Generalized-Analytic Coverings in the Spectrum of a Uniform Algebra

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We examine some conditions under which certain parts of a uniform algebra spectrum admit special structures so that Gelfand transforms of algebra elements are generalized-analytic functions.

The theory of analytic functions of one complex variable is naturally connected with the semigroup \( \mathbb{Z} \) of nonnegative integers. Analogous functions can be considered with arbitrary subsemigroups of nonnegative real numbers instead of \( \mathbb{Z} \). Though not very well developed, there are various constructions of such functions. We shall deal with generalized-analytic functions in the sense of Arens-Singer, connected with the semigroup \( \mathbb{Q}^+ \) of nonnegative rational numbers in the same natural way as \( \mathbb{Z} \) is connected with usual analytic functions. In this article we examine some conditions under which certain parts of a uniform algebra spectrum admit special structures, so that Gelfand transforms of algebra elements are generalized-analytic functions on them.

1. Basic definitions and results

Let \( S = \{ p \} \) be an additive subsemigroup of \( \mathbb{Q}^+ = \text{Rat } [0, \infty) \) containing zero. Denote by \( G \) the group generated by \( S \cup (-S) \) and provided with discrete topology and by \( \hat{G} \) — the dual group of \( G \). The group \( G \) is compact and connected and \( \hat{G} \cong G \).

The big plane is the cone \( \mathcal{C}_0 = [0, \infty) \times G/[0] \times G \) over \( G \) with the peak \( * = [0] \times G/[0] \times G \) and the big disc with radius \( c > 0 \) in it is the set \( \mathcal{D}_c(c) = \{ (\lambda, g) \in \mathcal{C}_0 \mid \lambda < c \} \).

We call generalized polynomials the linear combinations over \( \mathcal{C}_0 \) of functions \( \chi_p \), where \( \chi_p \in \hat{G} \) are the characters \( \chi_p(g) = g(p) \) for all \( g \in G \). Because of \( S \subseteq \mathbb{Q}^+ \), we have \( \chi_p(\lambda, g) = (\chi(\lambda, g))^p \), so that all the functions \( \chi_p \) have arbitrary powers \( p \in S \). Given an open set \( U \subseteq \mathcal{C}_0 \), we denote by \( A\mathcal{C}(U) \) the algebra of all generalized-analytic functions, i.e., the algebra of all complex valued functions on \( U \) that are approximable locally by generalized polynomials on \( U \). For a compact set \( K \subseteq \mathcal{C}_0 \) we denote by \( A\mathcal{C}(K) \) the algebra of all continuous functions on \( K \) that are generalized-analytic on \( \text{Int } K \). The corresponding algebra for \( K = \overline{A\mathcal{C}(1)} \) was historically the first of this type which attracted the attention of mathematicians.
It was introduced by Arens and Singer in 1956. They considered $A_{c0}(\beta C(1))$ as a uniform algebra on $G$ in a slightly more general setting. In [4–6, 9] the generalized-analytic functions are examined on arbitrary sets of $C_0$. Taking $G \cong S^1$, $C_0 \cong C^1$ and $\chi^\Delta(\lambda, e^{i\theta}) = \lambda^m e^{im\theta}$ (or equivalently: $\chi^\Delta(\lambda e^{i\theta}) = \lambda^m e^{im\theta}$), i.e. $\chi^\Delta(z) = z^m$.

Let $D$ be a domain in $C_0$ and $A$ be a subset of $D$. We call $A$ a negligible set if $A$ is nowhere dense and if for any subdomain $D' \subset D$ every generalized-analytic function $f$ on $D' \setminus A$, locally bounded in $D'$, admits a unique generalized-analytic extension on the whole domain $D'$. We call generalized-analytic covering any triple $(X, \pi, U)$ for which:

1. $X$ is a locally compact Hausdorff space;
2. $U$ is a domain in $C_0$;
3. $\pi$ is a proper continuous mapping of $X$ onto $U$, for which the set $\pi^{-1}(\lambda, g)$ is discrete for any $(\lambda, g) \in U$;
4. there exist a negligible set $A \subset U$ and an integer $m$, so that $\pi$ is a $m$-sheeted covering mapping of $X \setminus \pi^{-1}(A)$ onto $U \setminus A$;
5. the set $X \setminus \pi^{-1}(A)$ is dense in $X$.

