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Standing Solitary Euler-Korteweg Waves are Unstable

Johannes Höwing

Abstract. This note establishes instability of any planar standing wave in the Euler-Korteweg system.

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1. The result

The Euler-Korteweg system is given by the equations

$$\begin{aligned} V_t - U_x &= 0, \\ U_t + p(V)_x &= -(\kappa(V)V_{xx} + \frac{1}{2}(\kappa(V))_x V_x)_x, \end{aligned} \tag{1}$$

with $\kappa(V) > 0$. System (1), and notably its solitary waves

$$\begin{pmatrix} V \\ U \end{pmatrix}(x,t) = \begin{pmatrix} v \\ u \end{pmatrix}(x - ct) \quad \text{with} \quad \begin{pmatrix} v \\ u \end{pmatrix}(\pm\infty) = \begin{pmatrix} v_* \\ u_* \end{pmatrix},$$

appear in a number of contexts, cf. below. This paper is concerned with the stability of such solitary waves which is defined as follows.

Definition 1.1 ([3])**.** A traveling wave (v,u) of (1) is called orbitally stable if for each $\varepsilon > 0$, there exists a $\delta > 0$ such that for any solution $(V,U) \in (v,u) + C([0,T); H^3(\mathbb{R}) \times H^2(\mathbb{R}))$ of (1), closeness at initial time,

$$\|(V,U)(\cdot,0) - (v,u)(\cdot)\|_{H^1 \times L^2} < \delta$$

implies closeness at any time

$$\inf_{\sigma \in \mathbb{R}} \|(V,U)(\cdot,t) - (v,u)(\cdot + \sigma)\|_{H^1 \times L^2} < \varepsilon \quad \text{for all } t > 0.$$

J. Höwing: University of Hamburg, Department of Mathematics, 20146 Hamburg, Germany; Johannes.Hoewing@gmail.com

The following is the point of this short note.

Theorem 1.2. *All non-trivial solitary Euler-Korteweg waves with speed $c = 0$ are not orbitally stable.*

This finding has various applications. When $\kappa(V) \equiv 1$, equation (1) reduces to the Boussinesq equation, describing water waves (Bona and Sachs [4]), multi- (Benzoni et al. [3]) and one- (Höwing [13]) phase fluids with capillarity, and, as discovered more recently by Heimburg and Jackson [12], signal propagation in nerves (cf. also Freistühler and Höwing [9]). When $\kappa(V) = \frac{1}{4V^4}$, equation (1) is the hydrodynamic version of the generalized Gross-Pitaevskii equation

$$i\Psi_t + \frac{1}{2}\Psi_{xx} + \Psi G(|\Psi|^2) = 0, \quad \text{with} \quad \rho G'(\rho) = P'(\rho),\ P(\rho) = p\left(\frac{1}{\rho}\right),$$

used in the description of Bose-Einstein condensates.

Our interest in waves of vanishing speed stems from the fact that they take a special position since, (1) referring to Lagrangian coordinates, they represent structures that are "frozen" in the material.

There is an enormous interest in (in-)stability of (standing) waves in these equations; it is beyond the scope of this note to present the existing results, we refer here to the recent survey [2] for the Euler-Korteweg system and to [6] for the special case of the Gross-Pitaevskii equation.

In some of the above-mentioned examples, the assertion of Theorem 1.2 is well known; in particular certain of the results of de Bouard [7], and Pelinovsky and Kevrekidis [15] cover the Gross-Pitaevskii case. Certain results of Zumbrun [17] and Liu [14] are interesting special cases for the Bona-Sachs case [4]. In this respect, the achievment of Theorem 1.2 is its generality.

Finally, note that Theorem 1.2 does not cover those planar standing waves in the Gross-Pitaevskii equation considered for example by Cazenave and Lions [5], and de Bouard [7] which allow Ψ to vanish somewhere along its profile since in this case the Madelung transformation is not valid.

2. The proof

For fixed base state v_*, the solitary waves homoclinic to v_* occur in families (u^c, v^c) parametrized by their speed c. The proof of Theorem 1.2 is based on the moment of instability [11], in particular on the following result.

Lemma 2.1 ([1]). *A solitary wave (u^{c_*}, v^{c_*}) is orbitally unstable if the moment of instability*

$$m(c) = \int_{-\infty}^{\infty} \kappa(v) v'^2 \, d\xi$$

is not convex at $c = c_*$.

Theorem 1 follows from

Lemma 2.2. *$m(c)$ satisfies $m''(0) < 0$.*

Proof. To prove this lemma, we recall that with

$$F(v,c) = -f(v) + f(v_*) - p(v_*)(v - v_*) + \frac{1}{2}c^2(v - v_*)^2, \quad -\frac{df(v)}{dv} = p(v),$$

the profile equation

$$\kappa(v)v'' + \frac{1}{2}(\kappa(v))'v' = -\frac{\partial F(v,c)}{\partial v}$$

possesses (cf. [3]) a first integral given by

$$I(v,v') = \frac{1}{2}\kappa(v)v'^2 + F(v,c).$$

Now

$$m(c) = 2\int_{v_*}^{v_m(c)} \kappa(v)v' \, dv$$

with $v_*, v_m(c) > v_*$ consecutive zeros of $F(\cdot, c)$. Since $I(v,v') \equiv 0$ along solutions, we have

