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Two Non-Zero Solutions for Elliptic Dirichlet Problems

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Abstract. The aim of this note is to point out a two non-zero critical points theorem for differentiable functionals and, as an application, to obtain existence results of two positive solutions for elliptic Dirichlet problems by requiring, in particular, a suitable condition on the nonlinearity which is more general than the sublinearity at zero.

Keywords. Variational methods, multiple critical points, elliptic nonlinear equations, boundary value problems

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1. Introduction

In this paper, the existence of two non-zero critical points for an appropriate class of differentiable functionals is established. Our main tools are a local minimum theorem established in [5] and the powerful classical Ambrosetti-Rabinowitz theorem (see [4]). Our main result (Theorem 2.1) is not obtained as an immediate consequence of these previous critical point theorems, but it is indeed an appropriate combination of such results in order to obtain two non-zero critical points. In fact, once obtained the first non zero critical point by the local minimum theorem, a direct application of the mountain pass theorem allows to get the second critical point that in general can be zero. Instead, in Theorem 2.1, we verify that a local minimum actually is a global minimum for a suitable restriction of the functional and, hence, we prove that all the paths starting from it have a high level greater than zero, and this guarantees the second critical point must be non-zero (see the proof of Theorem 2.1).

It is worth noticing that Theorem 2.1 can be applied to a very wide class of nonlinear differential problems thus ensuring the existence of two non-zero solutions, both for ordinary and partial differential equations.

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In this note, we investigate nonlinear elliptic Dirichlet problems. To be precise, consider the following problem

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{D_λ^f}$$

where Ω is a non-empty bounded open subset of the real euclidian space $(\mathbb{R}^N, |\cdot|)$, $N \geq 3$, with boundary of class C^1, λ is a positive parameter and $f : \mathbb{R} \to \mathbb{R}$ is a function. On applications to elliptic differential problems, the main result of this paper is Theorem 3.1 which, under suitable assumptions on the primitive of the nonlinearity f, ensures the existence of at least two positive weak solutions. As an example, we present here the following special case.

Theorem 1.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a nonnegative continuous function for which there exists $\lim_{t\to+\infty} \frac{\int_0^t f(s)ds}{tf(t)}$ and one has $\lim_{t\to+\infty} \frac{f(t)}{t^r} = 0$ for some $r \in \left]1, \frac{N+2}{N-2}\right[$. Assume that*

$$\lim_{t\to+\infty} \frac{f(t)}{t^p} = +\infty \tag{1.1}$$

for some $p \in]1, r[$, and

$$\lim_{t\to0^+} \frac{f(t)}{t} = +\infty. \tag{1.2}$$

Then, there is $\lambda^* > 0$ such that, for each $\lambda \in]0, \lambda^*[$, problem (D_λ^f) admits at least two positive weak solutions.

The key assumptions of Theorem 1.1 is that f is more than superlinear at $+\infty$, that is (1.1), and sublinear at zero, that is (1.2). Assumption (1.1), together with the existence of the limit of $\frac{\int_0^t f(s)ds}{tf(t)}$ as t goes to $+\infty$, is a special case of the classical Ambrosetti-Rabinowitz condition (see Remark 3.10). Recently, such a condition, in order to get the existence of at least one non-zero solution, has been improved (see, for instance, [17]). Precisely, the existence of one non-zero solution has been obtained for a class of functions which are only superlinear at $+\infty$ (that is, $p = 1$ in (1.1)), by using the version of the mountain pass theorem with the Cerami condition instead of the Palais-Smale condition (see Remark 2.2). We observe that, by using the same methods, we can extend in Theorem 1.1 the assumption (1.1) to $p = 1$, at least for such a class of functions (see Remark 3.11). However, our result does not take the whole class of superlinear functions (see the end of Remarks 3.10 and 3.11) exactly as well as it happens, at the best of our knowledge, in the results of the existence of one non-zero solution obtained by a mountain pass theorem (see [2, Chapter 11] for an overview).

Assumption (1.2) is a direct consequence of the local minimum theorem obtained in [6] and it ensures the existence of a critical point which has negative energy. Actually, it can be formulated in a more general form (see (3.1) in Theorem 3.1) by including so, as well as all sublinear functions, also some classes of functions which can be also linear or superlinear at zero. Thus, in particular, we obtain the existence of two positive solutions for nonlinearities which are superlinear both at zero and at $+\infty$ (see Example 3.3). We recall that in the fine and classical work of Crandall-Rabinowitz [9], instead of (1.2), the more restrictive assumption $f(0) > 0$ is required in order to obtain the conclusion of Theorem 1.1 (see Remark 3.6). We also point out that, in particular, Theorem 1.1 can be used to study combined effects of concave and convex nonlinearities, that is, when the function f is of the type

$$f(t) = |t|^{\tilde{q}} + \mu|t|^{\tilde{s}}, \quad t \in \mathbb{R}, \tag{1.3}$$

where $0 \leq \tilde{s} < 1 < \tilde{q} < \frac{N+2}{N-2}$ and for all $\mu \in]0, \mu^*[$, for a suitable $\mu^* > 0$, by proving that, in this case, one has $\lambda^* > 1$, for which (D_1^f), with f as in (1.3), admits at least two positive weak solutions (see Corollary 3.7). This latest problem has been introduced and developed in the seminal and fundamental paper due to Ambrosetti-Brezis-Cerami [3], where also topological methods have been applied (see Remark 3.8). It is worth noticing that in our proof only variational methods are used and moreover, a wider class of nonlinearities can be considered (see Examples 3.3 and 3.5), including, as already said before, some cases where f may be not sublinear at zero. Furthermore, for more recent papers that deal with differential problems in direction of concave-convex nonlinearities we refer the reader to the fine works [10, 12–16], by observing that also such results cannot be applied to problems as in Example 3.3. An exhaustive overview and a complete bibliography on this subject can be found in [17].

