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Generalized-Analytic Coverings in the Spectrum of a Uniform Algebra

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Wir untersuchen einige Bedingungen, unter denen bestimmte Teile im Spektrum einer gleichmäßigen Algebra spezielle Strukturen zulassen, so daß die Gelfand-Transformationen von Elementen der Algebra verallgemeinerte analytische Funktionen sind.

Рассматриваются некоторые условия, при которых определенные части спектра равномерной алгебры допускают специальные структуры, таких, что преобразования Гельфанда элементов алгебры являются обобщенными аналитическими функциями.

We examine some conditions under which certain parts of a uniform algebra spectrum admit special structures so that Gelfand transforms of algebra elements are generalized-analytic functions.

The theory of analytic functions of one complex variable is naturally connected with the semigroup $\mathbf{Z}_+$ of nonnegative integers. Analogous functions can be considered with arbitrary subsemigroups of nonnegative real numbers instead of $\mathbf{Z}_+$. Though not very well developed, there are various constructions of such functions. We shall deal with generalized-analytic functions in the sense of Arens-Singer, connected with the semigroup $\mathbf{Q}_+$ of nonnegative rational numbers in the same natural way as $\mathbf{Z}_+$ is connected with usual analytic functions. In this article we examine some conditions under which certain parts of a uniform algebra spectrum admit special structures, so that Gelfand transforms of algebra elements are generalized-analytic functions on them.

1. Basic definitions and results

Let $S = \{p\}$ be an additive subsemigroup of $\mathbf{Q}_+ = \mathrm{Rat}\,[0, \infty)$ containing zero. Denote by Γ the group generated by $S \cup (-S)$ and provided with discrete topology and by G — the dual group of Γ. The group G is compact and connected and $\hat{G} \cong \Gamma$. The *big plane* is the cone $\mathbf{C}_G = [0, \infty) \times G/\{0\} \times G$ over G with the peak $* = \{0\} \times G/\{0\} \times G$ and the *big disc* with radius $c > 0$ in it is the set $\Delta_G(c) = \{(\lambda, g) \in \mathbf{C}_G \mid \lambda < c\}$. We call *generalized polynomials* the linear combinations over $\mathbf{C}$ of functions $\chi^p(\lambda, g) = \lambda^p \chi^p(g)$, $p \in S$, where $\chi^p \in \hat{G}$ are the characters $\chi^p(g) = g(p)$ for all $g \in G$. Because of $S \subset \mathbf{Q}_+$, we have $\chi^p(\lambda, g) = (\chi^1(\lambda, g))^p$, so that all the functions χ^p have arbitrary powers $p \in S$. Given an open set $U \subset \mathbf{C}_G$, we denote by $A_G(U)$ the algebra of all *generalized-analytic functions*, i.e. the algebra of all complex valued functions on U that are approximable locally by generalized polynomials on U. For a compact set $K \subset \mathbf{C}_G$ we denote by $A_G(K)$ the algebra of all continuous functions on K that are generalized-analytic on $\mathrm{Int}\, K$. The corresponding algebra for $K = \overline{\Delta_G(1)}$ was historically the first of this type which attracted the attention of mathematicians

It was introduced by Arens and Singer in 1956. They considered $A_G(\overline{\Delta_G(1)})$ as a uniform algebra on G in a slightly more general setting. In [4—6, 9] the generalized-analytic functions are examined on arbitrary sets of $\mathbf{C}_G$. Taking $S \cong \mathbf{Z}_+$, we obtain: $G \cong \mathbf{S}^1$, $\mathbf{C}_G \cong \mathbf{C}^1$ and $\chi^n(\lambda, e^{i\theta}) = \lambda^n e^{in\theta}$ (or, equivalently: $\chi^n(\lambda e^{i\theta}) = \lambda^n e^{in\theta}$), i.e. $\chi^n(z) = z^n$.