Sometimes $X$ is called a covering onto $U$ and $\Lambda$ — its critical space. We call generalized-holomorphic any complex valued function $f$, defined on an open subset $V$ of a generalized-analytic covering $X$, if for any open subset $V' \subset X \setminus \pi^{-1}(A)$ on which $\pi$ is homeomorphic the function $(f|_{V'}) \circ \pi^{-1}$ is generalized-analytic on $\pi(V')$.

Let $A$ be a uniform algebra on the compact Hausdorff space $X$ and let $sp A$ be its maximal ideal space. Let $\{m|_{\pi}(X)\}$ be a multiplicative semigroup of elements of $A$, where $S$ is as above. We call spectral mapping of $S$ the mapping $\Phi_S: sp A \to C(G)$, $\Phi_S(z) = (\langle f_\lambda | x \rangle, g_x)$, where $g_x \in G = \Gamma$ is defined as follows: $g_x(p) = \langle f_\lambda | x \rangle, g_x(-p) = g_x(p)$. It is easy to see that $\langle f_\lambda | x \rangle = \chi^\Delta(\Phi_S(x))$, where $x \in sp A$ and $\chi^\Delta$ stands for the Gelfand transform of $f_\lambda$. In the classical case, when $S \cong \mathbb{Z}^+$, $\Phi_S(\varphi) = \{\varphi\}$; the last property is simply: $\langle f_\lambda | x \rangle = (\chi^\Delta(\varphi))^n$. We call spectrum $\sigma(S)$ of a semigroup $S \subset C_0$ the image of $\Phi_S$, i.e. $\sigma(S) = \Phi_S(sp A)$. In the sequel we shall omit the index $S$. In [6, 9] several aspects of the spectrum $\sigma(S)$ have been discussed.

Further we assume that $S = \mathbb{Q}_+$: In [10] we have found conditions assuring that some neighbourhood of certain infinitely generated linear multiplicative functional of a uniform algebra is homeomorphic to a generalized-analytic covering and the restrictions of algebra elements are generalized-holomorphic on it. Actually, we have obtained there the following results.

Theorem 1.1: Let $W$ be a component of $\text{Int } \sigma(S) \setminus \Phi(X)$ and let the spectral mapping $\Phi$ be one-to-one on $\Phi^{-1}(W)$. Then $h \circ \Phi^{-1}$ is a generalized-analytic function on $W$ for any $h \in A$, i.e. $A|_{\Phi^{-1}(W)} \subset A_{c0}(W)$.

In the sequel we shall denote the number of elements of the set $E$, as usual, by $\# E$.

Theorem 1.2: Let $X \supset \Phi^{-1}(\langle f_\lambda | x \rangle, \varphi) \setminus A$, $W$ be a connected component of $C_0 \setminus \Phi(X)$ and let there exist a $k < \infty$, such that $\# \Phi^{-1}(\lambda, g) \leq k$ for any $(\lambda, g) \in W$. Then the set $\Phi^{-1}(W)$ has the structure of a $k$-sheeted $(k_1 \leq k)$ generalized-analytic covering over $W$ and the functions $f, f \in A$, are generalized-holomorphic there.

For the proof of Theorem 1.1 in [10] we made use of a result by S. Grigorjan, announced in [4]. Since no proof was given afterwards, we give one here. Some preliminary facts: The big plane $C_0$ can be presented as $\lim \limits_{n \to \infty} (C_n, \pi_m^n)$, where $C_n \cong C_0$, where
\[\pi_m(z_m) = z_m^n\] and \(m > n\) iff \(m = nk\) for some \(k \in \mathbb{N}\), \(z_m \in C_m\) [8]. Now \(\pi_n = \chi^{1/n}\).

We provide \(C\) with the weak topology with respect to the family of functions \(\{\chi^{1/n}\}_n\), where the base neighbourhoods are the sets of the type:

\[Q((\lambda_0, g_0), \varepsilon, n) = \{(\lambda, g) \in C | |\chi^{1/n}(\lambda, g) - \chi^{1/n}(\lambda_0, g_0)| < \varepsilon, \varepsilon > 0\},\]

\(n = 1, 2, \ldots, (\lambda_0, g_0) \in C\). If \(\lambda_0 = 0\), the corresponding base neighbourhood

\[Q(\ast, \varepsilon, n) = \{(\lambda, g) \in C | |\chi^{1/n}(\lambda, g)| < \varepsilon\} = \Delta_C(z^n)\]

is homeomorphic to some big disc.

