Topological Realizations of Calkin Algebras on Frechet Domains of Unbounded Operator Algebras

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Let \( D \) be a dense linear subspace of a separable complex Hilbert space \( X \) endowed with the graph topology \( \tau_G \) (see Section 1 for precise definitions). Suppose \( X \) is a Frechet space with respect to the graph topology of \( L^+(D) \). Let \( \mathcal{E}(D) \) denote the set of all operators in \( L^+(D) \) which map each bounded subset of \( D[1] \) into a relatively compact subset of \( D[1] \). Then \( \mathcal{E}(D) \) is a \( \tau_G \)-closed two-sided \( \star \)-ideal of \( L^+(D) \) which contains the finite rank operators in \( \mathcal{F}(D) \) as a dense subset [15, 7]. (Note that in [15] the ideal \( \mathcal{E}(D) \) is denoted by \( \text{Vol} (t, t) \).) The quotient algebra \( \mathcal{A}(D) := L^+(D)/\mathcal{E}(D) \) is called the Calkin algebra on the domain \( D \). Let \( \pi \) denote the quotient topology on \( \mathcal{A}(D) \) of \( L^+(D) \) [\( \tau_G \)]. Obviously, \( \mathcal{A}(D) \) is a topological \( \star \)-algebra. If \( D = X \), then \( \mathcal{A}(D) = \mathcal{A}(X) \) is the usual Calkin algebra on the Hilbert space \( X \). It should be mentioned that if \( D[1] \) is a Montel space, then \( \mathcal{E}(D) = L^+(D) \) and hence the Calkin algebra \( \mathcal{A}(D) \) is trivial.

In his classical paper [3] CALKIN constructed a class of faithful isometric \( \star \)-representations of the C*-algebra \( \mathcal{A}(X) \) (see [11] for a modern treatment). In this paper we investigate the corresponding problem for the Calkin algebra \( \mathcal{A}(D) \) on the Frechet domain \( D[1] \): Does there exist a faithful \( \star \)-representation \( \pi \) of \( \mathcal{A}(D) \) which is a homeomorphism of \( \mathcal{A}(D) \) [\( \pi \)] onto \( \pi(\mathcal{A}(D)) \) [\( \tau_G \)? For the domain \( l_2 \otimes d \), \( d \) the space of all finite complex sequences, this problem has been considered in [9]. Note that \( l_2 \otimes d[t] \) is not a Frechet space.
Let us briefly describe our main results concerning the above question.

Given a free ultrafilter \( \mathcal{U} \) on \( \mathbb{N} \), we define in Section 2 a \(*\)-representation \( \pi_\mathcal{U} \) of \( \mathcal{A}(\mathcal{D}) \) in a similar way as in the case \( \mathcal{D} = \mathcal{H} \). We show that \( \pi_\mathcal{U} \) is faithful and that \( \pi_\mathcal{U}^{-1} \) is continuous (Theorem 2.1). Let \( \tau_\mathcal{U} \) denote the finest locally convex topology on \( \mathcal{L}^+(\mathcal{D}) \) for which the positive cone \( \mathcal{L}^+(\mathcal{D})_+ \) is normal [12]. If \( \tau_\mathcal{U} = \tau_\mathcal{D} \) on \( \mathcal{L}^+(\mathcal{D}) \), then each \(*\)-representation \( \pi_\mathcal{U} \) is continuous and hence a homeomorphism (Theorem 2.2).

In Section 3 we obtain a converse of the latter in some sense. Suppose that the graph topology \( \tau_{\mathcal{D}} \) on \( \mathcal{D} \) is generated by a sequence of strongly commuting self-adjoint operators whose restrictions to \( \mathcal{D} \) are in \( \mathcal{A}(\mathcal{D}) \). Under this additional assumption we prove that if \( \tau_\mathcal{U} \neq \tau_\mathcal{D} \) on \( \mathcal{L}^+(\mathcal{D}) \), then there is no continuous faithful \(*\)-representation of \( \mathcal{A}(\mathcal{D}) \) [13] (Theorem 3.1).

1. Preliminaries

In this section we collect some definitions and notations (see e.g. [8, 10]) needed later and we prove some preliminary lemmas.

1.1 Let \( \mathcal{D} \) be a dense linear subspace of a complex Hilbert space \( \mathcal{H} \) and let \( \mathcal{L}^+(\mathcal{D}) := \{a \in \text{End} \mathcal{D} : \mathcal{D} \subseteq \mathcal{D}(a^*) \text{ and } a^* \mathcal{D} \subseteq \mathcal{D} \} \). \( \mathcal{L}^+(\mathcal{D}) \) is a \(*\)-algebra endowed with the involution \( a \mapsto a^* := a^\dagger \) \( \mathcal{D} \). An \textit{Op*algebra} \( \mathcal{B} \) on \( \mathcal{D} \) is a \(*\)-subalgebra of \( \mathcal{L}^+(\mathcal{D}) \). In what follows we assume that \( \mathcal{B} \) is an \textit{Op*algebra} on \( \mathcal{D} \). Define \( \mathcal{D}(\mathcal{B}) = \cap \{ \mathcal{D}(b) : b \in \mathcal{B} \} \), where \( \mathcal{D}(b) \) is the closure of the operator \( b \).

Let \( \{\phi_n : n \in \mathbb{N}) \) be a sequence of vectors \( \phi_n \in \mathcal{H} \) and let \( \mathcal{B} \subseteq \mathcal{X} \). Suppose \( \mathcal{U} \) is a filter on \( \mathbb{N} \). We write \( \phi = \text{w-lim}_\mathcal{U} \phi_n \) if \( \lim_{\mathcal{U}} \langle \phi_n, \psi \rangle = \langle \phi, \psi \rangle \) for all \( \psi \in \mathcal{H} \) and \( \phi = \text{w-lim}_\mathcal{U} \phi_n \) if \( \lim_{\mathcal{U}} \langle \phi_n, \psi \rangle = \langle \phi, \psi \rangle \) for all \( \psi \in \mathcal{H} \).

Lemma 1.1: Suppose \( \mathcal{U} \) is an ultrafilter on \( \mathbb{N} \). Let \( \{\phi_n : n \in \mathbb{N}\) be a bounded sequence of vectors of \( \mathcal{D}(\mathcal{B}) \). Let \( \phi := \text{w-lim}_\mathcal{U} \phi_n \).

