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# A Modification of the Lipschitz Condition in the Newton-Kantorovich Theorem

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**Abstract.** We analyse the semilocal convergence of Newton's method in Banach spaces under a modification of the classic Lipschitz condition on the first derivative of the operator involved in Kantorovich's theory. For this, we use a technique based on recurrence relations instead of the well-known majorant principle of Kantorovich. We illustrate this analysis with an application where a Hammerstein nonlinear integral equation of the second kind is involved.

**Keywords.** Newton's method, semilocal convergence, the Newton-Kantorovich theorem, recurrence relations, error estimates, order of convergence, nonlinear integral equation

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## 1. Introduction

Newton's method is the most used iterative method to solve nonlinear operator equations $F(x) = 0$ in a Banach space. In this case, the equation $F(x) = 0$ can represent a large number of problems: an ordinary differential equation, a boundary value problem, an integral equation, a problem of variational calculus, etc. So, we consider that $F$ is a nonlinear operator, $F : \Omega \subseteq X \longrightarrow Y$, defined on a non-empty open convex domain $\Omega$ of a Banach space $X$ with values in a Banach space $Y$.

It is well-known that Newton's method is defined by the following algorithm:

$$x_0 \text{ given in } \Omega, \quad x_{n+1} = x_n - [F'(x_n)]^{-1}F(x_n), \quad n = 0, 1, 2 \ldots \tag{1}$$

In this work, we focus our attention on the analysis of the semilocal convergence of Newton's method. The best known semilocal convergence result for Newton's

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method is the Newton-Kantorovich theorem [9], whose best known variant is the result given by Ortega in [11], which is also known as the Newton-Kantorovich theorem and is established under the following conditions:

(A1) There exists $\Gamma_0 = [F'(x_0)]^{-1} \in \mathcal{L}(Y,X)$, for some $x_0 \in \Omega$, with $\|\Gamma_0\| \leq \beta$ and $\|\Gamma_0 F(x_0)\| \leq \eta$, where $\mathcal{L}(Y,X)$ is the set of bounded linear operators from $Y$ to $X$,

(A2) There exists a constant $L \geq 0$ such that $\|F'(x) - F'(y)\| \leq L\|x-y\|$ for $x, y \in \Omega$,

(A3) $L\beta\eta \leq \frac{1}{2}$ and $B(x_0, t^*) \subset \Omega$, where $t^* = \frac{1-\sqrt{1-2L\beta\eta}}{L\beta}$ is the smallest positive zero of the polynomial $p(t) = \frac{L}{2}t^2 - \frac{t}{\beta} + \frac{\eta}{\beta}$.

**Theorem 1.1** (The Newton-Kantorovich theorem)**.** *Let $F : \Omega \subseteq X \longrightarrow Y$ be a continuously differentiable operator defined on a non-empty open convex domain $\Omega$ of a Banach space $X$ with values in a Banach space $Y$. Suppose that conditions* (A1)–(A3) *are satisfied. Then Newton's sequence, given by* (1)*, converges to a solution $x^*$ of the equation $F(x) = 0$, starting at $x_0$, and $x_n, x^* \in \overline{B(x_0, t^*)}$, for all $n = 0, 1, 2, \ldots$ Moreover, if $L\beta\eta < \frac{1}{2}$, $x^*$ is the unique solution of $F(x) = 0$ in $B(x_0, t^{**}) \cap \Omega$, where $t^{**} = \frac{1+\sqrt{1-2L\beta\eta}}{L\beta}$, and if $L\beta\eta = \frac{1}{2}$, $x^*$ is unique in $\overline{B(x_0, t^*)}$. Furthermore,*

$$\|x_{n+1} - x_n\| \leq t_{n+1} - t_n \quad \textit{and} \quad \|x^* - x_n\| \leq t^* - t_n, \quad \textit{for all} \quad 0, 1, 2, \ldots,$$

*where $t_n = t_{n-1} - \frac{p(t_{n-1})}{p'(t_{n-1})}$ and $n \in \mathbb{N}$.*

Observe that, as in all results of semilocal convergence (see for example [1, 3, 5, 6]), there are conditions on the operator involved (condition (A2)), the starting point (condition (A1)) and the relationship between both (condition (A3)). If we pay special attention to condition (A3), we observe that if the value of the parameter $L$ is large, this implies that the starting point $x_0$ must be very close to a solution of the equation $F(x) = 0$, so that condition (A3) is satisfied.

The main aim of this paper is to establish, from a modification of condition (A2), a new semilocal convergence result for Newton's method, which modifies the domain of starting points that is obtained from the Newton-Kantorovich theorem. In addition, in two particular cases, our new semilocal convergence result is reduced to the Newton-Kantorovich theorem and the semilocal convergence result given in [8] for operators $F'$ Hölder continuous in $\Omega$.

This paper is organized as follows. In Section 2, we study the difficulties presented by the fulfilment of conditions (A1)–(A3) when considering certain operators $F$ and propose a consistent alternative to condition (A2). Next, in Section 3, we prove the semilocal convergence of Newton's method under the new convergence conditions and using a technique based on recurrence relations. Some comments on the previous analysis are given in Section 4. Finally, in Section 5, we present a study on certain nonlinear Hammerstein integral equations of the second kind, where domains of existence and uniqueness of solution are given, a graphical analysis is done for a particular equation and solutions are approximated by Newton's method when the Newton-Kantorovich theorem cannot guarantee the semilocal convergence of the method.

Throughout the paper we denote $\overline{B(x,\varrho)} = \{y \in X; \|y-x\| \leq \varrho\}$ and $B(x,\varrho) = \{y \in X; \|y-x\| < \varrho\}$.

## 2. Motivation

We consider nonlinear Hammerstein integral equations of the second kind of the type [7]

$$x(s) = u(s) + \lambda \int_a^b G(s,t)\, x(t)^m dt, \quad s \in [a,b],\ \lambda \in \mathbb{R},\ m \in \mathbb{N}, \tag{2}$$

where $u$ is a continuous function and the kernel $G$ is continuous and nonnegative in $[a,b] \times [a,b]$.

Note that if $G(s,t)$ is the Green function in $[a,b] \times [a,b]$, then equation (2) is equivalent to the following boundary value problem:

$$\begin{cases} x''(t) = -\lambda x(t)^m, \\ x(a) = \upsilon(a),\ x(b) = \upsilon(b). \end{cases}$$

Observe that solving equation (2) is equivalent to solve $F(x) = 0$, where $F : \mathcal{C}([a,b]) \longrightarrow \mathcal{C}([a,b])$ and

$$[F(x)](s) = x(s) - u(s) - \lambda \int_a^b G(s,t)\, x(t)^m dt, \quad \lambda \in \mathbb{R},\ m \in \mathbb{N}. \tag{3}$$

In addition, we have

$$\begin{aligned} [F'(x)y](s) &= y(s) - m\lambda \int_a^b G(s,t) x(t)^{m-1} y(t)\, dt, \\ [(F'(x) - F'(y))z](s) &= -m\lambda \int_a^b G(s,t) \left((x(t)^{m-1} - y(t)^{m-1}\right) z(t)\, dt. \end{aligned} \tag{4}$$

As a consequence,

$$\begin{aligned} &\|F'(x) - F'(y)\| \\ &\leq m|\lambda|\ell \left(\|x\|^{m-2} + \|x\|^{m-3}\|y\| + \cdots + \|x\|\|y\|^{m-3} + \|y\|^{m-2}\right) \|x-y\| \end{aligned} \tag{5}$$

where the max-norm is used and $\ell = \max_{[a,b]} \int_a^b |G(s,t)|\, dt$.

Obviously, condition (A2) of the Newton-Kantorovich theorem is not satisfied in the previous situation if we cannot locate in advance a solution of the equation $F(x) = 0$ in some domain of the form $\Omega = B(\nu, \rho)$ with $\nu \in \mathcal{C}([a, b])$ and $\rho > 0$, where $\|x\|, \|y\| \leq \|\nu\| + \rho$. Then, we can only calculate the constant $L$ appearing in the Newton-Kantorovich theorem, $L = m(m-1)|\lambda|\ell\left(\|\nu\| + \rho\right)^{m-2}$, if we can first locate a solution of $F(x) = 0$.

