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Asymptotic Almost Periodicity to Some Evolution Equations

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Abstract. In this paper, we introduce a new notion of semi-Lipschitz continuity for the class of asymptotically almost periodic functions and establish new existence theorems for asymptotically almost periodic mild solutions to some semilinear abstract evolution equations upon making some suitable assumptions. As one would expect, the results presented here would generalize and improve some recent results in this area.
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1. Introduction

As a natural extension of almost periodicity, the notion of asymptotic almost periodicity was introduced in the works of Fréchet [9, 10] in the early 1940s. Since then, this notion has found several developments and has been generalized into different directions. In particular, the study of the existence of asymptotically almost periodic solutions is an attractive topic in the qualitative theory of differential equations due to their significance and applications in physics, mathematical biology, control theory, and so on. For significant works along this line, we refer readers to Arendt and Batty [3] for inhomogeneous Cauchy problems on $\mathbb{R}^+$, Agarwal et al. [1] for some evolution equations in Banach spaces, de Andrade and Lizama [2] and Lizama et al. [11] for a class of nonlinear damped wave equations and strongly damped semilinear wave equations respectively, and Cushing [7] for predator-prey systems with or without hereditary effects.

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For more information concerning the motivations and relevant developments of this study, please see Cheban [5], and Ruess and Phong [13] and the references therein.

Stimulated by the works above, the main purpose of this paper is to study asymptotically almost periodic solutions of semilinear evolution equations of the form

$$u'(t) = Au(t) + f(t, u(t)), \quad t \in \mathbb{R} \tag{1}$$

in the Banach space $\{X, \|\cdot\|\}$ by introducing a new notion of semi-Lipschitz continuity for the class of asymptotically almost periodic functions. Here, the operator $A : D(A) \subset X \to X$ is the infinitesimal generator of a hyperbolic C_0-semigroup $\{T(t)\}_{t\geq 0}$ on X and $f : \mathbb{R} \times X \to X$ is a given function to be specified later.

Moreover, we extend this result to the semilinear neutral evolution equation in the form

$$\frac{d}{dt}[u(t) - h(t, u(t)] = Au(t) + f(t, u(t)), \quad t \in \mathbb{R}, \tag{2}$$

where $h : \mathbb{R} \times X \to Y$ is a given function to be specified later ($Y \subset X$ is a Banach space).

Some new existence results of asymptotically periodic mild solutions to equations (1) and (2) are established, without imposing (locally) Lipschitz condition on the nonlinearity f with respect to the second variable. As can be seen, the hypotheses in our result are reasonably weak.

In our main results, we require $T(t)$ to satisfy a compactness assumption, but do not require the Lipschitz condition of f as a whole with respect to the second variable (see Remark 2.3, Theorem 2.8 and Theorem 3.5 below). However, in many previous papers on asymptotically periodic solutions such as [1, 17], the nonlinearity as a whole is assumed to satisfy a (locally) Lipschitz condition and hence the Banach contraction principle becomes one of the key tools in the study of the corresponding problems. So our results generalize essentially those in [1,17] and related research and have more broad applications.

To begin with, we recall that a C_0-semigroup $\{T(t)\}_{t\geq 0}$ is said to be hyperbolic if there exists a projection $\mathcal{P}$ on X and constants $M, \delta > 0$ such that $T(t)$ commutes with $\mathcal{P}$, $T(t)N(\mathcal{P}) = N(\mathcal{P})$, $T(t) : Im(\mathcal{Q}) \to Im(\mathcal{Q})$ is invertible and for $x \in X$,

$$\|T(t)\mathcal{P}x\| \leq Me^{-\delta t}\|x\|, \quad t \geq 0,$$
$$\|T(t)\mathcal{Q}x\| \leq Me^{\delta t}\|x\|, \quad t \leq 0,$$

where $N(\mathcal{P})$ is the kernel of $\mathcal{P}$, $\mathcal{Q} := I - \mathcal{P}$ and $T(t) := T(-t)^{-1}$ for $t < 0$. We would like to mention that $\{T(t)\}_{t\geq 0}$ is hyperbolic if and only if

$$\sigma(T(t)) \cap \{\lambda \in \mathbb{C}; |\lambda| = 1\} = \emptyset \quad \text{for one/all } t > 0.$$

Especially, if $\mathcal{P} = I$, then $\{T(t)\}_{t\geq 0}$ is said to be uniformly exponentially stable, that is to say,

$$\|T(t)\| \leq Me^{-\delta t} \quad \text{for all } t \geq 0.$$

For more information on spectral characterizations of hyperbolicity, we refer to the monograph of Engel and Nagel [8, Chapter V].

In the sequel, we fix some notations and definitions. By $C_b(\mathbb{R};X)$ denote the Banach space of all bounded, continuous functions from $\mathbb{R}$ to X, equipped with the supreme norm $\|\cdot\|_\infty := \sup_{t\in\mathbb{R}} \|\cdot(t)\|$. Let $C_0(\mathbb{R};X)$ be the closed subspace of $C_b(\mathbb{R};X)$ consisting of functions vanishing at infinity. Additionally, similar notations as above also apply to $C_b(\mathbb{R};Y)$ and $C_0(\mathbb{R};Y)$, where $Y \subset X$ is a Banach space.

We present the following compactness criterion, which is a special case of the general compactness result of [14, Theorem 2.1].

Lemma 1.1. *A set $D \subset C_0(\mathbb{R};X)$ is relatively compact if*

(1) *D is equicontinuous;*
(2) $\lim_{|t|\to+\infty} u(t) = 0$ *uniformly for $u \in D$;*
(3) *the set $D(t) := \{u(t); u \in D\}$ is relatively compact in X for every $t \in \mathbb{R}$.*

Definition 1.2. A continuous function $u : \mathbb{R} \to X$ is said to be almost periodic if for each $\epsilon > 0$ there exists a positive number $l = l(\epsilon)$ such that every interval $[a, a+l]$ $(a \in \mathbb{R})$ of length l contains at least one number τ with the property

$$\|u(t+\tau) - u(t)\| < \epsilon \quad \text{for every } t \in \mathbb{R}.$$

The number τ is called an ϵ almost period of the function u. Denote by $AP(\mathbb{R};X)$ the set of such functions.

Definition 1.3. A continuous function $u : \mathbb{R} \to X$ is said to be asymptotically almost periodic if it can be decomposed as

$$u = u_1 + u_2,$$

where $u_1 \in AP(\mathbb{R};X)$ and $u_2 \in C_0(\mathbb{R};X)$. Denote by $AAP(\mathbb{R};X)$ the set of such functions.