(i) Then, \( \phi \in \mathcal{D}(\mathcal{B}) \) and \( b\phi = \text{w-lim}_\mathcal{U} b\phi_n \). In particular, if \( 0 = \text{w-lim}_\mathcal{U} \phi_n \), then \( 0 = \text{w-lim}_\mathcal{U} \phi_n \) for each \( b \in \mathcal{B} \).

(ii) If \( \lim_{\mathcal{U}} \|\phi_n\| = 0 \), then \( \text{lim}_{\mathcal{U}} \|b\phi_n\| = 0 \) for each \( b \in \mathcal{B} \).

(iii) If \( \phi = 0 \) and if the set \( \{\phi_n\} \) is relatively compact in \( \mathcal{D}(\mathcal{B}) \), then \( \lim_{\mathcal{U}} \|b\phi_n\| = 0 \) for each \( b \in \mathcal{B} \).

Proof: (i) Suppose \( b \in \mathcal{B} \). Since the set \( \{b\phi_n\} \) is bounded, \( \lim_{\mathcal{U}} \langle \phi_n, \psi \rangle = \langle \phi, \psi \rangle \) for all \( \psi \in \mathcal{H} \) and \( \phi \in \mathcal{D}(\mathcal{B}) \), this gives

\[
\langle \phi, \psi \rangle = \lim_{\mathcal{U}} \langle \phi_n, \psi \rangle = \lim_{\mathcal{U}} \langle \phi_n, b^*\psi \rangle = \langle \phi, b^*\psi \rangle.
\]

Therefore, \( \phi \in \mathcal{D}(\mathcal{B}^{**}) = \mathcal{D}(\mathcal{B}) \) and \( \phi_n = b^{**}\phi^* = b\phi_n \). Since \( b \in \mathcal{B} \) is arbitrary, \( \phi \in \cap \{ \mathcal{D}(b) : b \in \mathcal{B} \} = \mathcal{D}(\mathcal{B}) \).

(ii) Since \( \{\phi_n\} \) is \( \mathcal{B} \)-bounded, \( C_b := \sup \{\|b^*\phi_n\| : n \in \mathbb{N}\} < \infty \) for \( b \in \mathcal{B} \). Now the assertion follows from

\[
(\lim_{\mathcal{U}} \|b\phi_n\|)^2 = \lim_{\mathcal{U}} \langle b^*b\phi_n, \phi_n \rangle \leq C_b (\lim_{\mathcal{U}} \|\phi_n\|) = 0.
\]

(iii) Let \( b \in \mathcal{B} \). Since \( \{\phi_n\} \) is relatively compact in \( \mathcal{D}(\mathcal{B}) \), the set \( \{b\phi_n\} \) is relatively compact in \( \mathcal{H} \). Given \( \varepsilon > 0 \), there is a finite rank projection \( F_\varepsilon \) on \( \mathcal{H} \) such that \( \|(I - F_\varepsilon)\phi_n\| \leq \varepsilon \) for \( n \in \mathbb{N} \). Since \( 0 = \text{w-lim}_\mathcal{U} b\phi_n \) because of (i) and hence \( \lim_{\mathcal{U}} \|F_\varepsilon b\phi_n\| = 0 \), we have \( \lim_{\mathcal{U}} \|b\phi_n\| \leq \lim_{\mathcal{U}} \|(I - F_\varepsilon) b\phi_n\| \leq \varepsilon \), thus \( \lim_{\mathcal{U}} \|b\phi_n\| = 0 \).

The following corollary is of some interest in itself:
Corollary 1.2: Suppose \( \varphi \in \mathcal{H} \). If there is a bounded sequence \((\varphi_n : n \in \mathbb{N})\) in \( \mathcal{D}[\mathcal{B}] \) such that \( \varphi = \text{w-lim}_n \varphi_n \), then \( \varphi \in \mathcal{D}(\mathcal{B}) \).

Proof: Take an ultrafilter \( \mathcal{U} \) on \( \mathbb{N} \) which contains all sets \( \{n \in \mathbb{N} : n \geq k\}, k \in \mathbb{N} \). Then \( \varphi = \text{w-lim}_n \varphi_n \) and Lemma 1.1 (i) applies.

1.2 Next we briefly discuss the topologization of the Op*-algebra \( \mathcal{B} \) on \( \mathcal{D} \). Let \( \mathcal{B}_b := \{b \in \mathcal{B} : b = b^*\} \). Suppose \( b, b_1, b_2 \in \mathcal{B}_b \). We write \( b_1 \geq b_2 \) if \( \langle b_1 \varphi, \varphi \rangle \geq \langle b_2 \varphi, \varphi \rangle \) for all \( \varphi \in \mathcal{D} \). Define \( \mathcal{B}_+ := \{b \in \mathcal{B} : b \geq 0\} \) and \( \{b_1, b_2\} := \{b \in \mathcal{B}_b : b_1 \leq b \leq b_2\} \). The uniform topology \( \tau_{\mathcal{B}} \) is the locally convex topology on \( \mathcal{B} \) defined by the seminorms

\[ p_{\mathcal{B}}(x) := \sup \{|\langle x \varphi, \varphi \rangle| : \varphi, \varphi \in \mathcal{M}\}, \mathcal{M} \subset \mathcal{D}[\mathcal{B}] \text{ bounded}. \]

It has been introduced in \([8]\). We denote by \( \tau_{\mathcal{D}} \) the finest locally convex topology on \( \mathcal{D} \) for which the positive cone \( \mathcal{B}_+ \) is normal. (All notions and facts concerning ordered vector spaces we need can be found in \([12]\).) Since \( \mathcal{B}_+ \) is \( \tau_{\mathcal{B}} \)-normal \([13]\), we have \( \tau_{\mathcal{B}} \subseteq \tau_{\mathcal{D}} \). Let \( \tau_0 \) denote the finest locally convex topology on \( \mathcal{D} \) for which every order interval \( \{b, \varphi : b \geq 0\} \) is bounded. Since \( \mathcal{B}_+ \) is \( \tau_{\mathcal{B}} \)-normal, all order intervals are \( \tau_{\mathcal{B}} \)-bounded \([12; p. 216]\) and hence \( \tau_0 \subseteq \tau_{\mathcal{B}} \). In \([1]\) the topology \( \tau_0 \) is called the \( \mathcal{G} \)-topology.