In addition, it is clear the fact that the condition $L\beta\eta \leq \frac{1}{2}$ is satisfied depends on the value of $\rho$, which leads us to locate a starting point $x_0$ very close to the solution (this is not easy).

On the other hand, in the worst case, if we cannot locate previously a solution of the equation, the mentioned difficulty cannot be solved.

To avoid the previous two problems, we consider in this work the following condition:

$$\|F'(x) - F'(y)\| \leq \omega(\|x\|, \|y\|)\, \|x - y\|^p, \quad x, y \in \Omega,\ p \in [0, 1], \tag{6}$$

where $\omega : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a nondecreasing continuous function in both arguments and such that $\omega(0, 0) \geq 0$. Notice that we include the parameter $p$ to be considered operators with Hölder continuous Fréchet derivative; namely, if $\omega(s, t) = K$, $F'$ is $(K, p)$-Hölder continuous in $\Omega$.

From condition (6), we prove the semilocal convergence of Newton's method by using an alternative technique to the well-known majorant principle used by Kantorovich [9] and Ortega [11]. This technique is based on recurrence relations and requires, from the starting point $x_0$, that Newton's sequence $\{x_n\}$ is included in a ball $B(x_0, R) \subset \Omega$, $R > 0$, to be determined. The conditions of our semilocal convergence result do not depend directly on the domain $\Omega$. This result pays attention to the existence of the value $R$, which depends on the starting point $x_0$, but it avoids the difficulties set out by the Newton-Kantorovich theorem.

We analyse all the above mentioned with examples that clarify the different possibilities that may occur. In addition, we observe the modification of the domain of starting points given by the Newton-Kantorovich theorem that we obtain from our semilocal convergence result.

## 3. Convergence analysis

In this section, we prove the semilocal convergence of Newton's method under condition (6). First, we establish a system of recurrence relations, from the real parameters that are introduced under some conditions for the pair $(F, x_0)$, where a sequence of positive real numbers is involved. After that, we can guarantee the semilocal convergence of Newton's method in the Banach space $X$.

**3.1. Recurrence relations.** We suppose:

(C1) There exists $\Gamma_0 = [F'(x_0)]^{-1} \in \mathcal{L}(Y, X)$, for some $x_0 \in \Omega$, with $\|\Gamma_0\| \leq \beta$ and $\|\Gamma_0 F(x_0)\| \leq \eta$,

(C2) There exists a function $\omega : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that $\|F'(x) - F'(y)\| \leq \omega(\|x\|, \|y\|)\, \|x - y\|^p$, $x, y \in \Omega$, $p \in (0, 1]$, nondecreasing continuous in both arguments and $\omega(0, 0) \geq 0$.

In addition, we also suppose that the equation

$$\varphi(t) = ((1+p) - (2+p)Q(t)\beta\eta^p)\, t - (1+p)\,(1 - Q(t)\beta\eta^p)\,\eta = 0,$$

where $Q(t) = \omega(\|x_0\|+t, \|x_0\|+t)$, has at least one positive real root and denote the smallest one by $R$.

From the above, we denote $a_0 = Q(R)\beta\eta^p$ and define the scalar sequence:

$$a_{n+1} = a_n f(a_n)^{1+p} g(a_n)^p, \quad n = 0, 1, 2, \ldots, \tag{7}$$

where

$$f(x) = \frac{1}{1-x} \quad \text{and} \quad g(x) = \frac{x}{1+p}. \tag{8}$$

Note that an obvious problem results if $a_0 = 0$, so that we only consider $a_0 > 0$.

Next, we prove the following recurrence relations for sequences (7) and $\{x_n\}$:

$$\|\Gamma_1\| = \|[F'(x_1)]^{-1}\| \leq f(a_0)\|\Gamma_0\|, \tag{9}$$

$$\|x_2 - x_1\| \leq f(a_0)g(a_0)\|x_1 - x_0\|, \tag{10}$$

$$Q(R)\|\Gamma_1\|\|x_2 - x_1\|^p \leq a_1, \tag{11}$$

provided that

$$x_1 \in \Omega \quad \text{and} \quad a_0 < 1. \tag{12}$$

If $x_1 \in \Omega$, then $\|I - \Gamma_0 F'(x_1)\| \leq \|\Gamma_0\|\|F'(x_0) - F'(x_1)\| \leq \beta\, \omega(\|x_0\|, \|x_1\|)$ $\|x_1 - x_0\|^p \leq Q(R)\beta\eta^p = a_0 < 1$. In addition, by the Banach lemma on invertible operators, it follows that there exists the operator $\Gamma_1$ and

$$\|\Gamma_1\| \leq \frac{\|\Gamma_0\|}{1 - \|I - \Gamma_0 F'(x_1)\|} \leq f(a_0)\|\Gamma_0\|.$$

Next, from Taylor's series and the sequence $\{x_n\}$, we have

$$\begin{aligned}
\|F(x_1)\| &= \left\| \int_0^1 (F'(x_0 + \tau(x_1 - x_0)) - F'(x_0))(x_1 - x_0)\, d\tau \right\| \\
&\leq \left( \int_0^1 \omega(\|x_0 + \tau(x_1 - x_0)\|, \|x_0\|)\, \tau^p dt \right) \|x_1 - x_0\|^{1+p} \\
&\leq \frac{Q(R)\eta^p}{1+p} \|x_1 - x_0\|.
\end{aligned}$$

As a consequence,

$$\|x_2 - x_1\| \le \|\Gamma_1\| \|F(x_1)\| \le f(a_0)g(a_0)\|x_1 - x_0\|,$$
$$Q(R)\|\Gamma_1\|\|x_2 - x_1\|^p \le a_0 f(a_0)^{1+p} g(a_0)^p = a_1.$$

Moreover, if $f(a_0)g(a_0) < 1$, then

$$\|x_2 - x_0\| \le (1 + f(a_0)g(a_0))\|x_1 - x_0\| < \frac{(1+p)(1-a_0)}{(1+p)-(2+p)a_0}\eta = R. \tag{13}$$

Later, we generalize the previous recurrence relations to every point of the sequence $\{x_n\}$, so that we can guarantee that $\{x_n\}$ is a Cauchy sequence from them. For this, we first analyse the sequence $\{a_n\}$ in the next section.

**3.2. Analysis of the scalar sequence.** Now, we analyse the scalar sequence defined in (7) in order to prove later the semilocal convergence of the sequence $\{x_n\}$ in the Banach space $X$. For this, it suffices to see that $\{x_n\}$ is a Cauchy sequence and (12) is true for all $x_n$ and $a_{n-1}$ with $n \ge 2$. First, we give a technical lemma whose proof is trivial.

**Lemma 3.1.** *Let $f$ and $g$ be the two real functions given in* (8). *Then*

(a) *$f$ is increasing and $f(x) > 1$ in $(0,1)$,*
(b) *$g$ is increasing,*
(c) *for $\gamma \in (0,1)$, we have $f(\gamma x) < f(x)$ if $x \in [0,1)$ and $g(\gamma x) = \gamma g(x)$.*

Then, we prove some properties of scalar sequence (7). For this, we consider the auxiliary function

$$h(x) = (1+p)^p(1-x)^{1+p} - x^p, \quad p \in (0,1], \tag{14}$$

which has only one zero $\xi$ in the interval $(0, \frac{1}{2}]$, since $h(0) = (1+p)^p > 0$, $h(\frac{1}{2}) \le 0$ and $h'(x) < 0$ in $(0, \frac{1}{2}]$. Notice that function (14) arises from the analysis of the value $f(a_0)^{1+p}g(a_0)^p = \frac{a_0^p}{(1+p)^p(1-a_0)^{1+p}}$.

**Lemma 3.2.** *Let $f$ and $g$ be the two scalar functions defined in* (8). *If $a_0 \in (0,\xi)$, then*

(a) *$f(a_0)^{1+p}g(a_0)^p < 1$,*
(b) *the sequence $\{a_n\}$ is strictly decreasing,*
(c) *$a_n < 1$, for all $n \ge 0$.*

*If $a_0 = \xi$, then $a_n = a_0 < 1$ for all $n \ge 1$.*

*Proof.* We first consider the case $a_0 \in (0,\xi)$. Then, item (a) follows with strict inequality, since $h(a_0) > 0$. Item (b) is proved by mathematical induction on $n$.

