Convolution in Rearrangement-Invariant Spaces Defined in Terms of Oscillation and the Maximal Function

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Abstract. We characterize boundedness of a convolution operator with a fixed kernel between the classes $S^p(v)$, defined in terms of oscillation, and weighted Lorentz spaces $\Gamma^q(w)$, defined in terms of the maximal function, for $0 < p, q \leq \infty$. We prove corresponding weighted Young-type inequalities of the form

$$\|f * g\|_{\Gamma^q(w)} \leq C\|f\|_{S^p(v)}\|g\|_Y$$

and characterize the optimal rearrangement-invariant space $Y$ for which these inequalities hold.

Keywords. Convolution, Young inequality, weighted Lorentz spaces, oscillation

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1. Introduction

The classical Young inequality

$$\|f * g\|_q \leq \|f\|_p\|g\|_r,$$

where $1 \leq p, q, r \leq \infty$, $\frac{1}{p} + \frac{1}{r} = 1 + \frac{1}{q}$ and $f * g$ is the convolution given by

$$(f * g)(t) = \int_{-\infty}^{\infty} f(x)g(t-x)\,dx, \quad t \in \mathbb{R},$$

is one of the fundamental results related to the convolution and function spaces. It has been already modified and generalized for classes of function spaces that

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are wider than the Lebesgue spaces in the original Young inequality. O’Neil [14] extended the result for the two-parametric Lorentz spaces \( L_{p,q} \). Precisely, he proved that, for \( 1 < p,q,r < \infty \) and \( 1 \leq a,b,c \leq \infty \) such that \( 1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r} \) and \( \frac{1}{a} = \frac{1}{b} + \frac{1}{c} \), the inequality
\[
\|f \ast g\|_{L_{q,a}} \leq C \|f\|_{L_{p,b}} \|g\|_{L_{r,c}}, \quad f \in L_{p,b}, \ g \in L_{r,c},
\]
holds. This problem was further studied e.g. in [3,10,18] and the result was also improved up to the range \( 1 < p,q,r < \infty \) and \( 0 < a,b,c \leq \infty \). Nursultanov and Tikhonov [13] recently studied the same question considering convolution of periodic functions.

In the preceding paper [11] the author studied the boundedness of the operator \( T_g \) given by
\[
T_g f(t) := (f \ast g)(t)
\]
between weighted Lorentz spaces \( \Lambda^p(v) \) and \( \Gamma^q(w) \) with given weights \( v,w \) and exponents \( p,q \). It turned out that the result could be expressed by Young-type inequalities of the form
\[
\|f \ast g\|_{\Gamma^q(w)} \leq C \|f\|_{\Lambda^p(v)} \|g\|_Y, \quad f \in \Lambda^p(v), \ g \in Y,
\]
where the best r.i. space \( Y \), such that this inequality holds, was characterized.

In this paper we deal with similar questions with \( S^p(v) \) in place of \( \Lambda^p(v) \). The class \( S^p(v) \) is defined in terms of \( f^{**} - f^* \), where \( f^* \) is the nonincreasing rearrangement of \( f \) and \( f^{**} \) is the maximal function of \( f \) (for precise definitions see Section 2 below). The quantity \( f^{**} - f^* \) naturally represents the oscillation of \( f \) (see the fundamental paper of Bennett, DeVore and Sharpley [1]) and has appeared in numerous applications, particularly within the theory of Sobolev embeddings (see e.g. [4] and the references therein).

We are going to solve the following problems: At first, given exponents \( p,q \in (0, \infty] \) and weights \( v,w \), we provide conditions on the kernel \( g \in L^1 \) under which \( T_g \) is bounded between \( S^p(v) \) and \( \Gamma^q(w) \), written \( T_g : S^p(v) \to \Gamma^q(w) \). Precisely, we will show that there exists an r.i. space \( Y \) such that \( T_g : S^p(v) \to \Gamma^q(w) \) if (and in reasonable cases also only if) \( g \in Y \) and characterize the norm of \( Y \). Next, we write these results in the form of Young-type convolution inequalities
\[
\|f \ast g\|_{\Gamma^q(w)} \leq C \|f\|_{S^p(v)} \|g\|_Y, \quad f \in S^p(v), \ g \in L^1 \cap Y. \tag{1}
\]
The constant \( C \) here in general depends on \( p,q \) but is independent of \( f,g,v,w \). We will also show that the space \( Y \) we obtained is the essentially largest (optimal) r.i. space for which the inequality (1) is valid.

To get the desired results, we employ a similar technique as in [11]. We represent the investigated convolution-related inequalities by certain Hardy-type weighted inequalities and then treat the problem by working with the latter ones. This is done in Section 3. The final result shaped as the Young-type inequality (1) is presented in Section 4.
2. Preliminaries

Let us present some definitions and technical results we are going to use. The set of all measurable functions on $\mathbb{R}$ is denoted by $\mathcal{M}(\mathbb{R})$. The symbols $\mathcal{M}_+(0, \infty)$ and $\mathcal{M}_+(\mathbb{R})$ stand for the sets of all nonnegative measurable functions on $(0, \infty)$ and $\mathbb{R}$, respectively. If $p \in (1, \infty)$, we define $p' := \frac{p}{p-1}$. The notation $A \lesssim B$ means that $A \leq CB$ where $C$ is a positive constant independent of relevant quantities. Unless specified else, $C$ actually depends only on the exponents $p$ and $q$, if they are involved. If $A \lesssim B$ and $B \lesssim A$, we write $A \simeq B$. The optimal constant $C$ in an inequality $A \leq CB$ is the least $C$ such that the inequality holds. By writing inequalities in the form $A(f) \lesssim B(f)$, $f \in X$, we always mean that $A(f) \lesssim B(f)$ is satisfied for all $f \in X$.

A weight is any nonnegative function on $(0, \infty)$ such that $0 < W(t) < \infty$ for all $t > 0$, where $W(t) := \int_0^t w(s) \, ds$.

