Zeitschrift für Analysis und ihre Anwendungen
Journal of Analysis and its Applications
Volume 35 (2016), 489–504
DOI: 10.4171/ZAA/1575

Bounds for the Weighted Hardy-Cesàro Operator and its Commutator on Morrey-Herz Type Spaces

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Abstract. In this paper we will prove the boundedness of weighted Hardy-Cesàro operator on Herz and Morrey-Herz spaces with absolutely homogeneous weights. The corresponding operator norms are obtained. We also prove the boundedness of the commutators of Hardy-Cesàro operator on weighted spaces of Morrey-Herz type, with their symbols belong to Lipschitz space. By these we extend and strengthen previous results due to Kuang [15], Fu et al. [9] and Tang et al. [20].

Keywords. Hardy-Cesàro operator, Hardy-Littlewood average operator, Morrey-Herz spaces

Mathematics Subject Classification (2010). Primary 47P05 , secondary 47G10, 42B35

1. Introduction

The Hardy inequality and its various generalizations play an important role in various branches of analysis such as approximation theory, partial differential equations, theory of function spaces etc. Therefore during the last twenty years a huge amount of papers has been devoted to Hardy and Hardy type inequalities in various spaces. The main results and their applications are given in the books [2, 6, 18] and references therein.

In the following, we present some of these results that serve and motivate the contents of this paper. Let $\psi : [0,1] \to [0,\infty)$ be a measurable function.

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The weighted Hardy operator U_ψ is defined on all complex-valued measurable functions f on $\mathbb{R}^n$ as

$$U_\psi f(x) = \int_0^1 f(tx)\psi(t)dt.$$

When $\psi = 1$, this operator is reduced to the usual Hardy operator S defined by

$$Sf(x) = \frac{1}{x}\int_0^x f(t)dt.$$

Results on the boundedness of U_ψ on $L^p(\mathbb{R}^n)$ were first proved by Carton-Lebrun and Fosset [1]. Under certain conditions on ψ, the authors also found that U_ψ is bounded from $BMO(\mathbb{R}^n)$ into itself. Furthermore, U_ψ commutes with the Hilbert transform in the case $n = 1$ and with a certain Calderón-Zygmund singular integral operator (and thus with the Riesz transform) in the case $n \geq 2$. A deep result for the boundeness of U_ψ on L^p space were given by J. Xiao [21] read as.

Theorem 1.1 (Xiao [21, p. 662]). *Let $1 < p < \infty$ and $\psi : [0,1] \to [0,\infty)$ be a measurable function. Then, U_ψ is bounded on $L^p(\mathbb{R}^n)$ if and only if*

$$\int_0^1 t^{-\frac{n}{p}}\psi(t)dt < \infty.$$

Furthermore,

$$\|U_\psi\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} = \int_0^1 t^{-\frac{n}{p}}\psi(t)dt < \infty.$$

Theorem 1.1 implies immediately the following celebrated integral inequality, due to Hardy [11]

$$\|Sf\|_{L^p(\mathbb{R})} \leq \frac{p}{p-1}\|f\|_{L^p(\mathbb{R})}.$$

We remind that the commutator of U_ψ, in the sense of Coifman-Rochberg-Weiss [5], is defined as

$$U_\psi^b f = bU_\psi f - U_\psi(bf).$$

Then, there is a deep result on U_ψ^b obtained by Fu et al. [10], where they showed that U_ψ^b is bounded on $L^p(\mathbb{R}^n)$ for all $b \in BMO(\mathbb{R}^n)$ if and only if

$$\int_0^1 t^{-\frac{n}{p}}\psi(t)\log\frac{2}{t}\,dt < \infty.$$

On the other hand, Chuong and Hung [3] considered a more general operator of weighted Hardy operator as follows.

Definition 1.2 ([3, p. 698])**.** Let $\psi : [0,1] \to [0,\infty)$, $s : [0,1] \to \mathbb{R}^n$ be measurable functions. We define the generalized Hardy-Cesàro operator $U_{\psi,s}$, associated to the parameter curve $s(x,t) := s(t)x$, as

$$U_{\psi,s}f(x) = \int_0^1 f\left(s(x,t)\right)\psi(t)\,dt. \tag{1}$$

for a measurable complex valued function f on $\mathbb{R}^n$.

Definition 1.3 ([3, p. 699])**.** Let $\psi : [0,1] \to [0,\infty)$, $s : [0,1] \to \mathbb{R}$ and $b : \mathbb{R}^n \to \mathbb{C}$ be measurable functions. Define the commutator of the weighted Hardy-Cesàro operator $U^b_{\psi,s}$ as

$$U^b_{\psi,s}f = bU_{\psi,s} - U_{\psi,s}(bf), \tag{2}$$

for a measurable complex valued function f on $\mathbb{R}^n$.

It turns out that such operators are still keeping almost all nice properties as the weighted Hardy-Littlewood average operators [10,21]. For examples, the authors in [3] obtained a sufficient and necessary condition on $\psi(t)$ and $s(t)$ such that $U_{\psi,s}$ is bounded on L^p and BMO spaces. The corresponding operator norms are also worked out. The authors also give necessary and sufficient conditions on ψ, s such that the commutator of $U_{\psi,s}$ is bounded on weighted L^p, with symbol b in BMO. This is interesting to notice that, the p-adic version of $U_{\psi,s}$ has a surprising application to discrete Hardy inequalities. More details, by using the boundedness of p-adic Hardy-Cesàro, the author in [12] proved the following inequality

Theorem 1.4 (Hung [12, Corollary 3.2, p. 870])**.** *Let $(x_j)_{j\in\mathbb{Z}}$ and $(y_k)_{k\geq 0}$ be two nonnegative sequences. For any positive integer β and for any $1 \leq r < \infty$, the following Hardy inequality holds*

$$\left(\sum_{j\in\mathbb{Z}}\left(\sum_{n=0}^{\infty} x_{j+\beta n}\, y_n\right)^r\right)^{\frac{1}{r}} \leq \left(\sum_{j\in\mathbb{Z}} x_j^r\right)\left(\sum_{n=0}^{\infty} y_n\right)^{\frac{1}{r}}. \tag{3}$$

For further readings on U_ψ and $U_{\psi,s}$ operators or even more on Hausdorff operators in Morrey-Herz spaces, Campanato spaces, Hardy spaces, ..., the reader may find in [3,4,8–10,12,13,15,19,20]. Now we are ready to describe our main goals of this paper.