As $f(a_0)^{1+p}g(a_0)^p < 1$, we have $a_1 < a_0$. If we now suppose that $a_j < a_{j-1}$, for $j = 1, 2, \dots, n$, then

$$a_{n+1} = a_n f(a_n)^{1+p} g(a_n)^p < a_n f(a_0)^{1+p} g(a_0)^p < a_n,$$

since $f$ and $g$ are increasing. As a result, the sequence $\{a_n\}$ is strictly decreasing. To see item (c), we have $a_n < a_0 < 1$, for all $n \geq 0$, from the fact that the sequence $\{a_n\}$ is strictly decreasing and $a_0 \in (0, \xi)$.

Furthermore, if $a_0 = \xi$, then $f(a_0)^{1+p}g(a_0)^p = 1$ and, consequently, $a_n = a_0 = \xi < 1$, for all $n \geq 0$. □

**Lemma 3.3.** *Let $f$ and $g$ be the two scalar functions defined in* (8)*. If $a_0 \in (0, \xi)$, we define $\gamma = \frac{a_1}{a_0}$, and then*

(a) $a_n < \gamma^{(1+p)^{n-1}} a_{n-1}$ *and* $a_n < \gamma^{\frac{(1+p)^n - 1}{p}} a_0$, *for all* $n \geq 2$,

(b) $f(a_n)g(a_n) < \gamma^{\frac{(1+p)^n - 1}{p}} f(a_0)g(a_0) = \dfrac{\gamma^{\frac{(1+p)^n}{p}}}{f(a_0)^{\frac{1}{p}}}$, *for all* $n \geq 1$.

*If $a_0 = \xi$, then $f(a_n)g(a_n) = f(a_0)g(a_0) = f(a_0)^{-\frac{1}{p}}$, for all $n \geq 1$.*

*Proof.* We only prove the case $a_0 \in (0, \xi)$, since the case $a_0 = \xi$ follows analogously to the former. The proof of item (a) follows by an induction process. If $n = 2$, by item (b) of Lemma 3.1, we have

$$a_2 = a_1 f(a_1)^{1+p} g(a_1)^p = \gamma a_0 f(\gamma a_0)^{1+p} g(\gamma a_0)^p < \gamma^{1+p} a_1 = \gamma^{2+p} a_0.$$

We now suppose that

$$a_{n-1} < \gamma^{(1+p)^{n-2}} a_{n-2} < \gamma^{\frac{(1+p)^{n-1} - 1}{p}} a_0.$$

Then, by the same reasoning,

$$\begin{aligned} a_n &= a_{n-1} f(a_{n-1})^{1+p} g(a_{n-1})^p \\ &< \gamma^{(1+p)^{n-2}} a_{n-2} f\left(\gamma^{(1+p)^{n-2}} a_{n-2}\right)^{1+p} g\left(\gamma^{(1+p)^{n-2}} a_{n-2}\right)^p \\ &< \gamma^{(1+p)^{n-1}} a_{n-1} \\ &< \gamma^{\frac{(1+p)^n - 1}{p}} a_0. \end{aligned}$$

To prove item (b), we observe, for $n \geq 1$,

$$f(a_n)g(a_n) < f\left(\gamma^{\frac{(1+p)^n - 1}{p}} a_0\right) g\left(\gamma^{\frac{(1+p)^n - 1}{p}} a_0\right) < \gamma^{\frac{(1+p)^n - 1}{p}} f(a_0)g(a_0) = \frac{\gamma^{\frac{(1+p)^n}{p}}}{f(a_0)^{\frac{1}{p}}}.$$

The proof is complete. □

**3.3. A semilocal convergence result.** Notice that the equation $\varphi(t) = 0$ arises from imposing that all the points $x_n$ are in the ball $B(x_0, R)$. In addition, we consider the following condition:

(C3) $a_0 = Q(R)\beta\eta^p \in (0, \xi]$, where $R$ is the smallest positive real root of $\varphi(t) = 0$, $\xi$ is the unique zero of function (14) in the interval $(0, \frac{1}{2}]$, $p \in (0, 1]$, and $B(x_0, R) \subset \Omega$.

We are now ready to prove the semilocal convergence of Newton's method when is applied to differentiable operators $F$ such that $F'$ satisfies a condition of type (6).

**Theorem 3.4.** *Let $X$ and $Y$ be two Banach spaces and $F : \Omega \subseteq X \longrightarrow Y$ a continuously differentiable operator on a nonempty open convex domain $\Omega$. Suppose that* (C1)–(C3) *are satisfied. Then, Newton's sequence $\{x_n\}$ converges to a solution $x^*$ of $F(x) = 0$ starting at $x_0$ and $x_n, x^* \in \overline{B(x_0, R)}$. Moreover, if the equation*

$$\beta\,\omega(\|x_0\|, \|x_0\| + t)\left(t^{1+p} - R^{1+p}\right) = 1 + p$$

*has at least one positive real root and the smallest positive one is denoted by $r$, then $x^*$ is unique in $B(x_0, r) \cap \Omega$. Furthermore, the sequence $\{x_n\}$ has R-order of convergence at least $1 + p$ if $a_0 \in (0, \xi)$, or at least one if $a_0 = \xi$, and*

$$\|x^* - x_n\| \le \left(\gamma^{\frac{(1+p)^n - 1}{p^2}}\right) \frac{\Delta^n}{1 - \gamma^{\frac{(1+p)^n}{p}}\Delta}\eta, \quad n \ge 0, \tag{15}$$

*where $\gamma = \frac{a_1}{a_0}$ and $\Delta = (1 - a_0)^{\frac{1}{p}}$.*

*Proof.* We start proving the case $a_0 \in (0, \xi)$. First, we prove that the following four items are satisfied, for $n \ge 2$, by the sequence $\{x_n\}$:

(I) There exists $\Gamma_{n-1} = [F'(x_{n-1})]^{-1}$ and $\|\Gamma_{n-1}\| \le f(a_{n-2})\|\Gamma_{n-2}\|$,
(II) $\|x_n - x_{n-1}\| \le f(a_{n-2})g(a_{n-2})\|x_{n-1} - x_{n-2}\|$,
(III) $Q(R)\|\Gamma_{n-1}\|\|x_n - x_{n-1}\|^p \le a_{n-1}$,
(IV) $x_n \in \Omega$.

First, as $\eta < R$, then $x_1 \in \Omega$. Then, from (9), (10), (11) and (13), we have that the previous items hold for $n = 2$. If we now suppose that (I)–(III) are true for some $n-1$, it follows by analogy to the case where $n = 2$, by induction, that (I)–(III) also hold for $n$. Notice that $a_n < 1$ for all $n \ge 0$. Now, we prove (IV).

Observe

$$
\begin{aligned}
\|x_n - x_0\| &\leq \|x_n - x_{n-1}\| + \|x_{n-1} - x_{n-2}\| + \cdots + \|x_1 - x_0\| \\
&\overset{(\mathrm{II})}{\leq} \left(1 + \sum_{i=0}^{n-2} \left( \prod_{j=0}^{i} f(a_j) g(a_j) \right)\right) \|x_1 - x_0\| \\
&< \left(1 + \sum_{i=0}^{n-2} \left( \prod_{j=0}^{i} f(a_0) g(a_0) \gamma^{\frac{(1+p)^j - 1}{p}} \right)\right) \|x_1 - x_0\| \quad \text{(Lemma 3.3(b))} \\
&= \left(1 + \sum_{i=0}^{n-2} \left( \prod_{j=0}^{i} \left( \gamma^{\frac{(1+p)^j}{p}} \Delta \right)\right)\right) \|x_1 - x_0\| \\
&= \left(1 + \sum_{i=0}^{n-2} \left( \gamma^{\frac{(1+p)^{1+i} - 1}{p^2}} \Delta^{1+i} \right)\right) \|x_1 - x_0\|,
\end{aligned}
$$

where $\gamma = \frac{a_1}{a_0} < 1$ and $\Delta = \frac{f(a_0)g(a_0)}{\gamma^{\frac{1}{p}}} = \frac{1}{f(a_0)^{\frac{1}{p}}} = (1 - a_0)^{\frac{1}{p}} < 1$. By Bernoulli's inequality, it follows $\gamma^{\frac{(1+p)^{1+i}-1}{p^2}} = \gamma^{\frac{1}{p}} \gamma^{\frac{1+p}{p^2}((1+p)^i - 1)} \leq \gamma^{\frac{1}{p}} \gamma^{\frac{1+p}{p} i}$, and therefore

$$
\|x_n - x_0\| < \left(1 + \gamma^{\frac{1}{p}} \Delta \sum_{i=0}^{n-2} \gamma^{\frac{1+p}{p} i} \Delta^i \right) \|x_1 - x_0\| < \frac{\eta}{1 - \gamma^{\frac{1}{p}} \Delta} = R,
$$

so that $x_n \in B(x_0, R)$. As $B(x_0, R) \subset \Omega$, then $x_n \in \Omega$, for all $n \geq 0$. Note that the conditions required in (12) are now satisfied for all $x_n$ and $a_{n-1}$, with $n \geq 2$.