Finally, we point out that Theorem 1.1 holds true also for ordinary differential problems, that is, $N = 1$ and $r > 1$ (see Remark 3.12). However, in this case, a more general result can be proved (see Theorem 3.13), where the polynomial growth ($\lim_{t\to+\infty} \frac{f(t)}{t^r} = 0$, $r > 1$) is not requested and a more general condition than the sublinearity at zero (see (3.1″) in Theorem 3.13) is assumed. We also observe that Theorem 3.13 can be used for nonlinear problems (see Example 3.16 and Remark 3.17) to which the fundamental and seminal work of Amann [1] cannot be applied (see Remark 3.18).

In conclusion, the main aim of this paper is to establish the existence of two positive solutions to a class of nonlinear differential problems for which classical very powerful results as Crandall-Rabinowitz [9], Ambrosetti-Brezis-Cerami [3], Amann [1] cannot be always applied (see Examples 3.3, 3.5 and 3.16).

This paper is arranged as follows. In Section 2, we present a two non-zero critical points theorem (Theorem 2.1), while in Section 3, we establish our main result (Theorem 3.1), its consequence (Corollary 3.7) and its version in ordinary case (Theorem 3.13). Furthermore in Section 3, concrete examples and remarks illustrate the obtained results.

2. A two non-zero critical points theorem

The main result of this section is the following theorem on the existence of two non-zero critical points for differentiable functionals. It is a consequence of a local minimum theorem obtained in [5] (see also [6, Theorem 2.3]) and the classical Ambrosetti-Rabinowitz theorem established in [4] (see also [19, Theorem 2.2]). First, we recall the definition of (PS)-condition. Let X be a real Banach space and let $I : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional. We say that I satisfies the Palais-Smale condition (in short (PS)-condition) if any sequence $\{u_n\}$ such that $\{I(u_n)\}$ is bounded and $\{I'(u_n)\}$ is convergent to 0 in X^* admits a subsequence which is convergent in X.

Theorem 2.1. *Let X be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$. Assume that there are $r \in \mathbb{R}$ and $\tilde{u} \in X$, with $0 < \Phi(\tilde{u}) < r$, such that*

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}{r} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})} \tag{2.1}$$

and, for each $\lambda \in \left]\frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}\right[$, the functional $I_\lambda = \Phi - \lambda\Psi$ satisfies (PS)-condition and it is unbounded from below.

Then, for each $\lambda \in \left]\frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}\right[$, the functional I_λ admits at least two non-zero critical points $u_{\lambda,1}$, $u_{\lambda,2}$ such that $I_\lambda(u_{\lambda,1}) < 0 < I_\lambda(u_{\lambda,2})$.

Proof. Fix λ as in the conclusion. Since I_λ satisfies (PS)-condition, then it satisfies $(PS)^{[r]}$-condition (see [5, Chapter 2]). Moreover, owing to (2.1), in particular, one has

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)}\Psi(u)}{r} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})},$$

with $0 < \Phi(\tilde{u}) < r$. Therefore, from [6, Theorem 2.3], there is $u_{\lambda,1} \in \Phi^{-1}(]0, r[)$ (hence, $u_{\lambda,1} \neq 0$) such that $I_\lambda(u_{\lambda,1}) \leq I_\lambda(u)$ for all $u \in \Phi^{-1}(]0, r[)$ and $I'_\lambda(u_{\lambda,1}) = 0$. We observe that, in addition, one also has $I_\lambda(u_{\lambda,1}) \leq I_\lambda(u)$ for all $u \in \Phi^{-1}(]-\infty, r])$ and $I_\lambda(u_{\lambda,1}) < 0$. In fact, since $\lambda > \frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}$, one has $\Phi(\tilde{u}) - \lambda\Psi(\tilde{u}) < 0 = \Phi(0) - \lambda\Psi(0)$, for which $I_\lambda(u_{\lambda,1}) \leq I_\lambda(\tilde{u}) < I_\lambda(0) = 0$. Moreover, for all $\bar{u} \in X$ such that $\Phi(\bar{u}) = r$, taking into account that $\lambda < \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}$, one has $\Phi(\bar{u}) - \lambda\Psi(\bar{u}) \geq \Phi(\bar{u}) - \lambda \sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u) > \Phi(\bar{u}) - r = 0$, that is $I_\lambda(\bar{u}) > I_\lambda(0) > I_\lambda(u_{\lambda,1})$. So, our claim is proved.

Now, since I_λ is unbounded from below there is $\bar{u}_{\lambda,2} \in X$ such that

$$I_\lambda(\bar{u}_{\lambda,2}) < I_\lambda(u_{\lambda,1}).$$

Clearly, being $u_{\lambda,1}$ a global minimum for I_λ in $\Phi^{-1}(]-\infty, r])$, must be $\Phi(\bar{u}_{\lambda,2}) > r$. It is easy to verify that all the assumptions of the mountain pass theorem (see, for instance, [11, Corollary 5.11]) are satisfied for which there exists $u_{\lambda,2} \in X$ such that $I'_\lambda(u_{\lambda,2}) = 0$ and $I_\lambda(u_{\lambda,2}) = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\lambda(\gamma(t))$, where $\Gamma = \{\gamma \in C([0,1]) : \gamma(0) = u_{\lambda,1}, \gamma(1) = \bar{u}_{\lambda,2}\}$.