Kuang [15] generalizes Xiao's results to Herz spaces. He obtained some necessary and sufficient conditions for the weighted Cesàro mean operators V_ψ to be bounded on Herz spaces, where

$$V_\psi f(x) = \int_0^1 f\left(\frac{x}{t}\right) t^n \psi(t)dt, \tag{4}$$

for a measurable complex-valued function f on $\mathbb{R}^n$. Fu and Lu [9]gave a necessary and sufficient condition on the weight function for the boundedness of U_ψ on the Morrey-Herz space. Our first aim of this paper is to generalize results by Kuang [15], and by Fu, Lu [9] to the weighted Hardy-Cesàro operator on the weighted Herz-type spaces. Notice that, for the boundedness on Herz spaces, we could remove the condition of concavity of the function $t \mapsto t^{-n(1-\frac{1}{q})}\psi(t)$, which were needed in [15, Theorem 1-2], in case $1 \leq p < \infty$.

On the other hand, in 2011, Tang et al. [20] gave a necessary condition on $\psi(t)$ such that U_ψ^b is bounded on Morrey-Herz spaces with symbol b belongs to a Lipschitz space (see Section 2 for its definition). They obtained the following Theorem.

Theorem 1.5 (Tang, Xue, Zhou [20, p. 269]). *Let $\psi : [0;1] \to [0;\infty)$ be a measurable function, $0 < \beta < 1$, $b \in \mathrm{Lip}^\beta(\mathbb{R}^n)$, $1 \leq q_2 \leq q_1 < \infty$. If*

$$\mathcal{C} = \int_0^1 t^{-\left(\gamma_1 - \lambda - \frac{n}{q_1}\right)}\psi(t)dt < \infty, \tag{5}$$

then U_ψ^b is bounded from $M\dot{K}_{p,q_1}^{\gamma_1,\lambda}$ to $M\dot{K}_{p,q_2}^{\gamma_2,\lambda}$, where $\gamma_1 = \gamma_2 + \beta + n\left(\frac{1}{q_2} - \frac{1}{q_1}\right)$.

Our second purpose of this paper, is to show that the condition (5) could be replaced by a weaker condition, and such result still holds when considering the commutator of $U_{\psi,s}$ in two-weighted Morrey-Herz spaces.

Our paper is organized as follows. In the first part of Section 2, we introduce necessary preliminaries on Morrey-Herz spaces and on a class of homogeneous weights. Our main theorems are given in Section 3. The main proofs for theorems are given in Section 4 and Section 5.

2. Notations and definitions

Throughout the whole paper, n denotes the dimensional number of the Euclidean space $\mathbb{R}^n$. By $\|T\|_{X\to Y}$, we denote the norm of T between two normed vector spaces X, Y. C denotes a positive geometric constant which is independent of the main parameters, but may change from line to line. For any measurable set E, we denote by χ_E its characteristic function, by $|E|$ its Lebesgue measure, and by $\omega(E)$ the integral $\int_E \omega(x)dx$. For any $a \in \mathbb{R}^n$ and $r > 0$, we shall denote by $B(a,r)$ the ball centered at a with radius r. The symbol $f \simeq g$ means that f is equivalent to g (i.e. $C^{-1}f \leq g \leq Cf$).

Let $\omega(x)$ be a weight function, that is a nonnegative locally integrable function on $\mathbb{R}^n$. The weighted $L^p(\omega)$ space is defined as the set of all measurable functions f such that

$$\|f\|_{p,\omega} = \left(\int_{\mathbb{R}^n} |f(x)|^p\omega(x)dx\right)^{\frac{1}{p}} < \infty.$$

Definition 2.1. Let $0 < \beta \leq 1$. The Lipschitz space $\mathrm{Lip}^{\beta}(\mathbb{R}^n)$ is defined as the set of all functions $f : \mathbb{R}^n \to \mathbb{C}$ such that $\|f\|_{\mathrm{Lip}^{\beta}(\mathbb{R}^n)} < \infty$, where

$$\|f\|_{\mathrm{Lip}^{\beta}(\mathbb{R}^n)} := \sup_{x,y\in\mathbb{R}^n,\, x\neq y} \frac{|f(x)-f(y)|}{|x-y|^{\beta}} < \infty. \tag{6}$$

We now present some notations and definitions from the theory of Herz and Morrey-Herz spaces, which are necessary for understanding this paper. They are taken mainly from the book [16] (for definitions and applications of two-weighted Morrey-Herz spaces). For $k \in \mathbb{Z}$, let $B_k = \{x \in \mathbb{R}^n : |x| \leq 2^k\}$, $D_k = B_k \setminus B_{k-1}$ and χ_k denotes the characteristic function of the set D_k.

Definition 2.2 (see [9,16])**.** Let $\gamma \in \mathbb{R}$, $0 < p < \infty$, $\lambda \geq 0$ and ω be weight function. We denote $M\dot{K}^{\gamma,\lambda}_{p,q}(\omega)$ by the space of all functions $f \in L^q_{loc}(\mathbb{R}^n\backslash\{0\},\omega)$ such that $\|f\|_{M\dot{K}^{\gamma,\lambda}_{p,q}(\omega)} < \infty$, where

$$\|f\|_{M\dot{K}^{\gamma,\lambda}_{p,q}(\omega)} = \sup_{k_0\in\mathbb{Z}} \left\{ 2^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} 2^{k\gamma p} \|f\chi_k\|^p_{q,\omega} \right)^{\frac{1}{p}} \right\}.$$

Definition 2.3 (see [17])**.** Let $\gamma \in \mathbb{R}$, $0 < p \leq \infty$, $\lambda \geq 0$ and ω_1, ω_2 be weight functions. We denote $M\dot{K}^{\gamma,\lambda}_{p,q}(\omega_1,\omega_2)$ by the space of all functions $f \in L^q_{loc}(\mathbb{R}^n \setminus \{0\}, \omega_2)$ such that $\|f\|_{M\dot{K}^{\gamma,\lambda}_{p,q}(\omega_1,\omega_2)} < \infty$, where

$$\|f\|_{M\dot{K}^{\gamma,\lambda}_{p,q}(\omega_1,\omega_2)} = \sup_{k_0\in\mathbb{Z}} \left\{ \omega_1\left(B_{k_0}\right)^{-\frac{\lambda}{n}} \left(\sum_{k=-\infty}^{k_0} \omega_1\left(B_k\right)^{\gamma\frac{p}{n}} \|f\chi_k\|^p_{q,\omega_2} \right)^{\frac{1}{p}} \right\}.$$

When $\omega_1(x) = c^{-1}$, where $c = |B_0|$ then $M\dot{K}^{\gamma,\lambda}_{p,q}(\omega_1,\omega_2)$ is the usual weighted Herz space $MK^{\alpha,\lambda}_{p,q}(\omega)$. We shall not discuss further the applications of these spaces here, but refer to [16,17].