If $f \in \mathcal{M}(\mathbb{R})$, we define the nonincreasing rearrangement of $f$ by

$$f^*(t) := \inf \{s > 0; \{x \in \mathbb{R}; |f(x)| > s\} \leq t\}, \quad t > 0,$$

and the Hardy-Littlewood maximal function of $f$ by

$$f^{**}(t) := \frac{1}{t} \int_0^t f^*(s) \, ds, \quad t > 0.$$

If $u$ is a weight, then a generalized version of the maximal function is defined by

$$f^{**}_u(t) := \frac{1}{U(t)} \int_0^t f^*(s)u(s) \, ds, \quad t > 0.$$

By $L^1$ we denote the Lebesgue-integrable functions on $\mathbb{R}$. The symbol $L^1_{\text{loc}}$ stands for locally integrable functions on $\mathbb{R}$. If $q \in (0, \infty]$ and $w$ is a weight, then $L^q(w)$ denotes the Lebesgue $L^q$-space over the interval $(0, \infty)$ with the measure $w(t) \, dt$.

Let $\varrho : \mathcal{M}(\mathbb{R}) \to [0, \infty]$ be a functional with the following properties:

(i) $E \subset \mathbb{R}$, $|E| < \infty \Rightarrow \varrho(\chi_E) < \infty$,
(ii) $f \in \mathcal{M}(\mathbb{R})$, $c \geq 0 \Rightarrow \varrho(cf) = c\varrho(f)$ (positive homogeneity),
(iii) $f, g \in \mathcal{M}(\mathbb{R})$, $0 \leq f \leq g$ a.e. \, $\Rightarrow \varrho(f) \leq \varrho(g)$ (lattice property),
(iv) $f, g \in \mathcal{M}(\mathbb{R})$, $f^* = g^* \Rightarrow \varrho(f) = \varrho(g)$ (r.i. property).

The set $X = X(\varrho) := \{f \in \mathcal{M}(\mathbb{R}); \varrho(f) < \infty\}$ is called a rearrangement-invariant (r.i.) lattice. For such $X$ we define $\|f\|_X := \varrho(|f|)$ for all $f \in X$.

For the definition of a rearrangement-invariant space see [2, p. 59].
Let \( p \in (0, \infty] \) and \( u, v \) be weights. The weighted Lorentz spaces are defined by what follows:

\[
\Lambda^p(v) := \left\{ f \in \mathcal{M}(\mathbb{R}); \| f \|_{\Lambda^p(v)} := \left( \int_0^\infty (f^*(t))^p v(t) \, dt \right)^{\frac{1}{p}} < \infty \right\}, \quad p \in (0, \infty),
\]

\[
\Lambda^\infty(v) := \left\{ f \in \mathcal{M}(\mathbb{R}); \| f \|_{\Lambda^\infty(v)} := \text{ess sup}_{t > 0} f^*(t) v(t) < \infty \right\}, \quad p = \infty,
\]

\[
\Gamma^p_u(v) := \left\{ f \in \mathcal{M}(\mathbb{R}); \| f \|_{\Gamma^p_u(v)} := \left( \int_0^\infty (f_u^*(t))^p v(t) \, dt \right)^{\frac{1}{p}} < \infty \right\}, \quad p \in (0, \infty),
\]

\[
\Gamma^\infty_u(v) := \left\{ f \in \mathcal{M}(\mathbb{R}); \| f \|_{\Gamma^\infty_u(v)} := \text{ess sup}_{t > 0} f_u^*(t) v(t) < \infty \right\}, \quad p = \infty.
\]

If \( u \equiv 1 \), we write just \( \Gamma^p(v) \), \( \Gamma^\infty(v) \). Next, we denote

\[
A := \left\{ f \in \mathcal{M}(\mathbb{R}); f^*(\infty) = 0 \right\}.
\]

Clearly, any function \( f \in A \) satisfies \( f^{**}(\infty) = 0 \).

The class \( S^p(v) \) is given by

\[
S^p(v) := \left\{ f \in A; \| f \|_{S^p(v)} := \left( \int_0^\infty (f^{**}(t) - f^*(t))^p v(t) \, dt \right)^{\frac{1}{p}} < \infty \right\}, \quad p \in (0, \infty),
\]

\[
S^\infty(v) := \left\{ f \in A; \| f \|_{S^\infty(v)} := \text{ess sup}_{t > 0} (f^{**}(t) - f^*(t)) v(t) < \infty \right\}, \quad p = \infty.
\]

The \( \Gamma \)-spaces with \( u \equiv 1 \) are linear and the functional \( \| \cdot \|_{\Gamma^p(v)} \) is at least a quasi-norm. In fact, for \( p \in [1, \infty] \) it is a norm. The key property is the sublinearity of the maximal function (see e.g. [2, p. 54]), i.e.

\[
(f + g)^{**}(t) \leq f^{**}(t) + g^{**}(t), \quad t > 0.
\]

On the other hand, the rearrangement itself is not sublinear and the \( \Lambda \)-“spaces” need not to be linear [7]. However, they are always at least r.i. lattices.

In contrast with that, \( S^p(v) \) in general does not even have the lattice property. A detailed study of this and other functional properties of \( S^p(v) \) was published in [4].

Obviously, \( \Gamma^p(v) \subset S^p(v) \) for any \( p \in (0, \infty] \) and any weight \( v \). In case of \( p \in (0, \infty) \), we will work with weights \( v \) satisfying the conditions

\[
\int_\varepsilon^\infty \frac{v(t)}{t^p} \, dt < \infty \text{ for every } \varepsilon > 0 \quad \text{and} \quad \int_0^\infty \frac{v(t)}{t^p} \, dt = \infty.
\]

(2)

It can be checked easily that if the first part of (2) is not satisfied, then \( \Gamma^p(v) = S^p(v) = \{0\} \), while failing the other part implies that \( L^1 \subset \Gamma^p(v) \subset S^p(v) \). By
the symbol $\mathcal{V}_p$ we denote the set of all weights $v$ satisfying (2) with $p \in (0, \infty)$. Similarly, $\mathcal{V}_\infty$ stands for the set of all weights satisfying
\[
\text{ess sup}_{t > \varepsilon} \frac{v(t)}{t} < \infty \text{ for every } \varepsilon > 0 \quad \text{and} \quad \text{ess sup}_{t > 0} \frac{v(t)}{t} = \infty.
\]

A useful tool for investigation of convolution inequalities is the O’Neil inequality [14, Lemma 2.5]:

**Lemma 2.1.** Let $f, g \in L^1_{\text{loc}}$. Then, for every $t \in (0, \infty)$ it holds
\[
(f * g)**(t) \leq tf**(t)g**(t) + \int_t^\infty f**(s)g**(s) \, ds.
\]

We are going to use this inequality with an alternative expression of its right-hand side from [11, Proposition 4.1]:

**Lemma 2.2.** Let $f, g \in L^1_{\text{loc}}$. Then for every $t \in (0, \infty)$ it holds
\[
tf**(t)g**(t) + \int_t^\infty f**(s)g**(s) \, ds
= \limsup_{s \to \infty} sf**(s)g**(s) + \int_t^\infty (f**(s) - f**(s))(g**(s) - g**(s)) \, ds.
\]

In particular, if $f \in \mathcal{A}$ and $g \in L^1$, then $\lim_{s \to \infty} sf**(s)g**(s) = 0$. Thus, Lemmas 2.1 and 2.2 together yield
\[
(f * g)**(t) \leq \int_t^\infty (f**(s) - f**(s))(g**(s) - g**(s)) \, ds, \quad t > 0. \tag{3}
\]

As observed already in [14], O’Neil inequality has also a converse form (for the proof of the following statement see e.g. [11, Lemma 2.3]).

**Lemma 2.3.** Let $f, g \in L^1_{\text{loc}}$ be nonnegative even functions which are nonincreasing on $(0, \infty)$. Then for every $t \in (0, \infty)$ it holds
\[
tf**(t)g**(t) + \int_t^\infty f**(s)g**(s) \, dy \leq 12(f * g)**(t).
\]

From now on we denote the “positive symmetrically decreasing” functions by
\[
PSD := \{ f; f \in \mathcal{M}_+(\mathbb{R}), \text{ f is even, } f \text{ is nonincreasing on } (0, \infty) \}.
\]

Applying Lemmas 2.2, 2.3 and the observation (3), we reach the following conclusion: Let $f \in \mathcal{A}$, $g \in L^1$ and assume that both $f, g \in PSD$. Then
\[
\int_t^\infty (f**(s) - f**(s))(g**(s) - g**(s)) \, ds \leq 12(f * g)**(t), \quad t > 0. \tag{4}
\]

The last preliminary result is the proposition below (cf. e.g. [16, Lemma 1.2], [5, Proposition 7.2]).
Proposition 2.4. Let \( h \) be a nonnegative and nonincreasing real-valued function on \((0, \infty)\). Then there exists a sequence \( \{ f_n \}_{n \in \mathbb{N}} \) of functions \( f_n \in \mathcal{M}(\mathbb{R}) \) such that for a.e. \( t > 0 \) it holds
\[
\frac{f_n^*(\frac{1}{t}) - f_n^*(\frac{1}{t})}{t} \uparrow h(t), \quad n \to \infty.
\]

Proof. There exists a nonnegative Radon measure \( \nu \) on \((0, \infty)\) such that for a.e. \( t > 0 \) it is
\[
h(t) = \int_{[t, \infty)} \frac{d\nu(x)}{x}.
\]
For any \( n \in \mathbb{N} \) we can find a function \( f_n \in \mathcal{M}(\mathbb{R}) \) such that
\[
f_n^*(t) = \int_{(0, \frac{1}{t})} \chi(x, \infty) \, d\nu(x)
\]
for all \( t > 0 \). Now choose any \( t > 0 \) such that (5) holds. By Fubini theorem,
\[
\frac{f_n^*(\frac{1}{t}) - f_n^*(\frac{1}{t})}{t} = \int_0^\frac{1}{t} \int_{(0, \frac{1}{t})} \chi(x, \infty) \, d\nu(x) \, ds - \frac{1}{t} \int_{(0, t]} \chi(x, \infty) \, d\nu(x)
\]
\[
= \int_{(0, \frac{1}{t})} \int_{0}^{\min\left\{ \frac{1}{t}, \frac{1}{n^2} \right\}} \chi(x, \infty) \, d\nu(x) - \frac{1}{t} \int_{(0, t]} \chi(x, \infty) \, d\nu(x)
\]
\[
= \int_{[t, \infty)} \chi(x, \infty) \, d\nu(x) \uparrow h(t), \quad n \to \infty. \quad \Box
\]

3. Inequalities with \( f^{**} - f^* \) and boundedness of the convolution operator

As mentioned in the introduction, we are going to describe when \( T_g : S^p(v) \to \Gamma^q(w) \) is bounded and, above all, what is the optimal r.i. space \( Y \) such that the inequality \( \| f * g \|_{\Gamma^q(w)} \lesssim \| f \|_{S^p(v)} \| g \|_Y \) holds for all \( f \in S^p(v) \) and \( g \in L^1 \cap Y \).

The problem is connected to inequalities involving the expression \( f^{**} - f^* \) which are shown in the following lemma. It is a direct consequence of the O’Neil inequality (3).

Lemma 3.1. Let \( p, q \in (0, \infty] \). Let \( v, w \) be weights, \( v \in \mathcal{K}_p \). Let \( g \in L^1 \).

(i) If \( p, q \in (0, \infty) \) and
\[
\left( \int_0^\infty \left( \int_x^\infty (f^{**}(t) - f^*(t)) (g^{**}(t) - g^*(t)) \, dt \right)^q w(x) \, dx \right)^{\frac{1}{q}} \leq C(6) \left( \int_0^\infty (f^{**}(x) - f^*(x))^p v(x) \, dx \right)^{\frac{1}{p}}, \quad f \in S^p(v),
\]

(ii) If \( p \in (0, \infty) \) and \( q = \infty \) then
\[
\left( \int_0^\infty \left( \int_x^\infty (f^{**}(t) - f^*(t)) (g^{**}(t) - g^*(t)) \, dt \right) w(x) \, dx \right)^{\frac{1}{p}} \leq C(6) \left( \int_0^\infty (f^{**}(x) - f^*(x))^p v(x) \, dx \right)^{\frac{1}{p}}, \quad f \in S^p(v),
\]