Let D be a domain in $\mathbf{C}_G$ and Λ be a subset of D. We call Λ a *negligible set* if Λ is nowhere dense and if for any subdomain $D' \subset D$ every generalized-analytic function f on $D' \setminus \Lambda$, locally bounded in D', admits a unique generalized-analytic extension on the whole domain D'. We call *generalized-analytic covering* any triple (X, π, U) for which:

1. X is a locally compact Hausdorff space;
2. U is a domain in $\mathbf{C}_G$;
3. π is a proper continuous mapping of X onto U, for which the set $\pi^{-1}(\lambda, g)$ is discrete for any $(\lambda, g) \in U$;
4. there exist a negligible set $\Lambda \subset U$ and an integer m, so that π is a m-sheeted covering mapping of $X \setminus \pi^{-1}(\Lambda)$ onto $U \setminus \Lambda$;
5. the set $X \setminus \pi^{-1}(\Lambda)$ is dense in X.

Sometimes X is called a covering onto U and Λ — its *critical space*. We call *generalized-holomorphic* any complex valued function f, defined on an open subset V of a generalized-analytic covering X, if for any open subset $V' \subset X \setminus \pi^{-1}(\Lambda)$ on which π is homeomorphic the function $(f|_{V'}) \circ \pi^{-1}$ is generalized-analytic on $\pi(V')$.

Let A be a uniform algebra on the compact Hausdorff space X and let $\mathrm{sp}\, A$ be its maximal ideal space. Let $\{f^p\}_{p \in S}$ be a multiplicative semigroup of elements of A, where S is as above. We call *spectral mapping* of S the mapping Φ_S: $\mathrm{sp}\ A \to C_G$, $\Phi_S(x) = (|f_1(x)|, g_x)$, where $g_x \in G = \hat{\Gamma}$ is defined as follows: $g_x(p) = f^p(x)/|f^p(x)|$, $g_x(-p) = \overline{g_x(p)}$. It is easy to see that $\hat{f}^p(x) = \chi^p(\Phi_S(x))$, where $x \in \mathrm{sp}\ A$ and $\hat{f}^p$ stands for the Gelfand transform of f^p. In the classical case, when $S \cong \mathbf{Z}_+$, $\Phi_S(\varphi) = \hat{f}^1(\varphi)$; the last property is simply: $\hat{f}^n(x) = (\hat{f}^1(x))^n$. We call *spectrum* $\sigma(S)$ of a semigroup $S \subset A$ the image of Φ_S, i.e. $\sigma(S) = \Phi_S(\mathrm{sp}\ A)$. In the sequel we shall omit the index S. In [6, 9] several aspects of the spectrum $\sigma(S)$ have been discussed.

Further we assume that $S = \mathbf{Q}_+$. In [10] we have found conditions assuring that some neighbourhood of certain infinitely generated linear multiplicative functional of a uniform algebra is homeomorphic to a generalized-analytic covering and the restrictions of algebra elements are generalized-holomorphic on it. Actually, we have obtained there the following results.

Theorem 1.1: *Let W be a component of* $\mathrm{Int}\, \sigma(S) \setminus \Phi(X)$ *and let the spectral mapping Φ be one-to-one on $\Phi^{-1}(W)$. Then $\hat{h} \circ \Phi^{-1}$ is a generalized-analytic function on W for any $h \in A$, i.e. $A|_{\Phi^{-1}(W)} \subset A_G(W)$.*

In the sequel we shall denote the number of elements of the set E, as usual, by $\# E$.

Theorem 1.2: *Let $X \supset \Phi^{-1}(b\Phi(\mathrm{sp}\ A)) \cup \partial A$, W be a connected component of $\mathbf{C}_G \setminus \Phi(X)$ and let there exist a $k < \infty$, such that $\# \Phi^{-1}(\lambda, g) \leq k$ for any $(\lambda, g) \in W$. Then the set $\Phi^{-1}(W)$ has the structure of a k_1-sheeted ($k_1 \leq k$) generalized-analytic covering over W and the functions $\hat{f}$, $f \in A$, are generalized-holomorphic there.*