1.3 Let \( \mathcal{A} \) be a *-algebra with unit element denoted by 1. By a *-representation of \( \mathcal{A} \) on \( \mathcal{D} \) we mean a *-homomorphism \( \pi \) of \( \mathcal{A} \) into \( \mathcal{L}^* (\mathcal{D}) \) satisfying \( \pi(1) = 1 \), where 1 is the identity map of \( \mathcal{D} \). We then write \( \pi(\mathcal{D}) \) for \( \mathcal{D}(\mathcal{A}) \) as the graph topology of \( \mathcal{A}[\pi] \) on \( \mathcal{D}(\mathcal{A}) \). Suppose \( \pi \) is a *-representation of \( \mathcal{A} \) on \( \mathcal{D}(\mathcal{A}) \). \( \pi \) is called weakly continuous if for each \( \varphi \in \mathcal{D}(\mathcal{A}) \) the linear functional \( \langle \varphi(\cdot), \varphi \rangle \) is continuous on \( \mathcal{A} \). If \( \pi \) is a continuous mapping of \( \mathcal{A} \) onto \( \mathcal{D}(\mathcal{A}) \), we say \( \pi \) is continuous.

As above, let \( \mathcal{B} \) be an Op*-algebra on \( \mathcal{D} \). Let \( \pi \) be a *-representation of \( \mathcal{B} \) on \( \mathcal{D}(\mathcal{A}) \). We say \( \pi \) is positive if \( \pi(b) \geq 0 \) on \( \mathcal{D} \), i.e., if \( b, b_1 \in \mathcal{B} \) and \( b \geq 0 \) on \( \mathcal{D} \) always implies that \( \pi(b) \geq 0 \) on \( \mathcal{D} \). A linear functional \( f \) on \( \mathcal{B} \) is called positive if \( f(b) \geq 0 \) for all \( b \in \mathcal{B} \).

Lemma 1.3: Each positive *-representation \( \pi \) of the Op*-algebra \( \mathcal{B} \) is a continuous mapping of \( \mathcal{A}[\pi] \) onto \( \pi(\mathcal{D}) \).

Proof: By the polarization formula it is easy to see \([13]\) that the uniform topology \( \tau_{\mathcal{B}} \) on \( \pi(\mathcal{D}) \) is generated by the family of seminorms

\[ p_{\mathcal{B}}(x) := \sup \{|\langle x \varphi, \varphi \rangle| : \varphi, \varphi \in \mathcal{M}\}, \mathcal{M} \subset \mathcal{D}[\mathcal{B}] \text{ bounded}. \]

Fix the bounded set \( \mathcal{M} \). Since the set \( \{x \in \mathcal{B} : p_{\mathcal{B}}(x) \leq 1\} \) is absolutely convex and \( \tau_{\mathcal{B}} \)-saturated, it is a \( \tau_0 \)-neighborhood of zero in \( \mathcal{B} \). This proves the continuity of \( \pi \).

Lemma 1.4: Suppose that \( \mathcal{D}(\mathcal{A}) \) is a Frechet space. Let \( \mathcal{A} \) be a weakly continuous *-representation of \( \mathcal{L}^* (\mathcal{D}) \) on \( \mathcal{D}(\mathcal{A}) \). Then:

(i) \( \pi \) is positive.
(ii) If \( x \in \mathcal{D}(\mathcal{A}) \) is bounded, then \( \pi(x) \) is bounded on \( \mathcal{D}(\mathcal{A}) \) and \( \|\pi(x)\| \leq \|x\| \).
(iii) Suppose \( x_n \in \mathcal{L}^* (\mathcal{D}) \) for \( n \in \mathbb{N} \). If \( \{\|\cdot\|_{\tau_{\mathcal{B}}} : n \in \mathbb{N}\} \) is a generating family for the graph topology \( \tau_{\mathcal{B}} \) on \( \mathcal{D}(\mathcal{A}) \), then \( \{\|\cdot\|_{\tau_{\mathcal{B}}} : n \in \mathbb{N}\} \) is a generating family of seminorms for the graph topology \( \tau_{\mathcal{B}} \) on \( \mathcal{D}(\mathcal{A}) \).

Proof: (i) Suppose \( x \in \mathcal{L}^* (\mathcal{D}) \) and \( \varphi \in \mathcal{D}(\mathcal{A}) \). By \([6; \text{Theorem 6.1}]\) there is a net \( \{q_j\} \) of orthogonal projections \( q_j \in \mathcal{L}^* (\mathcal{D}) \) (that is, \( q_j = q_j^* \) and \( q_j = q_j^2 \)) such that \( q_j \mathcal{H} \subseteq \mathcal{D} \) for all \( j \) and \( x = \tau_{\mathcal{B}} \lim_j q_j xq_j \). Let \( x \) denote the operator \( q_j xq_j \) on the Hilbert
space $g_j \mathcal{H}$. Since $x \in \mathcal{L}^+(\mathcal{D})$, $x_j$ is closed and hence bounded. Let $y_j$ denote the positive square root of the bounded self-adjoint operator $x_j$ on the Hilbert space $g_j \mathcal{H}$. Then $y_j g_j \in \mathcal{L}^+(\mathcal{D})$ and

$$||x(y_j g_j)\varphi||^2 = \langle x(y_j g_j^2 y_j) \varphi, \varphi \rangle = \langle x(y_j x y_j) \varphi, \varphi \rangle \geq 0.$$  

Since $\pi$ is weakly continuous, $\langle x(\varphi, \varphi) \rangle = \lim \langle x(y_j^2 y_j) \varphi, \varphi \rangle \geq 0$. That is, $\pi(x) \geq 0$ on $\mathcal{D}(x)$.