(iii) If \( p = \infty \) and \( q \in (0, \infty) \) then
\[
\left( \int_0^\infty \left( \int_x^\infty (f^{**}(t) - f^*(t)) (g^{**}(t) - g^*(t)) \, dt \right)^q w(x) \, dx \right)^{\frac{1}{q}} \leq C(6) \left( \int_0^\infty (f^{**}(x) - f^*(x))^p v(x) \, dx \right)^{\frac{1}{p}}, \quad f \in S^p(v),
\]

(iv) If \( p, q \in (0, \infty) \) and \( q \neq \infty \) then
\[
\left( \int_0^\infty \left( \int_x^\infty (f^{**}(t) - f^*(t)) (g^{**}(t) - g^*(t)) \, dt \right)^q w(x) \, dx \right)^{\frac{1}{q}} \leq C(6) \left( \int_0^\infty (f^{**}(x) - f^*(x))^p v(x) \, dx \right)^{\frac{1}{p}}, \quad f \in S^p(v),
\]

(v) If \( p = \infty \) and \( q = \infty \) then
\[
\left( \int_0^\infty \left( \int_x^\infty (f^{**}(t) - f^*(t)) (g^{**}(t) - g^*(t)) \, dt \right) w(x) \, dx \right)^{\frac{1}{p}} \leq C(6) \left( \int_0^\infty (f^{**}(x) - f^*(x))^p v(x) \, dx \right)^{\frac{1}{p}}, \quad f \in S^p(v),
\]
then $T_g : S^p(v) \to \Gamma^q(w)$ and, moreover, the optimal constant $C_{(6)}$ satisfies
\[\|T_g\|_{S^p(v) \to \Gamma^q(w)} \leq C_{(6)}.\]

(ii) If $0 < p < \infty = q$ and
\[
\begin{align*}
\text{ess sup}_{x > 0} \int_x^\infty (f^{**}(t) - f^*(t))(g^{**}(t) - g^*(t)) dt \ w(x) \\
& \leq C_{(7)} \left( \int_0^\infty (f^{**}(x) - f^*(x)) p v(x) \ dx \right)^{\frac{1}{p}}, \ f \in S^p(v),
\end{align*}
\]

then $T_g : S^p(v) \to \Gamma^q(w)$ and, moreover, the optimal constant $C_{(8)}$ satisfies
\[\|T_g\|_{S^p(v) \to \Gamma^q(w)} \leq C_{(8)}.
\]

(iii) If $0 < q < \infty = p$ and
\[
\begin{align*}
\left( \int_0^\infty \left( \int_x^\infty (f^{**}(t) - f^*(t))(g^{**}(t) - g^*(t)) dt \right)^q w(x) \ dx \right)^{\frac{1}{q}} \\
& \leq C_{(8)} \text{ess sup}_{x > 0} (f^{**}(x) - f^*(x)) v(x), \ f \in S^\infty(v),
\end{align*}
\]

then $T_g : S^\infty(v) \to \Gamma^q(w)$ and, moreover, the optimal constant $C_{(7)}$ satisfies
\[\|T_g\|_{S^\infty(v) \to \Gamma^q(w)} \leq C_{(7)}.
\]

(iv) If $p = q = \infty$ and
\[
\begin{align*}
\text{ess sup}_{x > 0} \int_x^\infty (f^{**}(t) - f^*(t))(g^{**}(t) - g^*(t)) dt \ w(x) \\
& \leq C_{(9)} \text{ess sup}_{x > 0} (f^{**}(x) - f^*(x)) v(x), \ f \in S^\infty(v),
\end{align*}
\]

then $T_g : S^\infty(v) \to \Gamma^\infty(w)$ and, moreover, the optimal constant $C_{(9)}$ satisfies
\[\|T_g\|_{S^\infty(v) \to \Gamma^\infty(w)} \leq C_{(9)}.
\]

The next result is inverse to the previous lemma, showing that the validity of the inequalities with $f^{**} - f^*$ from that lemma is also necessary for the boundedness of $T_g$, given that $g \in PSD$.

Lemma 3.2. Let $p, q \in (0, \infty]$. Let $v, w$ be weights, $v \in \mathcal{V}_p$. Let $g \in L^1 \cap PSD$.

(i) If $p, q \in (0, \infty)$ and $T_g : S^p(v) \to \Gamma^q(w)$, then (6) holds and the optimal constant $C_{(6)}$ satisfies $C_{(6)} \lesssim \|T_g\|_{S^p(v) \to \Gamma^q(w)}$.

(ii) If $0 < p < \infty = q$ and $T_g : S^p(v) \to \Gamma^\infty(w)$, then (7) holds and the optimal constant $C_{(7)}$ satisfies $C_{(7)} \lesssim \|T_g\|_{S^p(v) \to \Gamma^\infty(w)}$.

(iii) If $0 < q < \infty = p$ and $T_g : S^\infty(v) \to \Gamma^q(w)$, then (8) holds and the optimal constant $C_{(8)}$ satisfies $C_{(8)} \lesssim \|T_g\|_{S^\infty(v) \to \Gamma^q(w)}$.

(iv) If $p = q = \infty$ and $T_g : S^\infty(v) \to \Gamma^\infty(w)$, then (9) holds and the optimal constant $C_{(9)}$ satisfies $C_{(9)} \lesssim \|T_g\|_{S^\infty(v) \to \Gamma^\infty(w)}$.
Proof. Let us show (i), the other cases are analogous. By (4), for the optimal constant $C(6)$ we get

$$C(6) = \sup_{\|f\|_{\mathcal{F}(\nu)} \leq 1} \left( \int_0^\infty \left( \int_x^\infty (f^{**}(t) - f^*(t))(g^{**}(t) - g^*(t)) \, dt \right)^q w(x) \, dx \right)^{\frac{1}{q}}$$

$$= \sup_{\|f\|_{\mathcal{F}(\nu)} \leq 1, f \in \mathcal{P}D} \left( \int_0^\infty \left( \int_x^\infty (f^{**}(t) - f^*(t))(g^{**}(t) - g^*(t)) \, dt \right)^q w(x) \, dx \right)^{\frac{1}{q}}$$

$$\leq 12 \sup_{\|f\|_{\mathcal{F}(\nu)} \leq 1, f \in \mathcal{P}D} \left( \int_0^\infty ((f \ast g)^{**}(t))^q w(t) \, dt \right)^{\frac{1}{q}}$$

$$\leq \|T_g\| s_{\nu(v) \rightarrow \Gamma} = 0(w). \quad \square$$

Now we characterize under which conditions on weights and exponents the inequalities of Lemma 3.1 are satisfied.