Let ω be any measurable function on $\mathbb{R}^n$. Let ρ be the measure on $(0,\infty)$ so that $\rho(E) = \int_E r^{n-1}dr$ and the map $\Phi(x) = \left(|x|, \frac{x}{|x|}\right)$. Then there exists an unique Borel measure σ on S_n such that $\rho\times\sigma$ is the Borel measure induced by Φ from Lebesgue measure on $\mathbb{R}^n$ $(n > 1)$. (see [7, p. 78] or [14] for more details). In one dimension case, it it conventional that $\int_{S_n} \omega(x)d\sigma(x)$ refers to $2\omega(1)$. In this paper, we shall denote $2\omega(1)$ by $\int_{S_n} \omega(x)d\sigma(x)$ in case $n = 1$.

Definition 2.4 ([3, p. 700])**.** Let α be a real number. Let $\mathcal{W}_\alpha$ be the set of all functions ω on $\mathbb{R}^n$, which are measurable, $\omega(x) > 0$ for almost everywhere $x \in \mathbb{R}^n$, $0 < \int_{S_n} \omega(y)d\sigma(y) < \infty$, and are absolutely homogeneous of degree α, that is $\omega(tx) = |t|^{\alpha}\omega(x)$, for all $t \in \mathbb{R}\setminus\{0\}$, $x \in \mathbb{R}^n$, where $S_n = \{x \in \mathbb{R}^n : |x| = 1\}$.

Some basic examples and properties of $\mathcal{W}_\alpha$, see [3]. We also have the following property for $\mathcal{W}_\alpha$, whose proof is trivial and left to the reader.

Lemma 2.5. *If $\omega \in \mathcal{W}_\alpha$, $\alpha > -n$, then there exists a constant $C = C(\omega, n) > 0$ such that $\omega(B_k) = C|B_k|^{\frac{n+\alpha}{n}}$ and*

$$\omega(D_k) = \left(1 - 2^{-\alpha-n}\right)\omega(B_k),$$

for any integer k.

3. Statement of the results

In [15], Kuang gives some necessary and sufficient conditions for the weighted Cesàro mean operators V_ψ (see (4)) to be bounded on $K^\alpha_{p,q}(\mathbb{R}^n)$. In [9], Fu and Lu obtained a necessary and sufficient condition on the weight function for the boundedness of U_ψ on the Morrey-Herz space. The corresponding operator norm inequalities are also obtained. We will extend these results to the Hardy-Cesàro operators on the Morrey-Herz spaces with homogeneous weights. Our first main results are the following theorems concerning the boundedness and bounds of $U_{\psi,s}$ on the weighted Morrey-Herz spaces.

Theorem 3.1. *Let α, β be arbitrary real numbers, and $p, q \in [1, \infty)$. Suppose that $s(t) \neq 0$ for almost everywhere $t \in [0, 1]$ and $\omega \in \mathcal{W}_\beta$.*

(i) *If*

$$\int_0^1 |s(t)|^{-\alpha-\frac{n+\beta}{q}}\psi(t)dt < \infty, \tag{7}$$

then $U_{\psi,s}$ is a bounded operator on $K^\alpha_{p,q}(\omega)$. Moreover,

$$\|U_{\psi,s}\|_{K^\alpha_{p,q}(\omega)\to K^\alpha_{p,q}(\omega)} \leq 2\left(1 + 2^{|\alpha|}\right)\int_0^1 |s(t)|^{-\alpha-\frac{n+\beta}{q}}\psi(t)dt \tag{8}$$

(ii) *Conversely, suppose that $U_{\psi,s}$ is bounded on the space $K^\alpha_{p,q}(\omega)$. If $|s(t)| \geq c|t|^\epsilon$ for almost everywhere $t \in [0, 1]$, where c, ϵ are some positive constants, then (7) holds. Furthermore,*

$$\|U_{\psi,s}\|_{K^\alpha_{p,q}(\omega)\to K^\alpha_{p,q}(\omega)} \geq \int_0^1 |s(t)|^{-\alpha-\frac{n+\beta}{q}}\psi(t)dt. \tag{9}$$

Theorem 3.2. *Let α, β be arbitrary real numbers, $\lambda > 0$, and $p, q \in [1, \infty)$. Suppose that $s(t) \neq 0$ for almost everywhere $t \in [0, 1]$ and $\omega \in \mathcal{W}_\beta$. Then the operator $U_{\psi,s}$ is bounded from the space $MK^{\alpha,\lambda}_{p,q}(\omega)$ into itself if and only if*

$$\int_0^1 |s(t)|^{\lambda-\alpha-\frac{n+\beta}{q}}\psi(t)dt < \infty. \tag{10}$$

Moreover, when (10) holds, we have

$$\|U_{\psi,s}\|_{MK^{\alpha,\lambda}_{p,q}(\omega)\to MK^{\alpha,\lambda}_{p,q}(\omega)} \simeq \int_0^1 |s(t)|^{\lambda-\alpha-\frac{n+\beta}{q}}\psi(t)dt. \tag{11}$$

When $s(t) = t$, $\omega = 1$, one obtains immediately [9, Theorem 1-3]. When $s(t) = \frac{1}{t}$, $\omega(x) = |x|^\beta$, and replacing $\psi(t)$ by $t^{-n}\psi(t)$ one gets [9, Theorem 4] for $\beta = 0$ and [15, Theorem 1-2] for any β, and in case $1 \le p < \infty$.

On the other hand, Fu, Liu and Lu [10] established a sufficient and necessary condition on the weight function ψ to ensure the L^p $(1 < p < \infty)$ boundedness of U_ψ^b when $b \in BMO$. In [8], Fu proved the (L^p, L^q) boundedness of the classical Hardy operator when symbols $b \in \mathrm{Lip}^\beta(\mathbb{R}^n)$. Recently, Tang et al. [20] found a sufficient condition on the weight function ψ to get the boundedness of U_ψ^b on Herz type spaces when $b \in \mathrm{Lip}^\beta(\mathbb{R}^n)$. We will extend that result to the Hardy-Cesàro operators. More details, we give a sufficient condition on functions $\psi(t)$ and $s(t)$ for which $U_{\psi,s}^b$ is bounded on the weighted Morrey-Herz type spaces with symbols $b \in \mathrm{Lip}^\beta(\mathbb{R}^n)$.