(ii): First let $x \in \mathcal{L}^+(\mathcal{D})$. Since $\pi$ is positive by (i) and $\pi(I) = I$, inf $\{ \lambda \in \mathbb{R} : -\lambda I \leq \pi(x) \leq \lambda I \}$ $\leq$ inf $\{ \lambda \in \mathbb{R} - \lambda I \leq x \leq \lambda I \} = ||x||$, which implies that $\pi(x)$ is bounded and $||\pi(x)|| \leq ||x||$. For arbitrary $x \in \mathcal{L}^+(\mathcal{D})$ the assertion follows from $||\pi(x)||^2 = ||\pi(x^* x)|| \leq ||x^* x|| = ||x||^2$.

(iii): Suppose $x \in \mathcal{L}^+(\mathcal{D})$. By assumption, there are a positive constant $C$ and a natural number $s$ such that

$$||x \varphi||^2 \leq C \left( ||\varphi||^2 + \sum_{n=1}^s ||x_n \varphi||^2 \right) \text{ for all } \varphi \in \mathcal{D}.$$  

Therefore,

$$y := C \left( I + \sum_{n=1}^s x_n + x_n^* \right) - x^* x \in \mathcal{L}^+(\mathcal{D}) \text{ and } \pi(y) \geq 0 \text{ on } \mathcal{D}(\pi).$$  

The latter implies that

$$||\pi(x) \varphi||^2 \leq C \left( ||\varphi||^2 + \sum_{n=1}^s ||\pi(x_n) \varphi||^2 \right) \text{ for all } \varphi \in \mathcal{D}(\pi).$$  

1.4 From now on we assume that $\mathcal{D}(\pi)$ is a Frechet space and that the underlying Hilbert space $\mathcal{H}$ is separable. To simplify the notation we adopt the following notational convention: We shall denote an operator whose domain contains $\mathcal{D}$ and its restriction to $\mathcal{D}$ by the same symbol. This will be mainly used in Section 3. Let $\mathcal{F}(\mathcal{D})$ denote the finite rank operators contained in $\mathcal{L}^+(\mathcal{D})$. For a linear subspace $\mathcal{D}_1$ of $\mathcal{H}$, let $\mathcal{F}(\mathcal{H}, \mathcal{D}_1)$ be the set of all bounded finite-ranked operators on $\mathcal{H}$ mapping $\mathcal{H}$ into $\mathcal{D}_1$. Moreover, we let $\mathcal{B}_{\mathcal{D}_1} := \{ \varphi \in \mathcal{D}_1 : ||\varphi|| \leq 1 \}$.

2. Generalized Calkin representations of $\mathcal{A}(\mathcal{D})$

2.1 Suppose that $\mathcal{U}$ is an ultrafilter on $\mathbb{N}$. Let $\mathcal{D}_\mathcal{U}$ denote the set of all bounded sequences $(q_n : n \in \mathbb{N}) = (q_n)$ in the locally convex space $\mathcal{D}(\pi)$ satisfying $0 = w$-$\lim \varphi_n$. Let $\mathcal{H}_\mathcal{U}$ be the set of all bounded sequences $(q_n)$ in $\mathcal{H}$ with $0 = w$-$\lim \varphi_n$. $\mathcal{D}_\mathcal{U}$ and $\mathcal{H}_\mathcal{U}$ are vector spaces in the obvious way. Let $\mathcal{N}_\mathcal{U}$ be the set of all $(q_n) \in \mathcal{H}_\mathcal{U}$ with $w$-$\lim \varphi_n = 0$. We define a scalar product on the quotient space $\mathcal{D}_\mathcal{U} := \mathcal{D}_\mathcal{U}/\mathcal{D}_\mathcal{U} \cap \mathcal{N}_\mathcal{U}$ by $\langle (q_n), (p_n) \rangle := \lim u (q_n, p_n)$. In the same way, the quotient space $\mathcal{H}_\mathcal{U} := \mathcal{H}_\mathcal{U}/\mathcal{N}_\mathcal{U}$ becomes a Hilbert space (see e.g. [11: Section 2]). By an abuse of notation we denote the elements of the quotient spaces again by $(q_n)$. Since $\mathcal{D} \subseteq \mathcal{H}$, $\mathcal{D}_\mathcal{U}$ can be considered as a linear subspace of $\mathcal{H}_\mathcal{U}$.

Define $\sigma_\mathcal{U}(x)(q_n) := (x q_n)$ for $(q_n) \in \mathcal{D}_\mathcal{U}$, and $x \in \mathcal{L}^+(\mathcal{D})$. Each operator $x \in \mathcal{L}^+(\mathcal{D})$ maps a bounded sequence in $\mathcal{D}(\pi)$ into a bounded sequence. By Lemma 1.1, (i) and (ii), $x \mathcal{N}_\mathcal{U} \subseteq \mathcal{N}_\mathcal{U}$ and $x \mathcal{D}_\mathcal{U} \subseteq \mathcal{D}_\mathcal{U}$. Therefore, the above definition makes sense and defines a linear operator $\sigma_\mathcal{U}(x)$ which maps $\mathcal{D}_\mathcal{U}$ into $\mathcal{D}_\mathcal{U}$. It is straightforward to check that the mapping $x \rightarrow \sigma_\mathcal{U}(x)$ is a positive *-representation of $\mathcal{L}^+(\mathcal{D})$ on $\mathcal{D}_\mathcal{U}$.

Let $\mathcal{J}$ denote the quotient map of $\mathcal{L}^+(\mathcal{D})$ onto $\mathcal{A}(\mathcal{D}) = \mathcal{L}^+(\mathcal{D})/\mathcal{B}(\mathcal{D})$. Suppose $x \in \mathcal{E}(\mathcal{D})$ and $(q_n) \in \mathcal{D}_\mathcal{U}$. Then the set $\{x q_n\}$ is relatively compact in $\mathcal{D}(\pi)$ and hence $\lim u ||x q_n|| = 0$ by Lemma 1.1 (iii). This shows that $\mathcal{E}(\mathcal{D}) \subseteq \ker \sigma_\mathcal{U}$. Therefore, $\pi_\mathcal{U}(\mathcal{J}(x)) := \sigma_\mathcal{U}(x)$ for $x \in \mathcal{L}^+(\mathcal{D})$ defines a *-representation of the *-algebra $\mathcal{A}(\mathcal{D})$ on $\mathcal{D}_\mathcal{U} = \mathcal{D}(\pi_\mathcal{U})$. 


2.2 Recall that an ultrafilter on \( \mathbb{N} \) is said to be \textit{free} if the intersection of all its members is empty.