Clearly, $AP(\mathbb{R};X)$ and $AAP(\mathbb{R};X)$, endowed with the supremum norm $\|\cdot\|_\infty$, turn out to be Banach spaces.

Let $C_0(\mathbb{R}\times X;X)$ be the set of all continuous functions g from $\mathbb{R}\times X$ to X vanishing at infinity uniformly in any compact subset of X, in other words,

$$\lim_{|t|\to+\infty} \|g(t,x)\| = 0 \quad \text{uniformly for } x \in \mathbb{K},$$

where $\mathbb{K}$ is an any compact subset of X.

Definition 1.4. Let $f : \mathbb{R} \times X \to X$ be a continuous function.

(a) f is called almost periodic if $f(t, x)$ is almost periodic in $t \in \mathbb{R}$ uniformly for x in any compact subset of X. The class of such functions will be denoted by $AP(\mathbb{R} \times X; X)$.

(b) f is called asymptotically almost periodic if it can be decomposed as

$$f = f_1 + f_2,$$

where $f_1 \in AP(\mathbb{R}\times X; X)$ and $f_2 \in C_0(\mathbb{R}\times X; X)$. Denote by $AAP(\mathbb{R}\times X; X)$ the set of such functions.

Concerning the composition of almost periodic functions, we have the following result.

Lemma 1.5. (see, e.g., [2,16]) *Let $f \in AP(\mathbb{R} \times X; X)$ and $u \in AP(\mathbb{R}; X)$, then $f(\cdot, u(\cdot))$ belongs to $AP(\mathbb{R}; X)$.*

2. Main results

Throughout this section, it is assumed that $\{T(t)\}_{t\geq 0}$ is hyperbolic and compact. In this section, we are going to derive some sufficient conditions under which equation (1) has at least one asymptotically almost periodic mild solution.

We begin, by introducing a new notion of semi-Lipschitz continuity for the class of almost periodic functions.

Definition 2.1. Given $f = f_1 + f_2 \in AAP(\mathbb{R} \times X; X)$ with $f_1 \in AP(\mathbb{R} \times X; X)$, $f_2 \in C_0(\mathbb{R} \times X; X)$. f is said to be semi-Lipschitz continuous if there exists a constant $L > 0$ such that

$$\|f_1(t, x) - f_1(t, y)\| \leq L\|x - y\|$$

for all $t \in \mathbb{R}$ and $x, y \in X$.

Remark 2.2. Notice in particular that if f_1 satisfies the Lipschitz condition stated above and $f_1(t, x)$ is almost periodic on $\mathbb{R}$ for all $x \in X$, then f_1 is almost periodic uniformly for $x \in X$ ranging over compact subsets of X (see [6, Lemma 2.6] and [4, Theorem 3.12]), which means that the Lipschitz condition of Definition 2.1 for f_1 would work to reduce the almost periodic property of f_1 to simply asking for $f_1(t, x)$ being almost periodic on $\mathbb{R}$ for all $x \in X$.

Remark 2.3. Assuming that $f \in AAP(\mathbb{R}\times X; X)$ is semi-Lipschitz continuous, it is noted that f as a whole does not necessarily verify the Lipschitz continuity with respect to the second variable. Such class of functions is more complicated than those with Lipschitz continuity with respect to the second variable and little is known about them.

Lemma 2.4. *Given* $f = f_1 + f_2 \in AAP(\mathbb{R}\times X;X)$ *with* $f_1 \in AP(\mathbb{R}\times X;X)$, $f_2 \in C_0(\mathbb{R}\times X;X)$. *Then it yields that*

$$\sup_{t\in\mathbb{R}} \|f_1(t,x) - f_1(t,y)\| \le \sup_{t\in\mathbb{R}} \|f(t,x) - f(t,y)\|, \quad x, y \in X,$$

which implies that f *is semi-Lipschitz continuous when* f *is Lipschitz continuous with respect to the second variable uniformly for* $t \in \mathbb{R}$.

Proof. To show our result, it suffices to verify that

$$\{f_1(t,x) - f_1(t,y); t \in \mathbb{R}\} \subset \overline{\{f(t,x) - f(t,y); t \in \mathbb{R}\}}, \quad x, y \in X.$$

In fact, if this is not the case, then for fixed $x, y \in X$, there exist some $t_0 \in \mathbb{R}$ and $\epsilon > 0$ such that

$$\|(f_1(t_0,x) - f_1(t_0,y)) - (f(t,x) - f(t,y))\| \ge 3\epsilon \quad \text{for all } t \in \mathbb{R}.$$

We assume, without loss of generality, that $t_0 \ge 0$, since the case when $t_0 < 0$ can be treated in a similar way.

It is clear that $\lim_{|t|\to\infty} \|f_2(t,x) - f_2(t,y)\| = 0$, which implies that there exists positive number T such that

$$\|f_2(t,x) - f_2(t,y)\| < \epsilon \tag{3}$$

whenever $t \ge T$. Since $f_1 \in AP(\mathbb{R}\times X;X)$ one can take $l = l(\epsilon) > 0$ such that $[T, T+l]$ of length l contains at least a τ with the properties

$$\|f_1(t_0+\tau,x) - f_1(t_0,x)\| < \epsilon \quad \text{and} \quad \|f_1(t_0+\tau,y) - f_1(t_0,y)\| < \epsilon,$$

which enables us to find that

$$\begin{aligned}
&\|f_2(t_0+\tau,x) - f_2(t_0+\tau,y)\| \\
&\ge \|f(t_0+\tau,x) - f(t_0+\tau,y) - f_1(t_0,x) + f_1(t_0,y)\| \\
&\quad - \|f_1(t_0+\tau,x) - f_1(t_0,x)\| - \|f_1(t_0+\tau,y) - f_1(t_0,y)\| \\
&> \epsilon,
\end{aligned}$$

which contradicts (3) (noticing $t_0 + \tau \ge T$), completing the proof. □

We also need the following composition result concerning asymptotically almost periodic functions.

Lemma 2.5. *Given* $f = f_1 + f_2 \in AAP(\mathbb{R}\times X;X)$ *with* $f_1 \in AP(\mathbb{R}\times X;X)$, $f_2 \in C_0(\mathbb{R}\times X;X)$ *and* $u = u_1 + u_2 \in AAP(\mathbb{R};X)$ *with* $u_1 \in AP(\mathbb{R};X)$, $u_2 \in C_0(\mathbb{R};X)$. *If* f *is semi-Lipschitz continuous, then* $f(\cdot, u(\cdot))$ *belongs to* $AAP(\mathbb{R};X)$ *with* $f_1(\cdot, u_1(\cdot)) \in AP(\mathbb{R};X)$, $f_1(\cdot, u(\cdot)) - f_1(\cdot, u_1(\cdot)) \in C_0(\mathbb{R};X)$, *and* $f_2(\cdot, u(\cdot)) \in C_0(\mathbb{R};X)$.