Theorem 3.3. *Let* $1 \le q_2 \le q_1 < \infty$, $0 < \lambda < \infty$, $\alpha > -n$, $\omega_i \in \mathcal{W}_\alpha$ *with* $i = 1, 2$ *and* $b \in \mathrm{Lip}^\beta(\mathbb{R}^n)$, $0 < \beta \le 1$. *Suppose that* $s(t) \ne 0$ *for almost everywhere* $t \in [0, 1]$ *and*

$$\mathcal{A} = \int_0^1 |s(t)|^{-(n+\alpha)\left(\frac{\gamma_1-\lambda}{n} - \frac{1}{q_1}\right)} |1 - s(t)|^\beta \psi(t) dt < \infty. \tag{12}$$

Then for any $\lambda > 0$ *and* $0 < p < \infty$ *or* $\lambda = 0$ *and* $1 \le p < \infty$, *the commutator* $U_{\psi,s}^b$ *is determined as a bounded operator from* $M\dot{K}_{p,q_1}^{\gamma_1,\lambda}(\omega_1, \omega_2)$ *to* $M\dot{K}_{p,q_2}^{\gamma_2,\lambda}(\omega_1, \omega_2)$, *where* $\gamma_1 = \gamma_2 + \frac{n\beta}{n+\alpha} + n\left(\frac{1}{q_2} - \frac{1}{q_1}\right)$.

When $\omega_1 = \frac{1}{|B_0|}$, $\omega_2 = 1$, $s(t) = t$ we obtain the following result.

Corollary 3.4. *Let* $\psi : [0; 1] \to [0; \infty)$, $0 < \beta \le 1$, $b \in \mathrm{Lip}^\beta(\mathbb{R}^n)$ *and* $1 \le q_2 \le q_1 < \infty$. *If*

$$\mathcal{B} = \int_0^1 t^{-(\gamma_1 - \lambda - \frac{n}{q_1})} (1 - t)^\beta \psi(t) dt < \infty, \tag{13}$$

then U_ψ^b *is bounded from* $M\dot{K}_{p,q_1}^{\gamma_1,\lambda}$ *to* $M\dot{K}_{p,q_2}^{\gamma_2,\lambda}$, *where* $\gamma_1 = \gamma_2 + \beta + n\left(\frac{1}{q_2} - \frac{1}{q_1}\right)$.

In [20], to obtain the boundedness of U_ψ^b from $M\dot{K}_{p,q_1}^{\gamma_1,\lambda}$ to $M\dot{K}_{p,q_2}^{\gamma_2,\lambda}$, the authors required a sufficient condition condition on ψ that

$$\mathcal{C} = \int_0^1 t^{-\left(\gamma_1 - \lambda - \frac{n}{q_1}\right)} \psi(t) dt < \infty.$$

Since $0 \le t \le 1$, then $\mathcal{B} \le \mathcal{C}$. In fact by choosing $\psi(t) = \frac{t}{(1-t)^{1+\frac{\beta}{2}}}$, $\gamma_1 - \lambda - \frac{n}{q_1} = 1$, since $0 < \beta \le 1$, it is easy to see that $\mathcal{C} = \infty$ but $\mathcal{B} < \infty$. Thus our result extends and strengthens result due to Tang et al. [20].

4. Proofs of Theorem 3.1 and Theorem 3.2

Suppose that (10) holds. It is enough to show that $U_{\psi,s}$ is bounded on $MK^{\alpha,\lambda}_{p,q}(\omega)$, where $\lambda \geq 0$ (thus part (i) of Theorem 3.1 follows directly since $MK^{\alpha,0}_{p,q}(\omega) = K^{\alpha}_{p,q}(\omega)$). Fix $k \in \mathbb{Z}$, using the Minkowski inequality and change of variable $y = s(t)x$, we have

$$\begin{aligned}\|(U_{\psi,s}f)\chi_k\|_{q,\omega} &= \left(\int_{D_k}\left|\int_0^1 f(s(t)x)\psi(t)dt\right|^q \omega(x)dx\right)^{\frac{1}{q}}\\ &\leq \int_0^1 \left(\int_{D_k} |f(s(t)x)|^q \omega(x)dx\right)^{\frac{1}{q}} \psi(t)dt\\ &= \int_0^1 \left(\int_{S(k,t)} |f(y)|^q \omega(y)dy\right)^{\frac{1}{q}} |s(t)|^{-\frac{n+\beta}{q}}\psi(t)dt,\end{aligned}$$

where $S(k,t) = \{y \in \mathbb{R}^n : 2^{k-1}|s(t)| < |y| \leq 2^k|s(t)|\}$. For each $t \in [0,1]$ which $s(t) \neq 0$, one can find an integer number $m = m(t)$ such that $2^{m-1} < |s(t)| \leq 2^m$. This implies $S(k,t)$ is a subset of $D_{k+m-1} \cup D_{k+m}$. Thus we obtain that

$$\begin{aligned}\|(U_{\psi,s}f)\chi_k\|_{q,\omega} &\leq \int_0^1 \left(\int_{D_{k+m-1}\cup D_{k+m}} |f(y)|^q \omega(y)dy\right)^{\frac{1}{q}} |s(t)|^{-\frac{n+\beta}{q}}\psi(t)dt\\ &\leq \int_0^1 \left(\|f\chi_{k+m-1}\|_{q,\omega} + \|f\chi_{k+m}\|_{q,\omega}\right)|s(t)|^{-\frac{n+\beta}{q}}\psi(t)dt.\end{aligned}$$

So by the definition of Morrey-Herz spaces, one has

$$\begin{aligned}&\|U_{\psi,s}f\|_{MK^{\alpha,\lambda}_{p,q}(\omega)}\\ &= \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\|(U_{\psi,s}f)\chi_k\|^p_{q,\omega}\right)^{\frac{1}{p}}\\ &\leq \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\left(\int_0^1 \left(\|f\chi_{k+m-1}\|_{q,\omega}\right.\right.\right.\\ &\quad + \left.\left.\left.\|f\chi_{k+m}\|_{q,\omega}\right)|s(t)|^{-\frac{n+\beta}{q}}\psi(t)dt\right)^p\right)^{\frac{1}{p}}\\ &\leq \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\int_0^1\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\|f\chi_{k+m-1}\|^p_{q,\omega}|s(t)|^{-(\frac{n+\beta}{q})p}\right)^{\frac{1}{p}} |s(t)|^{-\frac{n+\beta}{q}}\psi(t)dt\\ &\quad + \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\int_0^1\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\|f\chi_{k+m}\|^p_{q,\omega}\right)^{\frac{1}{p}} |s(t)|^{-\frac{n+\beta}{q}}\psi(t)dt\end{aligned}$$