Second, we prove that $\{x_n\}$ is a Cauchy sequence. For this, we follow an analogous procedure to the latter. So, for $m \geq 1$ and $n \geq 1$, we have

$$
\begin{aligned}
\|x_{n+m} - x_n\| &\leq \sum_{i=n}^{n+m-1} \|x_{i+1} - x_i\| \\
&\overset{(\mathrm{II})}{\leq} \sum_{i=n-1}^{n+m-2} \left( \prod_{j=0}^{i} f(a_j) g(a_j) \right) \|x_1 - x_0\| \\
&< \sum_{i=n-1}^{n+m-2} \left( \prod_{j=0}^{i} f(a_0) g(a_0) \, \gamma^{\frac{(1+p)^j - 1}{p}} \right) \|x_1 - x_0\| \quad \text{(Lemma 3.3(b))} \\
&= \sum_{i=n-1}^{n+m-2} \left( \prod_{j=0}^{i} \left( \gamma^{\frac{(1+p)^j}{p}} \Delta \right)\right) \|x_1 - x_0\| \\
&= \sum_{i=0}^{m-1} \left( \gamma^{\frac{(1+p)^{n+i} - 1}{p^2}} \Delta^{n+i} \right) \|x_1 - x_0\|.
\end{aligned}
$$

Taking into account Bernoulli's inequality, it follows

$$\gamma^{\frac{(1+p)^{n+i}-1}{p^2}} = \gamma^{\frac{(1+p)^n-1}{p^2}} \gamma^{\frac{(1+p)^n}{p^2}((1+p)^i-1)} \leq \gamma^{\frac{(1+p)^n-1}{p^2}} \gamma^{\frac{(1+p)^n}{p}i},$$

so that

$$\begin{aligned} \|x_{n+m}-x_n\| &< \left(\sum_{i=0}^{m-1}\left(\gamma^{\frac{(1+p)^n}{p}i}\Delta^i\right)\right)\gamma^{\frac{(1+p)^n-1}{p^2}}\Delta^n\|x_1-x_0\| \\ &< \frac{1-\left(\gamma^{\frac{(1+p)^n}{p}}\Delta\right)^m}{1-\gamma^{\frac{(1+p)^n}{p}}\Delta}\gamma^{\frac{(1+p)^n-1}{p^2}}\Delta^n\eta. \end{aligned} \tag{16}$$

Thus, $\{x_n\}$ is a Cauchy sequence.

Third, we prove that $x^*$ is a solution of equation $F(x)=0$. As $\|\Gamma_n F(x_n)\|\to 0$ when $n\to\infty$, if we take into account that $\|F(x_n)\| \leq \|F'(x_n)\|\|\Gamma_n F(x_n)\|$ and $\{\|F'(x_n)\|\}$ is bounded, since

$$\|F'(x_n)\| \leq \|F'(x_0)\| + \omega(\|x_0\|, \|x_n\|)\|x_n - x_0\|^p < \|F'(x_0)\| + \omega(\|x_0\|, \|x_0\| + R)R^p,$$

it follows that $\|F(x_n)\| \to 0$ when $n\to\infty$. As a consequence, we obtain $F(x^*)=0$ by the continuity of $F$ in $\overline{B(x_0,R)}$.

To prove the uniqueness of the solution $x^*$, we suppose that $y^*$ is another solution of $F(x)=0$ in $B(x_0,r)\cap\Omega$. Then, from the approximation

$$0 = F(y^*) - F(x^*) = \int_{x^*}^{y^*} F'(x)dx = \int_0^1 F'(x^* + \tau(y^*-x^*))\,d\tau(y^*-x^*),$$

it follows that $x^*=y^*$, provided that the operator $\int_0^1 F'(x^*+\tau(y^*-x^*))\,d\tau$ is invertible. To prove this, we prove equivalently that there exists the operator $J^{-1}$, where $J=\Gamma_0\int_0^1 F'(x^*+\tau(y^*-x^*))\,d\tau$. Indeed, as

$$\begin{aligned} \|I-J\| &\leq \|\Gamma_0\| \int_0^1 \|F'(x_0) - F'(x^*+\tau(y^*-x^*))\|\,d\tau \\ &\leq \beta\int_0^1 \omega(\|x_0\|, \|x^*+\tau(y^*-x^*)\|)\,\|x_0-(x^*+\tau(y^*-x^*))\|^p d\tau \\ &\leq \beta\,\omega(\|x_0\|, \|x_0\|+r)\int_0^1 ((1-\tau)\|x^*-x_0\| + \tau\|y^*-x_0\|)^p\,d\tau \\ &< \beta\,\omega(\|x_0\|, \|x_0\|+r)\int_0^1 ((1-\tau)R+\tau r)^p\,d\tau \\ &= 1, \end{aligned}$$

the operator $J^{-1}$ exists by the Banach lemma on invertible operators.

Finally, by letting $m \to \infty$ in (16), we obtain (15) for all $n \geq 0$. Moreover, from (15), it follows that the $R$-order of convergence of sequence $\{x_n\}$ is at least $1+p$, since

$$\|x^* - x_n\| \leq \frac{\eta}{\gamma^{\frac{1}{p^2}}\left(1-\gamma^{\frac{1}{p}}\Delta\right)} \left(\gamma^{\frac{1}{p^2}}\right)^{(1+p)^n}, \quad n \geq 0.$$

For the second case, $a_0 = \xi$, we have $a_n = a_0 = \xi$, for all $n \geq 0$. Then, following an analogous procedure to the previous one, we obtain the same results, now taking into account that $\gamma = 1$ and $\Delta = f(a_0)g(a_0) < 1$, except for the $R$-order of convergence; in this case, the $R$-order of convergence is at least one. □

## 4. Comments

In this section, some remarkable comments are given. First, we provide a convergence result for Newton's method when $p = 0$ in (6), which was not included in Section 3. Second, we discuss the case $p = 1$ in (6). Third, the assumptions required in Theorem 3.4 are compared with those required by Keller in [10] for the convergence of Newton's method. Finally, we give another analysis of scalar sequence (7), different from that appearing in Section 3.2, that leads to the same results.

**4.1. Case** $p = 0$**.** If $p = 0$, condition (C2) is reduced to

$(\widetilde{C2})$ $\|F'(x) - F'(y)\| \leq \omega(\|x\|, \|y\|)$, $x, y \in \Omega$, where $\omega : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a nondecreasing continuous function in both arguments and such that $\omega(0,0) \geq 0$.

In this case, $a_0 = Q(R)\beta$, where $Q(t) = \omega(\|x_0\| + t, \|x_0\| + t)$ and $R$, if there exists, is the smallest positive real root of the equation $\varphi(t) = 0$ with $p = 0$. Following a procedure similar to that of case $p \in (0,1]$, we have

- if $a_0 < 1$ and $x_n \in \Omega$ $(n \geq 1)$, then $\|\Gamma_n\| \leq \frac{\|\Gamma_0\|}{1-a_0}$, for all $n \geq 1$,
- $\|x_{n+1} - x_n\| \leq \left(\frac{a_0}{1-a_0}\right)^n \|x_1 - x_0\|$, $n \geq 0$,

so that the semilocal convergence result for Newton's method is now the following.