We now claim that $I_\lambda(u_{\lambda,2}) > 0$. To this end, first put

$$k = r - \lambda \sup_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u)$$

and then observe that, since $\lambda < \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u)}$, one has $k > 0$. Now, let $\gamma \in \Gamma$. Since $\Phi(\gamma(0)) < r$ and $\Phi(\gamma(1)) > r$, there is $\bar{t} \in]0,1[$ such that $\Phi(\gamma(\bar{t})) = r$. So, setting $\bar{u} = \gamma(\bar{t})$, one has

$$\Phi(\bar{u}) - \lambda\Psi(\bar{u}) \geq \Phi(\bar{u}) - \lambda \sup_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u) = r - \lambda \sup_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u) = k,$$

that is $I_\lambda(\gamma(\bar{t})) > k$. It follows that $\max_{t\in[0,1]} I_\lambda(\gamma(t)) > k$ for each $\gamma \in \Gamma$. Hence, one has $I_\lambda(u_{\lambda,2}) = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\lambda(\gamma(t)) \geq k > 0$ for which our claim is proved and the conclusion is achieved. □

Remark 2.2. In Theorem 2.1, we can assume the Cerami condition or (C)-condition as introduced by Cerami in [8], instead of (PS)-condition, provided that the coercivity of Φ is assumed. The (C)-condition is slightly weaker than (PS)-condition and we refer to [21, p. 80] and [17, Chapter 5] for the definition and more details on it. Here, even if for most purposes it suffices to use the standard (PS), we point out that Theorem 2.1 holds again true by assuming (C)-condition instead of (PS)-condition. Indeed, it is enough to observe that (PS)-condition and (C)-condition coincide for bounded sequences and so, taking into account that Φ is coercive, also (C)-condition implies the $(PS)^{[r]}$-condition for all $r > 0$. Therefore, the same proof of Theorem 2.1 ensures our claim, by applying the version of mountain pass theorem with the (C)-condition (see, for instance, [17, Theorem 5.40]).

Remark 2.3. In Theorem 2.1, it is assumed that I_λ is unbounded from below. When the functional I_λ is bounded from below we refer to [5] (see also [6, Theorem 1.2]) for an existence result of three distinct critical points. We recall that such a result is based again on the local minimum theorem established in [5] and the mountain pass theorem as given by Pucci-Serrin in [18].

Remark 2.4. If in Theorem 2.1 condition (2.1) is not assumed, existence of two distinct critical points for I_λ, for each $\lambda \in \left]0, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r[)}\Psi(u)}\right[$ and for all $r > 0$, is ensured by [6, Theorem 3.2]. However, in this case, one of the two critical points may be zero.

We also recall that a first version of a local minimum theorem has been given in [20], where, contrary to [5], assumptions involving the weak topology are made.

3. Two positive solutions for elliptic Dirichlet problems

Consider the problem (D^f_λ), where $f : \mathbb{R} \to \mathbb{R}$ is a function which is nonnegative and continuous in $[0, +\infty[$. Assume that

(h) *there exist* $s \in [1, 2[$, $q \in \left]2, \frac{2N}{N-2}\right[$ *and two positive constants* a_s, a_q *such that*

$$f(t) \leq a_s|t|^{s-1} + a_q|t|^{q-1} \quad \textit{for all } t \geq 0.$$

Without loss of generality, we assume $f(t) = f(0)$ for all $t < 0$. Moreover, put $X = H^1_0(\Omega)$ endowed with the norm $\|u\| = \left(\int_\Omega |\nabla u(x)|^2 dx\right)^{\frac{1}{2}}$ and $2^* = \frac{2N}{N-2}$. We recall that one has

$$\|u\|_{L^{2^*}(\Omega)} \leq T\|u\| \quad \forall\, u \in X, \quad \text{where} \quad T = \frac{1}{\sqrt{N(N-2)\pi}} \left(\frac{N!}{2\Gamma(1+\frac{N}{2})}\right)^{\frac{1}{N}}$$

is the best constant (see [22]) and Γ is the gamma function. So, owing to Hölder's inequality, it follows

$$\|u\|_{L^p(\Omega)} \leq T|\Omega|^{\frac{2^*-p}{2^*p}}\|u\| \quad \forall\, u \in X, \forall\, p \in [1, 2^*[.$$

Now, put $F(\xi) = \int_0^\xi f(t)dt$ for every $\xi \in \mathbb{R}$,

$$\Phi(u) = \frac{\|u\|^2}{2}, \quad \Psi(u) = \int_\Omega F(u(x))dx \quad \text{and} \quad I_\lambda(u) = \Phi(u) - \lambda\Psi(u)$$

for all $u \in X$ and $\lambda > 0$. As it is well known, critical points of I_λ are the weak solutions to (D^f_λ).

Moreover, put $R(x) = \sup\{\delta : B(x, \delta) \subseteq \Omega\}$ for all $x \in \Omega$, and $R = \sup_{x\in\Omega} R(x)$, for which there exists $x_0 \in \Omega$ such that $B(x_0, R) \subseteq \Omega$. Finally, put

$$K = \frac{R^2}{2(2^N-1)}\frac{1}{2T^2|\Omega|^{\frac{2}{N}}}, \quad \Xi_\delta = \frac{1}{K}\frac{1}{2T^2|\Omega|^{\frac{2}{N}}}\frac{\delta^2}{F(\delta)}, \quad \Lambda_\gamma = \frac{1}{2T^2|\Omega|^{\frac{2}{N}}}\frac{1}{\frac{a_s}{s}\gamma^{s-2} + \frac{a_q}{q}\gamma^{q-2}}$$

where γ, δ are positive constants.