$$\leq \int_0^1 \sup_{k_0\in\mathbb{Z}} 2^{-(k_0+m-1)\lambda}\left(\sum_{k=-\infty}^{k_0+m-1} 2^{k\alpha p}\|f\chi_k\|_{q,\omega}^p\right)^{\frac{1}{p}} 2^{(m-1)(\lambda-\alpha)}|s(t)|^{-\frac{n+\beta}{q}}\psi(t)dt$$

$$+\int_0^1 \sup_{k_0\in\mathbb{Z}} 2^{-(k_0+m)\lambda}\left(\sum_{k=-\infty}^{k_0+m} 2^{k\alpha p}\|f\chi_k\|_{q,\omega}^p\right)^{\frac{1}{p}} 2^{m(\lambda-\alpha)}|s(t)|^{-\frac{n+\beta}{q}}\psi(t)dt$$

$$\leq (1+2^{\lambda})\|f\|_{MK_{p,q}^{\alpha,\lambda}(\omega)}\int_0^1 (2^{-(m-1)\alpha}+2^{-m\alpha})|s(t)|^{\lambda-\frac{n+\beta}{q}}\psi(t)dt.$$

$$\leq (1+2^{\lambda})(1+2^{|\alpha|})\left(\int_0^1 |s(t)|^{\lambda-\alpha-\frac{n+\beta}{q}}\psi(t)dt\right)\|f\|_{MK_{p,q}^{\alpha,\lambda}(\omega)}.$$

Hence, by (7) and (10), we get that $U_{\psi,s}$ is bounded on $MK_{p,q}^{\alpha,\lambda}(\omega)$ and

$$\|U_{\psi,s}\|_{MK_{p,q}^{\alpha,\lambda}\to MK_{p,q}^{\alpha,\lambda}} \leq (1+2^{\lambda})(1+2^{|\alpha|})\int_0^1 |s(t)|^{\lambda-\alpha-\frac{n+\beta}{q}}\psi(t)dt. \tag{14}$$

Part (ii). Conversely, suppose that $U_{\psi,s}$ is bounded on the $MK_{p,q}^{\alpha,\lambda}(\omega)$, where $\lambda \geq 0$. We will consider two cases as follows.

Case 1: $\lambda > 0$. In this case, we set

$$f_0(x) = |x|^{\lambda-\alpha-\frac{n+\beta}{q}}.$$

It is obviously that $f_0 \in L^q_{\mathrm{loc}}(\omega, \mathbb{R}^n\setminus\{0\})$. Since $\omega(rx) = r^{\beta}\omega(x)$ for any $r > 0$,

$$\begin{aligned}\|f_0\chi_k\|_{q,\omega}^q &= \int_{D_k} |x|^{q\lambda-q\alpha-n-\beta}\omega(x)dx\\ &= \omega(S_n)\int_{2^{k-1}}^{2^k} r^{q\lambda-q\alpha-1}dr\\ &= \begin{cases}\omega(S_n)\ln 2, & \text{if } \alpha=\lambda,\\ \left|\dfrac{1-2^{q(\alpha-\lambda)}}{(\alpha-\lambda)q}\right|\omega(S_n)2^{-k(\alpha-\lambda)q}, & \text{if } \alpha\neq\lambda.\end{cases}\end{aligned}$$

Therefore

$$\begin{aligned}\|f_0\|_{MK_{p,q}^{\alpha,\lambda}(\omega)} &= \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\|f_0\chi_k\|_{q,\omega}^p\right)^{\frac{1}{p}}\\ &\lesssim \omega(S_n)^{\frac{1}{q}}\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}2^{kp(\lambda-\alpha)}\right)^{\frac{1}{p}}\\ &< \infty.\end{aligned}$$

On the other hand, notice that $\|f_0\|_{MK^{\alpha,\lambda}_{p,q}(\omega)} > 0$ and

$$U_{\psi,s}f_0(x) = f_0(x)\int_0^1 |s(t)|^{\lambda-\alpha-\frac{n+\beta}{q}}\psi(t)dt.$$

Then

$$\int_0^1 |s(t)|^{\lambda-\alpha-\frac{n+\beta}{q}}\psi(t)dt \leq \|U_{\psi,s}\|_{MK^{\alpha,\lambda}_{p,q}(\omega)\to MK^{\alpha,\lambda}_{p,q}(\omega)}, \tag{15}$$

Combining (14) and (15), then

$$\|U_{\psi,s}\|_{MK^{\alpha}_{p,q}(\omega)\to MK^{\alpha}_{p,q}(\omega)} \simeq \int_0^1 |s(t)|^{-\alpha-\frac{n+\beta}{q}}\psi(t)dt,$$

which completes the proof of Theorem 3.2.

Case 2: $\lambda = 0$. In this case, we set for any $m \in \mathbb{Z}$,

$$f_m(x) = \begin{cases} 0, & \text{if } |x| < 1, \\ |x|^{-\alpha-\frac{n+\beta}{q}-\frac{1}{2^m}}, & \text{if } |x| \geq 1. \end{cases}$$

First we have $f_m\chi_k = 0$ when $k < 0$. Let k be any nonnegative integer, we choose m large enough such that $\alpha + \frac{1}{2^m} \neq 0$. This gives

$$\begin{aligned} \|f_m\chi_k\|^q_{q,\omega} &= \int_{D_k} |x|^{-\alpha q-n-\beta-\frac{q}{2^m}}\omega(x)dx \\ &= \omega(S_n)\int_{2^{k-1}}^{2^k} r^{-q\alpha-\frac{q}{2^m}-1}dr \\ &= \left|\frac{1-2^{q(\alpha+\frac{1}{2^m})}}{q\left(\alpha+\frac{1}{2^m}\right)}\right| 2^{-kq\left(\alpha+\frac{1}{2^m}\right)}\omega(S_n). \end{aligned}$$

Hence,

$$\begin{aligned} \|f_m\|_{K^{\alpha}_{p,q}(\omega)} &= \left(\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\|f_m\chi_k\|^p_{q,\omega}\right)^{\frac{1}{p}} \\ &= \left|\frac{1-2^{q(\alpha+\frac{1}{2^m})}}{q(\alpha+\frac{1}{2^m})}\right|^{\frac{1}{q}} \omega(S_n)^{\frac{1}{q}} \left(\sum_{k=0}^{\infty} 2^{k\alpha p}2^{-kp(\alpha+\frac{1}{2^m})}\right)^{\frac{1}{p}} \\ &< \infty. \end{aligned}$$