**Theorem 2.1:** Suppose that \( \mathcal{U} \) is a free ultrafilter on \( \mathbb{N} \). Then \( \pi_\mathcal{U} \) is a faithful \( * \)-representation of the Calkin algebra \( \mathcal{A}(\mathcal{D}) \). Its inverse \( \pi_\mathcal{U}^{-1} \) is a continuous mapping of \( \pi_\mathcal{U}(\mathcal{A}(\mathcal{D})) \) onto \( \mathcal{A}(\mathcal{D}) \).

**Proof:** The quotient topology \( \hat{\tau} \) on \( \mathcal{A}(\mathcal{D}) \) is generated by the seminorms

\[
\rho_{\mathcal{M}}(\{x\}) := \inf \{p_{\mathcal{M}}(x + c) : c \in \mathcal{C}(\mathcal{D})\}, \quad \mathcal{M} \subset \mathcal{D}[1] \text{ bounded.}
\]

Fix such a set \( \mathcal{M} \). Suppose for a moment we have shown that there exists a bounded subset \( \mathcal{R} \) (depending on \( \mathcal{M} \)) of \( \mathcal{D}[\pi_\mathcal{U}] \) such that

\[
\rho_{\mathcal{M}}(\{x\}) \leq \rho_{\mathcal{M}}(\pi_\mathcal{U}(x)) \quad \text{for all } x \in \mathcal{L}^+(\mathcal{D}).
\]

The latter means that

\[
\rho_{\mathcal{M}}(a) \leq \rho_{\mathcal{M}}(\pi_\mathcal{U}(a)) \quad \text{for all } a \in \mathcal{A}(\mathcal{D}).
\]

Since \( \mathcal{C}(\mathcal{D}) \) is \( \tau_\mathcal{D} \)-closed in \( \mathcal{L}^+(\mathcal{D}) \) and hence \( \hat{\tau} \) is Hausdorff, it follows from (2) that \( \ker \pi_\mathcal{U} = \{0\} \), that is, \( \pi_\mathcal{U} \) is faithful. Moreover, (2) proves the continuity of \( \pi_\mathcal{U}^{-1} \), and the proof would be complete.

It remains to show that there is a bounded set \( \mathcal{R} \) in \( \mathcal{D}[\pi_\mathcal{U}] \) such that (1) is satisfied. According to [6: Theorem 4.1] there is a bounded self-adjoint operator \( z \) on \( \mathcal{K} \) such that \( \ker z = \{0\} \), \( z\mathcal{K} \subseteq \mathcal{D} \) and \( \mathcal{M} \subseteq z\mathcal{B}_{\mathcal{K}} \). If \( x \in \mathcal{L}^+(\mathcal{D}) \), then \( zx \) is a closed operator defined on \( \mathcal{K} \) and hence bounded. Now fix an operator \( z \in \mathcal{L}^+(\mathcal{D}) \). Since \( \mathcal{F}(\mathcal{D}) \) is \( \tau_\mathcal{D} \)-dense in \( \mathcal{C}(\mathcal{D}) \), we obtain

\[
\rho_{\mathcal{M}}(\{x\}) \leq \inf_{c \in \mathcal{F}(\mathcal{D})} \rho_{\mathcal{M}}(x + c) 
= \inf_{c \in \mathcal{F}(\mathcal{D})} \sup_{\varphi \in \mathcal{K}} |(x + c) \varphi| = \inf_{c \in \mathcal{F}(\mathcal{D})} \|z(x + c) z\|. 
\]

Since \( \ker z = \{0\} \), we have \( \{cz : c \in \mathcal{F}(\mathcal{D})\} = \mathcal{F}(\mathcal{D}) \). Moreover, \( \{cz : c \in \mathcal{F}(\mathcal{D})\} \) is norm dense in \( \mathcal{F}(\mathcal{K}) \). Using these facts, we get

\[
\rho_{\mathcal{M}}(\{x\}) \leq \inf_{c \in \mathcal{F}(\mathcal{K})} \|zxz + cz\| 
= \inf_{c \in \mathcal{F}(\mathcal{K})} \|zxz + c\| = \inf_{c \in \mathcal{F}(\mathcal{K})} \|zxz + c\|. 
\]

On the other hand, let \( \omega_{\mathcal{H}} \) denote the \( * \)-representation of \( \mathcal{B}(\mathcal{K}) \) on \( \mathcal{K}_{\mathcal{U}} \) defined by

\[
\omega_{\mathcal{H}}(y) := (y \varphi_n) \quad \text{for } (\varphi_n) \in \mathcal{K}_{\mathcal{U}} \text{ and } y \in \mathcal{B}(\mathcal{K}).
\]

Since \( \omega_{\mathcal{H}} \) obviously annihilates \( \mathcal{C}(\mathcal{K}) \); \( \omega_{\mathcal{H}} \) defines a \( * \)-representation of the C*-algebra \( \mathcal{A}(\mathcal{K}) \) on \( \mathcal{K}_{\mathcal{U}} \) (see [11: Section 2]). Since \( \mathcal{U} \) is assumed to be free and \( \mathcal{A}(\mathcal{K}) \) is simple, this \( * \)-representation of the C*-algebra \( \mathcal{A}(\mathcal{K}) \) is faithful and hence isometric. Since \( zxz \in \mathcal{B}(\mathcal{K}) \), this yields

\[
\|\omega_{\mathcal{H}}(zxz)\| = \inf \|zxz + c\| : c \in \mathcal{C}(\mathcal{K}) \].
\]

By (3), we obtain

\[
\rho_{\mathcal{M}}(\{x\}) \leq \|\omega_{\mathcal{H}}(zxz)\| \quad \text{for all } x \in \mathcal{L}^+(\mathcal{D}).
\]

Now define

\[
\mathcal{R} := \omega_{\mathcal{H}}(z) \mathcal{K}_{\mathcal{U}} = \{(z \varphi_n) : (\varphi_n) \in \mathcal{K}_{\mathcal{U}} \text{ and } \|(\varphi_n)\|_{\mathcal{K}_{\mathcal{U}}} \leq 1\}.
\]

If \( (\varphi_n) \in \mathcal{K}_{\mathcal{U}} \) and if \( x \in \mathcal{L}^+(\mathcal{D}) \), then \( zx \) is bounded on \( \mathcal{K} \) and thus

\[
\sup_{n \in \mathbb{N}} \|z \varphi_n\| \leq \|zx\| \sup_{n \in \mathbb{N}} \|\varphi_n\| < \infty.
\]
This implies $\mathcal{R} \subseteq D_{\mathcal{U}}$. From

$$\|\omega_{\mathcal{U}}(x)(z\varphi_n)\| = \|z\varphi_n\| = \lim_{n \to \infty} \|z\varphi_n\| = \|z\varphi\|$$

we see that $\mathcal{R}$ is bounded in $D_{\mathcal{U}}[\mathcal{U}]$.

