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Comparison Theorems for Conjugate Points of Sturm-Liouville Differential Equations

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Es werden Vergleichssätze für Lösungen Sturm-Liouvillescher Differentialgleichungen bezüglich ihrer Nullstellen auf einem endlichen Intervall bewiesen.

Доказываются теоремы сравнения решений дифференциальных уравнений Штурма-Лиувилля относительно их нулей на конечном интервале.

Comparison theorems for solutions of Sturm-Liouville equations are proved concerning their zeros on a finite interval.

Consider the differential equations

$$-(P(x)\,u')' + Q(x)\,u = 0, \qquad -a \leqq x \leqq a, \tag{1}$$

and

$$-(p(x)\,u')' + q(x)\,u = 0, \qquad -a \leqq x \leqq a, \tag{2}$$

$a \in \mathbf{R}$, where the coefficients are real-valued continuous functions and, additionally, p, P are positive piecewise and continuously differentiable. The points $x_1, x_2 \in [-a, a]$ are said to be *conjugate points* for equation (1) or (2) if there exists a nontrivial solution u of the corresponding equation with $u(x_1) = 0 = u(x_2)$. We prove the following

Theorem 1: *Assume the following*:

(i) *P and Q are even functions on* $[-a, a]$,

(ii) $p \leqq P$ *on* $[-a, a]$,

(iii) $\int_{[-\sigma,\sigma]} q\,dx \leqq \int_{[-\sigma,\sigma]\setminus(-\tau,\tau)} Q\,dx$ *for all σ, τ with* $0 < \tau < \sigma < a$.

If there exists a solution u of (1) *with $u(-a) = 0 = u(a)$ and $u(x) > 0$, $-a < x < a$, then there exists a pair of conjugate points for equation* (2) *on* $[-a, a]$.

Proof: The solution u of (1) belongs to $C^1[-a, a]$ (cp. [7: p. 25], for instance). By using the Sturm comparison theorem (cp. [7]) it is easily seen that such u is an even function. By means of u we construct a test function for the quadratic form of equation (2). Let $\alpha \in [0, a)$ be a point with $u'(\alpha) = 0$ and $u' \leqq 0$ on $[\alpha, a]$. (Note that there can exist several points α with this property.) It is easily seen that the function

$$w(x) = \begin{cases} u(\alpha), & x \in (-\alpha, \alpha) \\ u(x), & x \in [-a, a] \setminus (-\alpha, \alpha) \end{cases}$$

belongs to the Sobolev spaces $\mathring{W}_2^1(-a, a)$ and $W_2^2(-a, a)$[1]. The sequilinear forms of the equations (1) and (2) defined on $\mathring{W}_2^1(-a, a)$ are closed. In the following, by means of w, the quadratic form of (2) will be estimated. Thus, by using Fubini's theorem, the function

$$h(y) := \sup \{x \in [0, a] \mid w^2(x) \geqq y\}, \qquad 0 \leqq y \leqq u^2(\alpha),$$

and the hypotheses (ii) and (iii), we obtain

$$\begin{aligned}
\int_{-a}^{a} [p(w')^2 + qw^2]\, dx &= \int_{-a}^{a} [(p - P)(w')^2 + (q - Q)\, w^2]\, dx + \int_{-a}^{a} [P(w')^2 + Qw^2]\, dx \\
&\leqq \int_{-a}^{a} (q - Q)\, w^2\, dx + Pu'u|_{-a}^{-\alpha} + \int_{-a}^{-\alpha} [-(Pu')' + Qu]\, u\, dx \\
&\quad + u^2(\alpha) \int_{-\alpha}^{\alpha} Q\, dx + Pu'u|_{\alpha}^{a} + \int_{\alpha}^{a} [-(Pu')' + Qu]\, u\, dx \\
&= \int_{-a}^{a} (q - Q)\, w^2\, dx + u^2(\alpha) \int_{-\alpha}^{\alpha} Q\, dx \\
&= \int_{-a}^{a} \int_{0}^{w^2(x)} [q(x) - Q(x)]\, dy\, dx + u^2(\alpha) \int_{-\alpha}^{\alpha} Q(x)\, dx \\
&= \int_{0}^{u^2(\alpha)} \left(\int_{-h(y)}^{h(y)} [q(x) - Q(x)]\, dx + \int_{-\alpha}^{\alpha} Q(x)\, dx \right) dy \\
&= \int_{0}^{u^2(\alpha)} \left(\int_{[-h(y), h(y)]} q(x)\, dx - \int_{[-h(y), h(y)] \setminus (-\alpha, \alpha)} Q(x)\, dx \right) dy \leqq 0.
\end{aligned}$$

Therefore,

$$\inf \left\{ \int_{-a}^{a} (p\, |\varphi'|^2 + q\, |\varphi|^2)\, dx : \varphi \in C_0^\infty, \|\varphi\| = 1 \right\}$$

is either less or equal zero.