Now, we present our main result.

Theorem 3.1. *Assume that* (h) *holds. Moreover, assume that there are two positive constants* γ *and* δ*, with* $\delta < \gamma$*, such that*

$$\frac{a_s}{s}\gamma^{s-2} + \frac{a_q}{q}\gamma^{q-2} < K\frac{F(\delta)}{\delta^2} \tag{3.1}$$

and there are two constants $m > 2$ *and* $l > 0$ *such that, for all* $\xi \geq l$*, one has*

$$0 < mF(\xi) \leq \xi f(\xi). \tag{AR}$$

Then, for each $\lambda \in]\Xi_\delta, \Lambda_\gamma[$*, problem* (D_λ^f) *admits at least two positive weak solutions.*

Proof. Fix $\lambda \in]\Xi_\delta, \Lambda_\gamma[$, taking into account that from (3.1) one has $]\Xi_\delta, \Lambda_\gamma[\neq \emptyset$. From (AR), by standard computations, one has that I_λ is unbounded from below and satisfies the Palais-Smale condition (see, for instance, [19]). So, in order to apply Theorem 2.1, it is enough to verify condition (2.1). To this end, fix

$$r = \frac{|\Omega|^{\frac{2}{2^*}}}{2T^2}\gamma^2.$$

From (h) one has $F(\xi) \leq \frac{a_s}{s}|\xi|^s + \frac{a_q}{q}|\xi|^q$ for all $\xi \in \mathbb{R}$. Therefore, one has

$$\begin{aligned}
\frac{\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}{r} &\leq \frac{\sup_{u\in\Phi^{-1}(]-\infty,r])}\left(\frac{a_s}{s}\|u\|_{L^s}^s + \frac{a_q}{q}\|u\|_{L^q}^q\right)}{r} \\
&\leq \frac{\sup_{u\in\Phi^{-1}(]-\infty,r])}\left(\frac{a_s}{s}T^s|\Omega|^{\frac{2^*-s}{2^*}}\|u\|^s + \frac{a_q}{q}T^q|\Omega|^{\frac{2^*-q}{2^*}}\|u\|^q\right)}{r} \\
&\leq \frac{\left(\frac{a_s}{s}T^s|\Omega|^{\frac{2^*-s}{2^*}}(2r)^{\frac{s}{2}} + \frac{a_q}{q}T^q|\Omega|^{\frac{2^*-q}{2^*}}(2r)^{\frac{q}{2}}\right)}{r} \\
&= 2T^2|\Omega|^{\frac{2^*-2}{2^*}}\left(\frac{a_s}{s}\left(\frac{2T^2r}{|\Omega|^{\frac{2}{2^*}}}\right)^{\frac{s-2}{2}} + \frac{a_q}{q}\left(\frac{2T^2r}{|\Omega|^{\frac{2}{2^*}}}\right)^{\frac{q-2}{2}}\right) \\
&= 2T^2|\Omega|^{\frac{2}{N}}\left(\frac{a_s}{s}\gamma^{s-2} + \frac{a_q}{q}\gamma^{q-2}\right) \\
&= \frac{1}{\Lambda_\gamma}
\end{aligned}$$

that is,

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}{r} < \frac{1}{\lambda}. \tag{3.2}$$

Now, put

$$v_\delta(x) := \begin{cases} 0 & \text{if } x \in \Omega \setminus B(x_0, R) \\ \frac{2\delta}{R}(R - |x - x_0|) & \text{if } x \in B(x_0, R) \setminus B(x_0, \frac{R}{2}) \\ \delta & \text{if } x \in B(x_0, \frac{R}{2}). \end{cases}$$

Clearly, one has that $v_\delta \in X$,

$$\Phi(v_\delta) = \frac{1}{2}\frac{(2\delta)^2}{R^2}\frac{\pi^{\frac{N}{2}}}{\Gamma(1+\frac{N}{2})}\left(R^N - \left(\frac{R}{2}\right)^N\right) = \frac{2(2^N-1)}{2^N}R^{N-2}\frac{\pi^{\frac{N}{2}}}{\Gamma(1+\frac{N}{2})}\delta^2$$

and

$$\Psi(v_\delta) \geq \int_{B(x_0,\frac{R}{2})} F(\delta)\,dx = F(\delta)\frac{\pi^{\frac{N}{2}}}{\Gamma(1+\frac{N}{2})}\frac{R^N}{2^N}.$$

Hence, one has

$$\frac{\Psi(v_\delta)}{\Phi(v_\delta)} \geq \frac{R^2}{2(2^N-1)}\frac{F(\delta)}{\delta^2} = 2T^2|\Omega|^{\frac{2}{N}}K\frac{F(\delta)}{\delta^2} = \frac{1}{\Lambda_\delta} > \frac{1}{\lambda}, \tag{3.3}$$

that is,

$$\frac{\Psi(v_\delta)}{\Phi(v_\delta)} > \frac{1}{\lambda}. \tag{3.4}$$

Now, in order to prove that $\Phi(v_\delta) < r$, put

$$k = \left(\frac{2(2^N-1)}{2^N}R^{N-2}\frac{\pi^{\frac{N}{2}}}{\Gamma(1+\frac{N}{2})}\frac{2T^2}{|\Omega|^{\frac{2}{2^*}}}\right)^{\frac{1}{2}}$$

and observe that $\Phi(v_\delta) = k^2\frac{|\Omega|^{\frac{2}{2^*}}}{2T^2}\delta^2$. Therefore, one has

$$\frac{1}{k^2} = \frac{R^2}{2(2^N-1)}\frac{1}{\frac{\pi^{\frac{N}{2}}}{\Gamma(1+\frac{N}{2})}\left(\frac{R}{2}\right)^N}\frac{|\Omega|}{2T^2|\Omega|^{2/N}} = \frac{|\Omega|}{|B(x_0,\frac{R}{2})|}K \geq K.$$