For the proof of Theorem 1.1 in [10] we made use of a result by S. GRIGORJAN, announced in [4]. Since no proof was given afterwards, we give one here. Some preliminary facts: The big plane $\mathbf{C}_G$ can be presented as $\varprojlim_n \{\mathbf{C}_n, \pi_m{}^n\}$, where $C_n \cong \mathbf{C}$,

$\pi_n{}^m(z_m) = z_m{}^k$ and $m > n$ iff $m = nk$ for some $k \in \mathbf{N}$, $z_m \in \mathbf{C}_m$ [8]. Now $\pi_n = \chi^{1/n}$. We provide $\mathbf{C}_G$ with the weak topology with respect to the family of functions $\{\chi^{1/n}\}_n$, where the base neighbourhoods are the sets of the type

$$\begin{aligned} &Q\big((\lambda_0, g_0), \varepsilon, n\big) \\ &= \{(\lambda, g) \in \mathbf{C}_G \mid |\chi^{1/n}(\lambda, g) - \chi^{1/n}(\lambda_0, g_0)| < \varepsilon, \varepsilon > 0\}, \end{aligned}$$

$n = 1, 2, \ldots, (\lambda_0, g_0) \in \mathbf{C}_G$. If $\lambda_0 = 0$, the corresponding base neighbourhood

$$Q(*, \varepsilon, n) = \{(\lambda, g) \in \mathbf{C}_G \mid |\chi^{1/n}(\lambda, g)| < \varepsilon\} = \Delta_G(\varepsilon^n)$$

is homeomorphic to some big disc.

Proposition 1.3: *If $\lambda_0 \neq 0$ and $\varepsilon > 0$ is small enough, the neighbourhood $Q\big((\lambda_0, g_0), \varepsilon, n\big)$ is homeomorphic to the set* $(\operatorname{Ker} \chi^{1/n}) \times \Delta(1)$.

Proof: Let $* \notin Q = Q\big((\lambda_0, g_0), \varepsilon, n\big)$ and denote by $V\big((\lambda, g), n\big)$ the set $\pi_n(Q) \subset \mathbf{C}^1$, $(\lambda, g) \in \operatorname{Ker} \pi_n = \operatorname{Ker} \chi^{1/n}$. For any $m > n$ we denote by $V\big((\lambda, g), m\big)$ the component of $\pi_m(Q) \subset \mathbf{C}^1$ that contains $\pi_m(\lambda, g)$. Now the set $V(\lambda, g) = \varprojlim_{m > n} V\big((\lambda, g), m\big)$ contains (λ, g) and

$$Q\big((\lambda_0, g_0), \varepsilon, n\big) = \pi_n{}^{-1}\big(V((\lambda_0, g_0), n)\big) = \bigcup_{(\lambda, g) \in \operatorname{Ker} \chi^{1/n}} V(\lambda, g),$$

where $V(\lambda, g) \cap V(\lambda_1, g_1) = \emptyset$ for $(\lambda, g) \neq (\lambda_1, g_1)$. Because of $V(\lambda, g) \cong V(\lambda_1, g_1)$, then $Q\big((\lambda_0, g_0), \varepsilon, n\big) = V(\lambda_0, g_0) \times \operatorname{Ker} \chi^{1/n}$. The set $V(\lambda_0, g_0)$ is homeomorphic to the disc $\Delta(1) \subset \mathbf{C}$ with radius 1. In fact, let $\varphi_m \colon V\big((\lambda_0, g_0), m\big) \to \Delta(1)$ be the Riemann conformal mapping with $\varphi_m\big(\chi^{1/m}(\lambda_0, g_0)\big) = 0$, $\varphi_M'\big(\chi^{1/m}(\lambda_0, g_0)\big) > 0$ for any $m > n$. There arises the diagram

$$\begin{tikzcd} & \Delta & \\ V\big((\lambda_0, g_0), l\big) \arrow[ur, "\varphi_l"] \arrow[rr, "z^k"'] & & V\big((\lambda_0, g_0), m\big) \arrow[ul, "\varphi_m"'] \end{tikzcd}$$

