(t)$ (cf. [6]). This can be found by using the inequality $\boldsymbol{\xi}(t) \leqq \boldsymbol{\xi}_1(t)$ and the monotonicity of $\omega_k^i(t, \mathbf{s}_1, \mathbf{s}_2)$ with respect to s_2^j $(j \in I)$. Repetition of the proof with $\boldsymbol{\xi}_1$ instead of $\boldsymbol{\xi}$ provides the assertion ∎

In the case $\mu_k^{jj} < 0$ $(j \in I; k = 0, 1, 2)$ it may be better to use the following assumption instead of $(\mathbf{A}_2)$:

$$(\mathbf{A}_2') \qquad \left.\begin{aligned} &\sum_{j\in I} \mu_0^{ij} d^j + |d_x^i|\, K_1^i + |d_{xx}^i|\, K_0^i - d_t^i \leqq K, \\ &\sum_{j\in I} \mu_2^{ij} d^j + |d_x^i|\, K_2^i - d^i \leqq K \end{aligned}\right\} \quad (i \in I).$$

If only some few μ_k^{ij} are negative, then we use $(\mathbf{A}_2')$ only for these $j \in I$. One special case of Theorem 1 for $m = 1$ can be found in [9]. Further more specialized cases are regarded in [6].

6. Theorem 1 is the starting point for investigations of uniqueness as well as stability. For this, it is necessary to restrict the class $\mathscr{E}$.

Definition 5: By $\mathscr{E}^*$ we denote the set of $\boldsymbol{\omega} \in \mathscr{E}$ having the following property: For any quantity $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) \geqq 0$ such that for the functions $\bar{\boldsymbol{\delta}}_k$ in Definition 2 with $\bar{\delta}_k^i \leqq \delta$ $(i \in I; k = 0, 1, 2)$ a function $\boldsymbol{\rho}$ from Definition 2 satisfies the inequality $\varrho^i(t) \leqq \varepsilon$. Especially, for $\delta = 0$ the class $\mathscr{E}^*$ is to be denoted by $\mathscr{E}_0^*$.

Now, the solution $\mathbf{u}$ of the problem

$$\mathbf{P}\mathbf{u} = 0, \qquad \mathbf{R}_1\mathbf{u} = 0, \qquad \mathbf{R}_2\mathbf{u} = 0 \tag{7}$$

can be investigated. From Theorem 1 immediately follows

Theorem 2: *For any two functions* $\mathbf{u}, \mathbf{v} \in U(\boldsymbol{\lambda})$, *let the assumptions of Theorem* 1 *be fulfilled with* $\bar{\boldsymbol{\delta}}_k \equiv 0$ $(k = 0, 1, 2)$. *Let* $\boldsymbol{\omega} \in \mathscr{E}_0^*$. *Then* $\mathbf{u} \equiv \mathbf{v}$.

Finally, we obtain stability results from Theorem 1. Let $\mathbf{v} \in U(\boldsymbol{\lambda})$ be such a function that

$$|\tilde{\mathbf{P}}\mathbf{v}| \leqq \boldsymbol{\delta}_0, \qquad |\tilde{\mathbf{R}}_1\mathbf{v}| \leqq \boldsymbol{\delta}_1, \qquad |\tilde{\mathbf{R}}_2\mathbf{v}| \leqq \boldsymbol{\delta}_2 \tag{8}$$

and $(\mathbf{A}_3)$ hold. In the following Definition 6, four definitions are concentrated. In each case, those components of the expressions in the braces belong to one definition which are standing in the same places.

Definition 6: The problem (7) is called *stable with respect to*

$$\begin{Bmatrix} \mathbf{P} \\ \mathbf{R}_1 \\ \mathbf{R}_2 \\ \text{the solution} \end{Bmatrix} \text{ if for } \begin{Bmatrix} \boldsymbol{\delta}_0 = \boldsymbol{\delta}_k = \boldsymbol{\eta}_k = 0 & (k = 1, 2) \\ \boldsymbol{\delta}_1 = \boldsymbol{\delta}_k = \boldsymbol{\eta}_k = 0 & (k = 0, 2) \\ \boldsymbol{\delta}_2 = \boldsymbol{\delta}_k = \boldsymbol{\eta}_k = 0 & (k = 0, 1) \\ \boldsymbol{\eta}_k = 0 & (k = 0, 1, 2) \end{Bmatrix},$$

and any quantity $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon) \geqq 0$ such that

$$\left.\begin{cases} \eta_0 \leqq \delta \\ \eta_1 \leqq \delta \\ \eta_2 \leqq \delta \\ \delta_k \leqq \delta \quad (k = 0, 1, 2) \end{cases}\right\} \text{ implies } |u^i(t, x) - v^i(t, x)| \leqq \varepsilon \quad \big((t, x) \in G^i, i \in I\big).$$

The stability of the problem (7) with respect to the solution **u** could be broken down further. Thus for instance it is possible that a function v^i satisfies exactly the condition on $\Gamma_1{}^i$ ($\delta_1{}^i = 0$), but with the operators $R_1{}^j$ ($j \neq i$) it produces non-identically vanishing defects. It is obvious how to proceed in this case and shall not be described further.

Theorem 3: *Let the assumptions of Theorem* 1 *be fulfilled for the solution* $\mathbf{u} \in U(\boldsymbol{\lambda})$ *of* (7) *as well as* $\mathbf{v} \in U(\boldsymbol{\lambda})$ *satisfying* (8). *Let* $\boldsymbol{\omega} \in \mathcal{E}^*$. *Then the problem* (8) *is stable with respect to* **P**, $\mathbf{R}_1$, $\mathbf{R}_2$, *and the solution* **u**.

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VERFASSER:

Prof. Dr. Wolfgang Voigt
Sektion Mathematik der Bergakademie
Bernhard-von-Cotta-Str. 2
DDR-9200 Freiberg

Neue Adresse:
Sektion Allg. Grundlagen der Offiziershochschule „Ernst Thälmann"
PF 35802/T
DDR-8722 Löbau