**Theorem 3.3.** Let $p, q \in (0, \infty)$. Let $v, w$ be weights, $v \in \mathcal{V}_p$. Let $g \in L^1$.

(i) If $1 < p \leq q < \infty$, then (6) holds if and only if

$$A(10) := \sup_{x > 0} \left( \int_x^\infty (g^{**}(t))^q w(t) \, dt \right)^{\frac{1}{q}} \left( \int_x^\infty \frac{v(s)}{s^p} \, ds \right)^{-\frac{1}{p}} < \infty$$

and

$$A(11) := \sup_{x > 0} W^{\frac{1}{q}}(x) \left( \int_x^\infty (g^{**}(t))^{p'} \left( \int_t^\infty \frac{v(s)}{s^p} \, ds \right)^{-p'} v(t) \, dt \right)^{\frac{1}{p'}} < \infty.$$  

The optimal constant $C(6)$ satisfies $C(6) \simeq A(10) + A(11)$.

(ii) If $0 < p \leq 1$, $0 < p \leq q < \infty$, then (6) holds if and only if $A(10) < \infty$ and

$$A(12) := \sup_{x > 0} g^{**}(x) W^{\frac{1}{q'}}(x) \left( \int_x^\infty v(t) \, dt \right)^{\frac{1}{p'}} < \infty.$$  

The optimal constant $C(6)$ satisfies $C(6) \simeq A(10) + A(12)$.

(iii) If $1 < p < \infty$, $0 < q < p$, then (6) holds if and only if

$$A(13) := \left( \int_0^\infty \left( \int_x^\infty (g^{**}(t))^q w(t) \, dt \right)^{\frac{1}{q}} \left( \int_x^\infty \frac{v(t)}{t^p} \, dt \right)^{-\frac{1}{p}} v(x) \, dx \right)^{\frac{1}{q}} < \infty$$

and

$$A(14) := \left( \int_0^\infty W^{\frac{1}{q'}}(x) w(x) \right) \times \left( \int_x^\infty (g^{**}(t))^{p'} \left( \int_t^\infty \frac{v(s)}{s^p} \, ds \right)^{-p'} v(t) \, dt \right)^{\frac{1}{p'}} < \infty.$$  

The optimal constant $C_{(6)}$ satisfies $C_{(6)} \simeq A_{(13)} + A_{(14)}$.

(iv) If $0 < q < p \leq 1$, then (6) holds if and only if $A_{(13)} < \infty$ and

$$A_{(15)} := \left( \int_0^\infty \sup_{x \leq t < \infty} (g^{**}(t))^r \left( \int_0^t v(s) \, ds \right)^{-\frac{r}{p}} W^r(x)w(x) \, dx \right)^{\frac{1}{r}} < \infty.$$  

The optimal constant $C_{(6)}$ satisfies $C_{(6)} \simeq A_{(13)} + A_{(15)}$.

Proof. Let us show (i). After the change of variable $x \mapsto \frac{1}{x}$, inequality (6) is written as

$$\left( \int_0^\infty \left( \int_0^x \frac{f^{**}(\frac{1}{t}) - f^*(\frac{1}{t})}{t} \frac{g^{**}(\frac{1}{t}) - g^*(\frac{1}{t})}{t} \, dt \right)^{q} \frac{w\left(\frac{1}{x}\right)}{x^2} \, dx \right)^{\frac{1}{q}} \leq C_{(6)} \left( \int_0^\infty \varphi(t) \frac{v\left(\frac{1}{t}\right)}{x}^{p-2} \, dx \right)^{\frac{1}{p}}, \quad f \in \mathcal{M} (\mathbb{R}).$$  

Let us denote by $\mathcal{M}_+(0, \infty)$ the cone of nonnegative and nonincreasing functions on $\mathbb{R}$. We claim that (16) is true if and only if

$$\left( \int_0^\infty \left( \int_0^\varphi(t) - f^*(\frac{1}{t}) \, dt \right) \frac{v\left(\frac{1}{x}\right)}{x^2} \, dx \right)^{\frac{1}{p}} \leq C_{(6)} \left( \int_0^\infty \varphi(x) \frac{v\left(\frac{1}{x}\right)}{x}^{p-2} \, dx \right)^{\frac{1}{p}}, \quad \varphi \in \mathcal{M}_+(0, \infty).$$  

Indeed, every function $t \mapsto \frac{f^{**}(\frac{1}{t}) - f^*(\frac{1}{t})}{t}$ is nonnegative and nonincreasing on $(0, \infty)$, hence (17) implies (16). On the other hand, if $\varphi \in \mathcal{M}_+(0, \infty)$ is given, by Proposition 2.4 we find $f_n \in \mathcal{M} (\mathbb{R})$ such that

$$\frac{f^{**}(\frac{1}{t}) - f^*(\frac{1}{t})}{t} \uparrow \varphi(t) \text{ for a.e. } t \in (0, \infty).$$

Since (16) holds for every $f_n$ in place of $f$, by the monotone convergence theorem we get (17) for the given $\varphi$, hence (16) implies (17).

Inequality (17) defines the embedding

$$\Lambda^p (\tilde{v}) \hookrightarrow \Gamma_u^q (\tilde{w})$$

with

$$\tilde{v}(x) := v\left(\frac{1}{x}\right)x^{p-2}, \quad \tilde{w}(x) := w\left(\frac{1}{x}\right)x^{q-2}, \quad u(x) := \frac{g^{**}\left(\frac{1}{x}\right) - g^*(\frac{1}{x})}{x}.$$  

By [8, Theorem 3.1(iii)] or a modified version of [6, Theorem 4.1(i)], (18) (as well as (17)) holds if and only if $A_{(10)} + A_{(11)} < \infty$ and the optimal $C_{(6)}$ satisfies $C_{(6)} \simeq A_{(10)} + A_{(11)}$, which is the result.
In cases (ii)–(iv) we proceed in the same way, the only difference being the conditions characterizing (18) for different settings of \( p \) and \( q \). These characterizations of (18) may be found in [8, Theorem 3.1] or, alternatively, in [6, Theorem 4.1] for (ii) and (iii) and [5, Theorem 3.1] for (iv). Note that in [5,6] the results are given just for \( u = 1 \).

\[ \square \]

\textbf{Remark 3.4.} For \( 1 \leq p < \infty \), Theorem 3.3 can be alternatively obtained using the reduction theorem [9, Theorem 2.2] and Hardy inequalities for nonnegative functions (see e.g. [12,15]).