**Theorem 4.1.** *Let $X$ and $Y$ be two Banach spaces and $F : \Omega \subseteq X \longrightarrow Y$ a continuously differentiable operator on a nonempty open convex domain $\Omega$. Suppose that* (C1), $(\widetilde{C2})$, (C3) *with $p = 0$ are satisfied. Suppose also that $a_0 = Q(R)\beta \in \left(0, \frac{1}{2}\right]$ and $B(x_0, R) \subset \Omega$. Then, Newton's sequence $\{x_n\}$ converges to a solution $x^*$ of $F(x) = 0$ starting at $x_0$. Moreover, $x_n, x^* \in \overline{B(x_0, R)}$ and $x^*$ is unique in $B(x_0, r) \cap \Omega$, where $r$, if there exists, is the smallest positive real root of the equation $\beta\, \omega(\|x_0\|, \|x_0\| + t) = 1$.*

**4.2.** $F'$ **is Lipschitz continuous.** If $\omega(s,t)$ is a constant $L$ and $p = 1$ in the study carried out in Section 3, Theorem 3.4 is reduced to the Newton-Kantorovich theorem, see [11].

In addition, inequalities (I)–(III) appeared in the proof of Theorem 3.4 are reduced to equalities for the polynomial

$$\zeta(x) = \frac{L}{2}x^2 - \frac{x}{\beta} + \frac{\eta}{\beta},$$

so that (I)–(III) are optimal for it; namely (I)–(III) can be written with equalities. Taking into account this, we can improve the a priori error bounds given by other authors. Observe that the polynomial $\zeta$ is just the quadratical Kantorovich polynomial $p(t)$, which appears in condition (A3).

**4.3.** $F'$ **is Hölder continuous.** We compare the conditions required for the convergence of Newton's method in Theorem 3.4 if $\omega(s,t) = K$ and those appearing in Keller's theorem (Theorem 4 of [10]).

Under the same conditions (C1)–(C3) with $\omega(s,t) = K$, Keller's theorem requires that

$$a_0 \leq \frac{1}{2+p}\left(\frac{p}{1+p}\right)^p,$$

and, in Theorem 3.4, we require

$$a_0 \leq \xi,$$

where $\xi$ is the unique zero of function (14) in $(0, \frac{1}{2}]$. Note that the former condition for $a_0$ is more restrictive than the latter one if $p \in (0.2856\ldots, 1]$, since

$$\frac{1}{2+p}\left(\frac{p}{1+p}\right)^p < \xi.$$

As a consequence, the chances of finding starting points in the application of Newton's method for operators with $(K,p)$-Hölder continuous first Fréchet-derivative and $p \in (0.2856\ldots, 1]$, is higher if Theorem 3.4 is applied.

**4.4. Scalar sequence.** We can also analyse the scalar sequence (7) by studing the fixed points of the function $q(x) = xf(x)^{1+p}g(x)^p$. If $q(x) = x$, $x$ is a fixed point of $q$, i.e. if $x \neq 0, 1$:

$$\frac{x^{1+p}}{(1+p)^p(1-x)^{1+p}} = x \qquad \Longleftrightarrow \qquad (1+p)^p(1-x)^{1+p} - x^p = 0,$$

this agrees with $h(x) = 0$, where $h$ is defined in (14).

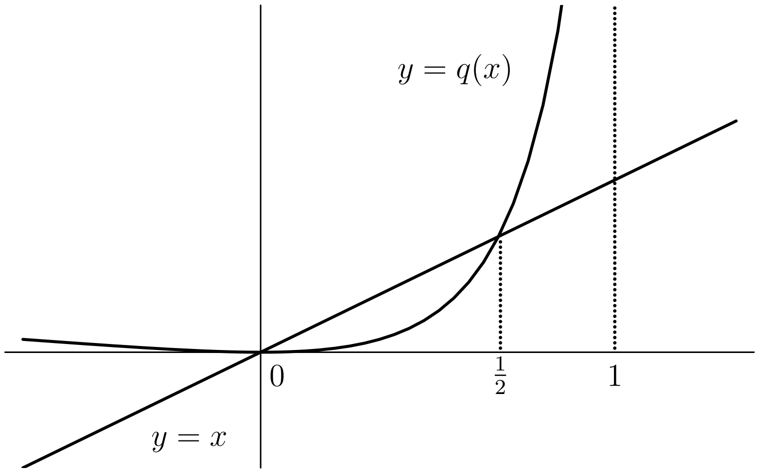


Figure 1: Fixed points of $q$ when $p = 1$

As we can see in Figure 1, $x = 0$ and $x = \frac{1}{2}$ are fixed points of $q$ in $[0, 1]$ if $p = 1$. It is easy to prove that the sequence $\{a_n = q(a_{n-1})\}$ only converges if $a_0 \leq \frac{1}{2}$, so that

$$a_n \searrow 0 \ (n \to \infty) \quad \text{if } a_0 < \frac{1}{2},$$
$$a_n = \frac{1}{2} \ (n \geq 0) \quad \text{if } a_0 = \frac{1}{2}.$$

Observe that both situations appear in Lemma 3.2.

On the other hand, note that if the value $p \in (0, 1)$ varies, the fixed points which appear are $x = 0$ and $x = \xi$ (zero of $h$) with $\xi < \frac{1}{2}$, see Figure 2.

In both cases, it is interesting to observe that if $a_0 > \xi = \frac{1}{2}$ $(p = 1)$ or $a_0 > \xi$ $(0 < p < 1)$, $n_0 \in \mathbb{N}$ exists such that $a_{n_0} > 1$, see Figure 3, so that the condition $a_n < 1$ for all $n \geq 0$ is not met.

## 5. Application to a nonlinear integral equation of Hammerstein type

In this section, we study an application where a nonlinear Hammerstein integral equations of the second kind is involved. First, we provide some results of existence and uniqueness of solution for the nonlinear Hammerstein integral equations of the second kind of type (2). Second, we analyse a particular equation of (2) from a graphical point of view, where we show that the domain of starting points given by the Newton-Kantorovich theorem for Newton's method is modified from Theorem 3.4. Third, we locate solutions of the particular equation. And, finally, fromTheorem 3.4, we guarantee the convergence of Newton's method and approximate two solutions.

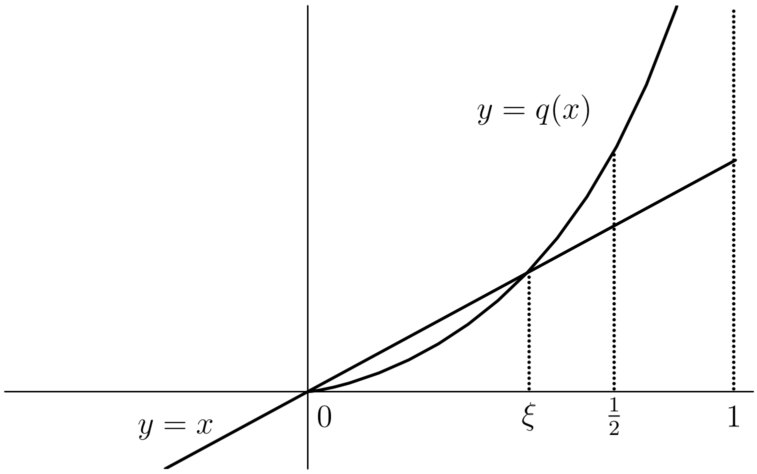


Figure 2: Fixed points of $q$ when $p \in (0, 1)$

**5.1. Existence and uniqueness of solution.** Remenber that solving equation (2) is equivalent to solve $F(x) = 0$, where $F$ is defined in (3). Now, we apply the study of Section 3 to obtain results on the existence and uniqueness of solution of equation (2).

We start calculating the parameters $\beta$ and $\eta$ that appear in the study. Firstly, from (4) and once $x_0(s)$ is fixed, we have

$$\|I - F'(x_0)\| \leq m|\lambda|\|x_0^{m-1}\|\ell.$$

By the Banach lemma on invertible operators, if $m|\lambda|\|x_0^{m-1}\|\ell < 1$, we obtain that there exists $\Gamma_0 = [F'(x_0)]^{-1}$ and

$$\|\Gamma_0\| \leq \frac{1}{1 - m|\lambda|\|x_0^{m-1}\|\ell}.$$

From (3), we have $\|F(x_0)\| \leq \|x_0 - u\| + |\lambda|\|x_0^m\|\ell$ and, therefore,

$$\|\Gamma_0 F(x_0)\| \leq \frac{\|x_0 - u\| + |\lambda|\|x_0^m\|\ell}{1 - m|\lambda|\|x_0^{m-1}\|\ell}.$$

On the other hand, from (5), it follows $\|F'(x) - F'(y)\| \leq \omega(\|x\|, \|y\|)\|x - y\|$, where

$$\omega(s, t) = m|\lambda|\ell \left(s^{m-2} + s^{m-3}t + \cdots + st^{m-3} + t^{m-2}\right). \qquad (17)$$

Once the parameters $\beta$ and $\eta$ are calculated and the function $\omega$ is known, we can already establish the following result on the existence of solution of equation (2) from Theorem 3.4.