In the first case it follows that there exists a nontrivial solution v of (2) having at least two zeros on $(-a, a)$ (cp. [5]). In the second case the (normalized) function w is realizing the infimum, and, consequently, it is a solution of (2). This can be proved as follows. Let A be the Friedrichs extension of the operator

$$A_0 \varphi = -(p\varphi')' + q\varphi, \qquad \varphi \in C_0^\infty(-a, a).$$

Because of $(A_0\varphi, \varphi) \geqq 0$[2], $\varphi \in C_0^\infty(-a, a)$, the operator $A^{1/2}$, $D(A^{1/2}) = \mathring{W}_2^1(-a, a)$, can be defined (cp. [3]). Then, it follows from

$$0 = \int_{-a}^{a} [p(w')^2 + qw^2]\, dx = \|A^{1/2} w\|^2$$

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[1]) $\mathring{W}_2^1(-a, a)$ is the completion of $C_0^\infty(-a, a)$ in the W_2^1-norm.
[2]) $(\cdot, \cdot)$ and $\|\cdot\|$ denote inner product and norm in the Hilbert space $L_2(-a, a)$, respectively.

that $A^{1/2}w = 0$ and $0 = A^{1/2}(A^{1/2}w) = Aw$. By the first representation theorem (see [3: p. 322]) we have

$$\int_{-a}^{a} (pw'\bar{v}' + qw\bar{v})\, dx = (Aw, v) = 0 \quad \text{for every } v \in \mathring{W}_2^1(-a, a),$$

and by integration by parts it follows that

$$\int_{-a}^{a} [-(pw')' + qw]\, \bar{v}\, dx = 0 \quad \text{for all } v \in \mathring{W}_2^1(-a, a).$$

Hence, we obtain $-(pw')' + qw = 0$. The solution w of (2) has zeros at the end points of the interval $[-a, a]$.

We state that in both cases there exists a pair of conjugate points for equation (2) on $[-a, a]$. This completes the proof of Theorem 1 ∎

If a point $\alpha \in [0, a)$ with the named properties is known, Theorem 1 can be modified as follows.

Corollary 1: *Assume the following:*
(i) *P and Q are even functions on $[-a, a]$,*
(ii) *$p \leqq P$ on $[-a, a]$.*
Assume that there exists a solution u of (1) with $u(-a) = 0 = u(a)$; $u(x) > 0$, $-a < x < a$; $u'(\alpha) = 0$ and $u' \leqq 0$ on $[\alpha, a]$. If, additionally,

$$\int_{[-\sigma,\sigma]} q\, dx \leqq \int_{[-\sigma,\sigma]\setminus(-\alpha,\alpha)} Q\, dx \quad \textit{for all } \sigma \in (\alpha, a),$$

then there exists a pair of conjugate points for equation (2) on $[-a, a]$.

Proof: Compare the proof of Theorem 1 ∎

In the special case that

$$\int_0^{\sigma} Q\, dx \leqq 0 \quad \text{when } \sigma \in (0, a) \tag{3}$$

the solution u is monotone decreasing on $[0, a]$. This can be seen as follows. By setting $v = -Pu'u^{-1}$ it follows from $uv' + u'v = (uv)' = -(Pu')' = -Qu$ that v satisfies the Riccati differential equation $v' = -Q + P^{-1}v^2$. Because of $u'(0) = 0$ we have $v(0) = 0$ and $v' = -Q + P^{-1}v^2$ implies

$$v(x) = -\int_0^x Q\, dt + \int_0^x P^{-1}v^2\, dt, \qquad 0 \leqq x \leqq a. \tag{4}$$

In view of (3) and (4) we obtain $v \geqq 0$ and, consequently, $u' \leqq$ on $[0, a]$. Therefore, by assuming (3), the point α can be chosen equal to zero. Thus, we obtain the following result from Corollary 1.