Proof. By the semi-Lipschitz continuity of f we observe that for all $t \in \mathbb{R}$, $\|f_1(t, u(t)) - f_1(t, u_1(t)\| \leq L\|u_2(t)\|$, which implies that $f_1(\cdot, u(\cdot)) - f_1(\cdot, u_1(\cdot) \in C_0(\mathbb{R}; X)$ due to $u_2 \in C_0(\mathbb{R}; X)$. Noticing this and Lemma 1.5, one can easily find, along the same lines as in the proof of [2, Lemma 2.8], the assertion of the lemma remains true. Here we omit the details. □

Before stating the existence theorem, we first prove the following auxiliary result.

Lemma 2.6. *Given $u_1 \in AP(\mathbb{R}; X)$ and $u_2 \in C_0(\mathbb{R}; X)$. Write*

$$G_1(t) := \int_{-\infty}^{t} T(t-s)\mathcal{P}u_1(s)\mathrm{d}s - \int_{t}^{+\infty} T(t-s)\mathcal{Q}u_1(s)\mathrm{d}s, \quad t \in \mathbb{R},$$
$$G_2(t) := \int_{-\infty}^{t} T(t-s)\mathcal{P}u_2(s)\mathrm{d}s - \int_{t}^{+\infty} T(t-s)\mathcal{Q}u_2(s)\mathrm{d}s, \quad t \in \mathbb{R}.$$

Then $G_1 \in AP(\mathbb{R}; X)$ and $G_2 \in C_0(\mathbb{R}; X)$.

Proof. From the hyperbolicity of C_0-semigroup $\{T(t)\}_{t\geq 0}$ it is clear that G_1 and G_2 are well-defined and continuous on $\mathbb{R}$.

Given $\epsilon > 0$. Since $u_1 \in AP(\mathbb{R}; X)$, one can take $l(\epsilon) > 0$ involved in Definition 1.2 such that every interval of length $l(\epsilon)$ contains a number τ with the property that $\|u_1(t+\tau) - u_1(t)\| < \epsilon$ for every $t \in \mathbb{R}$. The estimate

$$\begin{aligned}
\|G_1(t+\tau) - G_1(t)\| &\leq \left\| \int_{-\infty}^{t} T(t-s)\mathcal{P}\left(u_1(s+\tau) - u_1(s)\right)\mathrm{d}s \right\| \\
&\quad + \left\| \int_{t}^{+\infty} T(t-s)\mathcal{Q}\left(u_1(s+\tau) - u_1(s)\right)\mathrm{d}s \right\| \\
&\leq M \int_{-\infty}^{t} e^{-\delta(t-s)} \|u_1(\tau+s) - u_1(s)\|\mathrm{d}s \\
&\quad + M \int_{t}^{+\infty} e^{\delta(t-s)} \|u_1(\tau+s) - u_1(s)\|\mathrm{d}s \\
&\leq 2M\delta^{-1}\epsilon
\end{aligned}$$

is responsible for the fact that $G_1 \in AP(\mathbb{R}; X)$.

Since u_2 vanishes at infinity, one can choose an $N > 0$ such that $\|u_2(t)\| < \epsilon$ for all $t > N$. This enables us to conclude that for each $t > N$,

$$\begin{aligned}
\left\| \int_{-\infty}^{t} T(t-s)\mathcal{P}u_2(s)\mathrm{d}s \right\| &\leq \left\| \int_{-\infty}^{N} T(t-s)\mathcal{P}u_2(s)\mathrm{d}s \right\| + \left\| \int_{N}^{t} T(t-s)\mathcal{P}u_2(s)\mathrm{d}s \right\| \\
&\leq M\delta^{-1}e^{-\delta(t-N)}\|u_2\|_\infty + M\delta^{-1}\epsilon,
\end{aligned}$$

from which we see $\| \int_{-\infty}^{t} T(t-s)\mathcal{P}u_2(s)\mathrm{d}s\| \to 0$ as $t \to +\infty$. Also, for each $t \in \mathbb{R}$,

$$\left\| \int_{t}^{+\infty} T(t-s)\mathcal{Q}u_2(s)\mathrm{d}s \right\| \le M\delta^{-1} \sup_{s\ge t} \|u_2(s)\|,$$

which together with $u_2 \in C_0(\mathbb{R}; X)$ implies that $\int_t^{+\infty} T(t-s)\mathcal{Q}u_2(s)\mathrm{d}s \to 0$ in X as $t \to +\infty$. From this, we obtain $\lim_{t\to+\infty} \|G_2(t)\| = 0$. By a similar argument it follows readily that $\lim_{t\to-\infty} \|G_2(t)\| = 0$. The proof is then completed. □

To prove our main results, let us introduce the following assumption:

(H) $f = f_1 + f_2 \in AAP(\mathbb{R}\times X; X)$ with $f_1 \in AP(\mathbb{R}\times X; X)$ and $f_2 \in C_0(\mathbb{R}\times X; X)$ is semi-Lipschitz continuous with the Lipschitz constant L. Moreover, there exists a function $\beta \in C_0(\mathbb{R}, \mathbb{R}^+)$ and a nondecreasing function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $t \in \mathbb{R}$ and $x \in X$ satisfying $\|x\| \le r$,

$$\|f_2(t,x)\| \le \beta(t)\Phi(r), \quad \text{and} \quad \liminf_{r\to+\infty} \frac{\Phi(r)}{r} = \rho_1. \tag{4}$$

Let β be the function involved in assumption (H). Write

$$\sigma_1(t) := \int_{-\infty}^{t} e^{-\delta(t-s)}\beta(s)\mathrm{d}s, \qquad t \in \mathbb{R},$$
$$\sigma(t) := \int_{-\infty}^{t} e^{-\delta(t-s)}\beta(s)\mathrm{d}s + \int_{t}^{+\infty} e^{\delta(t-s)}\beta(s)\mathrm{d}s, \quad t \in \mathbb{R}.$$