In the case \( q = \infty \), i.e. for (7), we get

\textbf{Theorem 3.5.} Let \( p \in (0, \infty) \). Let \( v, w \) be weights, \( v \in V_p \). Let \( g \in L^1 \). Then

(i) If \( 0 < p \leq 1 \), then (7) holds if and only if

\[
A_{(19)} := \operatorname{ess sup}_{x > 0} w(x) \sup_{t > x} g^{**}(t) \left( \int_t^\infty \frac{v(s)}{s^p} \, ds \right)^{-\frac{1}{p}} < \infty. \tag{19}
\]

Moreover, the optimal constant \( C_{(7)} \) satisfies \( C_{(7)} \simeq A_{(19)} \).

(ii) If \( 1 < p < \infty \), then (7) holds if and only if

\[
A_{(20)} := \operatorname{ess sup}_{x > 0} w(x) \left[ \left( \int_x^\infty (g^{**}(t))^{p'} \left( \int_t^\infty \frac{v(s)}{s^p} \, ds \right)^{-p'} \frac{v(t)}{t^p} \, dt \right)^{\frac{1}{p'}} + g^{**}(x) \left( \int_x^\infty \frac{v(s)}{s^p} \, ds \right)^{-\frac{1}{p}} \right] < \infty. \tag{20}
\]

Moreover, the optimal constant \( C_{(7)} \) satisfies \( C_{(7)} \simeq A_{(20)} \).

\textbf{Proof.} Following the same reasoning as in the proof of Theorem 3.3, the inequality (7) is equivalent to

\[
\operatorname{ess sup}_{x > 0} \int_0^x \varphi(t) \frac{g^{**} \left( \frac{1}{t} \right) - g^* \left( \frac{1}{t} \right)}{t} \, dt \, w \left( \frac{1}{x} \right) \leq C_{(7)} \left( \int_0^\infty \varphi^p(x) v \left( \frac{1}{x} \right) x^{p-2} \, dx \right)^{\frac{1}{p}}, \quad \varphi \in M_+^+(0, \infty).
\]

Denote \( v_p(x) := v \left( \frac{1}{x} \right) x^{p-2} \). The optimal \( C_{(7)} \) satisfies

\[
C_{(7)} = \sup_{\|f\|_{L^p(v_p)} \leq 1} \operatorname{ess sup}_{x > 0} w \left( \frac{1}{x} \right) \int_0^x f^*(t) \frac{g^{**} \left( \frac{1}{t} \right) - g^* \left( \frac{1}{t} \right)}{t} \, dt \leq \operatorname{ess sup}_{x > 0} w \left( \frac{1}{x} \right) \sup_{\|f\|_{L^p(v_p)} \leq 1} \int_0^x f^*(t) \frac{g^{**} \left( \frac{1}{t} \right) - g^* \left( \frac{1}{t} \right)}{t} \, dt. \tag{21}
\]
In the following calculations, we are going to use the condition (2) without further comment.

(i) If $0 < p \leq 1$, [6, Theorem 3.1(i)] gives

$$\sup_{\|f\|_{\mathcal{L}(p)}} \int_0^x f^*(t) \frac{g^{**}(\frac{1}{x}) - g^*(\frac{1}{x})}{t} \, dt \simeq \sup_{t \in (0, x)} \int_0^t \frac{g^{**}(\frac{1}{x}) - g^*(\frac{1}{x})}{s} \, ds \left(\int_0^t v_p(s) \, ds\right)^{-\frac{1}{p}}.$$ 

Hence, we get

$$C(7) \simeq \sup_{x > 0} w \left(\frac{1}{x}\right) \sup_{t \in (0, x)} \int_0^t \frac{g^{**}(\frac{1}{x}) - g^*(\frac{1}{x})}{s} \, ds \left(\int_0^t v_p(s) \, ds\right)^{-\frac{1}{p}} \sup_{x > 0} w \left(\frac{1}{x}\right) \sup_{t \in (1, x]} g^{**}(t) \left(\int_t^\infty \frac{v(s)}{sp} \, ds\right)^{-\frac{1}{p}} = A(19).$$

(ii) If $1 < p < \infty$, by [6, Theorem 3.1(ii)] we have

$$\sup_{\|f\|_{\mathcal{L}(p)}} \int_0^x f^*(t) \frac{g^{**}(\frac{1}{x}) - g^*(\frac{1}{x})}{t} \, dt$$

$$\simeq \left(\int_0^x \left(\int_0^t \frac{g^{**}(\frac{1}{x}) - g^*(\frac{1}{x})}{s} \, ds\right)^{\frac{1}{p'}} \left(\int_0^t v_p(s) \, ds\right)^{-\frac{1}{p'}} v_p(t) \, dt\right)^{\frac{1}{p'}}$$

$$+ \left(\int_0^x \frac{g^{**}(\frac{1}{x}) - g^*(\frac{1}{x})}{s} \, ds \left(\int_x^\infty \left(\int_0^t v_p(s) \, ds\right)^{-\frac{1}{p'}} v_p(t) \, dt\right)\right)^{\frac{1}{p'}}$$

$$= \left(\int_{\frac{1}{x}}^\infty (g^{**}(t))^{\frac{1}{p'}} \left(\int_0^t \frac{v(s)}{sp} \, ds\right)^{-\frac{1}{p'}} \frac{v(t)}{t^{p'}} \, dt\right)^{\frac{1}{p'}}$$

$$+ \frac{g^{**}(\frac{1}{x})}{x} \left(\int_0^\frac{1}{x} \left(\int_0^t \frac{v(s)}{sp} \, ds\right)^{-\frac{1}{p'}} \frac{v(t)}{t^{p'}} \, dt\right)^{\frac{1}{p'}}$$

$$= \left(\int_{\frac{1}{x}}^\infty (g^{**}(t))^{\frac{1}{p'}} \left(\int_0^t \frac{v(s)}{sp} \, ds\right)^{-\frac{1}{p'}} \frac{v(t)}{t^{p'}} \, dt\right)^{\frac{1}{p'}} + g^{**}(\frac{1}{x}) \left(\int_{\frac{1}{x}}^\infty \frac{v(s)}{sp} \, ds\right)^{-\frac{1}{p'}}.$$

Hence, (21) implies $C(7) \simeq A(20)$ for the optimal $C(7)$.