Taking into account that $\delta < \gamma$, we claim that $k\delta < \gamma$. Indeed, from (3.1), taking (h) into account, one has

$$\frac{\frac{a_s}{s}\gamma^s + \frac{a_q}{q}\gamma^q}{\gamma^2} < K\frac{\frac{a_s}{s}\delta^s + \frac{a_q}{q}\delta^q}{\delta^2},$$

so, arguing by a contradiction and assuming $k\delta \geq \gamma$, one has

$$\frac{\frac{a_s}{s}\gamma^s + \frac{a_q}{q}\gamma^q}{\gamma^2} \geq \frac{\frac{a_s}{s}\gamma^s + \frac{a_q}{q}\gamma^q}{k^2\delta^2} \geq \frac{1}{k^2}\frac{\frac{a_s}{s}\delta^s + \frac{a_q}{q}\delta^q}{\delta^2} \geq K\frac{\frac{a_s}{s}\delta^s + \frac{a_q}{q}\delta^q}{\delta^2}$$

and this is an absurd. Therefore, our claim is proved and from $k\delta < \gamma$ it follows $\Phi(v_\delta) < r$.

So, owing to (3.2) and (3.4), one has

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}{r} < \frac{1}{\lambda} < \frac{\Psi(v_\delta)}{\Phi(v_\delta)}$$

with $0 < \Phi(v_\delta) < r$. Therefore, condition (2.1) is satisfied and Theorem 2.1 ensures that I_λ admits two non-zero critical points, which are, owing to the strong maximum principle, two positive weak solutions for (D_λ^f). Hence, the proof is complete. □

Now, setting

$$\lambda^* = \frac{1}{2T^2|\Omega|^{\frac{2}{N}}}\left(\frac{s}{a_s}\right)^{\frac{q-2}{q-s}}\left(\frac{q}{a_q}\right)^{\frac{2-s}{q-s}}\left(\frac{2-s}{q-2}\right)^{\frac{2-s}{q-s}}\frac{q-2}{q-s},$$

we point out the following consequence of Theorem 3.1.

Corollary 3.2. *Assume* (h), (AR) *and*

$$\limsup_{\xi\to 0^+}\frac{F(\xi)}{\xi^2} = +\infty. \tag{3.1'}$$

Then, for each $\lambda \in]0, \lambda^*[$, *problem* (D_λ^f) *admits at least two positive weak solutions.*

Proof. Fix $\lambda \in]0, \lambda^*[$. Taking into account that $\lambda^* = \sup_{\gamma>0}\Lambda_\gamma$, there is $\gamma > 0$ such that $\lambda < \Lambda_\gamma$. From $\limsup_{\xi\to 0^+} K2T^2|\Omega|^{\frac{2}{N}}\frac{F(\xi)}{\xi^2} = +\infty$ there is $\delta < \gamma$ such that $K2T^2|\Omega|^{\frac{2}{N}}\frac{F(\delta)}{\delta^2} > \frac{1}{\lambda}$ for which $\lambda \in]\Xi_\delta, \Lambda_\gamma[$ and (3.1) holds. Hence, Theorem 3.1 ensures the conclusion. □

Example 3.3. Let $\Omega = \{x \in \mathbb{R}^3 : |x| < 1\}$ and let $f : \mathbb{R} \to \mathbb{R}$ be a function defined as follows

$$f(t) = \begin{cases} (50)^3 t^2 & \text{if } t \le \left(\frac{1}{50}\right)^2, \\ \sqrt{t} & \text{if } \left(\frac{1}{50}\right)^2 < t \le 1, \\ t^2 & \text{if } t \ge 1. \end{cases}$$

Owing to Theorem 3.1, the problem

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0 \end{cases}$$

admits at least two positive weak solutions. Indeed, by setting $f(t) = 0$ for all $t < 0$, one has $f(t) \le |t|^{\frac{1}{2}} + |t|^2$ for all $t \in \mathbb{R}$. Moreover, since in this case $K = \frac{1}{7}\left(\frac{3^5\pi^2}{2^{14}}\right)^{\frac{1}{3}}$, by choosing $\delta = \left(\frac{1}{50}\right)^2$ and $\gamma = 1$, one has $\frac{2}{3}\frac{1}{\gamma^{\frac{1}{2}}} + \frac{1}{3}\gamma < K\frac{(50)^3\frac{\delta^3}{3}}{\delta^2}$ and $\delta < \gamma$, for which (3.1) is verified. So, taking also into account that $\frac{1}{K}\frac{1}{2T^2|\Omega|^{\frac{2}{N}}}\frac{\delta^2}{F(\delta)} < 1 < \frac{1}{2T^2|\Omega|^{\frac{2}{N}}}\frac{1}{\frac{a_s}{s}\gamma^{s-2}+\frac{a_q}{q}\gamma^{q-2}}$, our claim is proved.

It should be noted here that the function f is superlinear both at zero and infinity. Further, by setting $f_1(t) = (50)t$ if $0 \leq t \leq \left(\frac{1}{50}\right)^2$ and $f_1(t) = f(t)$ otherwise, same computations show that f_1 satisfies (3.1). So, our results can be applied also to functions which are linear at zero.