On the other hand,

$$U_{\psi,s}f_m(x) = \begin{cases} 0, & \text{if } |x| < 1, \\ |x|^{-\alpha-\frac{n+\beta}{q}-\frac{1}{2^m}}\int_{S(x)} |s(t)|^{-\alpha-\frac{n+\beta}{q}-\frac{1}{2^m}}\psi(t)dt, & \text{if } |x| \geq 1, \end{cases} \tag{16}$$

where $S(x) = \{t \in [0,1] : |s(t)x| \geq 1\}$. Since $|s(t)| \geq c|t|^{\epsilon}$ for almost every $t \in [0,1]$, there exists a measurable subset A with $|A| = 0$ satisfying

$$S(x) \supset \{t \in [0,1] : c|t|^{\epsilon}|x| \geq 1\} \setminus A.$$

For every $m \geq 1$, let

$$S_m = \left\{ t \in [0,1] : |t| \geq \frac{2^{-\frac{m}{\epsilon}}}{c^{\frac{1}{\epsilon}}} \right\}.$$

The sequence $\{S_m\}_{m\geq 0}$ is increasing and tends to $(0,1]$. From (16), for each $k \leq 0$, then $(U_{\psi,s} f_m)\, \chi_k = 0$. Let $k \geq m \geq 1$, then

$$\begin{aligned}\|(U_{\psi,s} f_m)\chi_k\|_{q,\omega}^q &\geq \int_{D_k} |x|^{-\alpha q - n - \beta - \frac{q}{2^m}} \omega(x) \left(\int_{S_k} |s(t)|^{-\alpha - \frac{n+\beta}{q} - \frac{1}{2^m}} \psi(t) dt \right)^q dx \\ &\geq \left(\int_{S_m} |s(t)|^{-\alpha - \frac{n+\beta}{q} - \frac{1}{2^m}} \psi(t) dt \right) \left(\int_{D_k} |x|^{-\alpha q - n - \beta - \frac{q}{2^m}} \omega(x) dx \right)^{\frac{1}{q}} \\ &= \left(\int_{S_m} |s(t)|^{-\alpha - \frac{n+\beta}{q} - \frac{1}{2^m}} \psi(t) dt \right) \|f_m \chi_k\|_{q,\omega}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\|U_{\psi,s} f_m\|_{K^{\alpha}_{p,q}(\omega)} \\ &\geq \left(\sum_{k=m}^{\infty} 2^{k\alpha p} \|f_m \chi_k\|_{q,\omega}^p \right)^{\frac{1}{p}} \left(\int_{D_m} |s(t)|^{-\alpha - \frac{n+\beta}{q} - \frac{1}{2^m}} \psi(t) dt \right) \\ &\geq \left| \frac{1 - 2^{q(\alpha + \frac{1}{2^m})}}{q(\alpha + \frac{1}{2^m})} \right|^{\frac{1}{q}} \omega(S_n)^{\frac{1}{q}} \left(\sum_{k=m}^{\infty} 2^{k\alpha p} 2^{-kp(\alpha + \frac{1}{2^m})} \right)^{\frac{1}{q}} \left(\int_{D_m} |s(t)|^{-\alpha - \frac{n+\beta}{q} - \frac{1}{2^m}} \psi(t) dt \right) \\ &= \|f_m\|_{K^{\alpha}_{p,q}(\omega)} \left(2^{-\frac{m}{2^m}} \int_{D_m} |s(t)|^{-\alpha - \frac{n+\beta}{q} - \frac{1}{2^m}} \psi(t) dt \right).\end{aligned}$$

So we have $\|U_{\psi,s}\|_{K^{\alpha}_{p,q}(\omega) \to K^{\alpha}_{p,q}(\omega)} \geq 2^{-\frac{m}{2^m}} \int_{D_m} |s(t)|^{-\alpha - \frac{n+\beta}{q} - \frac{1}{2^m}} \psi(t) dt$. Letting $m \to \infty$, one obtains

$$\|U_{\psi,s}\|_{K^{\alpha}_{p,q}(\omega) \to K^{\alpha}_{p,q}(\omega)} \geq \int_0^1 |s(t)|^{-\alpha - \frac{n+\beta}{q}} \psi(t) dt,$$

and hence Theorems 3.1 and 3.2 are proved.

5. Proof of Theorem 3.3

Let $r = \frac{1}{q_2} - \frac{1}{q_1}$. Suppose that $\mathcal{A}$ is finite. To obtain the boundedness of $U^b_{\psi,s}$ it suffices to prove for any $k_0 \in \mathbb{Z}$ that

$$\omega_1(B_{k_0})^{-\frac{\lambda}{n}} \left(\sum_{k=-\infty}^{k_0} \omega_1(B_k)^{\gamma_2 \frac{p}{n}} \|(U^b_{\psi,s} f)\chi_k\|_{q_2,\omega_2}^p \right)^{\frac{1}{p}} \lesssim \|b\|_{\mathrm{Lip}^{\beta}} \|f\|_{M\dot{K}^{\gamma_1,\lambda}_{p,q_1}} \mathcal{A}. \quad (17)$$

Since $b \in \mathrm{Lip}^{\beta}(\mathbb{R}^n)$,

$$|b(x) - b(s(t)x)| \lesssim \|b\|_{\mathrm{Lip}^{\beta}} |1 - s(t)|^{\beta}\, |x|^{\beta}.$$

For any $k \in \mathbb{Z}$, by Minkowski's and Hölder's inequalities we have

$$\begin{aligned}
&\| \left(U^b_{\psi,s} f\right) \chi_k\|_{q_2,\omega_2} \\
&= \left(\int_{D_k} \left| \int_0^1 (b(x) - b\,(s(t)x))\, f\,(s(t)x)\,\psi(t)dt \right|^{q_2} \omega_2(x)dx \right)^{\frac{1}{q_2}} \\
&\lesssim \|b\|_{\mathrm{Lip}^{\beta}(\mathbb{R}^n)} |B_k|^{\frac{\beta}{n}} \int_0^1 \left(\int_{D_k} |f(s(t)x)|^{q_2} \omega_2(x)dx \right)^{\frac{1}{q_2}} \widetilde{\psi}(t)dt \\
&\lesssim \|b\|_{\mathrm{Lip}^{\beta}(\mathbb{R}^n)} |B_k|^{\frac{\beta}{n}} \int_0^1 \left(\int_{D_k} |f(s(t)x)|^{q_1} \omega_2(x)dx \right)^{\frac{1}{q_1}} \left(\int_{D_k} \omega_2(x)dx \right)^{r} \widetilde{\psi}(t)dt \\
&\lesssim \|b\|_{\mathrm{Lip}^{\beta}(\mathbb{R}^n)}\, |B_k|^{\frac{\beta+(n+\alpha)r}{n}} \int_0^1 \|f\,(s(t)\cdot \circ)\,\chi_k\|_{q_1,\omega_2} \widetilde{\psi}(t)dt,
\end{aligned}$$