**Proposition 1.3:** If \(\lambda_0 \neq 0\) and \(\varepsilon > 0\) is small enough, the neighbourhood \(Q((\lambda_0, g_0), \varepsilon, n)\) is homeomorphic to the set \((\ker \chi^{1/n}) \times \Delta(1)\).

**Proof:** Let \(\ast \in Q = Q((\lambda_0, g_0), \varepsilon, n)\) and denote by \(V((\lambda, g), n)\) the set \(\chi_n(Q) \subseteq C\), \((\lambda, g) \in \ker \pi_n = \ker \chi^{1/n}\). For any \(m > n\) we denote by \(V((\lambda, g), m)\) the component of \(\pi_m(Q) \subseteq C\) that contains \(\pi_m(\lambda, g)\). Now the set \(V(\lambda, g) = \lim_{m \to n} V((\lambda, g), m)\) contains \((\lambda, g)\) and

\[Q((\lambda_0, g_0), \varepsilon, n) = \pi^{-1}(V((\lambda_0, g_0), n)) = \bigcup_{m \geq n} V((\lambda, g), m)\]

where \(V(\lambda, g) \cap V(\lambda_1, g_1) = \emptyset\) for \((\lambda, g) \neq (\lambda_1, g_1)\). Because of \(V(\lambda, g) = V(\lambda, g)\), then \(Q((\lambda_0, g_0), \varepsilon, n) = V((\lambda_0, g_0) \times \ker \chi^{1/n}\). The set \(V((\lambda_0, g_0)\) is homeomorphic to the disc \(\Delta(1) \subseteq C\) with radius 1. In fact, let \(\phi_m : V((\lambda_0, g_0), m) \to \Delta(1)\) be the Riemann conformal mapping with \(\phi_m(\chi^{1/m}(\lambda_0, g_0)) = 0\), \(\phi_m(\chi^{1/m}(\lambda, g)) > 0\) for any \(m > n\). There arises the diagram

\[
\begin{array}{ccc}
\varphi \downarrow & & \Delta \\
V((\lambda_0, g_0), l) & \xrightarrow{z^*} & V((\lambda_0, g_0), m) \\
\end{array}
\]

where \(m > l > n\), i.e. \(m = lk, k \in \mathbb{N}\). According to the definition of \(V((\lambda_0, g_0), m)\), the mapping \(z^*\) is one-to-one. Then \(\psi_m(z) = \phi(z^k)\) is also a one-to-one and conformal mapping from \(V((\lambda_0, g_0), m)\) onto \(\Delta(1)\), with \(\psi_m(\chi^{1/m}(\lambda_0, g_0)) = 0\), \(\psi_m(\chi^{1/m}(\lambda, g)) > 0\), i.e. \(\psi_m\) coincides with \(\varphi_m\). Hence the diagram is commutative and there exists a one-to-one and continuous mapping from \(\lim_{m \to n} V((\lambda_0, g_0), m) = V((\lambda_0, g_0)\) onto \(\lim_{m \to n} \Delta(1, 1) = \Delta(1)\).

Natural questions that arise in connection with Theorems 1.1 and 1.2 are: when is the spectral mapping \(\Phi\) one-to-one? When is the condition \(\Phi^{-1}(\lambda, g) \leq k\) fulfilled on \(W\)? Partial answers to these questions are given in [11]. Namely:

**Theorem 1.4:** Let \(W\) be a component of \(\operatorname{Int} \sigma(S) \setminus \Phi(X)\), containing the point \(\ast\), and \(\Delta_0(c)\) be a big disc in \(W\). If

\[\ker \varphi = \bigcup_{p \in S} \{\varphi - f^n(\varphi)\} A\]

for some \(p \in \Phi^{-1}(\Delta_0(c))\), then \(\Phi^{-1}\) is a homeomorphism of \(\Delta_0(c)\) into \(\operatorname{sp} A\).

**Theorem 1.5:** Let \(W\) be a component of \(\operatorname{Int} \sigma(S) \setminus \Phi(X)\), containing the point \(\ast\), and \(\Delta_0(c)\) be a big disc in \(W\). If for some \((\lambda_0, g_0) \in \Delta_0(c)\) the ideal

\[J((\lambda_0, g_0) = \bigcup_{p \in S} \{\varphi - \chi^p(\lambda_0, g_0)\} A\]
has codimension \( k < \infty \) in \( A \), then \( \# \Phi^{-1}(\lambda, g) \leq k \) for any \((\lambda, g)\) from the maximal big disc belonging to \( W \).