Remark 3.4. In Theorem 3.1 no condition at zero on the nonlinearity f is requested. In particular, Corollary 3.2 and Example 3.3 show that f may be sub-linear, linear or super-linear at zero and Theorem 3.1 can be applied. We recall that in order to obtain two positive solutions, in particular, a condition at zero on the nonlinearity is requested. Indeed, usually the hypothesis $f(0) > 0$ is assumed (see Remark 3.6) or, in some special case (as f in Corollary 3.7 below), the more general condition $\lim_{t\to 0^+} \frac{f(t)}{t} = +\infty$ is required.

Example 3.5. Owing to Corollary 3.2, for each $\lambda \in \left]0, \frac{\sqrt{3}}{4T^2|\Omega|^{\frac{2}{N}}}\right[$ the problem

$$\begin{cases} -\Delta u = \lambda \max\{\sqrt{u}, u^2\} & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

admits at least two positive weak solutions. Indeed, it is enough to pick $f(t) = \max\{\sqrt{t}, t^2\}$ if $t \geq 0$ and $f(t) = 0$ if $t < 0$, for which one has $f(t) \leq 1+|t|^2$ for all $t \in \mathbb{R}$, and verify all assumptions of the previous theorem. Moreover, in particular, if $\Omega = \{x \in \mathbb{R}^3 : |x| < 1\}$, the problem

$$\begin{cases} -\Delta u = \frac{1}{2} \max\{\sqrt{u}, u^2\} & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

admits at least two positive weak solutions, since $\frac{1}{2} < \lambda^* = \frac{9}{16} \sqrt[6]{3} \left(\frac{\pi}{2}\right)^{\frac{2}{3}}$.

Remark 3.6. A result of the type of Corollary 3.2 has been obtained in the fine and classical paper of Crandall-Rabinowitz (see [9, Theorem 2.1]), where, instead of (3.1'), the stronger assumption $f(0) > 0$ is assumed (see also Remark 3.18). Clearly, [9, Theorem 2.1] cannot be applied to Examples 3.3 and 3.5.

As a consequence of Theorem 3.1, we obtain a result for problems where the datum is given by combined effects of concave and convex nonlinearities. This type of study has been introduced and developed in the seminal paper of Ambrosetti-Brezis-Cerami [3]. Here, we obtain, in particular, the same type of result, but through a proof which is totally variational (see Remark 3.8). To be precise, by setting

$$\mu^* = \left(\frac{1}{2T^2|\Omega|^{\frac{2}{N}}}\right)^{\frac{q-s}{q-2}} s(q-2)q^{\frac{2-s}{q-2}} \left(\frac{(2-s)^{(2-s)}}{(q-s)^{(q-s)}}\right)^{\frac{1}{q-2}},$$

from Theorem 3.1 we obtain the following special case.

Corollary 3.7. *Fix* $1 \leq s < 2 < q < 2^*$. *Then, for each* $\mu \in]0, \mu^*[$ *problem*

$$\begin{cases} -\Delta u = \mu u^{s-1} + u^{q-1} & \text{in } \Omega, \\ u|_{\partial\Omega} = 0 \end{cases} \tag{D$_\mu$}$$

admits at least two positive weak solutions.

Proof. Fix $\mu \in]0, \mu^*[$ and put $f(t) = \mu t^{s-1} + t^{q-1}$ if $t \geq 0$ and $f(t) = f(0)$ if $t < 0$. One has

$$\lambda^* = \frac{1}{2T^2|\Omega|^{\frac{2}{N}}} \left(\frac{s}{\mu}\right)^{\frac{q-2}{q-s}} (q)^{\frac{2-s}{q-s}} \left(\frac{2-s}{q-2}\right)^{\frac{2-s}{q-s}} \frac{q-2}{q-s} > 1.$$

Hence, from Theorem 3.1 we obtain the conclusion. □

Remark 3.8. Corollary 3.7 is a particular case of the very nice result established in the fundamental and seminal work [3] by a clever combination of topological and variational methods. Precisely, in [3] the existence of a first positive solution, by using the method of sub- and super-solutions, is established and then, through a deep reasoning, by proving that this first solution is the minimum of a suitable functional associated to a modified problem, the mountain pass theorem is applied in order to obtain a second positive solution. However, in this type of proof, no numerical estimate of the superior, called Λ, of parameters μ for which the problem (D$_\mu$) admits such solutions is provided. We observe that our proof of Corollary 3.7 is totally variational. Indeed, the first positive solution is directly obtained as a local minimum and the second one is obtained by applying the mountain pass theorem but without modifying the functional in order to establish the positivity of the second solution.

In addition, we observe that the same proof of Corollary 3.7 gives precise numerical values μ for which (D$_\mu$) is solved (see Remark 3.9). Finally, we recall that in [3] also the critical case, that is $q = 2^*$, is considered and we refer to [7] for a proof which is totally variational for such a case.

Remark 3.9. We wish to highlight that Theorem 3.1 and the Ambrosetti-Brezis-Cerami result ([3, Theorem 2.3]) are mutually independent. Indeed, Examples 3.3 and 3.5 show functions for which we can apply our main result and we cannot apply [3, Theorem 2.3]. On the other hand, in the problem (D$_\mu$), we can apply actually both the previous results, but the value Λ obtained in [3], for which for each $\mu \in]0, \Lambda[$ the problem (D$_\mu$) admits two positive solutions, is the best.