where $m > l > n$, i.e. $m = lk$, $k \in \mathbf{N}$. According to the definition of $V\big((\lambda_0, g_0), m\big)$, the mapping z^k is one-to-one. Then $\psi_m(z) = \varphi_p(z^k)$ is also a one-to-one and conformal mapping from $V\big((\lambda_0, g_0), m\big)$ onto $\Delta(1)$, with $\psi_m\big(\chi^{1/m}(\lambda_0, g_0)\big) = 0$, $\psi_m'\big(\chi^{1/m}(\lambda_0, g_0)\big) > 0$, i.e. ψ_m coincides with φ_m. Hence the diagram is commutative and there exists a one-to-one and continuous mapping from $\varprojlim_{m > n} V\big((\lambda_0, g_0), m\big) = V(\lambda_0, g_0)$ onto $\varprojlim \{\Delta(1), \mathrm{id}\} = \Delta(1)$ ∎

Natural questions that arise in connection with Theorems 1.1 and 1.2 are: when is the spectral mapping Φ one-to-one? When is the condition $\# \Phi^{-1}(\lambda, g) \leq k$ fulfilled on W? Partial answers to these questions are given in [11]. Namely:

Theorem 1.4: *Let W be a component of* $\operatorname{Int} \sigma(S) \setminus \Phi(X)$, *containing the point $*$, and $\overline{\Delta_G(c)}$ be a big disc in W. If*

$$\operatorname{Ker} \varphi = \overline{\bigcup_{p \in S} \big(f^p - f^p(\varphi)\big) A}$$

for some $\varphi \in \Phi^{-1}\big(\Delta_G(c)\big)$, then Φ^{-1} is a homeomorphism of $\Delta_G(c)$ into sp A.

Theorem 1.5: *Let W be a component of* $\operatorname{Int} \sigma(S) \setminus \Phi(X)$, *containing the point $*$, and $\overline{\Delta_G(c)}$ be a big disc in W. If for some $(\lambda_0, g_0) \in \Delta_G(c)$ the ideal*

$$J(\lambda_0, g_0) = \overline{\bigcup_{p \in S} \big(f^p - \chi^p(\lambda_0, g_0)\big) A}$$

has codimension $k < \infty$ in A, then $\# \Phi^{-1}(\lambda, g) \leq k$ for any (λ, g) from the maximal big disc belonging to W.

The proofs of these two theorems made use of Kato's perturbation theory for semi-Fredholm pairs of closed linear subspaces of a Banach space.

Let us set $W_m = \{(\lambda, g) \in W \mid \# \Phi^{-1}(\lambda, g) = m\}$. If $\Delta_G(\eta) \cap \operatorname{Int} W_k \neq \emptyset$, then also $\# \Phi^{-1}(\lambda, g) \leqq k$ on any big disc $\Delta_G(c)$ belonging to W [11]. As a corollary, if $\Delta_G(\eta) \subseteq W$ and $\Delta_G(\eta) \cap \operatorname{Int} W_1 \neq \emptyset$ for some $\eta > 0$, then Φ is one-to-one on the et $\Phi^{-1}(\Delta_G(\varepsilon))$, where $\Delta_G(\varepsilon)$ is the maximal big disc in W_1. The most interesting is the case when $W = \Delta_G(\varepsilon)$ and $X = \operatorname{sp} A \setminus \Phi^{-1}(W)$.

2. Main results

Here we give new answers to the questions stated above.

Theorem 2.1: *Let A be a uniform algebra on X and $M = \operatorname{sp} A$. Let $\{f^p\}_{p \in S} \subset A$ be a multiplicative subsemigroup of A, isomorphic to $\mathbf{Q}_+$, and let $\Phi\colon X \to \mathbf{C}_G$ be the spectral mapping of $\{f^p\}$. Suppose that:*

a) $\Phi(X) \subset bQ$ for some base neighbourhood $Q = Q((\lambda_0, g_0), \varepsilon, q)$,

b) $(\lambda_0, g_0) \in \Phi(M)$ and

c) there exists a closed subset N of Q for which the set $\chi^p(N)$ has a non-zero Lebesgue measure for some (and any!) $p \in S$, such that Φ is one-to-one on the set $\Phi^{-1}(N)$.