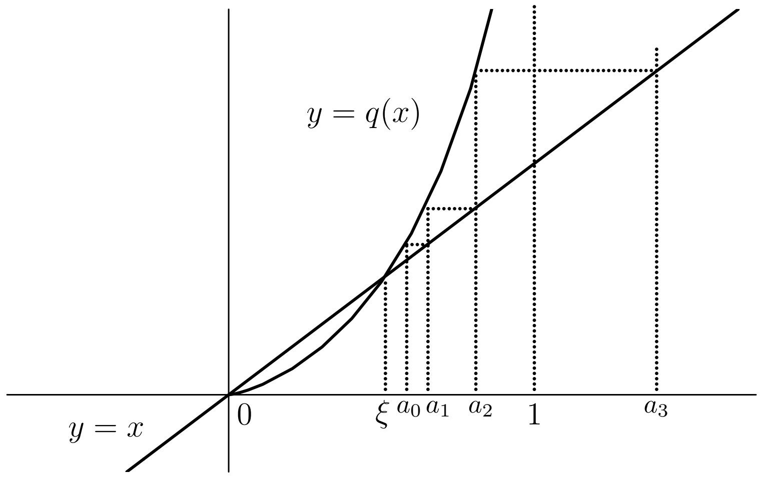


Figure 3: Situation when $a_0 > \xi$

**Theorem 5.1.** *Let $F$ be the operator defined in (3) and $x_0 \in \Omega$ a point such that there exists $[F'(x_0)]^{-1}$. If $m|\lambda|\|x_0^{m-1}\|\ell < 1$, the equation*

$$(2 - 3Q(t)\beta\eta)\, t - 2\,(1 - Q(t)\beta\eta)\,\eta = 0 \tag{18}$$

*where $Q(t) = \omega(\|x_0\| + t, \|x_0\| + t)$ and $\omega$ is defined in (17), has at least one positive real root and the smallest positive real root, denoted by $R$, satisfies that $Q(R)\beta\eta \in (0, \xi]$, where $\xi = \frac{1}{2}$ is the unique zero of function (14) with $p = 1$ in the interval $\left(0, \frac{1}{2}\right]$, and $B(x_0, R) \subset \Omega$, then a solution of (2) exists at least in $\overline{B(x_0, R)}$. Moreover, if the equation*

$$\beta\,\omega(\|x_0\|, \|x_0\| + t)\left(t^2 - R^2\right) = 2,$$

*has at least one positive real root and the smallest positive one is denoted by $r$, then $x^*$ is unique in $B(x_0, r)$.*

**5.2. Graphical analysis of a particular case.** If we consider the following particular case of equation (2)

$$x(s) = s + \frac{1}{2}\int_0^1 G(s,t)\, x(t)^5\, dt, \tag{19}$$

where $G(s,t)$ is the Green function in $[0,1] \times [0,1]$, and take into account the reasonable choice of the initial approximation $x_0(s) = \theta u(s)$, with $\theta \in \mathbb{R}$, [2,4], we see that the condition $m|\lambda|\|x_0^{m-1}\|\ell < 1$, that must be held to the operator $\Gamma_0$ exists, is reduced to $|\theta| < \sqrt[4]{\frac{16}{5}} = 1.3374\ldots$, since $m = 5$, $\lambda = \frac{1}{2}$, $\ell = \frac{1}{8}$, $u(s) = s$ and $\|x_0\| = |\theta|\|u\| = |\theta|$. See Figure 4, the values of $\|x_0\|$ are represented in the horizontal axis.

On the one hand, for the Newton-Kantorovich theorem can be applied, we have previously to locate a solution of integral equation (19) in order to obtain the value $L$ and be able to apply the theorem. So, taking into account that a solution $x^*(s)$ of (19) in $\mathcal{C}([a,b])$ must satisfy:

$$\|x^*\| - 1 - \frac{1}{16}\|x^*\|^5 \leq 0,$$

it follows that $\|x^*\| \leq \sigma_1 = 1.1012\ldots$ or $\|x^*\| \geq \sigma_2 = 1.5382\ldots$, where $\sigma_1$ and $\sigma_2$ are the two real positive roots of $\frac{t^5}{16} - t + 1 = 0$. Thus, from the Newton-Kantorovich theorem, we can only approximate one solution $x^*(s)$ by Newton's method, that which satisfies $\|x^*\| \in [0, \sigma_1]$, since we can consider $\Omega = B(0,\sigma)$, with $\sigma \in (\sigma_1, \sigma_2)$, where $F'(x)$ is Lipschitz continuous, and takes $x_0 \in B(0,\sigma)$ as starting point.

After that, taking into account what we have just mentioned, we consider $\Omega = B(0,\sigma)$ with $\sigma \in (\sigma_1, \sigma_2)$ and choose $\sigma = \frac{3}{2}$. Then, as $\|x_0(s) - u(s)\| = |\theta - 1|\|u(s)\| = |\theta - 1|$ in the upper bound of $\|\Gamma_0 F(x_0)\|$, we observe in Figure 5, where the horizontal axis represents the values of $\|x_0\|$, that the starting point $x_0(s)$ is such that $\|x_0\| = |\theta| \geq 2.1710\ldots$ for the condition $L\beta\eta \leq \frac{1}{2}$ of the Newton-Kantorovich theorem is true. Therefore, it is clear that there is no starting point which satisfies the two conditions simultaneously, so that the Newton-Kantorovich theorem is not applicable in this situation.

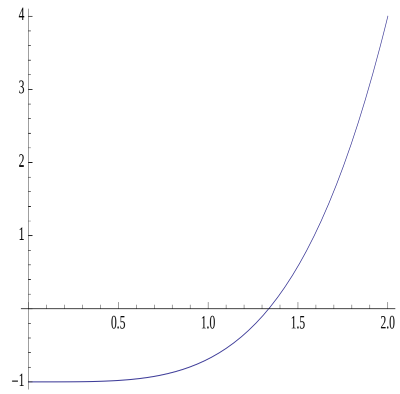


Figure 4: $\|x_0\| = |\theta| < 1.3374\ldots$ for the existence of $\Gamma_0$.

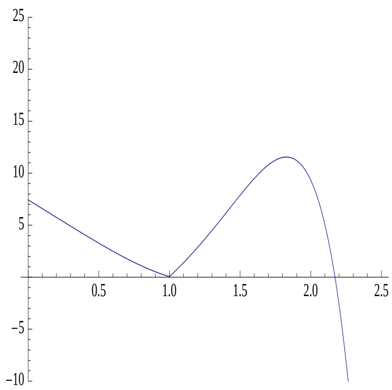


Figure 5: $\|x_0\| = |\theta| \geq 2.1710\ldots$ for Kantorovich's condition $L\beta\eta \leq \frac{1}{2}$.

On the other hand, if we consider Theorem 3.4, we see that, mainly, two conditions have to be held: first, the existence of at least one positive real root of the equation $\varphi(t) = 0$; and second, the condition $a_0 = Q(R)\beta\eta^p \in (0, \xi]$ on the smallest positive real root $R$ of $\varphi(t) = 0$, which now is reduced to $a_0 = Q(R)\beta\eta \in \left(0, \frac{1}{2}\right]$, since $p = 1$.

If, firstly, we focus our attention on the existence of at least one positive real root of the equation $\varphi(t) = 0$, from Figures 6 and 7, we suspect that

$\|x_0\| = |\theta| > 0.9$ (more or less), so that the equation $\varphi(t) = 0$ has positive real roots.

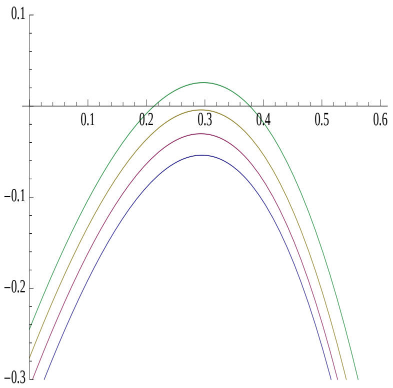


Figure 6: $\varphi(t)$ with $\|x_0\| = |\theta| = 0.85, 0.875, 0.9, 0.925$ (respectively: blue, red, yellow and green).