However, we observe that since Λ in [3] is expressed by a theoretical point of view, our result, in this case, can be used just as a complement of [3, Theorem 2.3] in order to give a numerical lower bound of Λ, that is, $\mu^* \leq \Lambda$.

A similar remark can be done also for a comparison between Theorem 3.1 and the Crandall-Rabinowitz result [9, Theorem 2.1], that is, they are mutually independent. Indeed, when $f(0) > 0$ and we can apply both results (see also Remark 3.6), the value Θ obtained in [9], for which for each $\lambda \in]0, \Theta[$ the problem (D_λ^f) admits two positive solutions, is the best.

Remark 3.10. Theorem 1.1 in Introduction is an immediate consequence of Corollary 3.2. In fact, it is enough to observe that (AR) is equivalent to the following two conditions:

(a_1) $\lim_{t\to+\infty} \frac{F(t)}{t^{p+1}} = +\infty$ *for some* $p > 1$;

(a_2) *if* $\limsup_{t\to+\infty} \frac{F(t)}{tf(t)} \geq \frac{1}{2}$ *then* $\liminf_{t\to+\infty} \frac{F(t)}{tf(t)} \geq \frac{1}{2}$.

Indeed, (a_1) and (a_2) imply that $\limsup_{t\to+\infty} \frac{F(t)}{tf(t)} < \frac{1}{2}$, for which (AR) condition holds. Precisely, arguing by a contradiction, assume $\limsup_{t\to+\infty} \frac{F(t)}{tf(t)} \geq \frac{1}{2}$. From ($a_2$) one has $\liminf_{t\to+\infty} \frac{F(t)}{tf(t)} \geq \frac{1}{2} > \frac{1}{p+1}$. It follows $\frac{F(t)}{t^{p+1}} < \frac{F(R)}{R^{p+1}}$ for all $t > R$ and for some $R > 0$, which is an absurd since (a_1) holds. Hence, our claim is proved.

In conclusion, roughly speaking, all the functions which are *more than superlinear* (that is, $\lim_{t\to+\infty} \frac{f(t)}{t^p} = +\infty$, $p > 1$) and for which the condition

$$\liminf_{t\to+\infty} \frac{F(t)}{tf(t)} < \frac{1}{2} \leq \limsup_{t\to+\infty} \frac{F(t)}{tf(t)}$$

is not verified, are satisfying the (AR) condition. Hence, the class of functions satisfying (AR) is a bit smaller than the class of functions which are *more than superlinear*. An example of function which is *more than superlinear* but for which (AR) does not hold true is $f(t) = t^2(\sin t + 2)$.

Remark 3.11. If in Theorem 3.1 we assume

(b_1) $\lim_{t\to+\infty} \frac{F(t)}{t^2} = +\infty$;

(b_2) *there exist* $\tau \in \left]\min\left\{\frac{(q-2)N}{2}, s\right\}, \frac{2N}{N-2}\right[$ *and* $\gamma_0 > 0$ *such that*

$$\liminf_{t\to+\infty} \frac{tf(t) - 2F(t)}{t^\tau} \geq \gamma_0;$$

instead of (AR), then the conclusion holds again true. Indeed, the energy functional I_λ in the proof of Theorem 3.1 is unbounded from below owing to (b_1) and it satisfies (C)-condition owing to (b_2), so from Theorem 2.1 and Remark 2.2 our claim follows. For a detailed proof showing that (b_2) implies (C)-condition for I_λ, we refer to [17, step 2, p. 358]. Moreover, simple computations show that conditions (a_1)–(a_2) imply condition (b_1)–(b_2). Indeed, if $\limsup_{t\to+\infty} \frac{F(t)}{tf(t)} < \frac{1}{2}$ then $\liminf_{t\to+\infty} \frac{tf(t)-2F(t)}{t^{p+1}} = \liminf_{t\to+\infty} \frac{F(t)}{t^{p+1}} \left[\frac{tf(t)}{F(t)} - 2\right] = +\infty$.

We observe that owing to such a remark we can obtain two positive solutions also for a class of functions which are *only* superlinear at infinity as for instance the following

$$f(t) = \begin{cases} (50)t & \text{if } t \leq \left(\frac{1}{50}\right)^2, \\ \sqrt{t} & \text{if } \left(\frac{1}{50}\right)^2 < t \leq 1, \\ \frac{t}{\log 2}\log(1+t) & \text{if } t \geq 1 \end{cases}$$

(see Example 3.3). Finally, we point out that the *more than superlinear* function given in Remark 3.10 does not satisfy (b_2) as well as (AR).

Remark 3.12. The assumption (h) is needed so that the energy functional is well defined. In our case, it is also used to determine the values Λ_γ and λ^*. We observe that Theorem 3.1 and Corollary 3.2 are again true also for $N = 1$ (clearly, in this case, in (h) we take $q \in]2, +\infty[$), provided that we choose Λ_γ and λ^* in an appropriate way. For instance, if $\Omega =]0, 1[$, taking into account that

$$\|u\|_{L^p(]0,1[)} \leq \frac{1}{2}\|u\| \quad \forall\, u \in X,\ \forall\, p \in [1, +\infty[,$$

Theorem 3.1 and Corollary 3.2 hold again true provided that

$$\Lambda_\gamma = \frac{2}{\frac{a_s}{s}\gamma^{s-2} + \frac{a_q}{q}\gamma^{q-2}}$$

and

$$\lambda^* = 2\left(\frac{s}{a_s}\right)^{\frac{q-2}{q-s}}\left(\frac{q}{a_q}\right)^{\frac{2-s}{q-s}}\left(\frac{2-s}{q-2}\right)^{\frac{2-s}{q-s}}\frac{q-2}{q-s}.$$

However, we point out below that, in the ordinary case, assumption (h) is not necessary, by choosing Λ_γ in a suitable way. Moreover, the assumption (3.1) can be expressed in a more simple and general form.