Corollary 2: *Assume the following:*
(i) *P and Q are even functions on $[-a, a]$,*
(ii) *$p \leqq P$ on $[-a, a]$,*
(iii) $\int_{-\sigma}^{\sigma} q\, dx \leqq \int_{-5}^{\sigma} Q\, dx \leqq 0$ *for all $\sigma \in (0, a)$.*

If there exists a solution u of (1) with $u(-a) = 0 = u(a)$ and $u(x) > 0$, $-a < x < a$, then there exists a pair of conjugate points for equation (2) on $[-a, a]$.

A similar result of Corollary 2 was obtained by FINK [2: Th. 2] under the hypothesis $p \equiv P$ (cp. [7: p. 186]). In the special case $p \equiv P \equiv 1$ and $Q \leqq 0$ Corollary 2 is a result of LEIGHTON [4: Th. 1.3].

Corollary 3: *Consider the equations (1) and (2) on $[0, a]$ and assume the following:*

(i) $p \leqq P$ *on* $[0, a]$,

(ii) $\int_0^\sigma q\,dx \leqq \int_\tau^\sigma Q\,dx$ *for all* σ, τ *with* $0 < \tau < \sigma < a$

or

(ii') $\int_0^\sigma q\,dx \leqq \int_0^\sigma Q\,dx \leqq 0$ *for all* $\sigma \in (0, a)$.

If there exists a solution u of (1) on $[0, a]$ with $u'(0) = u(a) = 0$ and $u(x) > 0$, $0 \leqq x < a$, then every solution v of (2) on $[0, a]$ with $v(0) > 0$ and $v'(0) \leqq 0$ has a zero on $(0, a]$.

Proof: By using the Sturm comparison theorem it is easily seen that it is sufficient to prove Corollary 3 under the hypothesis $v'(0) = 0$ in place of $v'(0) \leqq 0$. Then, extend the functions p, P, q, Q, and the solutions u and v as even functions on $[-a, a]$ and use Theorem 1 (or Corollary 2) with the aid of Sturm's comparison theorem ∎

In the following the hypothesis $p \leqq P$ is to be weakened. Henceforth, let P be continuously differentiable on $(0, a]$.

Theorem 2: *Assume the following:*

(i) *P and Q are even functions on* $[-a, a]$,

(ii) $Q, P' \leqq 0$ *on* $(0, a]$,

(iii) $\int_{[-a,a]\setminus(-\sigma,\sigma)} [P - p]\,dx \geqq 0$ *and* $\int_{-\sigma}^{\sigma} [Q - q]\,dx \geqq 0$ *for all* σ, $0 < \sigma < a$.

If there exists a solution u of (1) with $u(-a) = 0 = u(a)$ and $u(x) > 0$, $-a < x < a$, then there exists a pair of conjugate points for equation (2) on $[-a, a]$.

Proof: $Q \leqq 0$ implies inequality (3). Therefore, as shown in the proof of Corollary 1, we have $u' \leqq 0$ on $[0, a]$ and by means of $P' \leqq 0$ it follows from $Pu'' \equiv -P'u' + Qu$ on $(0, a]$, that $u'' \leqq 0$ on $(0, a]$. Consequently, the derivative u' is monotone decreasing on $[0, a]$. The function $(u')^2$ is even and monotone increasing on $[0, a]$. Hence, by means of the hypothesis (iii) we obtain

$$\int_{-a}^{a} (p - P)(u')^2\,dx = \int_{-a}^{a} \int_0^{[u'(x)]^2} [p(x) - P(x)]\,dy\,dx$$

$$= \int_0^{[u'(a)]^2} \int_{[-a,a]\setminus(-h(y),h(y))} [p(x) - P(x)]\,dx\,dy \leqq 0$$

where $h(y) = \sup\{x \in [0, a] \mid [u'(x)]^2 \leqq y\}$, $0 \leqq y \leqq [u'(a)]^2$. Analogously (compare the proof of Theorem 1), we get $\int_{-a}^{a} (q - Q)\, u^2\, dx \leqq 0$. Thus, it follows that

$$\int_{-a}^{a} [p(u')^2 + qu^2]\, dx = \int_{-a}^{a} [(p - P)(u')^2 + (q - Q)\, u^2]\, dx \leqq 0.$$

Finally, finish the proof as the proof of Theorem 1 ∎

Corollary 4: *Consider the equations* (1) *and* (2) *on* $[0, a]$ *and assume the following*:

(i) $Q, P' \leqq 0$ *on* $[0, a]$,

(ii) $\int_{\sigma}^{a} P\, dx \geqq \int_{\sigma}^{a} p\, dx$ *and* $\int_{0}^{\sigma} Q\, dx \geqq \int_{0}^{\sigma} q\, dx$ *for all* $\sigma \in (0, a)$.