For the last case, $p = \infty$, which covers the inequalities (8) and (9), we have the following theorem.
Theorem 3.6. Let \( v, w \) be weights, \( v \in \mathcal{V}_\infty \). Let \( g \in L^1 \). Then

(i) For \( 0 < q < \infty \), the inequality (8) holds and only if

\[
A_{(22)} := \left( \int_0^\infty \left( \int_0^\infty \frac{g^{**}(t) - g^*(t)}{t \operatorname{ess sup}_{s \in (t, \infty)} v(s)s^{-1}} \, dt \right)^{\frac{q}{q-1}} w(x) \, dx \right)^{\frac{1}{q}} < \infty. \tag{22}
\]

Moreover, the optimal constant \( C_{(8)} \) satisfies \( C_{(8)} \simeq A_{(22)} \).

(ii) The inequality (9) holds if and only if

\[
A_{(23)} := \operatorname{ess sup}_{x > 0} \int_x^\infty \frac{g^{**}(t) - g^*(t)}{t \operatorname{ess sup}_{s \in (t, \infty)} v(s)s^{-1}} \, dt \, w(x) < \infty. \tag{23}
\]

Moreover, the optimal constant \( C_{(9)} \) satisfies \( C_{(9)} \simeq A_{(23)} \).

Proof. Here we use the same technique as in Theorems 3.3 and 3.5. During the process we apply e.g. the result of [17, Proposition 2.7]. We omit the details. \( \square \)

Remark 3.7. In each of the particular settings of the exponents \( p, q \) in Theorem 3.3(i)–(iv), the functionals \( A_{(10)}, \ldots, A_{(15)} \) are r.i. norms of \( g \), with the following exceptions: In (iii) and (iv), if \( 0 < q < 1 \), then \( A_{(13)} \) is in general just an r.i. quasi-norm, the same applies to \( A_{(15)} \) in (iv) if \( r < 1 \). Similarly, the functionals \( A_{(19)} \) and \( A_{(20)} \) in Theorem 3.5 are r.i. norms of \( g \). For a detailed proof of this, see e.g. [11, Proposition 5.6].

In Theorem 3.6, the functional \( A_{(23)} \) acting on \( g \in L^1 \) is an r.i. norm of \( g \). The functional \( A_{(22)} \) is, in general, an r.i. quasi-norm, for \( q \geq 1 \) an r.i. norm. Let us prove the claim about \( A_{(22)} \). At first, since \( t \mapsto (\operatorname{ess sup}_{s \in (t, \infty)} v(s)s^{-1})^{-1} \) is nondecreasing, its derivative, which we denote by

\[
\delta(t) := \frac{d}{dt} \operatorname{ess sup}_{s \in (t, \infty)} v(s)s^{-1},
\]

exists and is nonnegative for a.e. \( t \in (0, \infty) \). Let \( x \in (0, \infty) \). Suppose that

\[
\int_x^\infty \frac{g^{**}(t) - g^*(t)}{t \operatorname{ess sup}_{s \in (t, \infty)} v(s)s^{-1}} \, dt < \infty.
\]

Then, by monotonicity of \( (\operatorname{ess sup}_{s \in (t, \infty)} v(s)s^{-1})^{-1} \), we have

\[
\frac{g^{**}(t)}{\operatorname{ess sup}_{s \in (t, \infty)} v(s)s^{-1}} = \frac{1}{\operatorname{ess sup}_{s \in (t, \infty)} v(s)s^{-1}} \int_t^\infty \frac{g^{**}(y) - g^*(y)}{y} \, dy \\
\leq \int_t^\infty \frac{g^{**}(y) - g^*(y)}{y \operatorname{ess sup}_{s \in (y, \infty)} v(s)s^{-1}} \, dy \xrightarrow{t \to \infty} 0.
\]
Hence, by partial integration and the previous, we get

\[
\int_{x}^{\infty} g^{**}(t) \delta(t) \, dt = \left[ \frac{g^{**}(t)}{\text{ess sup}_{s \in (t, \infty)} v(s) s^{-1}} \right]_{t=x}^{\infty} + \int_{x}^{\infty} \frac{g^{**}(t) - g^*(t)}{t \, \text{ess sup}_{s \in (t, \infty)} v(s) s^{-1}} \, dt
\]

\[
= \int_{x}^{\infty} \frac{g^{**}(t) - g^*(t)}{t \, \text{ess sup}_{s \in (t, \infty)} v(s) s^{-1}} \, dt - \frac{g^{**}(x)}{\text{ess sup}_{s \in (x, \infty)} v(s) s^{-1}} < \infty.
\]

Now assume, on the other hand, that \( \int_{x}^{\infty} g^{**}(t) \delta(t) \, dt < \infty \). Then,

\[
\int_{x}^{\infty} \frac{g^{**}(t) - g^*(t)}{t \, \text{ess sup}_{s \in (t, \infty)} v(s) s^{-1}} \, dt = \frac{g^{**}(x)}{\text{ess sup}_{s \in (x, \infty)} v(s) s^{-1}} + \int_{x}^{\infty} g^{**}(t) \delta(t) \, dt < \infty.
\]

Thus, we see that \( A_{(22)} \) is equal to

\[
\left( \int_{0}^{\infty} \left( \frac{g^{**}(x)}{\text{ess sup}_{s \in (x, \infty)} v(s) s^{-1}} + \int_{x}^{\infty} g^{**}(t) \delta(t) \, dt \right)^{\frac{q}{p}} w(x) \, dx \right)^{\frac{1}{q}}.
\]

This expression is an r.i. quasi-norm of \( g \), for \( q \geq 1 \) it is an r.i. norm. To check this, we refer again to [11].