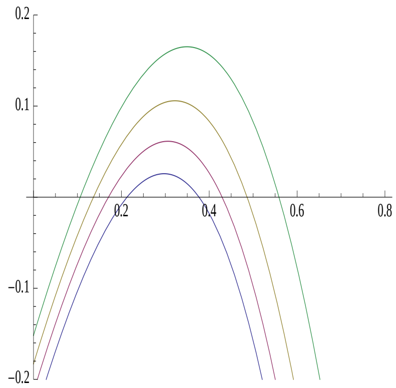


Figure 7: $\varphi(t)$ with $\|x_0\| = |\theta| = 0.925, 0.95, 0.975, 1$ (respectively: blue, red, yellow and green).

If we now focus our attention on the condition on $R$ of Theorem 3.4, $Q(R)\beta\eta \leq \frac{1}{2}$, we see graphically in Figures 8–11, where the curve is the function $\varphi(t)$ and the vertical line is $Q(t)\beta\eta = \frac{1}{2}$, which represents the condition $a_0 \leq \frac{1}{2}$, four different situations where the last condition is satisfied.

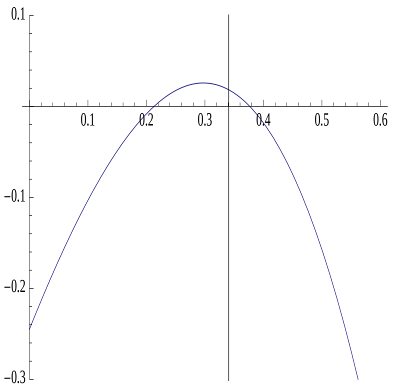


Figure 8: $\varphi(t)$ with $\|x_0\| = |\theta| = 0.925$: $R = 0.2128\ldots$ and $Q(R)\beta\eta = 0.2948\ldots \leq \xi = \frac{1}{2}$.

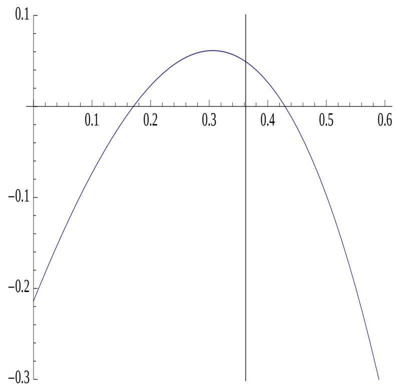


Figure 9: $\varphi(t)$ with $\|x_0\| = |\theta| = 0.95$: $R = 0.1704\ldots$ and $Q(R)\beta\eta = 0.2921\ldots \leq \xi = \frac{1}{2}$.

As a result, we have just seen four specific situations in which we can apply Theorem 3.4, but not the Newton-Kantorovich theorem. In addition, from the previous graphical analysis, the modification of the domain of starting points for Newton's method given by the Newton-Kantorovich theorem is seen.

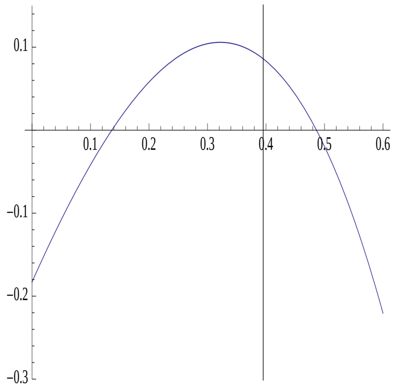


Figure 10: $\varphi(t)$ with $\|x_0\| = |\theta| = 0.975$: $R = 0.1363\dots$ and $Q(R)\beta\eta = 0.2668\dots \leq \xi = \frac{1}{2}$.

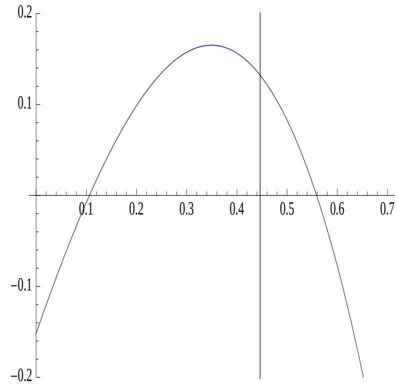


Figure 11: $\varphi(t)$ with $\|x_0\| = |\theta| = 1$, $R = 0.1062\dots$ and $Q(R)\beta\eta = 0.2237\dots \leq \xi = \frac{1}{2}$.

**5.3. Localization of solutions of equation (19).** If we now choose the starting point $x_0(s) = s$, choice analysed in the previous section, we have

$$\beta = \frac{16}{11}, \quad \eta = \frac{1}{11}, \quad \omega(s,t) = \frac{5}{16}\left(s^3 + s^2t + st^2 + t^3\right),$$

so that the smallest positive real root of equation (18), which is

$$-(0.1517\dots) + (1.5942\dots)t - (1.3974\dots)t^2 - (1.4575\dots)t^3 - (0.4958\dots)t^4 = 0,$$

is $R = 0.1062\dots$ As a consequence, $Q(R) = 1.6921\dots$, $Q(R)\beta\eta = 0.2237\dots \leq \xi = \frac{1}{2}$ and the hypotheses of Theorem 5.1 are then satisfied. Then (19) has a solution $x^*$ in the region $\{\nu \in \mathcal{C}([0,1]) : \|\nu - s\| \leq 0.1062\dots\}$, which is unique in $\{\nu \in \mathcal{C}([0,1]) : \|\nu - s\| < 0.6685\dots\}$.

On the other hand, as we have seen in the previous graphic study for the Newton-Kantorovich theorem, if we consider $\Omega = B\left(0, \frac{3}{2}\right)$, then condition $L\beta\eta \leq \frac{1}{2}$ is not satisfied if $x_0(s) = s$, since $L\beta\eta = 0.5578\dots > \frac{1}{2}$, so that we cannot apply the Newton-Kantorovich theorem in this case to guarantee the convergence of Newton's method to a solution of (19). In addition, we cannot draw conclusions about existence and uniqueness of solution for equation (19) from the Newton-Kantorovich theorem.

Moreover, if we want to approximate a solution $x^{**}(s)$ such that $\|x^{**}\| \geq \sigma_2 = 1.5382\dots$, we cannot apply the Newton-Kantorovich theorem because we cannot choose a domain where $x^{**}(s)$ lies, since if the domain is chosen at random, the domain could not contain $x^{**}(s)$ or cut it, in which case $F'(x)$ is not Lipschitz continuous.

We see in this example that we can consider the two situations from Theorem 3.4 if conditions (C1)–(C3) are assumed.

**5.4. An arithmetic model.** After that, we apply Newton's method for approximating a solution with the features mentioned above. For this, we use a discretization process. So, we approximate the integral of (19) by the following Gauss-Legendre quadrature formula with 8 nodes:

$$\int_0^1 v(t)\,dt \simeq \sum_{j=1}^{8} w_j v(t_j),$$

where the nodes $t_j$ and the weights $w_j$ are known. Moreover, we denote $x(t_i)$ by $x_i$, $i = 1, 2, \dots, 8$, so that equation (19) is now transformed into the following system of nonlinear equations:

$$x_i = t_i + \frac{1}{2}\sum_{j=1}^{8} a_{ij}\, x_j^5, \quad \text{where} \quad a_{ij} = \begin{cases} w_j t_j(1-t_i), & j \le i, \\ w_j t_i(1-t_j), & j > i. \end{cases}$$

Then, we write the above system in the following matrix form:

$$F(\mathbf{x}) \equiv \mathbf{x} - \mathbf{y} - \frac{1}{2}A\,\hat{\mathbf{x}} = 0, \quad F : \mathbb{R}^8 \longrightarrow \mathbb{R}^8, \tag{20}$$

where $\mathbf{x} = (x_1, x_2, \dots, x_8)^T$, $\mathbf{y} = (t_1, t_2, \dots, t_8)^T$, $A = (a_{jk})_{j,k=1}^8$ and $\hat{\mathbf{x}} = (x_1^5, x_2^5, \dots, x_8^5)^T$. Besides

$$F'(\mathbf{x}) = I - \frac{5}{2}A\,\mathrm{diag}\{x_1^4, x_2^4, \dots, x_8^4\}.$$