If there exists a solution u *of* (1) *with* $u'(0) = 0 = u(a)$ *and* $u(x) > 0$, $0 \leqq x < a$, *then every solution of* (2) *with* $v(0) > 0$ *and* $v'(0) \leqq 0$ *has a zero on* $(0, a]$.

Proof: Compare the proof of Corollary 3 ∎

Corollaries 3 and 4 are generalizations of theorems of Nehari [6] and Leighton [4: Th. 1.1].

Corollary 5: *If the inequality*

$$\sup_{\sigma \in (0,a)} \frac{1}{\sigma} \int_{0}^{\sigma} q\, dx \leqq -\frac{\pi^2}{4a^2} \sup_{\sigma \in (0,a)} \frac{1}{a - \sigma} \int_{\sigma}^{a} p\, dx \tag{5}$$

holds, then every solution v *of* (2) *on* $[0, a]$ *with* $v(0) > 0$ *and* $v'(0) \leqq 0$ *has a zero on* $(0, a]$.

Proof: Define

$$\varrho = \sup_{\sigma \in (0,a)} \frac{1}{a - \sigma} \int_{\sigma}^{a} p\, dx \tag{6}$$

and set $P \equiv \varrho$ and $Q \equiv -\pi^2\varrho/4a^2$. The function $u = \cos(\pi x/2a)$ is a solution of the differential equation

$$-\varrho u'' - \frac{\pi^2 \varrho}{4a^2} u = 0, \qquad 0 \leqq x \leqq a,$$

with the properties supposed in Corollary 4. Hypothesis (i) of Corollary 4 is fulfilled. It follows from (5) and (6) that (ii) also holds. This proves Corollary 5 ∎

Example: Consider the differential equation

$$-((1 - x^2)\, u')' - 6u = 0, \qquad 0 \leqq x \leqq a < 1. \tag{7}$$

Since

$$\frac{1}{a - \sigma} \int_{\sigma}^{a} (1 - x^2)\, dx \leqq \frac{1}{a} \int_{0}^{a} (1 - x^2)\, dx = 1 - \frac{a^2}{3}, \qquad \sigma \in (0, a),$$

we have $\varrho = 1 - a^2/3$. Inequality (5) holds when $a \geqq a_1 = (3\pi^2/(72 + \pi^2))^{1/2} \approx 0{,}6014$. Hence, every solution v of (7) with $v(0) > 0$ and $v'(0) \leqq 0$ has a zero on $(0, a_1]$. Because $v = 1 - 3x^2$ is a solution of (7) with $v'(0) = 0$, the smallest a is $a_0 = 3^{-1/2} \approx 0.5774$. We state that a_1 is a good approximate value for a_0.

REFERENCES

[1] BROWDER, F. E.: On the spectral theory of elliptic differential operators I. Math. Ann. **142** (1961), 22–130.
[2] FINK, A. M.: Comparison theorems for eigenvalues. Quart. Appl. Math. 28 (1970), 289–292.
[3] KATO, T.: Perturbation theory for linear operators. Berlin—Heidelberg—New York: Springer-Verlag 1966.
[4] LEIGHTON, W.: Some oscillation theory. Z. Angew. Math. Mech. **63** (1983), 303–315.
[5] MÜLLER-PFEIFFER, E.: On the existence of nodal domains for elliptic differential operators. Proc. Roy. Soc. Edinburgh **94 A** (1983), 287–299.
[6] NEHARI, Z.: Oscillation criteria for second-order linear differential equations. Trans. Amer. Math. Soc. 85 (1957), 428–445.
[7] REID, W. T.: Sturmian theory for ordinary differential equations (Applied Mathematical Sciences, Vol. 31). New York—Heidelberg—Berlin: Springer-Verlag 1980.

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