Then Φ is one-to-one on $\Phi^{-1}(Q_1)$, Q_1 being a base neighbourhood in Q.

Proof: We shall follow J. Wermer in his proving of a similar statement for the case $S \cong \mathbf{Z}_+$ (see [12]). That is why we shall not give all the details but only a sketch of the proof, emphasizing on the differences. Without loss of generality we may assume that $\varepsilon = 1$, $d\theta(N_1) > 0$, $N_1 = N \cap bQ$. If μ is a measure on X we denote by $\Phi(\mu)$ the induced measure on $C = bQ$, namely: $\Phi(\mu)(E) = \mu(\Phi^{-1}(E))$ for $E \subset C$. We know that $\Phi(M) \ni (\lambda_0, g_0)$ and $\Phi(X) \subset C$ according to a). Now, according to [10: Lemma 3], $\Phi(M) \supset Q$. Let φ_1 and φ_2 be such elements of M that $\Phi(\varphi_1) = \Phi(\varphi_2) = (\lambda_1, g_1) \in Q$. Supposing that $\varphi_1 \neq \varphi_2$ we can find such $p > 0$ that $\chi^p(\varphi_1) \neq \chi^p(\varphi_2)$. Let $p_0 \in S\colon p_0^{k_1} = p$, $p_0^{k_2} = q$, $k_1, k_2 \in \mathbf{N}$ and $Q_1 = Q((\lambda_0, g_0), 1, p_0) \subset Q$. Now again $\chi^{p_0}(\varphi_1) \neq \chi^{p_0}(\varphi_2)$. We can find also such a function $g(z) \in R(\chi^{p_0}(M))$ that $g(\chi^{p_0}(\varphi_1)) = 1$, $g(\chi^{p_0}(\varphi_2)) = 0$. If μ_1 and μ_2 are representing measures of φ_1 and φ_2, then it turns out that $\Phi(\mu_1) = \Phi(\mu_2)$. Because of the injectiveness of Φ on $\Phi^{-1}(N)$, μ_1 and μ_2 coincide on $\Phi^{-1}(N)$ and hence the measures $\nu_j = \Phi(g \cdot \mu_j)$, $j = 1, 2$, also coincide on N_1. Then, having in mind the choice of g, we obtain:

$$\int_C P\,d(\nu_1 - \nu_2) = P(\lambda_1, g_1) \tag{1}$$

for any generalized polynomial P on $\mathbf{C}_G$ of the type $P(\lambda, g) = \tilde{P}(\chi^{p_0}(\lambda, g))$, $\tilde{P}$ being a usual polynomial. Let us consider the measure $(\chi^{p_0} - \chi^{p_0}(\lambda_1, g_1))\,d(\nu_1 - \nu_2)$, orthogonal to all generalized polynomials. Denoting by $\nu_j{}^S$ the induced measures on the unit circle $S = \chi^{p_0}(C) - \chi^{p_0}(\lambda_0, g_0)$, defined as

$$\int_S f\,d\nu_j{}^S = \int_C f(\chi^{p_0} - \chi^{p_0}(\lambda_0, g_0))\,d\nu_j, \qquad f \in C(S),$$

we see that the measure

$$(\chi^{p_0} - \chi^{p_0}(\lambda_1, g_1))\,d(\nu_1 - \nu_2) = (z - \chi^{p_0}(\lambda_1, g_1))\,d(\nu_1{}^S - \nu_2{}^S)$$