Finally, by (4), if $x \in L^+(\mathcal{D})$, then

$$\hat{p}_{\mathcal{U}}(i(x)) \leq \|\omega_{\mathcal{U}}(x)z\| = \sup_{\varphi, \psi \in \mathcal{K}} |\langle \omega_{\mathcal{U}}(x) \varphi, \omega_{\mathcal{U}}(x) \psi \rangle|$$

$$= \sup_{\varphi_n, \psi_n \in \mathcal{K}} |\langle z\varphi_n, \psi_n \rangle|$$

$$= \sup_{\varphi_n, \psi_n \in \mathcal{K}} |\langle \omega_{\mathcal{U}}(x)z\varphi_n, \psi_n \rangle| = p_{\mathcal{U}}(\omega_{\mathcal{U}}(x)),$$

which proves (1). The proof of Theorem 2.1 is complete.

2.3 From Theorem 2.1 and Lemma 1.4 we obtain

**Theorem 2.2**: Suppose that $\tau_n = \tau_\mathcal{U}$ on $L^+(\mathcal{D})$. Let $\mathcal{U}$ be a free ultrafilter on $\mathcal{N}$. Then, $\pi_{\mathcal{U}}$ is a faithful $*$-representation of $A(\mathcal{D})$ and a homeomorphism of $A(\mathcal{D})[\mathcal{U}]$ onto $\pi_{\mathcal{U}}(A(\mathcal{D}))[\mathcal{U}]$.

1. In general the domain $D_{\mathcal{U}}$ is not dense in $\mathcal{K}_{\mathcal{U}}$. 2. If the domain is of the form $\mathcal{D} = \cap \{D(T): n \in \mathcal{N}\}$ for some self-adjoint operator $T$ on $\mathcal{K}$, then $\tau_n = \tau_\mathcal{U}$ on $L^+(\mathcal{D})$ (see also Section 3).

3. Existence of continuous faithful $*$-representations of $A(\mathcal{D})[\mathcal{U}]$

3.1 We first recall the setup of [14: Section 4]. However, the notation is slightly changed.

Suppose $a$ is a (bounded or unbounded) self-adjoint operator on the Hilbert space $\mathcal{K}$ with spectral decomposition $a = \int \lambda \lambda \lambda d\lambda(a)$. Let $(f_k(t): k \in \mathcal{N})$ be a sequence of real measurable functions on the spectrum $\sigma(a)$ of $a$. All measure-theoretic notions refer to the spectral measure of $a$. We assume that

$$f_1(t) = 1 \text{ and } f_k(t) \leq f_{k+1}(t) \text{ a.e. on } \sigma(a) \text{ for } k \in \mathcal{N}. \quad \text{(1)}$$

Set $a_k = f_k(a)$ and $\mathcal{D} = \cap \{D(a_k): k \in \mathcal{N}\}$. Then, by (1), the operators $a_k$ (more precisely, their restrictions to $\mathcal{D}$) are in $L^+(\mathcal{D})$ and the graph topology $l$ on $\mathcal{D}$ is generated by the seminorms $\|a_k, k \in \mathcal{N}$.

In our next theorem the following condition $(\ast)$ plays an important role:

For each sequence $\gamma = (\gamma_k: k \in \mathcal{N})$ of positive numbers $\gamma_k$ there is a $k = k_0 \in \mathcal{N}$ such that all functions $f_n$, $n \in \mathcal{N}$, are bounded on $R_k$, where

$$R_k := \{ t \in \sigma(a): f_1(t) \leq \gamma_1, \ldots, f_n(t) \leq \gamma_n \} \text{ for } n \in \mathcal{N}.$$

The following assertions are equivalent:

(i) Condition $(\ast)$ is fulfilled.

(ii) $\tau_0 = \tau_\mathcal{D}$ on $L^+(\mathcal{D})$.

(iii) $\tau_n = \tau_\mathcal{D}$ on $L^+(\mathcal{D})$.

(iv) Each positive linear functional on $L^+(\mathcal{D})$ is $\tau_\mathcal{D}$-continuous.
This is essentially [14: Theorem 4.1]. The equivalence of (i), (ii) and (iv) has been stated therein. Since $r_n \geq r_n \geq r_n$, (ii) $\implies$ (iii). Since each positive linear functional is $r_n$-continuous, we have (iii) $\implies$ (iv).

3.2 The following theorem may be considered as a supplement to [14: Theorem 4.1]. Among other things it shows that if $r_n \equiv r_n$, then there is no continuous faithful $*$-representation of $A(D)$ [1]. In particular, the $*$-representations $\pi_\gamma$ occurring in Theorem 2.1 are not continuous.

Theorem 3.1: Let $D$ be as above. Then (i) is equivalent to each of the following conditions:

(v) There exists a faithful $*$-representation $\pi$ of $A(D)$ which is a homeomorphism of $A(D)$ [\ref{1}] onto $\pi(A(D)) [\tau_D, n]$.

(v) There exists a continuous faithful $*$-representation of $A(D)$ [\ref{1}].

(vi) Each positive $*$-representation of $L^+(D) [\tau_D]$ is continuous.

(vi)' Each weakly continuous positive $*$-representation of $L^+(D) [\tau_D]$ is continuous.

Proof: Theorem 2.2 shows that (iii) $\implies$ (v). (iii) $\implies$ (vi) follows from Lemma 1.3. Since (v) $\Rightarrow$ (v)' and (vi) $\Rightarrow$ (vi)' are trivially fulfilled, it suffices to prove that (v)' $\Rightarrow$ (i) and (vi)' $\Rightarrow$ (i). Both proofs will be indirect (see e.g. the argument in [14: p. 366]).