Then an analogue argument used in Lemma 2.6 yields that $\sigma_1, \sigma \in C_0(\mathbb{R}, \mathbb{R}^+)$. Put

$$\rho_2 := \sup_{t\in\mathbb{R}} \sigma(t), \quad \rho_3 := \sup_{t\in\mathbb{R}} \sigma_1(t).$$

Definition 2.7. A continuous function $u : \mathbb{R} \to X$ is called an asymptotically almost periodic mild solution to equation (1) on $\mathbb{R}$ if $u \in AAP(\mathbb{R}; X)$ and it satisfies the integral equation of the form

$$u(t) = T(t-\tau)u(\tau) + \int_{\tau}^{t} T(t-s)f(s,u(s))\mathrm{d}s \quad \text{for all } t > \tau.$$

Now we are in a position to present our existence result:

Theorem 2.8. *Under the hypothesis* (H)*, equation* (1) *has at least one asymptotically almost periodic mild solution provided that*

$$2ML\delta^{-1} + M\rho_2\rho_1 < 1. \tag{5}$$

Proof. As in the hypothesis (H), $f = f_1 + f_2 \in AAP(\mathbb{R} \times X; X)$ with $f_1 \in AP(\mathbb{R} \times X; X)$ and $f_2 \in C_0(\mathbb{R} \times X; X)$ is semi-Lipschitz continuous.

To prove the existence of asymptotically almost periodic mild solution to equation (1), let us consider the coupled system of integral equations of form

$$\begin{cases} v(t) = \int_{-\infty}^{t} T(t-s)\mathcal{P} f_1(s, v(s))\mathrm{d}s - \int_{t}^{+\infty} T(t-s)\mathcal{Q} f_1(s, v(s))\mathrm{d}s, \quad t \in \mathbb{R}, \\ w(t) = \int_{-\infty}^{t} T(t-s)\mathcal{P}\left[f_1(s, v(s)+w(s)) - f_1(s, v(s))\right]\mathrm{d}s \\ \qquad - \int_{t}^{+\infty} T(t-s)\mathcal{Q}\left[f_1(s, v(s)+w(s)) - f_1(s, v(s))\right]\mathrm{d}s \\ \qquad + \int_{-\infty}^{t} T(t-s)\mathcal{P} f_2(s, v(s)+w(s))\mathrm{d}s \\ \qquad - \int_{t}^{+\infty} T(t-s)\mathcal{Q} f_2(s, v(s)+w(s))\mathrm{d}s, \quad t \in \mathbb{R}. \end{cases} \tag{6}$$

Note that if $(v, w) \in AP(\mathbb{R}; X) \times C_0(\mathbb{R}; X)$ is a solution to the coupled system (6), then $u := v + w$ belongs to $AAP(\mathbb{R}; X)$ and it is a solution to the integral equation in the form

$$u(\tau) = \int_{-\infty}^{\tau} T(\tau - s)\mathcal{P} f(s, u(s))\mathrm{d}s - \int_{\tau}^{+\infty} T(\tau - s)\mathcal{Q} f(s, u(s))\mathrm{d}s, \quad \tau \in \mathbb{R}.$$

Multiplying the both sides above by $T(t-\tau)$ with $t > \tau$, one finds that u is an asymptotically almost periodic mild solution to equation (1). Hence the beginning matters for the end is to show that the coupled system (6) has at least a solution in $AP(\mathbb{R}; X) \times C_0(\mathbb{R}; X)$.

We start by defining a mapping Λ on $AP(\mathbb{R}; X)$ as follows:

$$(\Lambda v)(t) = \int_{-\infty}^{t} T(t-s)\mathcal{P} f_1(s, v(s))\mathrm{d}s - \int_{t}^{+\infty} T(t-s)\mathcal{Q} f_1(s, v(s))\mathrm{d}s, \quad t \in \mathbb{R}.$$

From our hypotheses on f_1 and Lemma 2.5 it follows that $f_1(\cdot, v(\cdot)) \in AP(\mathbb{R}; X)$ for every $v \in AP(\mathbb{R}; X)$. This, together with Lemma 2.6, implies that Λ is well-defined and maps $AP(\mathbb{R}; X)$ into itself. Moreover, for any $v_1, v_2 \in AP(\mathbb{R}; X)$ we obtain by the semi-Lipschitz continuity of f,

$$\begin{aligned} &\|(\Lambda v_1)(t) - (\Lambda v_2)(t)\| \\ &\le ML\left(\int_{-\infty}^{t} e^{-\delta(t-s)}\|v_1(s) - v_2(s)\|\mathrm{d}s + \int_{t}^{+\infty} e^{\delta(t-s)}\|v_1(s) - v_2(s)\|\mathrm{d}s\right) \\ &\le 2ML\delta^{-1}\|v_1 - v_2\|_\infty. \end{aligned}$$

Together with (5), this proves that Λ is a contraction on $AP(\mathbb{R};X)$. Thus, applying the Banach's fixed point theorem, one finds that Λ has an unique fixed point $v \in AP(\mathbb{R};X)$.

For such v, we define a mapping $\Gamma := \Gamma^1 + \Gamma^2$ on $C_0(\mathbb{R};X)$ as

$$(\Gamma^1 w)(t) = \int_{-\infty}^{t} T(t-s)\mathcal{P}J_v'(s,w(s))\mathrm{d}s - \int_{t}^{+\infty} T(t-s)\mathcal{Q}J_v'(s,w(s))\mathrm{d}s, \quad t\in\mathbb{R},$$
$$(\Gamma^2 w)(t) = \int_{-\infty}^{t} T(t-s)\mathcal{P}J_v''(s,w(s))\mathrm{d}s - \int_{t}^{+\infty} T(t-s)\mathcal{Q}J_v''(s,w(s))\mathrm{d}s, \quad t\in\mathbb{R},$$

where the mappings $J_v', J_v'' \colon \mathbb{R}\times X \to X$ are defined by $J_v'(t,x) = f_1(t, v(t)+x) - f_1(t,v(t))$ and $J_v''(t,x) = f_2(t, v(t)+x)$ $(t \in \mathbb{R},\ x \in X)$. From (H) we observe

$$\begin{aligned} &\|J_v'(t,x)\| \le L\|x\|,\ \|J_v'(t,x) - J_v'(t,y)\| \le L\|x-y\| \quad \text{for } x,y\in X,\ t\in\mathbb{R}, \\ &\|J_v''(t,x)\| \le \beta(t)\Phi(r + \sup_{t\in\mathbb{R}} \|v(t)\|) \quad \text{for all } t\in\mathbb{R},\ x\in X \text{ with } \|x\|\le r, \end{aligned} \tag{7}$$

which imply that $J_v'(\cdot, w(\cdot))$ and $J_v''(\cdot, w(\cdot))$ belong to $C_0(\mathbb{R};X)$ for every $w \in C_0(\mathbb{R};X)$. Therefore, employing Lemma 2.6 one finds that Γ is well-defined and maps $C_0(\mathbb{R};X)$ into itself. To the end, it suffices to prove that Γ possesses at least one fixed point in $C_0(\mathbb{R};X)$. Set $\Omega_r := \{w \in C_0(\mathbb{R};X); \|w\|_\infty \le r\}$ for simplicity.