Since $x_0(s) = s$ has been chosen as starting point for the theoretical study, a reasonable choice of initial approximation for Newton's method seems to be the vector $\mathbf{x_0} = \mathbf{y}$. After three iterations, we obtain the numerical approximation to the solution $\mathbf{x}^* = (x_1^*, x_2^*, \dots, x_8^*)^T$ shown in Table 1 with a tolerance $10^{-16}$. Observe that $\|\mathbf{x}^*\| = 0.9816\ldots \le \sigma_1 = 1.1012\ldots$

| $i$ | $x_i^*$ | $i$ | $x_i^*$ |
|---|---|---|---|
| 1 | 0.02010275... | 5 | 0.59887545... |
| 2 | 0.10293500... | 6 | 0.77066535... |
| 3 | 0.24019309... | 7 | 0.90392935... |
| 4 | 0.41336490... | 8 | 0.98160632... |

Table 1: Numerical solution $\mathbf{x}^*$ of (20)

In Table 2 we show the errors $\|\mathbf{x}^* - \mathbf{x_n}\|$, using the stopping criterion $\|\mathbf{x_n} - \mathbf{x_{n-1}}\| < 10^{-16}$, and the sequence $\{\|F(\mathbf{x_n})\|\}$. Notice that the vector shown in Table 1 is a good approximation of the solution of system (20), since $\|F(\mathbf{x}^*)\| \le C \times 10^{-16}$.

| $n$ | $\|\mathbf{x}^* - \mathbf{x_n}\|$ | $\|F(\mathbf{x_n})\|$ |
|---|---|---|
| 0 | $7.8991\ldots \times 10^{-3}$ | $7.5595\ldots \times 10^{-3}$ |
| 1 | $6.7668\ldots \times 10^{-6}$ | $6.4814\ldots \times 10^{-6}$ |
| 2 | $5.2591\ldots \times 10^{-12}$ | $5.0502\ldots \times 10^{-12}$ |

Table 2: Absolute errors and $\{\|F(\mathbf{x_n})\|\}$

We now interpolate the points of Table 1. Taking into account that the solution of (19) satisfies $x(0) = 0$ and $x(1) = 1$, an approximation $x^I$ of the numerical solution is obtained, see Figure 12. Notice that the interpolated approximation $x^I$ lies within the existence domain of the solutions obtained previously.

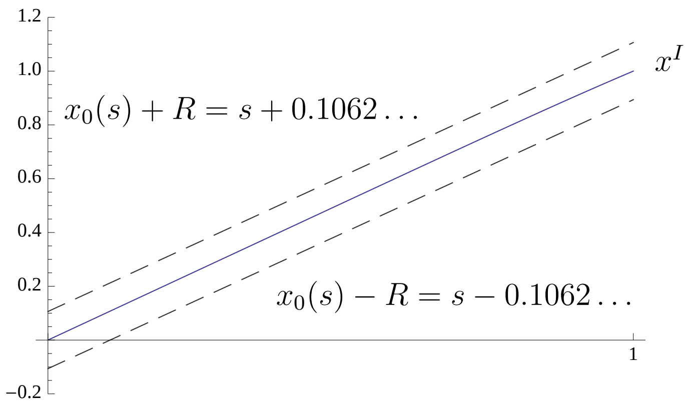


Figure 12: Approximated solution of equation (19)

We have seen previously that equation (19) can have a solution $x^{**}(s)$ such that $\|x^{**}(s)\| \geq \sigma_2 = 1.5382\ldots$, but the convergence of Newton's method cannot be guaranteed from Kantorovich's theory, since a domain where $x^{**}(s)$ lies and $F'$ is Lipschitz continuous cannot be fixed. However, the convergence of Newton's method can be guaranteed from Theorem 3.4, as we see in the following.

For example, if we choose the starting point $x_0(s) = 3s$, we observe that $\|x_0(s)\| = 3 > \sigma_2 = 1.5382\ldots$ In addition, a reasonable choice of initial approximation for Newton's method seems to be the vector $\mathbf{x_0} = 3\mathbf{y}$, but we cannot apply Theorem 3.4 either, since the real function $\varphi(t)$ cannot be defined. However, it seems clear that the conditions of Theorem 3.4 can be satisfied if the starting point is improved. So, starting at $\mathbf{x_0}$, by iterating with Newton's method, we obtain the following approximation

$$\mathbf{z_0} = \begin{pmatrix} 0.081669\ldots \\ 0.418183\ldots \\ 0.975714\ldots \\ 1.667559\ldots \\ 2.199495\ldots \\ 1.906819\ldots \\ 1.406840\ldots \\ 1.080048\ldots \end{pmatrix},$$

which is now used as new starting point for Newton's method. For this new starting point, we obtain

$$\varphi(t) = -(0.0108\ldots) + (0.9986\ldots)t - (1.3726\ldots)t^2 - (0.6250\ldots)t^3 - (0.0948\ldots)t^4,$$

and $R = 0.0110\ldots$, so that the hypotheses of Theorem 3.4 are then satisfied for $\mathbf{z_0}$. Observe that $\mathbf{z_0}$ satisfies $\|\mathbf{z_0}\| = 2.1994\ldots > \sigma_2 = 1.5382\ldots$ Choosing now the starting point $\mathbf{z_0}$ and iterating again with Newton's method, we obtain the approximated solution $\mathbf{x}^{**} = (x_1^{**}, x_2^{**}, \ldots, x_8^{**})^T$ given in Table 3, which is a solution that is beyond the scope of Kantorovich's theory. Observe that $\|\mathbf{x}^{**}\| = 1.8986\ldots > \sigma_2 = 1.5382\ldots$

| $i$ | $x_i^{**}$ | $i$ | $x_i^{**}$ |
|---|---|---|---|
| 1 | 0.081556... | 5 | 2.192901... |
| 2 | 0.417603... | 6 | 1.898637... |
| 3 | 0.974358... | 7 | 1.403096... |
| 4 | 1.665051... | 8 | 1.079315... |

Table 3: Numerical solution of (20)

In Table 4 we show the errors $\|\mathbf{x}^* - \mathbf{x_n}\|$, using the stopping criterion $\|\mathbf{x_n} - \mathbf{x_{n-1}}\| < 10^{-16}$, and the sequence $\{\|F(\mathbf{x_n})\|\}$. Notice that the vector shown in Table 3 is a good approximation of the solution of system (20), since $\|F(\mathbf{x}^*)\| \leq C \times 10^{-16}$.

By interpolating the values of Table 3 and taking into account that the solutions of (20) satisfy $x(0) = 0$ and $x(1) = 1$, we obtain the solution denoted by $\widehat{x}$ and drawn in Figure 13.

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| $n$ | $\|\mathbf{x}^* - \mathbf{x_n}\|$ | $\|F(\mathbf{x_n})\|$ |
|---|---|---|
| 0 | $8.1819\ldots \times 10^{-3}$ | $1.7865\ldots \times 10^{-2}$ |
| 1 | $7.9964\ldots \times 10^{-5}$ | $1.5537\ldots \times 10^{-4}$ |
| 2 | $7.2936\ldots \times 10^{-9}$ | $1.5207\ldots \times 10^{-8}$ |

Table 4: Absolute errors and $\{\|F(\mathbf{x_n})\|\}$

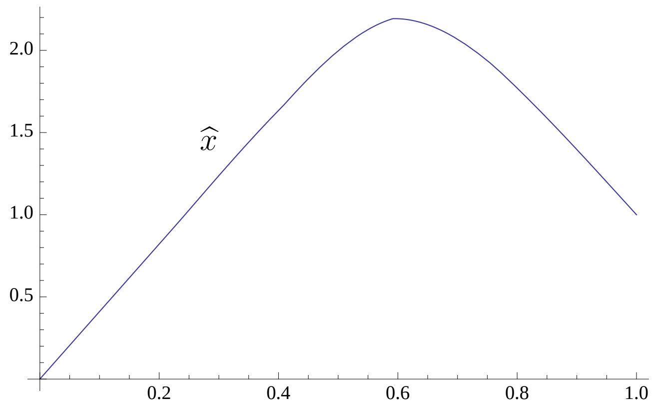


Figure 13: Approximated solution of equation (19)

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