where $\widetilde{\psi}(t) := |1 - s(t)|^{\beta}\psi(t)$. Since $s(t) \neq 0$ almost everywhere $t \in [0, 1]$, there exists an integer $m = m(t)$ such that $2^{-m-1} < |s(t)| \leq 2^{-m}$. Thus $s(t)x \in D_{k-m-1} \cup D_{k-m}$ for each $x \in B_k$, $t \in [0, 1]$. This implies that

$$\begin{aligned}
&\| \left(U^b_{\psi,s} f\right) \chi_k\|_{q_2,\omega_2} \\
&\lesssim \|b\|_{\mathrm{Lip}^{\beta}(\mathbb{R}^n)}\, |B_k|^{\frac{\beta+(n+\alpha)r}{n}} \int_0^1 \left(\sum_{i=0,1} \|f\chi_{k-m-i}\|_{q_1,\omega_2} \right) \cdot |s(t)|^{-\frac{n+\alpha}{q_1}} \widetilde{\psi}(t)dt.
\end{aligned}$$

If we put

$$S^i_k(t) = \|f\chi_{k-m-i}\|_{q_1,\omega_2} \cdot |s(t)|^{-\frac{n+\alpha}{q_1}} \widetilde{\psi}(t)dt,$$

where $i = 0, 1$ then

$$\| \left(U^b_{\psi,s} f\right) \chi_k\|_{q_2,\omega_2} \lesssim \|b\|_{\mathrm{Lip}^{\beta}(\mathbb{R}^n)}\, |B_k|^{\frac{\beta+(n+\alpha)r}{n}} \cdot \sum_{i=0,1} \int_0^1 S^i_k(t)dt.$$

Hence, we obtain

$$\begin{aligned}
&\omega_1(B_{k_0})^{-\frac{\lambda}{n}} \left(\sum_{k=-\infty}^{k_0} \omega_1(B_k)^{\gamma_2 \frac{p}{n}} \| \left(U^b_{\psi,s} f\right) \chi_k\|^p_{q_2,\omega_2} \right)^{\frac{1}{p}} \\
&\lesssim \|b\|_{\mathrm{Lip}^{\beta}(\mathbb{R}^n)}\, \omega_1(B_{k_0})^{-\frac{\lambda}{n}} \left(\sum_{k=-\infty}^{k_0} \left(\sum_{i=0,1} \omega_1(B_k)^{\frac{\gamma}{n}} |B_k|^{\frac{\beta+(n+\alpha)r}{n}} \int_0^1 S^i_k(t)dt \right)^{p} \right)^{\frac{1}{p}}
\end{aligned} \tag{18}$$

To estimate the right hand side of (18), we consider the following cases.

Case 1: $1 \le p < \infty$ and $\lambda \ge 0$. It follows from Lemma 2.5 that $\frac{|B_k|}{\omega_1(B_k)}$ is a constant and

$$\frac{\omega_1(B_k)}{\omega_1(B_{k-m-i})} = 2^{(m+i)n}$$

By using these remarks and Minkowski's inequality, we thus obtain

$$\begin{aligned}
&\omega_1\left(B_{k_0}\right)^{-\frac{\lambda}{n}}\left(\sum_{k=-\infty}^{k_0}\omega_1\left(B_k\right)^{\gamma_2\frac{p}{n}}\|\left(U^b_{\psi,s}f\right)\chi_k\|^p_{q_2,\omega_2}\right)^{\frac{1}{p}}\\
&\lesssim \sum_{i=0,1}\|b\|_{\mathrm{Lip}^\beta(\mathbb{R}^n)}\,\omega_1\left(B_{k_0}\right)^{-\frac{\lambda}{n}}\\
&\quad\times\int_0^1\left(\sum_{k=-\infty}^{k_0}\omega_1\left(B_k\right)^{\frac{\gamma_2 p}{n}}|B_k|^{\frac{\beta+(n+\alpha)r}{n}p}\|f\chi_{k-m-i}\|^p_{q_1,\omega_2}\right)^{\frac{1}{p}}|s(t)|^{-\frac{n+\alpha}{q_1}}\widetilde{\psi}(t)dt\\
&\lesssim \sum_{i=0,1}\|b\|_{\mathrm{Lip}^\beta(\mathbb{R}^n)}\int_0^1\omega_1\left(B_{k_0-m-i}\right)^{-\frac{\lambda}{n}}\left(\sum_{k=-\infty}^{k_0}\omega_1\left(B_{k-m-i}\right)^{\frac{\gamma_1 p}{n}}\|f\chi_{k-m-i}\|^p_{q_1,\omega_2}\right)^{\frac{1}{p}}\\
&\quad\times\left(\frac{\omega_1\left(B_k\right)}{\omega_1\left(B_{k-m-i}\right)}\right)^{\frac{\gamma_1}{n}}\left(\frac{\omega_1(B_{k_0})}{\omega_1\left(B_{k_0-m-i}\right)}\right)^{-\frac{\lambda}{n}}\left(\frac{|B_k|}{\omega_1\left(B_k\right)^{\frac{n}{n+\alpha}}}\right)^{\frac{\beta+(n+\alpha)r}{n}}|s(t)|^{-\frac{n+\alpha}{q_1}}\widetilde{\psi}(t)dt\\
&\lesssim \sum_{i=0,1}\|b\|_{\mathrm{Lip}^\beta}\|f\|_{M\dot{K}^{\gamma_1,\lambda}_{p,q_1}}\int_0^1 2^{(m+i)(n+\alpha)\frac{\gamma_1-\lambda}{n}}|s(t)|^{-\frac{n+\alpha}{q_1}}\widetilde{\psi}(t)dt.
\end{aligned}$$

Note now that for any $t \in [0,1]$ the inequality $\frac{1}{2|s(t)|} \le 2^{m+i} \le \frac{2}{|s(t)|}$, $(i = 0,1)$ holds. It follows that

$$\int_0^1 2^{(m+i)(n+\alpha)\frac{\gamma_1-\lambda}{n}}|s(t)|^{-\frac{n+\alpha}{q_1}}\widetilde{\psi}(t)dt \le \max\left\{2^{\frac{(n+\alpha)(\gamma_1-\lambda)}{n}}, 2^{\frac{(n+\alpha)(-\gamma_1+\lambda)}{n}}\right\}\mathcal{A}.$$

and therefore (17) is proved.