Firstly, by (4) and (5) it is not difficult to see that there exists a $k_0 > 0$ such that $2ML\delta^{-1}k_0 + M\Phi(k_0 + \sup_{t\in\mathbb{R}} \|v(t)\|)\rho_2 \le k_0$. This enables us to conclude that for any $t \in \mathbb{R}$ and $w_1, w_2 \in \Omega_{k_0}$,

$$\begin{aligned} &\|(\Gamma^1 w_1)(t) + (\Gamma^2 w_2)(t)\| \\ &\le M\int_{-\infty}^{t} e^{-\delta(t-s)}\|J_v'(s,w_1(s))\|\mathrm{d}s + M\int_{t}^{+\infty} e^{\delta(t-s)}\|J_v'(s,w_1(s))\|\mathrm{d}s \\ &\quad + M\int_{-\infty}^{t} e^{-\delta(t-s)}\|J_v''(s,w_2(s))\|\mathrm{d}s + M\int_{t}^{+\infty} e^{\delta(t-s)}\|J_v''(s,w_2(s))\|\mathrm{d}s \\ &\le 2ML\delta^{-1}k_0 + M\Phi(k_0 + \sup_{t\in\mathbb{R}}\|v(t)\|)\rho_2 \\ &\le k_0, \end{aligned}$$

which implies that $\Gamma^1 w_1 + \Gamma^2 w_2$ belongs to Ω_{k_0} for every pair $w_1, w_2 \in \Omega_{k_0}$.

In the sequel, we show that Γ^1 is a contraction on Ω_{k_0}. For $w_1, w_2 \in \Omega_{k_0}$ and $t \in \mathbb{R}$, by (7) we have

$$\begin{aligned} \big\|(\Gamma^1 w_1)(t) - (\Gamma^1 w_2)(t)\big\| &\le M\int_{-\infty}^{t} e^{-\delta(t-s)}\|J_v'(s,w_1(s)) - J_v'(s,w_2(s))\|\mathrm{d}s \\ &\quad + M\int_{t}^{+\infty} e^{\delta(t-s)}\|J_v'(s,w_1(s)) - J_v'(s,w_2(s))\|\mathrm{d}s \\ &\le 2ML\delta^{-1}\|w_1 - w_2\|_\infty. \end{aligned}$$

Thus, by virtue of (5), we obtain the conclusion.

From our assumptions it is clear that Γ^2 is a continuous mapping from Ω_{k_0} to Ω_{k_0}. Thus, to be able to apply a well known fixed point theorem of Krasnoselskii (see, e.g., [15]) to obtain a fixed point of Γ, we need to verify that Γ^2 is compact on Ω_{k_0}.

Firstly, as

$$\|(\Gamma^2 w)(t)\| \leq M\Phi(k_0 + \sup_{t\in\mathbb{R}} \|v(t)\|)\sigma(t) \quad \text{for all } w \in \Omega_{k_0} \text{ and } t \in \mathbb{R},$$

in view of (7), we conclude that $\lim_{|t|\to+\infty}(\Gamma^2 w)(t) = 0$ uniformly for $w \in \Omega_{k_0}$.

Let $t \in \mathbb{R}$ be fixed. For each $\epsilon_0 > 0$, from (7) it follows that

$$(\Gamma^2_{\epsilon_0} w)(t) := \int_{-\infty}^{t-\epsilon_0} T(t-\epsilon_0-s)\mathcal{P}J''_v(s, w(s))\mathrm{d}s - \int_{t-\epsilon_0}^{+\infty} T(t-\epsilon_0-s)\mathcal{Q}J''_v(s, w(s))\mathrm{d}s$$

is uniformly bounded for $w \in \Omega_{k_0}$ in X. This, together with the fact that $T(\epsilon_0)$ is compact, yields that the set $\{T(\epsilon_0)(\Gamma^2_{\epsilon_0} w)(t); w \in \Omega_{k_0}\}$ is relatively compact in X. Now, noticing

$$\begin{aligned}
&\|(\Gamma^2 w)(t) - T(\epsilon_0)(\Gamma^2_{\epsilon_0} w)(t)\| \\
&\leq \left\|\int_{t-\epsilon_0}^{t} T(t-s)\mathcal{P}J''_v(s, w(s))\mathrm{d}s\right\| + \left\|\int_{t-\epsilon_0}^{t} T(t-s)\mathcal{Q}J''_v(s, w(s))\mathrm{d}s\right\| \\
&\to 0 \quad \text{as } \epsilon_0 \to 0^+,
\end{aligned}$$

by (7), one finds, using the total boundedness, that the set $\{(\Gamma^2 w)(t); w \in \Omega_{k_0}\}$ is relatively compact in X for each $t \in \mathbb{R}$.

Next, we consider the equicontinuity of the set $\{\Gamma^2 w; w \in \Omega_{k_0}\}$. Given $\epsilon_1 > 0$. In view of (7) there exists an $\eta > 0$ such that

$$\left\|\int_{\tau}^{t} T(t-s)\mathcal{P}J''_v(s, w(s))\mathrm{d}s\right\| < \frac{\epsilon_1}{5},$$

$$\left\|\int_{\tau-\eta}^{\tau} (T(t-s) - T(\tau-s))\mathcal{P}J''_v(s, w(s))\mathrm{d}s\right\| < \frac{2\epsilon_1}{5}$$

are valid for all $w \in \Omega_{k_0}$ and $t \geq \tau$ with $t - \tau < \eta$. Also, one can choose a $\kappa > 0$ such that $2M\delta^{-1}\Phi(k_0 + \sup_{t\in\mathbb{R}} \|v(t)\|)e^{-\delta\kappa} \sup_{s\in\mathbb{R}} \beta(s) < \frac{\epsilon_1}{5}$, which yields that for all $w \in \Omega_{k_0}$ and $t \geq \tau$,

$$\left\|\int_{-\infty}^{\tau-\kappa} (T(t-s) - T(\tau-s))\mathcal{P}J''_v(s, w(s))\mathrm{d}s\right\| < \frac{\epsilon_1}{5}.$$