admits the Lebesgue decomposition $hd\theta + \nu_s$, where $h \in H^1(\Delta)$ and ν_s is a singular measure with respect to the Lebesgue measure on the unit circle. It follows from (1) that the same measure is orthogonal to all polynomials $P(\lambda, g) = \tilde{P}(\chi^{p_0}(\lambda, g))$ on Q_1 and at the same time — that the measure $(z - \chi^{p_0}(\lambda_1, g_1))\, d(\nu_1{}^S - \nu_2{}^S)$ is orthogonal to all polynomials of z on Δ. According to the theorem of F. and M. Riesz it follows that $\nu_s = 0$. Since $\nu_1|_{N_1} = \nu_2|_{N_1}$ on $\chi^{p_0}(N_1)$, we have $\nu_1{}^S = \nu_2{}^S$, from where $(z - \chi^{p_0}(\lambda_1, g_1))\, d(\nu_1{}^S - \nu_2{}^S) = hd\theta$ is identically zero, because the H^1-function h vanishes on the set $\chi^{p_0}(N_1)$ with $d\theta(\chi^{p_0}(N_1)) > 0$, (and hence $d\theta(\chi^p(N_1)) > 0$ for any $p \in S$). But $\chi^{p_0}(\lambda_1, g_1) \in \Delta$ and consequently $z - \chi^{p_0}(\lambda_1, g_1) \neq 0$ on S, from where $\nu_1{}^S = \nu_2{}^S$ in contradiction to the equality (1). Consequently, $\varphi_1 = \varphi_2$, and hence Φ is a one-to-one mapping from $\Phi^{-1}(Q_1)$ onto Q_1 ∎

Suppose that $\# \Phi^{-1}(\lambda, g) < \infty$ for any point of a measurable subset N of the maximal big disc in W with $dx\,dy\,\chi^p(N) > 0$ for some (and hence for any) $p \in S$. Let $W = \Delta_G(\varepsilon)$ and $X = \operatorname{sp} A \setminus \Phi^{-1}(\Delta_G(\varepsilon))$. The sets $\chi^p(N)$ and $N_j = N \cap W_j$ are also measurable. The proof of this statement for the classical case is due to J. Wermer [12] and holds true also for the generalized-analytic case, by replacing only $\mathbf{C}$ with $\mathbf{C}_G$, the Gelfand transform $\hat{f}$ — with the spectral mapping Φ, and Δ — with a base neighbourhood. Since $N = \bigcup_{j=1}^{\infty} N_j$, $\chi^{p_0}(N) = \bigcup_{j=1}^{\infty} \chi^{p_0}(N_j)$ for any fixed $p_0 \in S$, there exists such a k that $dx\,dy\,\chi^{p_0}(N_k) > 0$ and hence $dx\,dy\,\chi^p(N_k) > 0$ for any $p \in S$ (in fact, $dx\,dy\,\chi^{p_0}(N_k) > 0$ implies $dx\,dy\,\chi^{pp_0}(N_k) > 0$ and hence $dx\,dy\,\chi^p(N_k) > 0$). Now applying the diagonal principle, we can find such $(\lambda_0, g_0) \in N_k$ that for any $p \in S$ $\chi^p(\lambda_0, g_0)$ is a point of density for the set $\chi^p(N_k)$. Let $p_1, p_2, \ldots, p_k$ be all the points of $\Phi^{-1}(\lambda_0, g_0)$ and let Q be a standard neighbourhood: $Q = Q((\lambda_0, g_0), \varepsilon, p)$ for which $\Phi^{-1}(\bar{Q})$ splits into k disjoint closed subsets, any of which contains exactly one point of $\Phi^{-1}(\lambda_0, g_0) = \{p_1, p_2, \ldots, p_k\}$. For arbitrary small $\varepsilon_1 > 0$, the boundary bQ_1 of the base neighbourhood $Q_1 = Q((\lambda_0, g_0), \varepsilon_1, p)$ intersects N_k in a set L_k for which $\chi^p(L_k) \subset S(\varepsilon_1)$ and $d\theta(\chi^p(L_k)) > 0$. Let J_ν denote the component of $\Phi^{-1}(\bar{Q})$ containing the point p_ν $(\nu = 1, 2, \ldots, k)$. Now $dx\,dy\,\chi^p(J_\nu) > 0$ for any $p \in S$. According to [10: Lemma 4], we obtain that $\Phi^{-1}(Q_1) \subset \bigcup_{\nu=1}^{k} J_\nu$. We shall see that Φ maps $J_\nu \cap \Phi^{-1}(Q_1)$ injectively onto Q_1 for any ν. Supposing (λ_1, g_1) fixed in L_k, for any $\nu = 1, 2, \ldots, k$, we have: $(\lambda_0, g_0) = \Phi(p_\nu) \in \Phi(J_\nu)$, from where $\Phi(J_\nu) \supset \bar{Q}_1$ according to [10: Lemma 3], and consequently there exists at least one point (say q_ν) in every J_ν, with $\Phi(q_\nu) = (\lambda_1, g_1)$. Because $(\lambda_1, g_1) \in bQ_1$, we have $q_\nu \in \partial A(J_\nu)$. For a fixed ν we can assume $A(J_\nu)$ to be a uniform algebra on $\partial A(J_\nu)$. Theorem 2.1 gives us now that Φ is a one-to-one mapping between $J_\nu \cap \Phi^{-1}(Q_2)$ and $Q_2 \subset Q_1$, i.e. we obtain that $Q_2 \subset W_k$ and consequently — that $\operatorname{Int} W_k \neq \emptyset$. If now $\Delta_G(\eta) \subset W$, $\Delta_G(\eta) \cap \operatorname{Int} W_k \neq \emptyset$, Theorem 1.5 implies that $\# \Phi^{-1}(\lambda, g) \leq k$ for any (λ, g) belonging to the maximal big disc $\Delta_G(\varepsilon)$ in W. By applying Theorem 1.2 we obtain the following result:

Theorem 2.2: *Let A be a uniform algebra and $\{f^p\}_{p \in S}$ be a multiplicative subsemigroup in A, isomorphic to $\mathbf{Q}_+$. Let Φ be the spectral mapping of $\{f^p\}$ and W be a component of $\Phi(\operatorname{sp} A) \setminus \Phi(\partial A)$ for which $|f^p| = \mathrm{const}$ on $\partial A \setminus \Phi^{-1}(W)$ for some (and hence — for every) $p \in S$. Suppose that there exists a measurable subset $N \subset W$ such that $dx\,dy\,(\chi^p(N)) > 0$ for some (and hence — for every) $p \in S$ and that the set $\Phi^{-1}(\lambda, g) = \{p \in M \mid \Phi(p) = (\lambda, g)\}$ is finite for any $(\lambda, g) \in N$.*

Then the set $\Phi^{-1}(W)$ has the structure of a k-sheeted generalized-analytic covering over $\Delta_G(|f^1|)$ and for any function h of A, the function $\hat{h} \circ \Phi$ is generalized-holomorphic on this covering.

Note that we know from the discussions preceding Theorem 2.2 that the set W_k is open, so that the negligible set $W \setminus W_k$ there is closed. As shown recently by B. AUPETIT and J. WERMER [2], the conditions for the set N in Wermer's theorem can be weakened. Following them, the condition $d\theta(\chi^p(N)) > 0$ for N in Theorem 2.2 can be weakened as well, by requiring N to be of nonzero exterior capacity instead of nonzero measure. In the frame of theory built above, it is possible to insert also generalized-analytic analogues to n-dimensional boundaries for the algebra A as well as the corresponding results of R. BASENER [3] for existence of n-dimensional analytic manifold's structure in the spectrum of a uniform algebra A.

All the results hold for subsemigroups S of $\mathbf{Q}_+$ possessing the following property: for any p_1 and $p_2 \in S \cap [0, 1]$ there exists a $p_3 \in S \cap [0, 1]$ such that $p_3 > p_1$ and $p_3 > p_2$. $\mathbf{Q}_+$ and $\mathbf{Z}_+$ are particular cases of such semigroups.

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