(v)' $\Rightarrow$ (i): Suppose that $\pi$ is a continuous faithful $*$-representation of $A(D)$ [\ref{1}]. Then, $\varrho(x) := \pi(\{x\})$, $x \in L^+(D)$, defines a continuous $*$-representation of $L^+(D) [\tau_D]$. To prove (i), we assume the contrary, that is, condition (*) is not satisfied. Then there are a positive sequence $\gamma = (\gamma_k)$ and a sequence $(i_k)$ of natural numbers such that $f_{i_k}$ is not essentially bounded on the set $\mathcal{S}_{k,n}$ for each $k \in N$. There is no loss of generality if we assume that $\gamma_{k+1} > \gamma_k \geq k$ and $i_k = k$ for all $k \in N$. Then there are measurable subsets $\mathcal{S}_{k,n}, n \in N$, of $\mathcal{S}_k$ of non-zero measure such that $f_{i+1}(t) \equiv \gamma_n$ a.e. on $\mathcal{S}_{k,n}$ for all $k, n \in N$. Let $\varphi_{k,n}$ be a unit vector from $e(\mathcal{S}_{k,n}, D)$.

Let $A$ denote the family of all sequences $\delta = (\delta_k)$ of natural numbers $\delta_k$ satisfying $\delta_k \leq i + 2$ for $k \in N$. Fix a $\delta \in A$. We first show that for $r \in N$ and $\varphi \in D(\varphi)$

$$\|\varrho(a_r) \varphi \left( \bigcup_{k \geq r+1} \mathcal{S}_{k,n} \right) \varphi \| \leq r \|\varphi\| \quad (2)$$

and

$$\|\varrho(a_r+1) \varphi (e(\mathcal{S}_{r,n}, A)) \varphi \| \geq r_n \|\varrho(e(\mathcal{S}_{r,n}, A)) \varphi\|. \quad (3)$$

For let $\chi$ denote the characteristic function of the set $\bigcup \{ \mathcal{S}_{k,n}, k \geq r+1 \}$. By construction, $f_t(t) \chi(t) \leq r_\gamma$ a.e. on $\sigma(a)$. Define a function $g$ on $\sigma(a)$ by $g := (r_\gamma^2 - f_t^2 \chi)^{1/2}$. Obviously, $g(a) \in L^+(D)$. For $\varphi \in D(\varphi)$, $\langle \varrho(g(a))^2 \varphi, \varphi \rangle = \|\varrho(g(a)) \varphi\|^2 \geq 0$ and hence

$$\|\varphi\|^2 \gamma_n^2 = \langle \varrho(r_\gamma^2 I) \varphi, \varphi \rangle \geq \langle \varrho(f_t^2 \chi(a)) \varphi, \varphi \rangle = \|\varrho(a_r) \left( \bigcup_{k \geq r+1} \mathcal{S}_{k,n} \right) \varphi\|^2.$$

(2) follows by the same argument.

Let $q_\delta$ be the orthogonal projection onto the closure of $D_\delta := 1.h. \{ \varphi_{k,n} : k \in N \}$. Next we prove that $q_\delta D \subseteq D$. For let $r \in N$. Each $\varphi \in D_\delta$ can be written as a finite sum

$$\sum_{k=1}^s \lambda_k \varphi_{k,n} \varphi_{k,n}, \quad \lambda_1, \ldots, \lambda_s \in C \text{ and } s \in N, s \geq r.$$

Suppose, $k, n \in N, n > k$. Since $f_{i+1}(t) \equiv \gamma_n \geq \gamma_{k+2} > \gamma_{k+1}$ a.e. on $\mathcal{S}_{k,n}$ and $f_{i+1}(t) \leq \gamma_{k+1}$ on $\mathcal{S}_{n,n}$, it follows that $\mathcal{S}_{k,n} \cap \mathcal{S}_{n,n}$ has measure zero. Therefore, $\varphi_{k,n}$
Using the latter, we obtain
\[
\|a_r \varphi\|^2 = \sum_{k=1}^{r} |\lambda_k|^2 |a_r \varphi_{k, \delta_k}|^2 + \sum_{k=r+1}^{s} |\lambda_k|^2 |a_r \varphi_{k, \delta_k}|^2
\]
\[
\leq \max (\|a_r \varphi_{1, \delta_1}\|^2, \ldots, \|a_r \varphi_{r, \delta_r}\|^2, \gamma_r) \sum_{k=1}^{s} |\lambda_k|^2 = \max (\ldots) \| \varphi \|^2.
\]

This implies \( q_\delta \mathcal{H} \subseteq \mathcal{D}(a_r) \). Since \( \mathcal{D} = \cap \{ \mathcal{D}(a_r) : r \in \mathbb{N} \} \) by definition, this shows that \( q_\delta \mathcal{H} \subseteq \mathcal{D} \).

We define \( \mathcal{M} := \bigcup \{ q(\varphi) : \delta \in \Delta \} \), where \( \mathcal{B} := \mathcal{B}_{2(\varphi)} := \{ \varphi \in \mathcal{D}(\varphi) : \| \varphi \| \leq 1 \} \).

We prove that \( \mathcal{M} \) is bounded in \( \mathcal{D}(\varphi) \) [\( \ell_\varphi \)]. For take \( r \in \mathbb{N} \) and \( \delta \in \Delta \). Let \( c_{r, \delta} \) denote the orthogonal projection on \( \mathcal{H}^* \) with range \( \varphi_{1, \delta}, \ldots, \varphi_{r, \delta} \). Since obviously \( a_r c_{r, \delta} \in \mathcal{C}(\mathcal{D}) \), we have \( a_r c_{r, \delta} \in \text{ker } q_\delta \).