Thus, we see from the fact that the compactness of $\{T(t)\}_{t>0}$ implies its norm continuity that there exists an $\eta' \in (0, \eta)$ such that

$$\begin{aligned}
&\left\|\int_{-\infty}^{t} T(t-s)\mathcal{P}J_v''(s,w(s))\mathrm{d}s - \int_{-\infty}^{\tau} T(\tau-s)\mathcal{P}J_v''(s,w(s))\mathrm{d}s\right\| \\
&\leq \left\|\int_{\tau}^{t} T(t-s)\mathcal{P}J_v''(s,w(s))\mathrm{d}s\right\| + \left\|\int_{\tau-\eta}^{\tau} (T(t-s)-T(\tau-s))\mathcal{P}J_v''(s,w(s))\mathrm{d}s\right\| \\
&\quad + \left\|\int_{-\infty}^{\tau-\kappa} (T(t-s)-T(\tau-s))\mathcal{P}J_v''(s,w(s))\mathrm{d}s\right\| \\
&\quad + \left\|\int_{\tau-\kappa}^{\tau-\eta} (T(t-s)-T(\tau-s))\mathcal{P}J_v''(s,w(s))\mathrm{d}s\right\| \\
&< \epsilon_1
\end{aligned}$$

for every $w \in \Omega_{k_0}$ and $t \geq \tau$ with $t - \tau < \eta'$, that is, the family of functions $\{\int_{-\infty}^{\cdot} T(\cdot - s)\mathcal{P}J_v''(s,w(s))\mathrm{d}s; w \in \Omega_{k_0}\}$ is equicontinuous. A similar argument proves that $\{\int_{\cdot}^{+\infty} T(\cdot - s)\mathcal{P}J_v''(s,w(s))\mathrm{d}s; w \in \Omega_{k_0}\}$ is also equicontinuous. Hence, we obtain the equicontinuity of the set $\{\Gamma^2 w; w \in \Omega_{k_0}\}$. An application of Lemma 1.1 justifies the compactness of Γ^2 on Ω_{k_0}.

Finally, applying the Krasnoselskii's fixed point theorem yields that Γ has at least one fixed point in Ω_{k_0}. This proves that the coupled system (6) has at least one solution in $AP(\mathbb{R}; X) \times C_0(\mathbb{R}; X)$. This completes this proof. □

Remark 2.9. Given $f = f_1 + f_2 \in AAP(\mathbb{R} \times X; X)$ with $f_1 \in AP(\mathbb{R} \times X; X)$, $f_2 \in C_0(\mathbb{R} \times X; X)$. f is said to be locally semi-Lipschitz continuous if there exists a nondecreasing function $L : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|f_1(t,x) - f_1(t,y)\| \leq L(r)\|x - y\|$$

for all $t \in \mathbb{R}$ and $x, y \in X$ satisfying $\|x\|, \|y\| \leq r$. Let us note that Theorem 2.8 can be easily extended to the case of f being locally semi-Lipschitz continuous.

3. Neutral evolution equation

In this section, we extend the result obtained in Section 2 to evolution equation of neutral type (2).

Throughout this section, A is assumed to be the infinitesimal generator of a compact analytic semigroup $\{T(t)\}_{t\geq 0}$ and $0 \in \rho(A)$, which implies that $\{T(t)\}_{t\geq 0}$ is uniformly exponentially stable and allows us to define the fractional power $(-A)^{\alpha}$ for $0 \leq \alpha < 1$, as a closed linear operator on its domain $D((-A)^{\alpha})$ with inverse $(-A)^{-\alpha}$. Let X_{α} denote the Banach space $D((-A)^{\alpha})$ endowed with the graph norm $\|u\|_{\alpha} = \|(-A)^{\alpha}u\|$ for $u \in X_{\alpha}$.

Proposition 3.1 ([12, pp. 69–75])**.** *We have*

(1) $T(t) : X \to X_\alpha$ *for each* $t > 0$*, and* $(-A)^\alpha T(t)x = T(t)(-A)^\alpha x$ *for each* $x \in X_\alpha$ *and* $t \geq 0$*,*

(2) $(-A)^\alpha T(t)$ *is bounded on* X *for every* $t > 0$ *and there exists a* $M_\alpha > 0$ *such that* $\|(-A)^\alpha T(t)\| \leq \frac{M_\alpha}{t^\alpha} e^{-\delta t}$*,*

(3) $(-A)^{-\alpha}$ *is a bounded linear operator in* X *with* $D((-A)^\alpha) = Im((-A)^{-\alpha})$*,*

where the parameters $M_0(= M)$ *and* δ *are those of uniform exponential stability of the semigroup.*

Definition 3.2. A continuous function $u : \mathbb{R} \to X$ is said to be an asymptotically almost periodic mild solution of equation (2), if $u \in AAP(\mathbb{R}; X)$, the function $s \to AT(t-s)h(s, u(s))$ is integrable on $[\tau, t)$ for all $t > \tau$ and it satisfies the following integral equation

$$\begin{aligned} u(t) &= T(t-\tau)\big[u(\tau) - h(\tau, u(\tau))\big] + h(t, u(t)) \\ &\quad + \int_\tau^t AT(t-s)h(s, u(s))\mathrm{d}s + \int_\tau^t T(t-s)f(s, u(s))\mathrm{d}s \quad \text{for all } t > \tau. \end{aligned}$$

Let us introduce the following assumption:

(H') There exists an $\alpha \in (0,1)$ such that $h : \mathbb{R} \times X \to X_\alpha$ is continuous and $(-A)^\alpha h \in AP(\mathbb{R} \times X; X)$. Moreover, for all $x, y \in X$ and $t \in \mathbb{R}$,

$$\|h(t,x) - h(t,y)\|_\alpha \leq L_\alpha \|x - y\|,$$

where $L_\alpha > 0$ is a constant.

We also need the following composition results, which can be obtained using similar arguments as in the proofs of Lemma 2.5 and Lemma 2.6.