$$\begin{aligned} m(c) &= 2\int_{v_*}^{v_m(c)} (\kappa(v))^{\frac{1}{2}} \, (-2F(v,c))^{\frac{1}{2}} \, dv \\ &= 4\int_0^{(v_m(c)-v_*)^{\frac{1}{2}}} \left(\kappa(v_m(c) - w^2)\right)^{\frac{1}{2}} \left(-2F(v_m(c) - w^2, c)\right)^{\frac{1}{2}} \, w \, dw, \end{aligned}$$

where $w := (v_m(c) - v)^{\frac{1}{2}}$ (cf. [13]). The first derivative of m is (note that the integral limits are $w = 0$ and $w = (v_m(c) - v_*)^{\frac{1}{2}}$ unless otherwise stated)

$$\begin{aligned} m'(c) &= 4\int \frac{d}{dc}\left\{ \left(\kappa(v_m(c) - w^2)\right)^{\frac{1}{2}} \left(-2F(v_m(c) - w^2, c)\right)^{\frac{1}{2}} \right\} w \, dw \\ &= 4\int \frac{\kappa_v(v_m(c) - w^2)v_m'(c)}{2\left(\kappa(v_m(c) - w^2)\right)^{\frac{1}{2}}} \left(-2F(v_m(c) - w^2, c)\right)^{\frac{1}{2}} \, w \, dw \\ &\quad + 4\int \frac{\left(\kappa(v_m(c) - w^2)\right)^{\frac{1}{2}} \left(-F_v(v_m(c) - w^2, c)\right) v_m'(c)}{\left(-2F(v_m(c) - w^2, c)\right)^{\frac{1}{2}}} \, w \, dw \\ &\quad + 4\int \frac{\left(\kappa(v_m(c) - w^2)\right)^{\frac{1}{2}} \left(-F_c(v_m(c) - w^2, c)\right)}{\left(-2F(v_m(c) - w^2, c)\right)^{\frac{1}{2}}} \, w \, dw \end{aligned}$$

which, due to

$$\frac{\partial}{\partial v}\left((\kappa(v)(-2F(v,c)))^{\frac{1}{2}} \right) = \frac{\kappa_v(v)(-2F(v,c))^{\frac{1}{2}}}{2(\kappa(v))^{\frac{1}{2}}} - \frac{\kappa(v)^{\frac{1}{2}} F_v(v,c)}{(-2F(v,c))^{\frac{1}{2}}},$$

simplifies to

$$m'(c) = -4 \int \frac{F_c(v_m(c) - w^2, c)\,(\kappa(v_m(c) - w^2))^{\frac{1}{2}}}{(-2F(v_m(c) - w^2, c))^{\frac{1}{2}}}\, w\, dw$$
$$= -4 \int c\; \frac{(v_m(c) - w^2 - v_*)^2\,(\kappa(v_m(c) - w^2))^{\frac{1}{2}}}{(-2F(v_m(c) - w^2, c))^{\frac{1}{2}}}\, w\, dw.$$

The sign of $m''(c)$ evaluated at $c = 0$ is obviously negative since the integrand is a product of c and a positive function: differentiating and setting c to 0, only the positive function remains. Anyway, let us derive $m''(c)$ as the resulting formula might be useful also when not looking at the vanishing speed case. After a transformation back to the original variable $v = v_m(c) - w^2$, the second derivative of m can be written in the form

$$m''(c) = 2 \int_{v_*}^{v_m(c)} \frac{A(v,c) + B(v,c)}{(\kappa(v))^{\frac{1}{2}}\,(-2F(v,c))^{\frac{3}{2}}}\, dv$$

with $A(v,c) = F(v,c)\kappa_v(v)v_m'(c)F_c(v,c)$ and

$$\begin{aligned} B(v,c) = {}& \kappa(v)(v - v_*) \\ & \times \big(2F(v,c)\,((v - v_*) + 2cv_m'(c)) - c(v - v_*)\,(F_v(v,c)v_m'(c) + F_c(v,c))\big). \end{aligned}$$

As $F_c(v,0) = 0 = A(v,0)$ and

$$B(v,0) = 2\kappa(v)(v - v_*)^2 F(v,0) < 0 \quad \text{for all } v \in (v_*, v_m(0)),$$

we indeed have $m''(0) < 0$. □

Remark 2.3. The idea to extend Theorem 1.2 to the Navier-Stokes-Korteweg system,

$$\begin{aligned} V_t - U_x &= 0, \\ U_t + p(V)_x &= (\mu(V)U_x)_x - (\kappa(V)V_{xx} + \frac{1}{2}(\kappa(V))_x V_x)_x, \end{aligned} \tag{2}$$

which is (1) endowed with a non-constant viscosity term $\mu(V) > 0$ (cf., e.g., [8,10] and references therein), suggests itself. Existence of standing waves in (1) and (2) is certainly equivalent since for a standing wave $u' \equiv 0$. In the case that both capillarity and viscosity are constant, instability of standing solitary waves in (2) is well known [16].

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