From
\[
q_\delta - c_{r, \delta} = e \left( \bigcup_{k \geq r+1} \mathcal{S}_{k, \delta_k} \right) (q_\delta - e_{r, \delta})
\]
and (2) we therefore obtain
\[
\| \varphi(a_r) q(\varphi) \| = \| \varphi(a_r) q(\varphi) - c_{r, \delta} \varphi \|
\]
\[
= \| \varphi(a_r) \left( e \left( \bigcup_{k \geq r+1} \mathcal{S}_{k, \delta_k} \right) \right) (q_\delta - c_{r, \delta}) \varphi \|
\]
\[
\leq \gamma_r \| q(\varphi) - c_{r, \delta} \varphi \| \leq \gamma_r \text{ for each } \varphi \in \mathcal{B}.
\]

By Lemma 1.4 (iii) the graph topology \( t_\varphi \) on \( \mathcal{D}(\varphi) \) is generated by the seminorms \( \| \cdot \|_{\varphi(a_r)} \), \( r \in \mathbb{N} \). Therefore, the preceding proof shows that \( \mathcal{M} \) is bounded with respect to the graph topology \( t_\varphi \).

Since the \( * \)-representation \( \varphi \) of \( \mathcal{L}^+(\mathcal{D})[\tau_\varphi] \) is continuous, there exists a bounded subset \( \mathcal{M} \) of \( \mathcal{D}(\mathcal{I}) \) such that
\[
\mathcal{P}(\varphi(x)) \leq \mathcal{P}(\mathcal{M}) \text{ for all } x \in \mathcal{L}^+(\mathcal{D}). \tag{4}
\]

Since \( \mathcal{M} \) is \( l \)-bounded, \( C_r := \sup \{ \| \varphi(a_r) \| : \varphi \in \mathcal{M} \} < \infty \) for each \( r \in \mathbb{N} \). We choose natural numbers \( \delta_k \) such that \( \delta_k \geq k + 2 \) and \( \gamma_{\delta_k} \geq C_{k+1} 2^k \) for \( k \in \mathbb{N} \). This is possible because \( \gamma_n \geq n \) for \( n \in \mathbb{N} \). Define an operator \( x \) by \( x := e(\bigcup_{k \geq r+1} \mathcal{S}_{k, \delta_k}) \). Clearly, \( x \in \mathcal{L}^+(\mathcal{D}) \). Our aim is to show that for this operator \( x \) (4) is not true. By (3), we have
\[
\gamma_r \| \varphi(e(\mathcal{S}_{r, \delta_r})) \| \leq \| \varphi(a_{r+1}) (e(\mathcal{S}_{r+1, \delta_{r+1}})) \| \leq C_{r+1} \| \varphi(\mathcal{S}_{r, \delta_r}) \| \leq C_{r+1} \gamma_{\delta_r}^{-1} \text{ for } r \in \mathbb{N} \text{ and } \varphi \in \mathcal{M}.
\]

That is,
\[
\sup_{\varphi \in \mathcal{M}} \| \varphi(e(\mathcal{S}_{r, \delta_r})) \| \leq C_{r+1} \gamma_{\delta_r}^{-1} \text{ for } r \in \mathbb{N}.
\]

Using this inequality, we obtain
\[
\mathcal{P}(\mathcal{M}) = \sup_{\varphi \in \mathcal{M}} \| \varphi(e(\bigcup_k \mathcal{S}_{k, \delta_k})) \| \leq \sup_{\varphi \in \mathcal{M}} \sum_{k=1}^{\infty} \| \varphi(e(\mathcal{S}_{k, \delta_k})) \| \leq \sum_{k=1}^{\infty} C_{k+1} 2^k \leq \sum_{k=1}^{\infty} 2^{-2k} < 1. \tag{5}
\]

Since \( a_r q_\delta \) is \( a \)-bounded operator on \( \mathcal{H} \) for \( r \in \mathbb{N} \) as shown above, the sequence \( (q_\delta \varphi_{k, \delta_k} : k \in \mathbb{N}) \) is bounded in \( \mathcal{D}(\mathcal{I}) \). But the set \( \{ q_\delta \varphi_{k, \delta_k} \} = \{ \varphi_{k, \delta_k} \} \) is certainly not relatively compact-in \( \mathcal{D}(\mathcal{I}) \), since \( (\varphi_{k, \delta_k}) \) is an orthonormal sequence in \( \mathcal{H} \). This proves that
Comparing (5) and (6) with (4), we obtain the desired contradiction.

(vi)  → (i): This will be similar as the preceding proof. Again we assume that condition (⋆) is not fulfilled. We keep the notation introduced above. Let \( \mathcal{U} \) be an arbitrary free ultrafilter on \( \mathbb{N} \). As already mentioned in Section 2, \( \varphi_\mathcal{U} \) is a positive *-representation of \( \mathcal{L}^+(\mathcal{D}) \). It suffices to show that \( \varphi_\mathcal{U} \) is weakly continuous, but not continuous. Let \( \varphi = (\varphi_n) \in \mathcal{D}_\mathcal{U} \). By definition of \( \mathcal{D}_\mathcal{U} \), the set \( \mathcal{M} := \{\varphi_n\} \) is bounded in \( \mathcal{D}_\mathcal{U} \): If \( x \in \mathcal{L}^+(\mathcal{D}) \), then

\[
|x(\varphi_\mathcal{U}(x) \varphi, \varphi)| = \lim_{n \in \mathbb{N}} |x(\varphi_n, \varphi_n)| \leq \sup_{n \in \mathbb{N}} |x(\varphi_n, \varphi_n)| \leq p_M(x).
\]

That is, \( \varphi_\mathcal{U} \) is weakly continuous. From Theorem 2.1 we know that \( \ker \varphi_\mathcal{U} = \mathcal{C}(\mathcal{D}) \). Therefore, the preceding proof in the case \( \varphi = \varphi_\mathcal{U} \) shows that \( \varphi_\mathcal{U} \) is not \( \tau_\mathcal{D} \)-continuous.

Results similar to those proved in this paper are true for the topologies \( \tau(\mathcal{D}) \) and \( \tau^0 \) (see also [14]).

Addendum: After completing the manuscript the author has learned that in the case \( \mathcal{D} = n [\mathcal{D}(\mathbb{F}^n) : n \in \mathbb{N}], \) \( T \), a self-adjoint operator, the existence of a topological realization of \( \mathcal{A}(\mathcal{D}) \) [1] has been independently obtained by F. LÖFFLER and W. TIMMERMANN in “The Calkin representation for a certain class of algebras of unbounded operators”, Dubna-Preprint E 5-84-807, 1984.

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Manuskripteingang: 16. 01. 1985

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