Lemma 3.3. *Let the hypothesis* (H′) *be satisfied. Given* $u_1 \in AP(\mathbb{R}; X)$ *and* $u_2 \in C_0(\mathbb{R}; X)$*, then* $h(\cdot, u_1(\cdot)) \in AP(\mathbb{R}; X_\alpha)$ *and* $h(\cdot, u_1(\cdot) + u_2(\cdot)) - h(\cdot, u_1(\cdot)) \in C_0(\mathbb{R}; X_\alpha)$*.*

Lemma 3.4. *Let* $\alpha \in (0,1)$*,* $u_1 \in AP(\mathbb{R}; X_\alpha)$ *and* $u_2 \in C_0(\mathbb{R}; X_\alpha)$*. Set*

$$H_1(t) := \int_{-\infty}^t AT(t-s)u_1(s)\mathrm{d}s, \quad H_2(t) := \int_{-\infty}^t AT(t-s)u_2(s)\mathrm{d}s.$$

Then H_1 *and* H_2 *belong to* $AP(\mathbb{R}; X)$ *and* $C_0(\mathbb{R}; X)$*, respectively.*

Our main result in this section is the following.

Theorem 3.5. *Under the assumptions* (H) *and* (H')*, equation* (2) *has at least one asymptotically almost periodic mild solution provided that*

$$L_\alpha \|(-A)^{-\alpha}\| + M_{1-\alpha}\delta^{-\alpha}\Gamma(\alpha)L_\alpha + M\delta^{-1}L + M\rho_1\rho_3 < 1. \tag{8}$$

Proof. We consider the coupled system of integral equations in the form

$$\begin{cases} v(t) = h(t, v(t)) + \int_{-\infty}^{t} AT(t-s)h(s, v(s))\mathrm{d}s + \int_{-\infty}^{t} T(t-s)f_1(s, v(s))\mathrm{d}s, \\ w(t) = h(t, v(t) + w(t)) - h(t, v(t)) + \int_{-\infty}^{t} T(t-s)f_2(s, v(s) + w(s))\mathrm{d}s \\ \qquad + \int_{-\infty}^{t} T(t-s)[f_1(s, v(s) + w(s)) - f_1(s, v(s))]\mathrm{d}s \\ \qquad + \int_{-\infty}^{t} AT(t-s)[h(s, v(s) + w(s)) - h(s, v(s))]\mathrm{d}s \end{cases} \tag{9}$$

for $t \in \mathbb{R}$. Arguing as in the proof of Theorem 2.8, we observe that if $(v, w) \in AP(\mathbb{R}; X) \times C_0(\mathbb{R}; X)$ is a solution to the coupled system (9), then $u := v + w$ belongs to $AAP(\mathbb{R}; X)$ and it is a solution to the following integral equation

$$u(t) = h(t, u(t)) + \int_{-\infty}^{t} AT(t-s)h(s, u(s))\mathrm{d}s + \int_{-\infty}^{t} T(t-s)f(s, u(s))\mathrm{d}s, \quad t \in \mathbb{R},$$

which implies that u is an asymptotically almost periodic mild solution to equation (2). Hence, the solvability of asymptotically almost periodic mild solutions to equation (2) is transformed into the existence problem of solutions to coupled system (9) in $AP(\mathbb{R}; X) \times C_0(\mathbb{R}; X)$.

On $AP(\mathbb{R}; X)$ let us define a mapping Λ' as

$$(\Lambda' v)(t) := h(t, v(t)) + \int_{-\infty}^{t} AT(t-s)h(s, v(s))\mathrm{d}s + \int_{-\infty}^{t} T(t-s)f_1(s, v(s))\mathrm{d}s, \quad t \in \mathbb{R}.$$

Then it is easy to see that Λ' is well defined and Λ' maps $AP(\mathbb{R}; X)$ into itself in view of our hypotheses with Lemma 2.5, Lemma 2.6, Lemma 3.3 and Lemma 3.4. Moreover, for all $v_1, v_2 \in AP(\mathbb{R}; X)$, one has

$$\begin{aligned} &\|(\Lambda' v_1)(t) - (\Lambda' v_2)(t)\| \\ &\le \|(-A)^{-\alpha}\| \, \|h(t, v_1(t)) - h(t, v_2(t))\|_\alpha \\ &\quad + \int_{-\infty}^{t} \|(-A)^{1-\alpha}T(t-s)\| \|h(s, v_1(s)) - h(s, v_2(s))\|_\alpha \mathrm{d}s \\ &\quad + \int_{-\infty}^{t} \|T(t-s)\| \|f_1(s, v_1(s)) - f_1(s, v_2(s))\| \mathrm{d}s \\ &\le L_\alpha \|(-A)^{-\alpha}\| \|v_1(t) - v_2(t)\| + M_{1-\alpha} L_\alpha \int_{-\infty}^{t} (t-s)^{\alpha-1} e^{-\delta(t-s)} \|v_1(s) - v_2(s)\| \mathrm{d}s \\ &\quad + ML \int_{-\infty}^{t} e^{-\delta(t-s)} \, \|v_1(s) - v_2(s)\| \, \mathrm{d}s \\ &\le \big(L_\alpha \|(-A)^{-\alpha}\| + M_{1-\alpha}\delta^{-\alpha}\Gamma(\alpha)L_\alpha + M\delta^{-1}L\big) \, \|v_1 - v_2\|_\infty, \end{aligned}$$

which, together with (8), implies that Λ' is a contraction on $AP(\mathbb{R};X)$. The existence and uniqueness of fixed point v of Λ' follows from the Banach's fixed point theorem.

In the sequel, we introduce a mapping $\Gamma_0 := \Gamma_0^a + \Gamma_0^b$ on $C_0(\mathbb{R};X)$ as

$$(\Gamma_0^a w)(t) := H_v(t,w(t)) + \int_{-\infty}^t AT(t-s)H_v(s,w(s))\mathrm{d}s + \int_{-\infty}^t T(t-s)J_v'(s,w(s))\mathrm{d}s,$$
$$(\Gamma_0^b w)(t) := \int_{-\infty}^t T(t-s)J_v''(s,w(s))\mathrm{d}s,$$

where J_v', J_v'' are the mappings defined in the proof of Theorem 2.8 and

$$H_v(t,x) := h(t,x+v(t)) - h(t,v(t)), \quad t\in\mathbb{R},\ x\in X.$$

Observe that $J_v'(\cdot,w(\cdot))$ and $J_v''(\cdot,w(\cdot))$ belong to $C_0(\mathbb{R};X)$ for every $w\in C_0(\mathbb{R};X)$ and

$$\begin{aligned} \|H_v(t,x)\|_\alpha &\le L_\alpha\|x\|, \\ \|H_v(t,x)-H_v(t,y)\|_\alpha &\le L_\alpha\|x-y\| \quad \text{for all } t\in\mathbb{R},\ x,y\in X. \end{aligned} \tag{10}$$

This, together with Lemma 2.5, Lemma 2.6, Lemma 3.3 and Lemma 3.4, yields that Γ_0 is well-defined and maps $C_0(\mathbb{R};X)$ into itself. Also, noticing (8), a similar argument as in the proof of Theorem 2.8 proves that there exists a $k_0>0$ such that $\Gamma_0^a w_1 + \Gamma_0^b w_2$ belongs to Ω_{k_0} for every pair $w_1,w_2\in\Omega_{k_0}$.