The proofs of these two theorems made use of Kato's perturbation theory for semi-Fredholm pairs of closed linear subspaces of a Banach space.

Let us set \( W_m = \{(\lambda, g) \in W \mid \# \Phi^{-1}(\lambda, g) = m\} \). If \( \Delta(\eta) \cap \text{Int} \ W_k = \emptyset \), then \( \# \Phi^{-1}(\lambda, g) \leq k \) on any big disc \( \Delta(\eta) \) belonging to \( W \) [11]. As a corollary, if \( \Delta(\eta) \subset W \) and \( \Delta(\eta) \cap \text{Int} \ W_k = \emptyset \) for some \( \eta > 0 \), then \( \Phi \) is one-to-one on the \( \Phi^{-1}(\Delta(\eta)) \), where \( \Delta(\eta) \) is the maximal big disc in \( W_k \). The most interesting is the case when \( W = \Delta(\eta) \) and \( X = \text{sp} \ A \setminus \Phi^{-1}(W) \).

2. Main results

Here we give new answers to the questions stated above.

**Theorem 2.1:** Let \( A \) be a uniform algebra on \( X \) and \( M = \text{sp} \ A \). Let \( \{f\}_{p \in S} \subset A \) be a multiplicative subsemigroup of \( A \), isomorphic to \( \mathbb{Q} \), and let \( \Phi : X \to \mathbb{C} \) be the spectral mapping of \( \{f\} \). Suppose that:

a) \( \Phi(X) \subset bQ \) for some base neighbourhood \( Q = Q((\lambda_0, g_0), \varepsilon, q) \).

b) \( (\lambda_0, g_0) \in \Phi(M) \) and

c) there exists a closed subset \( N \) of \( Q \) for which the set \( \chi(N) \) has a non-zero Lebesgue measure for some (and any!) \( p \in S \), such that \( \Phi \) is one-to-one on the \( \Phi^{-1}(N) \).

Then \( \Phi \) is one-to-one on \( \Phi^{-1}(Q_1) \), \( Q_1 \) being a base neighbourhood in \( Q \).

**Proof:** We shall follow J. WERMER in his proving of a similar statement for the case \( S = \mathbb{Z}_+ \) (see [12]). That is why we shall not give all the details but only a sketch of the proof, emphasizing on the differences. Without loss of generality we may assume that \( \varepsilon = 1 \), \( d\theta(N_1) > 0 \), \( N_1 = N \cap bQ \). If \( \mu \) is a measure on \( X \) we denote by \( \Phi(\mu) \) the induced measure on \( C = bQ \), namely: \( \Phi(\mu)(E) = \mu(\Phi^{-1}(E)) \) for \( E \subset C \). We know that \( \Phi(M) \supset (\lambda_0, g_0) \) and \( \Phi(X) \subset C \) according to a). Now, according to [10: Lemma 3], \( \Phi(M) \supset Q \). Let \( \varphi_1 \) and \( \varphi_2 \) be such elements of \( M \) that \( \Phi(\varphi_1) = \Phi(\varphi_2) = (\lambda_1, g_1) \in Q \). Supposing that \( \varphi_1 \neq \varphi_2 \), we can find such \( p > 0 \) that \( \chi_p(\varphi_1) \neq \chi_p(\varphi_2) \). Let \( p_0 \in S \); \( p_0^q = p, p_0^q = q, k_1, k_2 \in N \) and \( Q_1 = Q((\lambda_0, g_0), 1, p_0) \subset Q \). Now again \( \chi_p(\varphi_1) = \chi_p(\varphi_2) \). We can find also such a function \( g(z) \in H(\chi_p(M)) \) that \( g(\chi_p(\varphi_1)) = 1 \), \( g(\chi_p(\varphi_2)) = 0 \). If \( \mu_1 \) and \( \mu_2 \) are representing measures of \( \varphi_1 \) and \( \varphi_2 \), then it turns out that \( \Phi(\mu_1) = \Phi(\mu_2) \). Because of the injectiveness of \( \Phi \) on \( \Phi^{-1}(N_1) \), \( \mu_1 \) and \( \mu_2 \) coincide on \( \Phi^{-1}(N) \) and hence the measures \( v_j = \Phi(g \cdot \mu_j), j = 1, 2 \), also coincide on \( N_1 \). Then, having in mind the choice of \( g \), we obtain:

\[
\int_C Pf(v_1 - v_2) = P(\lambda_1, g_1)
\]

for any generalized polynomial \( P \) on \( C \) of the type \( P(\lambda, g) = P(\chi_p(\lambda, g)) \), \( P \) being a usual polynomial. Let us consider the measure \( (\chi_p - \chi_p(\lambda_1, g_1)) dv_1 - dv_2 \), orthogonal to all generalized polynomials. Denoting by \( v_j^S \) the induced measures on the unit circle \( S = \chi_p(C) - \chi_p(\lambda_0, g_0) \), defined as

\[
\int_C fdv_j^S = \int_C f(\chi_p - \chi_p(\lambda_0, g_0)) dv_j, \quad f \in C(S),
\]

we see that the measure

\[
(\chi_p - \chi_p(\lambda_1, g_1)) dv_1 - dv_2 = (z - \chi_p(\lambda_1, g_1)) dv_1^S - dv_2^S
\]
admits the Lebesgue decomposition $h d \theta + v$, where $h \in H^1(\Delta)$ and $v$ is a singular measure with respect to the Lebesgue measure on the unit circle. It follows from (1) that the same measure is orthogonal to all polynomials $P(\cdot, g)$ with $P(\cdot, g) = P(\bar{z}^n(\cdot, g))$ on $Q_1$ and at the same time — that the measure $(z - \bar{z}^n(\cdot, g_1)) d(v_1^s - v_2^s)$ is orthogonal to all polynomials of $z$ on $\Delta$. According to the theorem of F. and M. Riesz it follows that $v_2 = 0$. Since $v_1|x| = v_1|y|$ on $\hat{P}(N_1)$, we have $v_1^s = v_2^s$. From where $(z - \bar{z}^n(\cdot, g_1)) d(v_1^s - v_2^s) = h d \theta$ is identically zero, because the $H^1$-function $h$ vanishes on the set $\bar{z}^n(N_1)$ with $d\theta(\bar{z}^n(N_1)) > 0$, (and hence $d\theta(\bar{z}^n(N_1)) > 0$ for any $p \in S$). But $\bar{z}^n(\cdot, g_1) \in \Delta$ and consequently $z - \bar{z}^n(\cdot, g_1) = 0$ on $S$, from where $v_1 = v_2^s$ in contradiction to the equality (1). Consequently, $\varphi_1 = \varphi_2$, and hence $\Phi$ is a one-to-one mapping from $\Phi^{-1}(Q_1)$ onto $Q_1$.