Case 2: $0 < p < 1$ and $\lambda > 0$. We first observe that

$$\begin{aligned}
\|f\chi_{k-m-i}\|_{q_1,\omega_2} &\le \omega_1(B_{k-m-i})^{\frac{\lambda-\gamma_1}{n}}\,\omega_1(B_{k-m-i})^{-\frac{\lambda}{n}}\left(\sum_{j=-\infty}^{k-m-i}\omega(B_j)^{\frac{\gamma_1 p}{n}}\|f\chi_j\|^p_{q_1,\omega_2}\right)^{\frac{1}{p}}\\
&\le \omega_1(B_{k-m-i})^{\frac{\lambda-\gamma_1}{n}}\|f\|_{M\dot{K}^{\gamma_1,\lambda}_{p,q_1}(\omega_1,\omega_2)}.
\end{aligned}$$

Combining with (17), this leads to

$$\begin{aligned}
&\omega_1\left(B_{k_0}\right)^{-\frac{\lambda}{n}}\left(\sum_{k=-\infty}^{k_0}\omega_1\left(B_k\right)^{\frac{\gamma_2 p}{n}}\|\left(U^b_{\psi,s}f\right)\chi_k\|^p_{q_2,\omega_2}\right)^{\frac{1}{p}}\\
&\lesssim\sum_{i=0,1}\|b\|_{\mathrm{Lip}^{\beta}(\mathbb{R}^n)}\,\omega_1\left(B_{k_0}\right)^{-\frac{\lambda}{n}}\\
&\quad\times\left(\sum_{k=-\infty}^{k_0}\left(\omega_1\left(B_k\right)^{\frac{\gamma_2}{n}}|B_k|^{\frac{\beta+(n+\alpha)r}{n}}\int_0^1\|f\chi_{k-m-i}\|_{q_1,\omega_2}\cdot|s(t)|^{-\frac{n+\alpha}{q_1}}\widetilde{\psi}(t)dt\right)^p\right)^{\frac{1}{p}}\\
&\lesssim\sum_{i=0,1}\|b\|_{\mathrm{Lip}^{\beta}(\mathbb{R}^n)}\cdot\|f\|_{M\dot{K}^{\gamma_1,\lambda}_{p,q_1}(\omega_1,\omega_2)}\times\mathbf{J}^{\frac{1}{p}}
\end{aligned}$$

where

$$\mathbf{J}:=\sum_{k=-\infty}^{k_0}\left(\int_0^1\left(\frac{\omega_1(B_k)}{\omega(B_{k-m-i})}\right)^{\frac{\gamma_1}{n}}\left(\frac{|B_k|}{\omega_1(B_k)^{\frac{n}{n+\alpha}}}\right)^{\frac{\beta+(n+\alpha)r}{n}}\left(\frac{\omega_1(B_{k-m-i})}{\omega_1(B_{k_0})}\right)^{\frac{\lambda}{n}}|s(t)|^{-\frac{n+\alpha}{q_1}}\widetilde{\psi}(t)dt\right)^p.$$

By Lemma 2.5, for any $k\leq k_0$, we have

$$\begin{aligned}
\left(\frac{\omega_1(B_k)}{\omega(B_{k-m-i})}\right)^{\frac{\gamma_1}{n}}\left(\frac{|B_k|}{\omega_1(B_k)^{\frac{n}{n+\alpha}}}\right)^{\frac{\beta+(n+\alpha)r}{n}}\left(\frac{\omega_1(B_{k-m-i})}{\omega_1(B_{k_0})}\right)^{\frac{\lambda}{n}}&\lesssim 2^{(m+i)(\gamma_1-\lambda)+(k-k_0)\lambda}\\
&\lesssim|s(t)|^{(\lambda-\gamma_1)}\cdot 2^{(k-k_0)\lambda}.
\end{aligned}$$

Finally, we obtain

$$\begin{aligned}
&\omega_1\left(B_{k_0}\right)^{-\frac{\lambda}{n}}\left(\sum_{k=-\infty}^{k_0}\omega_1\left(B_k\right)^{\frac{\gamma_2 p}{n}}\|\left(U^b_{\psi,s}f\right)\chi_k\|^p_{q_2,\omega_2}\right)^{\frac{1}{p}}\\
&\lesssim\|b\|_{\mathrm{Lip}^{\beta}(\mathbb{R}^n)}\cdot\|f\|_{M\dot{K}^{\gamma_1,\lambda}_{p,q_1}(\omega_1,\omega_2)}\left(\sum_{k=-\infty}^{k_0}2^{(k-k_0)p\lambda}\right)^{\frac{1}{p}}\int_0^1|s(t)|^{\lambda-\gamma_1-\frac{n+\alpha}{q_1}}\widetilde{\psi}(t)dt\\
&\lesssim\|b\|_{\mathrm{Lip}^{\beta}(\mathbb{R}^n)}\cdot\|f\|_{M\dot{K}^{\gamma_1,\lambda}_{p,q_1}(\omega_1,\omega_2)}\int_0^1|s(t)|^{\lambda-\gamma_1-\frac{n+\alpha}{q_1}}\widetilde{\psi}(t)dt.
\end{aligned}$$

Consequently, $U^b_{\psi,s}$ is determined as a bounded operator from $M\dot{K}^{\gamma_1,\lambda}_{p,q_1}(\omega_1,\omega_2)$ to $M\dot{K}^{\gamma_2,\lambda}_{p,q_2}(\omega_1,\omega_2)$ and

$$\|U^b_{\psi,s}\|_{M\dot{K}^{\gamma,\lambda}_{p,q_2}(\omega_1,\omega_2)\to M\dot{K}^{\gamma,\lambda}_{p,q_2}(\omega_1,\omega_2)}\lesssim\|b\|_{\mathrm{Lip}^{\beta}(\mathbb{R}^n)}\cdot\int_0^1|s(t)|^{\lambda-\gamma_1-\frac{n+\alpha}{q_1}}\widetilde{\psi}(t)dt.\qquad\square$$

Acknowledgement. The authors cordially thank the anonymous referees for their valuable comments which lead to the improvement of this paper. The first and second authors were supported by the Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.02-2014.51.

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Received June 18, 2014; revised April 5, 2016