Here, we point out a version of Theorem 3.1 for the ordinary case.

Theorem 3.13. *Let $f : [0, +\infty[\to [0, +\infty[$ be a continuous function and assume that* (AR) *holds. Moreover, assume that there are two positive constants γ, δ, with $\delta < \gamma$ such that*

$$\frac{F(\gamma)}{\gamma^2} < \frac{1}{4}\frac{F(\delta)}{\delta^2}. \tag{3.1''}$$

Then, for each $\lambda \in \left]\frac{8\delta^2}{F(\delta)}, \frac{2\gamma^2}{F(\gamma)}\right[$, the problem

$$\begin{cases} -u'' = \lambda f(u) & \text{in }]0,1[, \\ u(0) = u(1) = 0, \end{cases} \tag{P_λ^f}$$

admits at least two positive classical solutions.

Proof. Fix $\lambda \in \left]\frac{8\delta^2}{F(\delta)}, \frac{2\gamma^2}{F(\gamma)}\right[$. As said in the proof of Theorem 3.1, I_λ is unbounded from below and satisfies the (PS)-condition, so, it is enough to verify condition (2.1). To this end, fix $r = 2\gamma^2$ and v_δ as before by choosing $x_0 = R = \frac{1}{2}$. Taking into account that $\|u\|_\infty \leq \frac{1}{2}\|u\|$ for all $u \in X$, one has

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u)}{r} \leq \frac{\sup_{\|u\|\leq\sqrt{2r}} \int_0^1 F(u(t))dt}{r} \leq \frac{F(\gamma)}{2\gamma^2}$$

and (see (3.3)) $\frac{\Psi(v_\delta)}{\Phi(v_\delta)} \geq \frac{F(\delta)}{8\delta^2}$, so, from (3.1′′) it follows

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u)}{r} < \frac{\Psi(v_\delta)}{\Phi(v_\delta)}.$$

Moreover, from $\delta < \gamma$ and (3.1′′) one has $\sqrt{2}\delta < \gamma$, that is, $\Phi(v_\delta) < r$. Hence, our claim is proved and the conclusion is achieved. □

Remark 3.14. In Theorem 3.13 we can assume, instead of (AR), the condition (b_1)–(b_2) (see Remark 3.11), provided that also (h) is assumed. Indeed, to verify the Cerami condition a growth of polynomial type on f is needed (see [17, step 2, p. 358]).

Remark 3.15. If $\limsup_{\delta\to 0^+} \frac{F(\delta)}{\delta^2} = +\infty$ then (3.1′′) holds true and the interval becomes $\left]0, \bar{\lambda}\right[$, where

$$\bar{\lambda} = \sup_{\gamma>0} \frac{2\gamma^2}{F(\gamma)}.$$

The converse is not true as the example below shows.

Example 3.16. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined as follows

$$f(t) = \begin{cases} t^2 & \text{if } t < 1, \\ \sqrt{t} & \text{if } 1 \leq t < 10^2, \\ \frac{1}{10^3}t^2 & \text{if } t \geq 10^2. \end{cases}$$

Owing to Theorem 3.13, the problem

$$\begin{cases} -u'' = 5^2 f(u) & \text{in }]0,1[, \\ u(0) = u(1) = 0, \end{cases}$$

admits at least two positive classical solutions. It is enough to to verify $\frac{1}{2}\frac{F(3)}{3^2} < \frac{1}{5^2} < \frac{1}{8}\frac{F(1)}{1^2}$. We observe that in this case, the nonlinearity f is not sublinear at zero.

Remark 3.17. The value $\bar{\lambda}$ guaranteed from Theorem 3.13 (see Remark 3.15) can be greater than the value λ^* guaranteed from Corollary 3.2 (see Remark 3.12) as the function in Example 3.5 shows. In fact, in this case, we have $\lambda^* = \sqrt{3}$ and $\bar{\lambda} = 3$. So, for each $\lambda \in]0, 3[$ the problem

$$\begin{cases} -u'' = \lambda \max\{\sqrt{u}, u^2\} & \text{in }]0,1[, \\ u(0) = u(1) = 0, \end{cases}$$

admits at least two positive classical solutions.

Remark 3.18. Theorem 3.13 and the fundamental result of Amann [1, p. 208] are mutually independent. Indeed, in [1], given a continuous function $f : \mathbb{R} \to \mathbb{R}$ which is positive in $]0, +\infty[$, the following assumptions are made:

(H_1) $f(0) > 0$;

(H_2) $\lim_{t\to+\infty} \frac{f(t)}{t} = +\infty$.

The author obtains the existence of $\tilde{\lambda} > 0$ such that the problem (P^f_λ) has at least two positive solutions for $0 < \lambda < \tilde{\lambda}$, at least one for $\lambda = \tilde{\lambda}$ and none for $\lambda > \tilde{\lambda}$. However, this very powerful result does not provide an estimate of $\tilde{\lambda}$. Clearly, (H_1) is more restrictive than ($3.1''$) since it implies that $\lim_{t\to 0^+} \frac{f(t)}{t} = +\infty$ and the result of [1] cannot be applied, for instance, to examples in Remark 3.17 and Example 3.16. On the other hand, (H_2) allows to consider functions for which (AR) and (b_2) does not hold true (see Remarks 3.10 and 3.11).

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