Suppose that $\# \Phi^{-1}(\cdot, g) < \infty$ for any point of a measurable subset $N$ of the maximal big disc in $W$ with $dx dy \bar{z}^n(N) > 0$ for some (and hence for any) $p \in S$. Let $W = A_\sigma(e)$ and $X = sp A \setminus \Phi^{-1}(A_\sigma(e))$. The sets $\bar{z}^n(N)$ and $N_j = N \cap W_j$ are also measurable. The proof of this statement for the classical case is due to J. Weisner [12] and holds true also for the generalized-analytic case, by replacing only $C$ with $G_0$, the Gelfand transform $\int_\infty$ with the spectral mapping $\Phi$, and $A$ — with a base neighbourhood. Since $N = \bigcup_j N_j$, $\bar{z}^n(N) = \bigcup_j \bar{z}^n(N_j)$ for any fixed $p \in S$, there exists such a $k$ that $dx dy \bar{z}^n(N_k) > 0$ and hence $dx dy \bar{z}^n(N_k) > 0$ for any $p \in S$ (in fact, $dx dy \bar{z}^n(N_k) > 0$ implies $dx dy \bar{z}^n(N_k) > 0$ and hence $dx dy \bar{z}^n(N_k) > 0$). Now applying the diagonal principle, we can find such $(\lambda_0, g_0) \in N_k$ that for any $p \in S$ $\bar{z}^n(\lambda_0, g_0)$ is a point of density for the set $\bar{z}^n(N_k)$. Let $p_1, p_2, \ldots, p_k$ be all the points of $\Phi^{-1}(\lambda_0, g_0)$ and $\xi$ be a standard neighbourhood: $\xi = Q((\lambda_0, g_0), \varepsilon_1, \varepsilon_2)$ for which $\Phi^{-1}(\xi)$ splits into $k$ disjoint closed subsets, any of which contains exactly one point of $\Phi^{-1}(\lambda_0, g_0) = \{p_1, p_2, \ldots, p_k\}$. For arbitrary small $\varepsilon_1 > 0$, the boundary $bQ_1$ of the base neighbourhood $Q_1 = Q((\lambda_0, g_0), x, y)$ for which $\Phi^{-1}(\xi)$ contains the point $p_\nu$ ($\nu = 1, 2, \ldots, k$). Now $dx dy \bar{z}^n(J_\nu) > 0$ for any $p \in S$. According to [10: Lemma 4], we obtain that $\Phi^{-1}(Q_1) \subset \bigcup J_\nu$. We shall see that $\Phi$ maps $J_\nu \cap \Phi^{-1}(Q_1)$ injectively onto $Q_1$ for any $\nu$. Supposing $(\lambda_1, g_1)$ fixed in $L_k$, for any $\nu = 1, 2, \ldots, k$, we have: $(\lambda_0, g_0) = \Phi(p_\nu) \in \Phi(J_\nu)$, from where $\Phi(J_\nu) \supset Q_1$ according to [10: Lemma 3], and consequently there exists at least one point (say, $g_1$) in every $J_\nu$, with $\Phi(g_1) = (\lambda_1, g_1)$. Because $(\lambda_1, g_1) \in bQ_1$, we have $g_1 \in \partial A(J_\nu)$. For a fixed $\nu$ we can assume $A(J_\nu)$ to be a uniform algebra on $\partial A(J_\nu)$. Theorem 2.1 gives us now that $\Phi$ is a one-to-one mapping between $J_\nu \cap \Phi^{-1}(Q_2)$ and $Q_2 \subset Q_1$, i.e. we obtain that $Q_2 \subset W_k$ and consequently — that $\text{Int } W_k \neq \emptyset$. If now $A_\sigma(\eta) \subset W_k \subset \partial A(J_\nu)$, Theorem 1.5 implies that $\# \Phi^{-1}(\lambda, g) \leq k$ for any $(\lambda, g)$ belonging to the maximal big disc $A_\sigma(e)$ in $W$. By applying Theorem 1.2 we obtain the following result.

**Theorem 2.2:** Let $A$ be a uniform algebra and $\{f_p\}_{p \in S}$ be a multiplicative subsemigroup in $A$, isomorphic to $Q_1$. Let $\Phi$ be the spectral mapping of $\{f_p\}$ and $W$ be a component of $\Phi(sp A) \setminus \Phi(\partial A)$ for which $|f_p| = \text{const}$ on $\partial A \setminus \Phi^{-1}(W)$ for some (and hence — for every) $p \in S$. Suppose that there exists a measurable subset $N \subset W$ such that $dx dy \bar{z}^n(N^p) > 0$ for some (and hence — for every) $p \in S$ and that the set $\Phi^{-1}(\lambda, g) = \{p \in M | \Phi(p) = (\lambda, g)\}$ is finite for any $(\lambda, g) \in N$.

Then the set $\Phi^{-1}(W)$ has the structure of a $k$-sheeted generalized-analytic covering over $A_\sigma(\{|f_p|\})$ and for any function $h$ of $A$, the function $h \circ \Phi$ is generalized-holomorphic on this covering.
Note that we know from the discussions preceding Theorem 2.2 that the set \( W_k \) is open, so that the negligible set \( W \setminus W_k \) there is closed. As shown recently by B. Aupetit and J. Wermer [2], the conditions for the set \( N \) in Wermer's theorem can be weakened. Following them, the condition \( d\theta(\gamma^p(N)) > 0 \) for \( N \) in Theorem 2.2 can be weakened as well, by requiring \( N \) to be of nonzero exterior capacity instead of nonzero measure. In the frame of the theory built above, it is possible to insert also generalized-analytic analogues to \( n \)-dimensional boundaries, for the algebra \( A \) as well as the corresponding results of R. Basener [3] for existence of \( n \)-dimensional analytic manifold's structure in the spectrum of a uniform algebra \( A \).

All the results hold for subsemigroups \( S \) of \( \mathbb{Q}^+ \) possessing the following property: for any \( p_1 \) and \( p_2 \in S \cap [0, 1] \) there exists a \( p_3 \in S \cap [0, 1] \) such that \( p_3 > p_1 \) and \( p_3 \in S \setminus \mathbb{Q}^+ \). \( \mathbb{Q}^+ \) and \( \mathbb{Z}^+ \) are particular cases of such semigroups.

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