Next, to obtain a fixed point of Γ_0 with the help of the Krasnoselskii's fixed point theorem, we will show that Γ_0^a is a contraction and Γ_0^b is completely continuous. Since $\Gamma^2=\Gamma_0^b$ holds when $\mathcal{Q}=0$, recalling the arguments used in the proof of Theorem 2.8 yields that Γ_0^b is completely continuous. Also, from (7) and (10) it follows that for any $w_1,w_2\in\Omega_{k_0}$,

$$\begin{aligned} &\|(\Gamma_0^a w_1)(t) - (\Gamma_0^a w_2)(t)\| \\ &\le \|(-A)^{-\alpha}\|\,\|H_v(s,w_1(s)) - H_v(s,w_2(s))\|_\alpha \\ &\quad + \int_{-\infty}^t \|T(t-s)[J_v'(s,w_1(s)) - J_v'(s,w_2(s))]\|\mathrm{d}s \\ &\quad + \int_{-\infty}^t \|(-A)^{1-\alpha}T(t-s)\|\|H_v(s,w_1(s)) - H_v(s,w_2(s))\|_\alpha\mathrm{d}s \\ &\le L_\alpha\|(-A)^{-\alpha}\|\|w_1(t)-w_2(t)\| \\ &\quad + M_{1-\alpha}L_\alpha\int_{-\infty}^t (t-s)^{\alpha-1}e^{-\delta(t-s)}\|w_1(s)-w_2(s)\|\mathrm{d}s \\ &\quad + ML\int_{-\infty}^t e^{-\delta(t-s)}\,\|w_1(s)-w_2(s)\|\,\mathrm{d}s \\ &\le \big(L_\alpha\|(-A)^{-\alpha}\| + M_{1-\alpha}\delta^{-\alpha}\Gamma(\alpha)L_\alpha + M\delta^{-1}L\big)\,\|w_1-w_2\|_\infty, \end{aligned}$$

which shows that Γ_0^a is a contraction in view of (8). Hence, applying the Krasnoselskii's fixed point theorem yields that Γ_0 has at least one fixed point in Ω_{k_0}. This proves that the coupled system (9) has at least one solution in $AP(\mathbb{R};X) \times C_0(\mathbb{R};X)$. This completes the proof of theorem. □

4. An example

In this section we present an example as an application of our abstract result. Take $X = L^2[0,\pi]$ with the norm $\|\cdot\|$ and inner product $(\cdot,\cdot)_2$. Consider the partial differential equation of the form

$$\begin{cases} \dfrac{\partial u(t,\xi)}{\partial t} = \dfrac{\partial^2 u(t,\xi)}{\partial \xi^2} + au(t,\xi) + (\sin t + \sin \pi t)\sin u(t,\xi) \\ \qquad\qquad + e^{-|t|}u(t,\xi)\sin u^2(t,\xi), \quad (t,\xi) \in \mathbb{R}\times[0,\pi], \\ u(t,0) = u(t,\pi) = 0, \quad t \in \mathbb{R}, \end{cases} \tag{11}$$

where $a \in (p^2,(p+1)^2)$ (for some $p \in \mathbb{N}^+$) is a constant.

Define an operator $A : D(A) \subset X \to X$ by $Ax = \frac{\partial^2 x(\xi)}{\partial \xi^2} + ax(\xi)$ and

$$D(A) = \{x \in X; x, x' \text{ are absolutely continuous}, x'' \in X, \text{ and } x(0) = x(\pi) = 0\}.$$

It is clear that $\rho(A) = \mathbb{C}\backslash\{a - n^2; n \in \mathbb{N}^+\}$ and A generates a C_0-semigroup $\{T(t)\}_{t\geq 0}$ on X as

$$T(t)x = \sum_{n=1}^{+\infty} e^{(-n^2+a)t}(x,y_n)_2 y_n, \quad t \geq 0,\ x \in X,$$

where $y_n(\xi) = \sqrt{\frac{2}{\pi}}\sin(n\xi)$. Also, note that $T(t)$ is a nuclear operator for each $t > 0$, which gives the compactness of $T(t)$ for each $t > 0$. Moreover, it is clear that

$$\|T(t)\mathcal{P}x\| \leq e^{(a-(p+1)^2)t}\|x\|, \quad t \geq 0,\ x \in X,$$
$$\|T(t)\mathcal{Q}x\| \leq e^{(a-p^2)t}\|x\|, \quad t \leq 0,\ x \in X,$$

where $\mathcal{Q}$ is the projection onto the subspace spanned by $\{y_1,\ldots,y_p\}$ and $\mathcal{P} = I - \mathcal{Q}$, which implies that $\{T(t)\}_{t\geq 0}$ is hyperbolic with $M = 1$ and $\delta = \min\{a - p^2, (p+1)^2 - a\}$.

Define

$$f_1(t,x(\xi)) := (\sin t + \sin \pi t)\sin x(\xi), \quad t \in \mathbb{R},\ x \in X$$
$$f_2(t,x(\xi)) := e^{-|t|}x(\xi)\sin x^2(\xi), \quad t \in \mathbb{R},\ x \in X.$$

Then it is easy to verify that $f_1, f_2 : \mathbb{R}\times X \to X$ are continuous, $f_1 \in AP(\mathbb{R}\times X; X)$, and

$$\|f_1(t,x) - f_1(t,y)\| \le 2\|x-y\|, \quad \|f_2(t,x)\| \le e^{-|t|}\|x\|$$

for all $t \in \mathbb{R}, x, y \in X$, which implies that $f := f_1 + f_2 \in AAP(\mathbb{R}\times X; X)$ and it is semi-Lipschitz continuous with the Lipschitz constant $L = 2$.

Thus, (11) can be reformulated as the abstract problem (1) and assumption (H) holds with $L = 2$, $\Phi(r) = r$, $\beta(t) = e^{-|t|}$, $\rho_1 = 1$, $\rho_2 \le \frac{2}{\delta}$. Then from Theorem 2.8 it follows that when $\delta > 6$, equation (11) at least has one asymptotically almost periodic mild solution.

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