In the same way as above, we may show that \( A_{(23)} \) is an r.i. norm.

4. Young-type convolution inequalities with the class \( S \)
on the right-hand side

In the previous part we obtained the conditions for boundedness of \( T_g \). Let us now summarize these results and apply them to get the desired convolution inequalities. Note that, in what follows, if we define \( \| \cdot \|_Y \) first, then the space \( Y \) is naturally defined as \( Y := \{ f \in \mathcal{M}(\mathbb{R}); \| f \|_Y < \infty \} \).

**Theorem 4.1.** Let \( p, q \in (0, \infty] \). Let \( v, w \) be weights, \( v \in \mathcal{V}_p \). For \( g \in L^1 \) define \( \| g \|_Y \) by what follows:

\[
\| g \|_Y := \begin{cases} 
A_{(10)} + A_{(11)} & \text{if } 1 < p \leq q < \infty; \\
A_{(10)} + A_{(12)} & \text{if } 0 < p \leq 1, \ 0 < p \leq q < \infty; \\
A_{(13)} + A_{(14)} & \text{if } 1 < p \leq \infty, \ 0 < q < p; \\
A_{(13)} + A_{(15)} & \text{if } 0 < q \leq 1; \\
A_{(19)} & \text{if } 0 < p \leq 1, \ q = \infty; \\
A_{(20)} & \text{if } 1 < p \leq \infty, \ q = \infty; \\
A_{(22)} & \text{if } p = \infty, \ 0 < q < \infty; \\
A_{(23)} & \text{if } p = q = \infty.
\end{cases}
\]

Then
(i) If \( g \in Y \), then \( T_g : \mathcal{S}^p(v) \to \Gamma^q(w) \) and
\[
\|T_g\|_{\mathcal{S}^p(v) \to \Gamma^q(w)} \lesssim \|g\|_Y.
\]
(ii) If \( g \in \text{PSD} \) and \( T_g : \mathcal{S}^p(v) \to \Gamma^q(w) \), then \( g \in Y \) and
\[
\|g\|_Y \lesssim \|T_g\|_{\mathcal{S}^p(v) \to \Gamma^q(w)}.
\]
(iii) The inequality
\[
\|f \ast g\|_{\Gamma^q(w)} \lesssim \|f\|_{\mathcal{S}^p(v)}\|g\|_Y, \quad f \in \mathcal{S}^p(v), \ g \in L^1 \cap Y,
\]
(24)
is satisfied. Moreover, if \( \tilde{Y} \) is any r.i. lattice such that (24) holds with \( \tilde{Y} \) in place of \( Y \), then \( L^1 \cap \tilde{Y} \to L^1 \cap Y \).

**Proof.** Let us prove the assertions for the case \( 1 < p \leq q < \infty \). In the other cases, the only difference is that we work with another appropriate functional \( A(\_\_\_\_\_) \).

(i) Let \( g \in Y \), thus \( A_{(10)} + A_{(11)} < \infty \). Then, by Theorem 3.3(i), the inequality (6) holds. Thus, from Lemma 3.1(i) it follows that \( T_g : \mathcal{S}^p(v) \to \Gamma^q(w) \) and \( \|T_g\|_{\mathcal{S}^p(v) \to \Gamma^q(w)} \lesssim C(6) \approx \|g\|_Y \).

(ii) Assume that \( g \in \text{PSD} \) and \( T_g : \mathcal{S}^p(v) \to \Gamma^q(w) \). By Lemma 3.2(i), inequality (6) holds and the optimal \( C(6) \) satisfies \( C(6) \lesssim \|T_g\|_{\mathcal{S}^p(v) \to \Gamma^q(w)} \). Theorem 3.1(i) now yields that \( A_{(10)} + A_{(11)} < \infty \), i.e. \( g \in Y \). Moreover, we also get \( \|g\|_Y \approx C(6) \lesssim \|T_g\|_{\mathcal{S}^p(v) \to \Gamma^q(w)} \).

(iii) The inequality (24) follows from (i) and the relation \( \|T_gf\|_{\Gamma^q(w)} \leq \|T_g\|_{\mathcal{S}^p(v) \to \Gamma^q(w)}\|f\|_{\mathcal{S}^p(v)} \). Let us prove the optimality of \( Y \). Assume that \( \tilde{Y} \) is an r.i. lattice such that
\[
\|f \ast g\|_{\Gamma^q(w)} \lesssim \|f\|_{\mathcal{S}^p(v)}\|g\|_{\tilde{Y}}, \quad f \in \mathcal{S}^p(v), \ g \in L^1 \cap \tilde{Y}.
\]
(25)
Let \( h \in L^1 \cap \tilde{Y} \). We can find a function \( g \in L^1 \cap \tilde{Y} \cap \text{PSD} \) such that \( g^* = h^* \). The inequality (25) yields that \( \|T_g\|_{\mathcal{S}^p(v) \to \Gamma^q(w)} \lesssim \|g\|_{\tilde{Y}} \). Thus, \( T_g : \mathcal{S}^p(v) \to \Gamma^q(w) \) and by (ii) it holds \( \|g\|_Y \lesssim \|T_g\|_{\mathcal{S}^p(v) \to \Gamma^q(w)} \). Together we get
\[
\|g\|_Y \lesssim \|T_g\|_{\mathcal{S}^p(v) \to \Gamma^q(w)} \lesssim \|g\|_{\tilde{Y}}.
\]
The functionals \( \|\cdot\|_Y \) and \( \|\cdot\|_{\tilde{Y}} \) are r.i., thus we obtain
\[
\|h\|_Y \lesssim \|h\|_{\tilde{Y}}.
\]
Since \( h \) was chosen arbitrarily, we got the desired embedding \( L^1 \cap \tilde{Y} \to L^1 \cap Y \).

**Remark 4.2.** For given weights \( v, w \) and exponents \( p, q \), the optimal space \( Y \) may equal \( \{0\} \). (Let us formally consider \( \{0\} \) to be an r.i. space.) In that case, the operator \( T_g \) with a nonnegative kernel \( g \) is bounded between \( \mathcal{S}^p(v) \) and \( \Gamma^q(w) \) if and only if \( g = 0 \) a.e. (cf. [11, Corollary 3.3]).
References


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